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Classification of quasihomogeneous polynomials of corank three with inner modality ≤ 14

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Abstract

For quasihomogeneous polynomials with isolated singularity, V.I.Arnold introduced the notion of inner modality and classified them with inner modality $= 0, 1$. The next works were done for inner modality $= 2, 3, 4, 5$ by E.Yoshinaga, M.Suzuki in [11], [17] and further for inner modality $= 6$ by J. Estrada Sarlabous, J. Arocha and A. Fuentes in [4]. Recently the classification is developed for inner modality $= 7, 8, 9$ in [14]. In this article we will classify quasihomogeneous polynomials of corank $= 3$ with inner modality ≤ 14.

1. Introduction

In the classification of local analytic hypersurface singularities with isolated singularity, there is a fact that singularities with low complexity are (semi-) quasihomogeneous (see [2], [3], [10]). Even if complexities of singularities increase a little, it seems that they are the majority in singularities. In this sense, it is important to classify quasihomogeneous singularities. V. I. Arnold introduced the notion of “inner modality” for (semi-) quasihomogeneous singularities and classified them with inner modality $= 0, 1$.

V. I. Arnold introduced the notion of “modality” for more generic hypersurface singularities and he classified all singularities of modality $0, 1$ and 2. He named the singularities with modality $= 0, 1$ and 2 simple one, unimodal one and bimodal one respectively. Here “modality” means the moduli of singularities in their small perturbations (see [5]). It is no doubt that this concept has very important meaning for singularities. This work of him has a great influence in various areas of mathematics. But enormous calculation is necessary for further classification of singularities by modality, and it is difficult to execute any more classification. Many of singularities (simple, unimodal and bimodal) classified are

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(semi-) quasihomogeneous. So he introduced the notion of inner modality, which can be calculate algebraically, into quasihomogeneous singularities and classified them of inner modality 0,1 actually, and find out these classification to coincide with classification by modality, and he conjectured that modality=inner modality in general. His conjecture is shown affirmatively in [12] for the one-dimensional singularities and in [15] in general.

After V. I. Arnol'd, the classification of quasihomogeneous singularities was developed to inner modality ≤ 9 by E. Yoshinaga, M. Suzuki, J. Estrada Sarlabous, J. Arocha and A. Fuentes (see [17], [11], [4] and [14]). They classified singularities by using the formula that for inner modality ≤ 9 inner modality = "arithmetic inner modality", which is a notion introduced in the next section. This formula is not always true, but fortunately, the quasihomogeneous singularities with inner modality ≤ 9 satisfied it. Moreover for quasihomogeneous polynomials f, it was shown in [14] that 9 is the upper limit of μ which satisfies "If the inner modality of $f \leq \mu$, then the inner modality of f = the arithmetic inner modality of f".

The purpose of this article is to find out the upper limit of inner modality which satisfies this formula in the case corank = 3 and to classify all the quasihomogeneous singularities of corank = 3 with inner modality less than or equal to the upper limit. As a result we find out it to be 14 and we obtain all quasihomogeneous singularities with inner modality $= 10, 11, 12, 13$ and 14.

2. Preliminaries

In this article, we will investigate a classification of "quasihomogeneous" isolated singularities defined by analytic functions of three variables. In this section we explain the terms and the results used in this article. The details of our results will be stated in §3.

A local analytic function $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$, that is $f \in \mathfrak{M} \subset \mathbb{C}\{x_1, \cdots, x_n\}$, has *an isolated singularity* if

$$\left\{ x \;\middle|\; \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \right\} = \{\mathbf{0}\}$$

locally, where $\mathfrak{M}$ is the maximal ideal of the local ring $\mathbb{C}\{x_1, \cdots, x_n\}$. It is well known that a local analytic function with isolated singularity is a polynomial up to a suitable local coordinate transformation (see [6], [16]) and hence we will study "quasihomogeneous" polynomials with isolated singularity at the origin.

For positive rational numbers $r_1, \ldots, r_n \in \mathbb{Q}^+$, a monomial

$$\mathfrak{m} = x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \ldots, x_n] \quad (i_1, \ldots, i_n \in \mathbb{N} \cup \{0\})$$

has *generalized degree* d if $r_1 i_1 + \cdots + r_n i_n = d$ and we denote the generalized degree of $\mathfrak{m}$ by gdeg($\mathfrak{m}$). A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is *quasihomogeneous of type* $(d; r_1, \ldots, r_n)$ if each monomial term of f with non-zero coefficient has generalized degree d. Then we call the number d the generalized degree of f and call r_i's the weights of f.

A local analytic function $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ has *a quasihomogeneous singularity* at the origin if f becomes a quasihomogeneous polynomial after a suitable local coordinate transformation.

Theorem 2.1 ([9]). *Suppose that* $f \in \mathfrak{M} \subset \mathbb{C}\{x_1, \ldots, x_n\}$ *has a quasihomogeneous isolated singularity. Then there exist a coordinate system* $(y_1, \ldots, y_n)$ *in which* f *has the form*

$$f = h(y_1, \ldots, y_k) + y_{k+1}^2 + \cdots + y_n^2$$

with a quasihomogeneous polynomial $h \in \mathbb{C}[y_1, \ldots, y_k]$ *of type* $(1; s_1, \ldots, s_k)$ $(0 < s_i < \frac{1}{2}, i = 1, \ldots, k)$. *The natural number* k *and* $(s_1, \ldots, s_k)$ *are uniquely determined up to permutations of components.*

We call the number k *the corank of* f and call the polynomial h *the residual part* of f. We denote the corank of f by corank(f). In order to classify them, it is sufficient to classify their residual parts.

We denote the ideal $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ of the ring $\mathbb{C}\{x_1, \ldots, x_n\}$ by $\Delta(f)$ and denote the quotient ring $\mathbb{C}\{x_1, \ldots, x_n\}/\Delta(f)$ by $\mathcal{R}_f$. Note that if f has an isolated singularity at the origin, then by Hilbert's Nullstellensatz, $\mathfrak{M}^p \subset \Delta(f) \subset \mathbb{C}\{x_1, \ldots, x_n\}$ for some $p \in \mathbb{N}$. Then the dimension of $\mathcal{R}_f$ over $\mathbb{C}$ is finite.

From now on let f be a quasihomogeneous polynomial of type $(1; r_1, \ldots, r_n)$ with isolated singularity at the origin. V.I. Arnol'd showed in [1] that the number of basis monomials of $\mathcal{R}_f$ with given generalized degree (for given $(r_1, \ldots, r_n)$) is the same for all quasihomogeneous polynomials f of the same type as follows.

Theorem 2.2 ([1], [8]). *Let* $r_1, \ldots, r_n$ *be positive rational numbers for which* $r_i = A_i/N$ $(i = 1, \ldots, n)$, *where* N *and* A_i*'s are positive natural numbers. If* f *is a quasihomogeneous polynomial of type* $(1; r_1, \ldots, r_n)$ *with isolated singularity at the origin, then*

$$\sum \mu_j z^j = \prod_{j=1}^{n} \frac{z^{N-A_j} - 1}{z^{A_j} - 1},$$

where μ_i *is the number of basis monomials in* $\mathcal{R}_f$ *with generalized degree* i/N.

We denote the right side of the equation in the above theorem by $\chi_f(z)$ and we call it *the characteristic function of* f, and when it becomes a polynomial, it is especially called *the characteristic polynomial of* f. By the above theorem we can define the following.

Definition 2.1 ([1]). The number of basis monomials of $\mathcal{R}_f$ with generalized degree ≥ 1 is called the *inner modality* of f and it is denoted by $m(f)$.

We see that the highest degree of generalized degrees of basis monomials of $\mathcal{R}_f$ is $n - 2\sum r_i$ and it is denoted by d_f. The coefficients of $\chi_f(z)$ are symmetric because it is the product of cyclotomic polynomials. Hence we have

$$m(f) = \sum_{j \geq N} \mu_j = \sum_{j \leq D-N} \mu_j,$$

where $D = nN - 2\sum A_j = N\left(n - 2\sum r_j\right) = Nd_f$. Hence $m(f)$ is the number of basis monomials of $\mathcal{R}_f$ with generalized degree $\leq d_f - 1$.

For a quasihomogeneous polynomial f of type $(1; r_1, \ldots, r_n)$, we define the following invariant expressed in terms of their weights.

Definition 2.2. We define $m_0(f)$ to be

$$\sharp\{\mathfrak{m} \mid \mathfrak{m} \text{ is a monomial in } \mathbb{C}[x_1, \ldots, x_k],\ \mathrm{gdeg}(\mathfrak{m}) \leq d_f - 1\}$$

and we call it *the arithmetic inner modality* of f, where # denotes the number of elements of a set.

By the definition, we have $m(f) \leq m_0(f)$ in general. If the images of monomials in $\mathcal{R}_f$ with gdeg $\leq d_f - 1$ are linearly independent, we have $m(f) = m_0(f)$. In [14] the following result about $m(f)$ and $m_0(f)$ is given.

Proposition 2.3. *We have $m(f) = m_0(f)$ if and only if*

$$\mathrm{gdeg}\left(\frac{\partial f}{\partial x_i}\right) > d_f - 1$$

for any i $(i = 1, \ldots, k)$.

The following theorem is a key to perform the classification of quasihomogeneous singularities in the series of articles ([4], [11], [17] and [14]).

Theorem 2.4. *For every quasihomogeneous polynomial f with isolated singularity at the origin, if $m(f) \leq 9$, then $m(f) = m_0(f)$.*

Our purpose of this article is to find out the upper limit μ of inner modality $m(f)$ which $m(f) = m_0(f)$ holds in the case corank $= 3$ and to classify all quasihomogeneous polynomials of corank $= 3$ with inner modality $\leq \mu$.

3. Main results

In this section, we state our main results. In what follows we consider quasihomogeneous polynomials with isolated singularity at the origin. The following theorem is shown in [14].

Theorem 3.1. *Let f be a quasihomogeneous polynomial. Then*

(1) *if $m(f) \leq 9$, then $m(f) = m_0(f)$,*

(2) *there exist some f such that $10 = m(f) < m_0(f) = 11$.*

The above theorem shows that 9 is the upper limit of $m(f)$ which satisfies $m(f) = m_0(f)$. Moreover in [14] all quasihomogeneous polynomials with inner modality ≤ 9 are classified. The following proposition is also shown there to prove this theorem.

Proposition 3.2. *For every quasihomogeneous polynomial f, if $m(f) \leq 9$, then* $\operatorname{corank}(f) \leq 4$.

According to the results of [14], if $\operatorname{corank}(f) = 2$, then we have always $m(f) = m_0(f)$ and if $\operatorname{corank}(f) = 4$ and $m(f) \leq 9$, then we have $m(f) = m_0(f)$ and we have some examples of quasihomogeneous polynomials f with $m(f) = 10 < m_0(f) = 11$. But in the case of $\operatorname{corank}(f) = 3$ no upper bounds are indicated there. In the case of corank $= 3$, what is the upper limit of numbers μ such that $m(f) \leq \mu$ implies $m(f) = m_0(f)$? The following is an answer.

Theorem 3.3. *Let f be a quasihomogeneous polynomial of* corank $= 3$. *Then*

(1) *if $m(f) \leq 14$, then $m(f) = m_0(f)$,*

(2) *there exist some f such that $15 = m(f) < m_0(f) = 16$.*

The above theorem shows the upper limit in the case corank $= 3$ is 14. A proof of this theorem is given in §4.

By using this theorem, we obtain the list of all quasihomogeneous polynomials of corank $= 3$ of inner modality $= 10, 11, 12, 13$ and 14. In the following theorem, the classification is done up to right-equivalence, where $f, g : (\mathbb{C}^3, O) \to (\mathbb{C}, O)$ are right-equivalent if there exist a local biholomorphism $\phi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ such that $f = g \circ \phi$.

Theorem 3.4. *In the tables below, x, y and z are variables and p, q, r, s, t, u and v are parameters.*

(1) *Residual parts of quasihomogeneous polynomials of* corank $= 3$ *with inner modality $= 10$ are equivalent to one of the following polynomials:*

Type	*Normal Form*
$\left(1; \frac{1}{11}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^3 + x^{11}$
$\left(1; \frac{1}{15}, \frac{4}{15}, \frac{7}{15}\right)$	$z^2x + y^2z + x^{15} + sx^8z + tx^{11}y$

$\left(1; \frac{3}{49}, \frac{13}{49}, \frac{23}{49}\right)$	$z^2x + y^2z + x^{12}y$
$\left(1; \frac{1}{16}, \frac{17}{64}, \frac{15}{32}\right)$	$z^2x + y^2z + x^{16}$
$\left(1; \frac{2}{21}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^3 + x^7z + tz^2y$
$\left(1; \frac{1}{31}, \frac{1}{3}, \frac{15}{31}\right)$	$z^2x + y^3 + x^{31}$
$\left(1; \frac{1}{30}, \frac{1}{3}, \frac{29}{60}\right)$	$z^2x + y^3 + x^{30} + tx^{10}y^2$
$\left(1; \frac{2}{63}, \frac{1}{3}, \frac{61}{126}\right)$	$z^2x + y^3 + x^{21}y$
$\left(1; \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$z^4 + y^4 + x^4 + pxz^2y$ $+qxy^2z + rx^2yz + sx^2z^2 + ty^2z^2 + ux^2y^2$
$\left(1; \frac{1}{9}, \frac{8}{27}, \frac{1}{3}\right)$	$z^3 + y^3x + x^9 + tx^3z^2$
$\left(1; \frac{1}{32}, \frac{1}{3}, \frac{31}{64}\right)$	$z^2x + y^3 + x^{32}$
$\left(1; \frac{1}{7}, \frac{1}{4}, \frac{1}{3}\right)$	$z^3 + y^4 + x^7$
$\left(1; \frac{1}{14}, \frac{1}{4}, \frac{13}{28}\right)$	$z^2x + y^4 + x^{14} + tx^7y^2$
$\left(1; \frac{1}{15}, \frac{1}{4}, \frac{7}{15}\right)$	$z^2x + y^4 + x^{15}$
$\left(1; \frac{3}{44}, \frac{1}{4}, \frac{41}{88}\right)$	$z^2x + y^4 + x^{11}y$
$\left(1; \frac{1}{5}, \frac{1}{4}, \frac{4}{15}\right)$	$z^3x + y^4 + x^5$
$\left(1; \frac{1}{9}, \frac{1}{5}, \frac{4}{9}\right)$	$z^2x + y^5 + x^9$
$\left(1; \frac{4}{35}, \frac{1}{5}, \frac{31}{70}\right)$	$z^2x + y^5 + x^7y$
$\left(1; \frac{1}{5}, \frac{11}{45}, \frac{4}{15}\right)$	$z^3x + y^3z + x^5$

(2) *Residual parts of quasihomogeneous polynomials of* corank $= 3$ *with inner modality* $= 11$ *are equivalent to one of the following polynomials:*

$Type$	$NormalForm$
$\left(1; \frac{1}{10}, \frac{3}{10}, \frac{7}{20}\right)$	$z^2y + y^3x + x^{10} + sx^3z^2 + tx^4y^2$
$\left(1; \frac{1}{11}, \frac{10}{33}, \frac{23}{66}\right)$	$z^2y + y^3x + x^{11}$
$\left(1; \frac{3}{53}, \frac{14}{53}, \frac{25}{53}\right)$	$z^2x + y^2z + x^{13}y$
$\left(1; \frac{1}{17}, \frac{9}{34}, \frac{8}{17}\right)$	$z^2x + y^2z + x^{17} + tx^8y^2$
$\left(1; \frac{1}{18}, \frac{19}{72}, \frac{17}{36}\right)$	$z^2x + y^2z + x^{18}$
$\left(1; \frac{4}{43}, \frac{13}{43}, \frac{15}{43}\right)$	$z^2y + y^3x + x^7z$
$\left(1; \frac{5}{41}, \frac{9}{41}, \frac{16}{41}\right)$	$z^2y + y^4x + x^5z$
$\left(1; \frac{6}{41}, \frac{7}{41}, \frac{17}{41}\right)$	$z^2y + y^5x + x^4z$
$\left(1; \frac{5}{43}, \frac{8}{43}, \frac{19}{43}\right)$	$z^2x + y^3z + x^7y$
$\left(1; \frac{1}{33}, \frac{1}{3}, \frac{16}{33}\right)$	$z^2x + y^3 + x^{33} + tx^{11}y^2$
$\left(1; \frac{1}{35}, \frac{1}{3}, \frac{17}{35}\right)$	$z^2x + y^3 + x^{35}$
$\left(1; \frac{2}{69}, \frac{1}{3}, \frac{67}{138}\right)$	$z^2x + y^3 + x^{23}y$
$\left(1; \frac{1}{6}, \frac{2}{9}, \frac{1}{3}\right)$	$z^3 + y^3z + x^6 + sx^2z^2 + tx^2y^3$
$\left(1; \frac{7}{37}, \frac{9}{37}, \frac{10}{37}\right)$	$z^3x + y^3z + x^4y$
$\left(1; \frac{1}{34}, \frac{1}{3}, \frac{33}{68}\right)$	$z^2x + y^3 + x^{34}$
$\left(1; \frac{7}{45}, \frac{2}{9}, \frac{1}{3}\right)$	$z^3 + y^3z + x^5y$

$\left(1; \frac{1}{8}, \frac{1}{4}, \frac{3}{8}\right)$	$z^2y + y^4 + x^8 + qx^2z^2 + rx^2y^3 + sx^3yz + tx^4y^2$
$\left(1; \frac{2}{11}, \frac{1}{4}, \frac{3}{11}\right)$	$z^3x + y^4 + x^4z$
$\left(1; \frac{3}{16}, \frac{1}{4}, \frac{13}{48}\right)$	$z^3x + y^4 + x^4y$
$\left(1; \frac{2}{15}, \frac{1}{4}, \frac{1}{3}\right)$	$z^3 + y^4 + x^5z$
$\left(1; \frac{1}{5}, \frac{1}{4}, \frac{1}{4}\right)$	$z^4 + y^4 + x^5 + ty^2z^2$
$\left(1; \frac{1}{6}, \frac{1}{6}, \frac{5}{12}\right)$	$z^2x + y^6 + x^6 + py^5x + qx^2y^4 + rx^3y^3 + sx^4y^2$
$\left(1; \frac{1}{7}, \frac{1}{6}, \frac{3}{7}\right)$	$z^2x + y^6 + x^7$
$\left(1; \frac{5}{36}, \frac{1}{6}, \frac{31}{72}\right)$	$z^2x + y^6 + x^6y$
$\left(1; \frac{7}{48}, \frac{1}{6}, \frac{5}{12}\right)$	$z^2y + y^6 + x^4z$
$\left(1; \frac{1}{7}, \frac{6}{35}, \frac{29}{70}\right)$	$z^2y + y^5x + x^7$
$\left(1; \frac{1}{7}, \frac{1}{6}, \frac{5}{12}\right)$	$z^2y + y^6 + x^7$
$\left(1; \frac{1}{8}, \frac{7}{32}, \frac{25}{64}\right)$	$z^2y + y^4x + x^8$
$\left(1; \frac{1}{8}, \frac{3}{16}, \frac{7}{16}\right)$	$z^2x + y^3z + x^8 + sx^2y^4 + tx^5y^2$
$\left(1; \frac{1}{9}, \frac{1}{4}, \frac{3}{8}\right)$	$z^2y + y^4 + x^9$

(3) *Residual parts of quasihomogeneous polynomials of* corank $= 3$ *with inner modality* $= 12$ *are equivalent to one of the following polynomials:*

Type	*NormalForm*
$\left(1; \frac{1}{9}, \frac{5}{27}, \frac{4}{9}\right)$	$z^2x + y^3z + x^9 + tx^4y^3$
$\left(1; \frac{1}{10}, \frac{3}{10}, \frac{1}{3}\right)$	$z^3 + y^3x + x^{10} + tx^4y^2$

$\left(1; \frac{1}{37}, \frac{1}{3}, \frac{18}{37}\right)$	$z^2x + y^3 + x^{37}$
$\left(1; \frac{1}{36}, \frac{1}{3}, \frac{35}{72}\right)$	$z^2x + y^3 + x^{36} + tx^{12}y^2$
$\left(1; \frac{2}{75}, \frac{1}{3}, \frac{73}{150}\right)$	$z^2x + y^3 + x^{25}y$
$\left(1; \frac{2}{21}, \frac{19}{63}, \frac{1}{3}\right)$	$z^3 + y^3x + x^7z$
$\left(1; \frac{1}{38}, \frac{1}{3}, \frac{37}{76}\right)$	$z^2x + y^3 + x^{38}$
$\left(1; \frac{1}{6}, \frac{5}{24}, \frac{1}{3}\right)$	$z^3 + y^4x + x^6 + tx^2z^2$
$\left(1; \frac{3}{19}, \frac{4}{19}, \frac{1}{3}\right)$	$z^3 + y^4x + x^5y$
$\left(1; \frac{1}{16}, \frac{1}{4}, \frac{15}{32}\right)$	$z^2x + y^4 + x^{16} + sx^4y^3 + tx^8y^2$
$\left(1; \frac{1}{17}, \frac{1}{4}, \frac{8}{17}\right)$	$z^2x + y^4 + x^{17}$
$\left(1; \frac{3}{52}, \frac{1}{4}, \frac{49}{104}\right)$	$z^2x + y^4 + x^{13}y$
$\left(1; \frac{5}{48}, \frac{1}{4}, \frac{3}{8}\right)$	$z^2y + y^4 + x^6z$
$\left(1; \frac{1}{8}, \frac{1}{5}, \frac{2}{5}\right)$	$z^2y + y^5 + x^8$

(4) *Residual parts of quasihomogeneous polynomials of* corank $= 3$ *with inner modality* $= 13$ *are equivalent to one of the following polynomials:*

$Type$	$NormalForm$
$\left(1; \frac{1}{10}, \frac{11}{60}, \frac{9}{20}\right)$	$z^2x + y^3z + x^{10}$
$\left(1; \frac{1}{10}, \frac{1}{4}, \frac{3}{8}\right)$	$z^2y + y^4 + x^{10} + tx^5y^2$
$\left(1; \frac{1}{12}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^3 + x^{12} + ryz^2 + sx^4z^2 + tx^4y^2$
$\left(1; \frac{1}{12}, \frac{11}{36}, \frac{25}{72}\right)$	$z^2y + y^3x + x^{12}$

$\left(1; \frac{1}{13}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^3 + x^{13}$
$\left(1; \frac{1}{19}, \frac{5}{19}, \frac{9}{19}\right)$	$z^2x + y^2z + x^{19} + sx^5yz + tx^9y^2$
$\left(1; \frac{3}{61}, \frac{16}{61}, \frac{29}{61}\right)$	$z^2x + y^2z + x^{15}y$
$\left(1; \frac{3}{25}, \frac{1}{5}, \frac{2}{5}\right)$	$z^2y + y^5 + x^5z + tx^5y^2$
$\left(1; \frac{4}{49}, \frac{15}{49}, \frac{17}{49}\right)$	$z^2y + y^3x + x^8z$
$\left(1; \frac{5}{49}, \frac{9}{49}, \frac{22}{49}\right)$	$z^2x + y^3z + x^8y$
$\left(1; \frac{1}{20}, \frac{21}{80}, \frac{19}{40}\right)$	$z^2x + y^2z + x^{20}$
$\left(1; \frac{1}{11}, \frac{10}{33}, \frac{1}{3}\right)$	$z^3 + y^3x + x^{11} + tx^4yz$
$\left(1; \frac{1}{39}, \frac{1}{3}, \frac{19}{39}\right)$	$z^2x + y^3 + x^{39} + tx^{13}y^2$
$\left(1; \frac{1}{41}, \frac{1}{3}, \frac{20}{41}\right)$	$z^2x + y^3 + x^{41}$
$\left(1; \frac{2}{81}, \frac{1}{3}, \frac{79}{162}\right)$	$z^2x + y^3 + x^{27}y$
$\left(1; \frac{3}{16}, \frac{1}{4}, \frac{1}{4}\right)$	$z^4 + y^4 + x^4z + sy^2z^2 + ty^3z$
$\left(1; \frac{2}{23}, \frac{7}{23}, \frac{1}{3}\right)$	$z^3 + y^3x + x^8y$
$\left(1; \frac{1}{8}, \frac{1}{4}, \frac{1}{3}\right)$	$z^3 + y^4 + x^8 + sx^2y^3 + tx^4y^2$
$\left(1; \frac{1}{40}, \frac{1}{3}, \frac{39}{80}\right)$	$z^2x + y^3 + x^{40}$
$\left(1; \frac{4}{25}, \frac{1}{5}, \frac{1}{3}\right)$	$z^3 + y^5 + x^5y$
$\left(1; \frac{1}{6}, \frac{1}{5}, \frac{1}{3}\right)$	$z^3 + y^5 + x^6 + tx^2z^2$
$\left(1; \frac{1}{19}, \frac{1}{4}, \frac{9}{19}\right)$	$z^2x + y^4 + x^{19}$

$\left(1; \frac{1}{18}, \frac{1}{4}, \frac{17}{36}\right)$	$z^2x + y^4 + x^{18} + tx^9y^2$
$\left(1; \frac{3}{56}, \frac{1}{4}, \frac{53}{112}\right)$	$z^2x + y^4 + x^{14}y$
$\left(1; \frac{1}{6}, \frac{1}{4}, \frac{5}{18}\right)$	$z^3x + y^4 + x^6 + tx^3y^2$
$\left(1; \frac{1}{10}, \frac{1}{5}, \frac{9}{20}\right)$	$z^2x + y^5 + x^{10} + rx^2y^4 + sx^4y^3 + tx^6y^2$
$\left(1; \frac{1}{11}, \frac{1}{5}, \frac{5}{11}\right)$	$z^2x + y^5 + x^{11}$
$\left(1; \frac{4}{45}, \frac{1}{5}, \frac{41}{90}\right)$	$z^2x + y^5 + x^9y$
$\left(1; \frac{5}{42}, \frac{1}{6}, \frac{37}{84}\right)$	$z^2x + y^6 + x^7y$
$\left(1; \frac{1}{6}, \frac{13}{54}, \frac{5}{18}\right)$	$z^3x + y^3z + x^6$
$\left(1; \frac{1}{8}, \frac{1}{6}, \frac{7}{16}\right)$	$z^2x + y^6 + x^8 + tx^4y^3$
$\left(1; \frac{1}{7}, \frac{2}{9}, \frac{1}{3}\right)$	$z^3 + y^3z + x^7$

(5) *Residual parts of quasihomogeneous polynomials of* corank $= 3$ *with inner modality* $= 14$ *are equivalent to one of the following polynomials:*

Type	*NormalForm*
$\left(1; \frac{1}{10}, \frac{9}{40}, \frac{31}{80}\right)$	$z^2y + y^4x + x^{10}$
$\left(1; \frac{1}{14}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^2z + x^{14}$
$\left(1; \frac{3}{65}, \frac{17}{65}, \frac{31}{65}\right)$	$z^2x + y^2z + x^{16}y$
$\left(1; \frac{1}{21}, \frac{11}{42}, \frac{10}{21}\right)$	$z^2x + y^2z + x^{21} + tx^{10}y^2$
$\left(1; \frac{2}{27}, \frac{1}{3}, \frac{1}{3}\right)$	$z^3 + y^3 + x^9z + tz^2y$
$\left(1; \frac{5}{49}, \frac{11}{49}, \frac{19}{49}\right)$	$z^2y + y^4x + x^6z$

$\left(1;\dfrac{1}{22},\dfrac{23}{88},\dfrac{21}{44}\right)$	$z^2x+y^2z+x^{22}$
$\left(1;\dfrac{1}{12},\dfrac{11}{36},\dfrac{1}{3}\right)$	$z^3+y^3x+x^{12}+tx^4z^2$
$\left(1;\dfrac{1}{43},\dfrac{1}{3},\dfrac{21}{43}\right)$	$z^2x+y^3+x^{43}$
$\left(1;\dfrac{1}{42},\dfrac{1}{3},\dfrac{41}{84}\right)$	$z^2x+y^3+x^{42}+tx^{14}y^2$
$\left(1;\dfrac{2}{87},\dfrac{1}{3},\dfrac{85}{174}\right)$	$z^2x+y^3+x^{29}y$
$\left(1;\dfrac{1}{44},\dfrac{1}{3},\dfrac{43}{88}\right)$	$z^2x+y^3+x^{44}$
$\left(1;\dfrac{1}{7},\dfrac{3}{14},\dfrac{1}{3}\right)$	$z^3+y^4x+x^7+tx^4y^2$
$\left(1;\dfrac{3}{20},\dfrac{1}{4},\dfrac{17}{60}\right)$	$z^3x+y^4+x^5y$
$\left(1;\dfrac{1}{9},\dfrac{2}{9},\dfrac{7}{18}\right)$	$z^2y+y^4x+x^9+rx^2z^2+sx^3y^3+tx^5y^2$
$\left(1;\dfrac{1}{12},\dfrac{1}{5},\dfrac{11}{24}\right)$	$z^2x+y^5+x^{12}$
$\left(1;\dfrac{1}{8},\dfrac{7}{40},\dfrac{33}{80}\right)$	$z^2y+y^5x+x^8$
$\left(1;\dfrac{1}{9},\dfrac{1}{5},\dfrac{2}{5}\right)$	$z^2y+y^5+x^9$

In the above tables, for each normal form $f_{\mathbf{t}}$ with parameters there exist some proper algebraic subset Δ of $\mathbb{C}^\sigma$ ($\sigma = 1,2,3,4,5$) such that $f_{\mathbf{t}}$ has an isolated singularity at the origin for any $\mathbf{t} \in \mathbb{C}^\sigma - \Delta$.

The existence of Δ in the above theorem is guaranteed by Proposition 2.2 in [11] (see also [7]). A proof of this theorem is given in the following section.

4. Proofs of Theorems

4.1 Estimation of exponents

In this subsection we consider quasihomogeneous polynomials of corank $= 3$ with isolated singularity at the origin. Quasihomogeneous polynomials of three variables are covered by seven class (see Proposition 11.1 in [1] and [9]).

Proposition 4.1. *For every quasihomogeneous polynomial of corank = 3, its residual part contains at least one of the seven systems of monomials with non-zero coefficients in the following table with a suitable numbering of variables.*

Class	*monomials*	r_1	r_2	r_3	
I	x^a, y^b, z^c	$\frac{1}{a}$	$\frac{1}{b}$	$\frac{1}{c}$	$(a, b, c \geq 3)$
II	x^a, y^b, yz^c	$\frac{1}{a}$	$\frac{1}{b}$	$\frac{b-1}{bc}$	$(a, b \geq 3, c \geq 2)$
III	$x^a, y^b x, z^c x$	$\frac{1}{a}$	$\frac{a-1}{ab}$	$\frac{a-1}{ac}$	$(a \geq 3, b, c \geq 2)$
IV	$x^a, y^b z, z^c y$	$\frac{1}{a}$	$\frac{c-1}{bc-1}$	$\frac{b-1}{bc-1}$	$(a \geq 3, b, c \geq 2)$
V	$x^a, y^b z, z^c x$	$\frac{1}{a}$	$\frac{ac-a+1}{abc}$	$\frac{a-1}{ac}$	$(a \geq 3, b, c \geq 2)$
VI	$x^a y, y^b x, z^c x$	$\frac{b-1}{ab-1}$	$\frac{a-1}{ab-1}$	$\frac{(a-1)b}{(ab-1)c}$	$(a, b, c \geq 2)$
VII	$x^a y, y^b z, z^c x$	$\frac{bc-c+1}{abc+1}$	$\frac{ac-a+1}{abc+1}$	$\frac{ab-b+1}{abc+1}$	$(a, b, c \geq 2)$

In order to classify quasihomogeneous polynomials f of corank $= 3$ with $m(f) \leq 14$, we have to know upper bounds of the exponents a, b, c of each set of monomials in Proposition 4.1 which f contains with non-zero coefficients.

Lemma 4.2. *For every quasihomogeneous polynomial f, if $m(f) \leq 14$, then we have $a, b, c \leq 195$ for the exponents a, b, c in Proposition 4.1.*

Proof. Let f be quasihomogeneous of type $(1; r_1, r_2, r_3)$ and let $r_{min} = \min\{r_1, r_2, r_3\}$. We divide the proof into the following two cases.

Case 1. $14r_{min} > d_f - 1$.

Let f be one of seven classes in Proposition 4.1. Then note that $r_1 \leq \frac{1}{a}$, $r_2 \leq \frac{1}{b}$, $r_3 \leq \frac{1}{c}$ for any class.

First we consider the case where two of a, b and c are greater than or equal to 3. Suppose that $r_1 = r_{min}$. Then by the assumption, we have $14r_1 > d_f - 1 = 2 - 2(r_1 + r_2 + r_3)$. Hence for any k $(k = 1, 2, 3)$

$$16r_k \geq 16r_1 > 2 - 2(r_2 + r_3) \geq 2 - 2\left(\frac{1}{b} + \frac{1}{c}\right) \geq 2 - 2\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3}.$$

Hence $\frac{16}{a}, \frac{16}{b}, \frac{16}{c} > \frac{1}{3}$ and thus $a, b, c < 48$. Also in the case where r_2 or r_3 is the minimum of r_i's $(i = 1, 2, 3)$, we obtain the same result similarly.

Next we consider the case where two of a, b and c are equal to 2. In this case the class of f must be III, IV, V,VI or VII.

In the case where the class of f is III, we obtain $b = c = 2$ because $a \geq 3$. Hence $\frac{1}{a} = r_1 \leq r_2 = r_3 = \frac{a-1}{2a}$. Suppose that $r_1 < r_2 = r_3$. Then for any non negative integers m, n $(m+n \geq 3)$

$$mr_2 + nr_3 = (m+n)r_2 > 2r_2 + r_1 = 1.$$

It means that every term of f contains the variable x and f is reducible. But it contradicts the assumption that f has an isolated singularity at the origin. Thus $r_1 = r_2 = r_3 = \frac{1}{3}$ from $2r_2 + r_1 = 2r_3 + r_1 = 1$. Hence we have $a = 3$.

In the case where the class of f is IV, we obtain $b = c = 2$ because $a \geq 3$. Hence $\frac{1}{a} = r_1 \leq r_2 = r_3 = \frac{1}{3}$. Then by the assumption $14r_1 > 2 - 2(r_1 + r_2 + r_3)$ we have

$$\frac{16}{a} \geq 16r_1 \geq 2 - 2(r_2 + r_3) \geq 2 - 2\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}.$$

Hence $a < 24$.

In the case where the class of f is V, we obtain $b = c = 2$ because $a \geq 3$ and thus $\frac{1}{a} = r_1 \leq r_2 = \frac{a+1}{4a} \leq r_3 = \frac{a-1}{2a}$. Then $14r_1 > 2 - 2(r_1 + r_2 + r_3)$. Hence

$$\frac{16}{a} \geq 16r_1 > 2 - 2(r_2 + r_3) = 2 - 2 \times \frac{3a-1}{4a} > 2 - 2 \times \frac{3}{4} = \frac{1}{2}.$$

Thus $a < 32$.

In the case where the class of f is VI, first we consider the case $a = b = 2$. Then $\frac{2}{3c} = r_3 \leq r_1 = r_2 = \frac{1}{3}$ and $14r_3 > 2 - 2(r_1 + r_2 + r_3)$. Hence

$$\frac{32}{3c} = 16r_3 > 2 - 2(r_1 + r_2) = 2 - 2 \times \frac{2}{3} = \frac{2}{3}.$$

It follows that $c < 16$. Next we consider the case $b = c = 2$ and then $\frac{1}{2a-1} = r_1 \leq r_2 = r_3 = \frac{a-1}{2a-1}$. If $r_1 < r_2 = r_3$, then for any non negative integer m, n $(m+n \geq 3)$

$$mr_2 + nr_3 = (m+n)r_2 > 2r_2 + r_1 = 1.$$

It means that every term of f contains the variable x and f is reducible. But it contradicts the assumption that f has an isolated singularity at the origin. Thus $r_1 = r_2 = r_3 = \frac{1}{3}$ because $2r_2 + r_1 = 2r_3 + r_1 = 1$. Hence $a = 3$. Finally we consider the case $c = a = 2$ and then $\frac{1}{2b-1} = r_2 \leq r_3 = \frac{b}{2(2b-1)} \leq r_1 = \frac{b-1}{2b-1}$. Then $14r_2 > 2 - 2(r_1 + r_2 + r_3)$ and thus

$$\frac{16}{b} \geq 16r_2 > 2 - 2(r_1 + r_3) = 2 - 2 \times \frac{3b-2}{4b-2} > 2 - 2 \times \frac{3}{4} = \frac{1}{2}.$$

Hence $b < 32$.

In the case where the class of f is VII, because of the symmetry of x, y, z, it is enough to consider the case $b = c = 2$. If $a \geq 3$, $\frac{3}{4a+1} = r_1 \leq r_2 = \frac{a+1}{4a+1} \leq r_3 = \frac{2a-1}{4a+1}$. Then $14r_1 > 2 - 2(r_1 + r_2 + r_3)$ and thus

$$\frac{16}{a} \geq 16r_1 > 2 - 2(r_2 + r_3) = 2 - 2 \times \frac{3a}{4a+1} > 2 - 2 \times \frac{3}{4} = \frac{1}{2}.$$

Hence $a < 32$.

Therefore we have $a, b, c \leq 32$ for any class.

Next we consider the other case.

Case 2. $14r_{min} \leq d_f - 1$.

First we consider the case where $r_{min} = r_1 \leq r_2 \leq r_3$. Then we have

$$\mathrm{gdeg}(1),\ \mathrm{gdeg}(x_1),\ \mathrm{gdeg}(x_1^2), \ldots,\ \mathrm{gdeg}(x_1^{14}) \leq d_f - 1.$$

Hence if $\mathcal{M} := \{1, x_1, x_1^2, \ldots, x_1^{14}\}$ is linearly independent over $\mathbb{C}$ in $\mathcal{R}_f$, then $m(f) \geq 15$. But it contradicts the hypothesis and thus $\mathcal{M}$ is linearly dependent. Then there exist some scalars $\lambda_0, \ldots, \lambda_{14}$, not all zero, such that $\lambda_0 \cdot 1 + \lambda_1 x_1 + \lambda_2 x_1^2 + \cdots + \lambda_{14} x_1^{14} \in \Delta(f)$, Since

$$\mathrm{gdeg}\left(\frac{\partial f}{\partial x_3}\right) = 1 - r_3 > \frac{1}{2} > r_1 = \mathrm{gdeg}(x_1)$$

we have $\lambda_0 = \lambda_1 = 0$. If $\lambda_2, \ldots, \lambda_{13}$ or λ_{14} are not zero, then f contains the monomial $x_1^l x_i$ with non-zero coefficient for some integer l $(2 \leq l \leq 14)$ and some i $(i = 1, 2, 3)$. Thus $lr_1 + r_i = 1$ and

$$(l+1)r_3 \geq lr_1 + r_i = 1, \quad r_3 \geq \frac{1}{l+1}.$$

On the other hand, by Proposition 4.1 f contains the monomial $x_3^m x_j$ with non-zero coefficient for some integer m $(2 \leq m)$ and some j $(j = 1, 2, 3)$. Thus we have $mr_3 + r_j = 1$. Then

$$lr_1 = 1 - r_i \geq 1 - r_3 = (m-1)r_3 + r_j \geq (m-1)r_3 + r_1.$$

Thus for any p $(p = 1, 2, 3)$

$$(l-1)r_p \geq (l-1)r_1 \geq (m-1)r_3 \geq \frac{m-1}{l+1}.$$

Therefore we have

$$a, b, c \leq \frac{(l-1)(l+1)}{m-1} \leq \frac{(14-1)(14+1)}{2-1} = 195.$$

Similarly we obtain the same result regardless of the order of the weights r_1, r_2, r_3. This completes the proof of the lemma. □

4.2 Proofs of Theorem 3.3 (1) and Theorem 3.4

In this subsection, we give proofs of Theorem 3.3 (1) and Theorem 3.4 by using a computer. Before proofs of them we prepare the following propositions in order to increase the efficiency of calculations by a computer.

Let f be a quasihomogeneous polynomial of type $(1; r_1, r_2, r_3)$ $(0 < r_1 \le r_2 \le r_3 < \frac{1}{2})$ with isolated singularity at the origin.

Proposition 4.3. *For any set $\mathcal{M}$ of monomials in $\mathbb{C}[x_1, x_2, x_3]$,*

$$m(f) \ge \sharp\{\mathfrak{m} |\, \mathrm{gdeg}(\mathfrak{m}) \le d_f - 1, \mathrm{gdeg}(\mathfrak{m}) < 1 - r_3, \mathfrak{m} \in \mathcal{M}\}.$$

Proof. It follows easily that the set

$$\{\mathfrak{m} |\, \mathrm{gdeg}(\mathfrak{m}) \le d_f - 1, \mathrm{gdeg}(\mathfrak{m}) < 1 - r_3, \mathfrak{m} \in \mathcal{M}\}$$

is linearly independent in $\mathcal{R}_f$ because

$$1 - r_3 = \mathrm{gdeg}\left(\frac{\partial f}{\partial x_3}\right) \le \mathrm{gdeg}\left(\frac{\partial f}{\partial x_2}\right) \le \mathrm{gdeg}\left(\frac{\partial f}{\partial x_1}\right).$$

Hence we have the conclusion. $\square$

In order to calculate the arithmetic inner modality $m_0(f)$ (≤ 14) in the case corank $= 3$ by using a computer we need to find out a finite subset $\mathcal{N}$ of monomials in $\mathbb{C}[x_1, x_2, x_3]$ of a suitable size which satisfies

$$\sharp\{\mathfrak{m} |\, \mathrm{gdeg}(\mathfrak{m}) \le d_f - 1 \quad \mathfrak{m} \in \mathcal{N}\} \begin{cases} = & m_0(f) \quad (m_0(f) \le 14) \\ > & 14 \quad\quad (otherwise). \end{cases}$$

For this purpose the following diagram of magnitude correlation of generalized degrees of monomials is helpful.

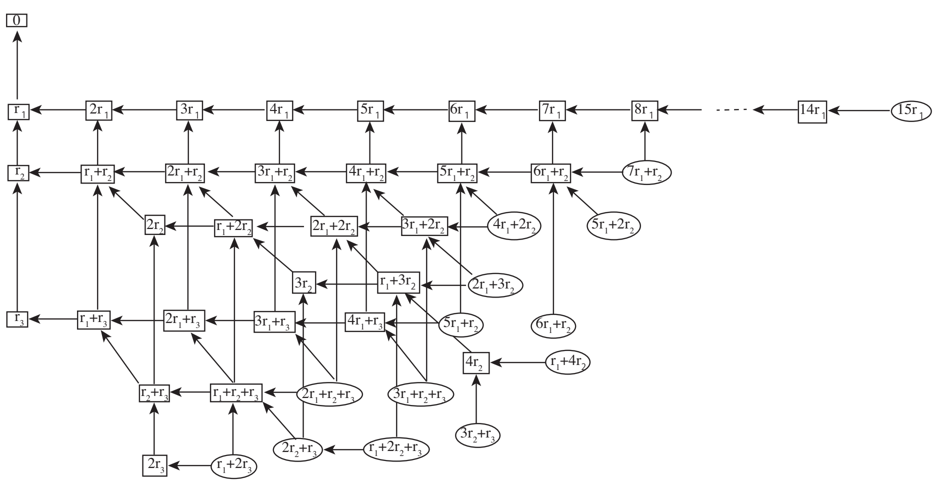


Figure 1

where $a \longleftarrow b$ means $a \leq b$. Let $\mathcal{N}_3$ be the sets of the monomials of variables "x_1, x_2, x_3" which have the generalized degrees surrounded by boxes in Figure 1 and let $\mathcal{B}_3$ be the sets of the monomials of variables "x_1, x_2, x_3" which have the generalized degrees surrounded by ovals in Figure 1. Then $\mathcal{N}_3$ is the subset $\mathcal{N}$ we are looking for.

Proposition 4.4. *We have*

$$\sharp\{\mathfrak{m} | \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1, \mathfrak{m} \in \mathcal{N}_3\} \begin{cases} = & m_0(f) \quad (m_0(f) \leq 14) \\ > & 14 \quad (\textit{otherwise}). \end{cases}$$

Proof. First, note that if all the monomials $\mathfrak{m} \in \mathbb{C}[x_1, x_2, x_3]$ with $\operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1$ belong to $\mathcal{N}_3$, then we have

$$\sharp\{\mathfrak{m} | \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1, \mathfrak{m} \in \mathcal{N}_3\} = m_0(f).$$

Now suppose that there exists some monomial $\mathfrak{m} \in \mathbb{C}[x_1, x_2, x_3]$ such that $\operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1$ and $\mathfrak{m} \notin \mathcal{N}_3$. Then from Figure 1 it follows that there exist monomials $1, \mathfrak{m}_1, \ldots, \mathfrak{m}_p, \mathfrak{n}$ $(p \geq 14)$ $(1, \mathfrak{m}_1, \ldots, \mathfrak{m}_p \in \mathcal{N}_3,\ \mathfrak{n} \in \mathcal{B}_3)$ such that

$$\operatorname{gdeg}(1),\ \operatorname{gdeg}(\mathfrak{m}_1), \cdots,\ \operatorname{gdeg}(\mathfrak{m}_p) \leq \operatorname{gdeg}(\mathfrak{n}) \leq \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1.$$

Thus we have

$$m_0(f) \geq \sharp\{\mathfrak{m} | \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1, \mathfrak{m} \in \mathcal{N}_3\} > 14.$$

Hence if $m_0(f) \leq 14$, there is no such $\mathfrak{m}$ and we have

$$\sharp\{\mathfrak{m} | \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1, \mathfrak{m} \in \mathcal{N}_3\} = m_0(f).$$

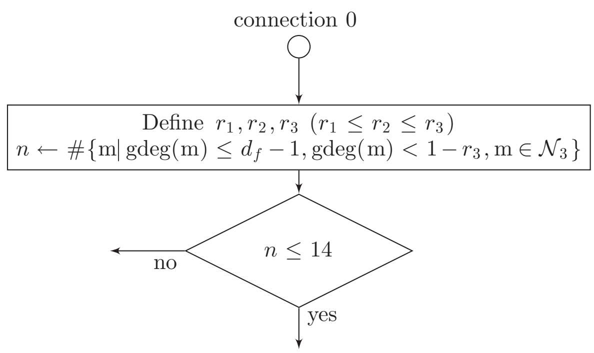
And if $m_0(f) > 14$, regardless of existence or nonexistence of such $\mathfrak{m}$, from the above arguments, we have

$$\sharp\{\mathfrak{m} | \operatorname{gdeg}(\mathfrak{m}) \leq d_f - 1, \mathfrak{m} \in \mathcal{N}_3\} > 14.$$

This completes the proof. □

A proof of Theorem 3.3 (1) and Theorem 3.4.

We classify quasihomogeneous polynomials of corank = 3 with inner modality = 10, 11, 12, 13 and 14. By calculation with a computer we prove that if $m(f) \leq 14$ then $m(f) = m_0(f)$. And also by calculation we perform the classification of quasihomogeneous polynomials of corank = 3 with inner modality = 10, 11, 12, 13 and 14. More precisely the classification is performed as follows. We know that quasihomogenous polynomials f of three variables contain at least one of the seven sets of monomials with non-zero coefficients in the table of Proposition 4.1. If $m(f) \leq 14$, then by Lemma 4.2 we also know upper bounds of exponents a, b, c of such monomials. For any a, b, c within the range, if $m(f) \leq 14$, by calculation with a computer, we can verify that $m(f) = m_0(f)$ without exception. And also by calculation, we can obtain all quasihomogeneous polynomials of corank $= 3$ with $m(f) = 10, 11, 12, 13$ and 14. We draw a procedure to perform these calculations in two flowcharts Figure 2 and Figure 3 below (see also Appendix in the last section for an example code). In the main process of the flowchart (Figure 3), the following decision procedure based on Proposition 4.4 plays a very important role.



We have to calculate $195^3 = 7414875$ cases to perform our purpose, which it is a very heavy job even for a computer. Especially it is difficult to calculate $\chi_f(z)$ for a large data within a reasonable time. However by the above procedure necessary calculations are reduced to less than 100 cases from 7414875 cases for each class in Proposition 4.1.

The part of "Main Process" in the above flowchart is as follows. Here we should note that Theorem 2.2 doesn't guarantee that $\chi_f(z)$ is a polynomial "only if" f has an isolated singularity at the origin. Since we just merely check whether $\chi_f(z)$ is a polynomial in the flowchart above mentioned, it is not guaranteed that f obtained as a result of calculation has an isolated singularity at the origin. Hence it is necessary to remove quasihomogeneous polynomials which have non-isolated singularities around the origin from the calculation result. But we see that all the quasihomogeneous polynomials which we get as a result of the calculations have an isolated singularity at the origin. Hence we have all the weights of quasihomogeneous polynomials of corank $= 3$ with inner modality $= 10, 11, 12, 13$, and 14. According to the weights we determine quasihomogeneous polynomials up to right-equivalence and we have the table in Theorem 3.4. This completes the proof of Theorem 3.3 (1) and Theorem 3.4. $\square$

4.3 Calculation example of a quasihomogeneous polynomial of one type

We explain a process to determine a polynomial of one type from a calculation results of a computer by an example. In the case $\mathrm{corank}(f) = 3$ and $m_0(f) = 13$,

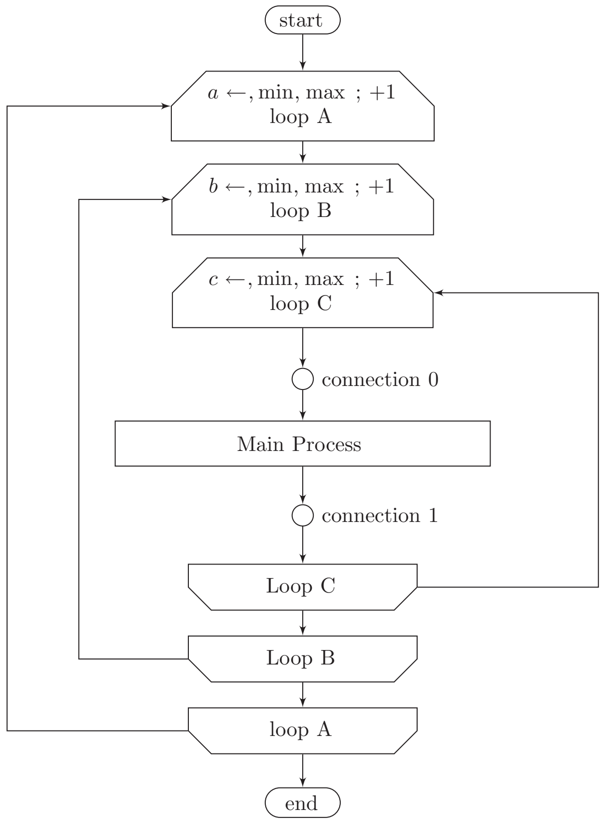


Figure 2 Flowchart

we have the type $\left(1; \frac{1}{12}, \frac{1}{3}, \frac{1}{3}\right)$ as the calculation result. For this type, there are 10 monomials with generalized degree $= 1$:

$$z^3,\ y^3,\ x^{12},\ z^2y,\ x^4z^2,\ x^4y^2,\ y^2z,\ x^4yz,\ x^8z,\ x^8y.$$

Hence the quasihomogeneous polynomial of this type is given by the following:

$$\begin{aligned} f(x,y,z) \ =\ & Az^3 + By^3 + Cx^{12} + Dz^2y + Ex^4z^2 + Fx^4y^2 + Gy^2z \\ & + Hx^4yz + Ix^8z + Jx^8y. \end{aligned}$$

Then by the local coordinate transformation

$$x = ax, \quad y = by + cz + dx^4, \quad z = ez + fy + gx^4 \quad abe \neq 0,$$

we can reduce f to $A = B = C = 1$, $G = H = I = J = 0$ and as a quasihomogeneous polynomial corresponding to the type $\left(1; \frac{1}{12}, \frac{1}{3}, \frac{1}{3}\right)$ we obtain

$$f_{r,s,t}(x,y,z) = z^3 + y^3 + x^{12} + rz^2y + sx^4z^2 + tx^4y^2.$$

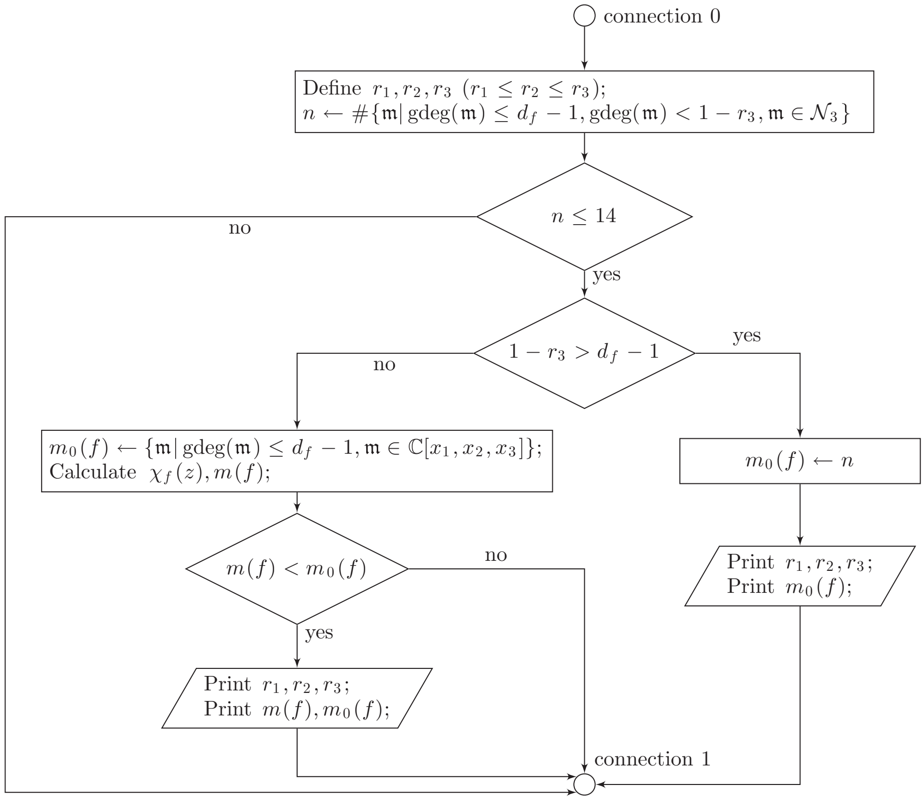


Figure 3 Flowchart for Main Process

When $r = s = t = 0$, we can see easily that $f_{0,0,0} = z^3 + y^3 + x^{12}$ has an isolated singularity at the origin. Thus $f_{r,s,t}$ is a normal form of the type $\left(1; \frac{1}{12}, \frac{1}{3}, \frac{1}{3}\right)$ with inner modality $= 13$. From Proposition 2.2 in [11] (see also [7]), we know the existence of a proper algebraic subvariety Δ of $\mathbb{C}^3$ such that $f_{r,s,t}$ has an isolated singularity at the origin for any $(r, s, t) \in \mathbb{C}^3 - \Delta$. Here we will try to find Δ concretely. The algebraic variety Δ of $\mathbb{C}^3$ is the set of (r, s, t) for which the equations of x, y, z

$$\begin{aligned}
\frac{\partial f_{r,s,t}}{\partial x} &= 12x^{11} + 4tx^3y^2 + 4sx^3z^2 = 0, \\
\frac{\partial f_{r,s,t}}{\partial y} &= 2tx^4y + 3y^2 + rz^2 = 0, \\
\frac{\partial f_{r,s,t}}{\partial z} &= 2sx^4z + 2ryz + 3z^2 = 0
\end{aligned}$$

have non-trivial solutions. If $z = 0$, then $x = y = 0$ and thus we may suppose $z \neq 0$. From the equation $\frac{\partial f_{r,s,t}}{\partial z} = 0$, we have $z = -\frac{2}{3}(sx^4 + ry)$ and by substituting it for the first and the second 2 equations, we have

$$\begin{aligned}\frac{\partial f_{r,s,t}}{\partial x} &= 12x^{11} + 4tx^3y^2 + \frac{16}{9}sx^3(sx^4+ry)^2 = 0,\\ \frac{\partial f_{r,s,t}}{\partial y} &= 2tx^4y + 3y^2 + \frac{4}{9}r(sx^4+ry)^2 = 0.\end{aligned}$$

Now we consider the branched covering $\rho : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $\rho(x,y) = (x, y^4)$. Then

$$\begin{aligned}\frac{\partial f_{r,s,t}}{\partial x} \circ \rho(x,y) &= \left(12 + \frac{16}{9}s^3\right)x^{11} + \frac{32}{9}rs^2x^7y^4 + \left(\frac{16}{9}r^2s + 4t\right)x^3y^8 = 0,\\ \frac{\partial f_{r,s,t}}{\partial y} \circ \rho(x,y) &= \frac{4}{9}rs^2x^8 + \left(\frac{8}{9}r^2s + 2t\right)x^4y^4 + \left(3 + \frac{4}{9}r^3\right)y^8 = 0.\end{aligned}$$

If $x = 0$ then we have $y = z = 0$ and we may assume that $x \neq 0$. We have the following equations by dividing the both sides of two equations by x^{11}, x^8 respectively.

$$\begin{aligned}\left(12 + \frac{16}{9}s^3\right) + \frac{32}{9}rs^2\left(\frac{y}{x}\right)^4 + \left(\frac{16}{9}r^2s + 4t\right)\left(\frac{y}{x}\right)^8 &= 0,\\ \frac{4}{9}rs^2 + \left(\frac{8}{9}r^2s + 2t\right)\left(\frac{y}{x}\right)^4 + \left(3 + \frac{4}{9}r^3\right)\left(\frac{y}{x}\right)^8 &= 0.\end{aligned}$$

Let h, g be the left-hand side polynomials of $\left(\frac{y}{x}\right)$ in the above equations respectively. Then the condition of r, s, t for these equations of $\frac{y}{x}$ to have common solutions is given by the resultant of h, g as follows.

$$\begin{aligned}&\mathrm{Res}(h,g) =\\ &\frac{1}{1853020188851841}(8503056 + 2519424r^3 + 186624r^6 + 2519424s^3 -\\ &373248r^3s^3 + 186624s^6 - 2519424rs^2t + 373248r^4s^2t - 373248rs^5t +\\ &\qquad 1679616r^2st^2 + 186624r^2s^4t^2 + 1259712t^3 + 186624s^3t^3)^4 = 0\end{aligned}$$

and it is the defining function of Δ.

In this way we obtain the tables of the quasihomogeneous polynomials in Theorem 3.3. This completes the proof of Theorem 3.3.

4.4 Proof of Theorem 3.3 (2)

A proof of Theorem 3.3 (2) is given by the following examples (see [13]).

Example 4.5. *The first example is* $f(x,y,z) = yz^2 + y^6 + x^8$. *Then* f *is quasihomogeneous of type* $\left(1; \frac{1}{8}, \frac{1}{6}, \frac{5}{12}\right)$ *with isolated singularity at the origin and* $m(f) = 15$, $m_0(f) = 16$.

Note that $d_f - 1 = \frac{7}{12}$ and $\left(1; \frac{1}{8}, \frac{1}{6}, \frac{5}{12}\right) = \left(1; \frac{3}{24}, \frac{4}{24}, \frac{10}{24}\right)$. Since $1 - r_3 = \frac{7}{12} = d_f - 1$ we have $m(f) < m_0(f)$ by Proposition 2.3. We will calculate $m_0(f)$, $m(f)$ actually.

In order to find monomials of x, y and z which has generalized degree $\leq d_f - 1$, we use the following diagram which shows magnitude correlation of the generalized degrees of them.

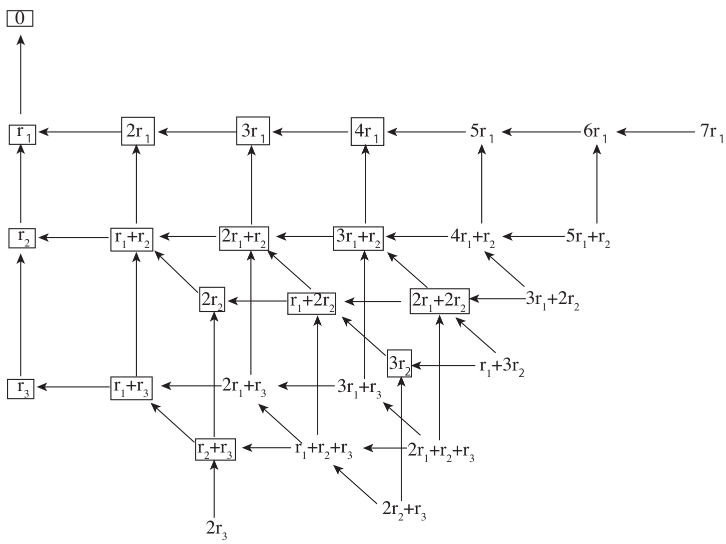


Figure 4

where the part enclosed with the square denotes the generalized degree that is less than or equal to $d_f - 1 = \dfrac{7}{12}$. Let $\mathcal{M}$ be the set of monomials

$$1,\ x,\ y,\ z,\ x^2,\ x^3,\ x^4,\ xy,\ x^2y,\ x^3y,\ y^2,\ xy^2,\ x^2y^2,\ y^3,\ xz,\ yz.$$

Then $\mathcal{M}$ is the biggest set containing monomials with gdeg $\leq d_f - 1$ and thus $m_0(f) = 16$.

On the other hand

$$\Delta(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (x^7,\ z^2 + 6y^5,\ yz),$$

and also

$$\operatorname{gdeg}\left(\frac{\partial f}{\partial x}\right) = \frac{21}{24},\ \operatorname{gdeg}\left(\frac{\partial f}{\partial y}\right) = \frac{20}{24},\ \operatorname{gdeg}\left(\frac{\partial f}{\partial z}\right) = \frac{14}{24}.$$

The set $\mathcal{M}$ is linearly dependent in $\mathcal{R}_f$ since $yz = \frac{1}{2}\frac{\partial f}{\partial z} = 0$ in $\mathcal{R}_f$. Let λ_i

$(i = 0, \dots, 9)$ be complex numbers such that

$$\lambda_0 1 + \lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 x^2 + \lambda_5 x^3 + \lambda_6 x^4 + \lambda_7 xy + \lambda_8 x^2 y + \lambda_9 x^3 y + \lambda_{10} y^2 + \lambda_{11} xy^2 + \lambda_{12} x^2 y^2 + \lambda_{13} y^3 + \lambda_{14} xz \in \Delta(f).$$

Then there exist some $a, b, c \in \mathbb{C}[x, y, z]$ such that

$$\lambda_0 1 + \lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 x^2 + \lambda_5 x^3 + \lambda_6 x^4 + \lambda_7 xy + \lambda_8 x^2 y + \lambda_9 x^3 y + \lambda_{10} y^2 + \lambda_{11} xy^2 + \lambda_{12} x^2 y^2 + \lambda_{13} y^3 + \lambda_{14} xz = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}.$$

Since the generalized order of the right hand side is greater than or equal to $\frac{14}{24}$ we have $\lambda_0 = \cdots = \lambda_{14} = 0$. Hence

$$\{1,\ x,\ y,\ z,\ x^2,\ x^3,\ x^4,\ xy,\ x^2y,\ x^3y,\ y^2,\ xy^2,\ x^2y^2,\ y^3,\ xz\}$$

is linearly independent in $\mathcal{R}_f$ and thus $m(f) = 15$.

Example 4.6. *The second example is* $f(x, y, z) = xz^2 + y^6 + x^9$. *Then* f *is quasihomogeneous of type* $\left(1; \frac{1}{9}, \frac{1}{6}, \frac{4}{9}\right)$ *with isolated singularity at the origin and* $m(f) = 15$, $m_0(f) = 16$.

The diagram of magnitude correlation of the generalized degrees of monomials is as follows:

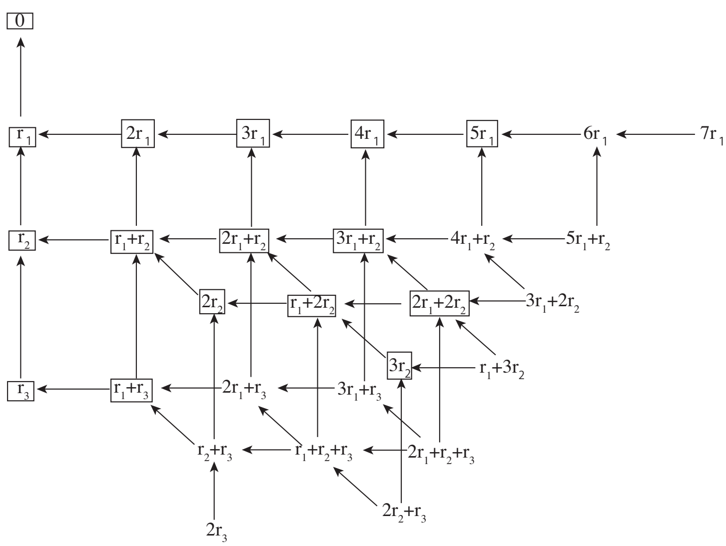


Figure 5

where the part enclosed with the square denotes the generalized degree that is less than or equal to $d_f - 1 = \frac{5}{9}$. By the same way as the first example we have $m(f) = 15$ and $m_0(f) = 16$.

Example 4.7. *The third example is* $f(x, y, z) = xz^2 + y^6 + x^8y$. *Then* f *is quasihomogeneous of type* $(1; \frac{5}{48}, \frac{1}{6}, \frac{43}{96})$ *with isolated singularity at the origin and* $m(f) = 15$, $m_0(f) = 16$.

We have $d_f - 1 = \frac{9}{16}$. The diagram of magnitude correlation of the generalized degrees of monomials is the same as the one in the previous example. By the same way as the first example we have $m(f) = 15$ and $m_0(f) = 16$.

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Appendix: Example of a program code for corank=3 and type I

The following program is written in Mathematica.

```
(* Var3-type01 *)
Do[
    sol=Solve[{a x==1,b y==1,c z==1},{x,y,z}];
    weights=Sort[{x/.sol[[1]],y/.sol[[1]],z/.sol[[1]]}];
    r1=weights[[1]];r2=weights[[2]];r3=weights[[3]];
    d=3-2*(r1+r2+r3);
    {q1,q2,q3}={Denominator[r1],Denominator[r2],Denominator[r3]};
    {p1,p2,p3}={Numerator[r1],Numerator[r2],Numerator[r3]};
    Q=LCM[q1,q2,q3];
    {A1,A2,A3}={p1*Quotient[Q,q1],p2*Quotient[Q,q2],p3*Quotient[Q,q3]};
    N3={0,r1,r2,r3,2r1,3r1,4r1,5r1,6r1,7r1,8r1,9r1,10r1,11r1,12r1,13r1,14r1
       ,r1+r2,2r1+r2,3r1+r2,4r1+r2,5r1+r2,6r1+r2
       ,2r2,r1+2r2,2r1+2r2, 3r1+2r2
       ,3r2,r1+3r2
       ,4r2
       ,r1+r3,2r1+r3,3r1+r3,4r1+r3
       ,r2+r3,r1+r2+r3
       ,2r3};
    NN=Length[Select[N3, # <= d-1 && # <1-r3 &]];
      If[ NN<=14,
          CC=CC+1;
          CPN=Expand[(z^{Q-A1}-1)(z^{Q-A2}-1)(z^{Q-A3}-1)];
          CPD=Expand[(z^{A1}-1)(z^{A2}-1)(z^{A3}-1)];
          CP=PolynomialQuotientRemainder[CPN[[1]],CPD[[1]],z];
             If[1-r3>d-1,
               If[CP[[2]]==0,
                  AIM=NN;
                  Print["Corank =3"];
                  Print["Inner Modality = Arithmetic Inner Modality"];
                  Print["Inner Modality=",AIM];
                  Print["Weights=",TeXForm[{r1,r2,r3}]];
                  EXP={ToRules[Reduce[x r1+y r2+z r3==1&&x>=0&&y>=0 && z>=0 ,{x,y,z},Integers]]};
                  L=Length[EXP];
                  Print["Number of terms = ",L];
                  Do[M=x^{x/.EXP[[i]]}y^{y/.EXP[[i]]}z^{z/.EXP[[i]]};
                     Print[TeXForm[M[[1]]]],{i,L}
                    ];
                  Print[";"];
               ];
              ,
                  AIM =Length[{ToRules[Reduce[(2+x) A1+(2+y) A2+(2+z) A3-2 Q<=0  &&  x>=0
                                              &&  y>=0  &&  z>=0 ,{x,y,z},Integers]]}];
                     If[CP[[2]]==0,
                         CF=Take[CoefficientList[CP[[1]],z],(d-1) Q+1];
                         IM=Sum[CF[[i]],{i,1,(d-1) Q+1}];;
                            If[IM <= 14 && IM < AIM,
                                   Print["Corank=3"];
                                   Print["Inner Modality < Arithmetic Inner Modality"];
                                   Print["Arithmetic Inner Modality=",AIM];
                                   Print["Inner Modality=",IM];
                                   Print["Weights=",TeXForm[{r1,r2,r3}]];
                                   Print[";"];
                 ];
           ];
         ];
       ];
          ,{a,3,195},{b,3,195},{c,3,195}
   ]
```

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