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From the blow-analytic equivalence to the arc-analytic equivalence: a survey

Jean-Baptiste Campesato

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Abstract

In the early eighties T.-C. Kuo introduced the blow-analytic equivalence in order to obtain a classification of real analytic function germs with no continuous moduli.

This new notion has been subsequently extensively investigated. In particular it led to the blow-Nash equivalence of G. Fichou and more recently to the arc-analytic equivalence. These notions may be seen as algebraic versions of the blow-analytic equivalence in order to classify Nash function germs with no continuous moduli.

This survey aims to give a self-contained introduction to this topic. It focuses on the construction of invariants of the above cited relations. In order to introduce some of these invariants, it is necessary to briefly recall some notions of motivic integration.

0. Introduction

The Whitney family

$$f_t(x, y) = xy(y - x)(y - tx),\ t \in (0, 1)$$

highlights the fact that the C^1-equivalence has moduli even for isolated singularities. This led T.-C. Kuo to introduce in the early eighties a new notion in order to classify real analytic function germs which are singular. The purpose of the first section of this survey is to introduce this notion called the blow-analytic equivalence. In particular we present some results showing that the blow-analytic equivalence behaves well with isolated singularities: it has no continuous moduli for such singularities.

The blow-analytic equivalence involves the notion of blow-analytic maps. One may show that these maps send real analytic arcs to real analytic arcs by composition. The second section of this survey is devoted to such maps, called arc-analytic maps. It is a notion introduced by K. Kurdyka. The close relation between blow-analyticity and arc-analyticity led T. Fukui to construct an invariant of the blow-analytic equivalence from which one may derive that the multiplicity is also an invariant.

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Then S. Koike and A. Parusiński continued in this direction by borrowing arguments coming from motivic integration in order to construct new invariants of the blow-analytic equivalence. The previous invariants, together with the Fukui ones, allowed them to classify entirely Brieskorn polynomials of two variables up to the blow-analytic equivalence. One main point of motivic integration consists in defining a measure on the space of arcs of an algebraic variety and hence the link with arc-analyticity led naturally to focus on this theory. The third section of this survey briefly introduces motivic integration in the non-singular case and the fourth section presents the invariants of S. Koike and A. Parusiński.

The motivic measure lies in a Grothendieck group which encodes all additive invariants of algebraic varieties. In order to construct their invariants, S. Koike and A. Parusiński realized this motivic measure through the Euler characteristic with compact support. Then G. Fichou brought a richer structure thanks to the virtual Poincaré polynomial, an additive invariant which encodes more information than the Euler characteristic with compact support. It is presented in section 5. For technical reasons G. Fichou had to restrict to Nash function germs (i.e. real analytic function germs with semialgebraic graphs) and to define the blow-Nash equivalence which is an algebraic version of the blow-analytic equivalence with no continuous moduli for Nash germs with isolated singularities. The sixth section is devoted to the blow-Nash equivalence and to the invariants introduced by G. Fichou. Thanks to these invariants, G. Fichou classified entirely Brieskorn polynomials in three variables up to the blow-Nash equivalence. More recently G. Fichou and T. Fukui used these invariants to prove that the weights of a non-degenerate (with respect to its Newton polyhedron) convenient weighted homogeneous polynomial is determined by its blow-Nash class.

Initially it was not obvious that the blow-Nash equivalence was an equivalence relation. The last section focuses on a characterization of the blow-Nash equivalence in terms of arc-analytic maps. This new definition, called the arc-analytic equivalence, allows one to prove that the blow-Nash equivalence is an equivalence relation as expected. A recent preprint by A. Parusiński and L. Paunescu announces that the arc-analytic equivalence admits no continuous moduli even for families of non-isolated singularities. We also present a motivic invariant of the arc-analytic equivalence which is based on an adaptation of a Grothendieck group defined by G. Guibert, F. Loeser and M. Merle. This invariant encodes the previous invariants of T. Fukui, of S. Koike and A. Parusiński and of G. Fichou. Moreover it has a convolution formula which allows one to show that the arc-analytic class of a Brieskorn polynomial determines its exponents.

1. The blow-analytic equivalence

The blow-analytic equivalence was introduced in the early eighties by T.-C. Kuo [51], [54], [52], [53] in order to get a classification of real analytic singularities with no moduli.

This new notion was necessary since the C^0-equivalence is too naive whereas the C^1-equivalence is already too fine to get a classification with no moduli for a family of isolated singularities. These phenomena are highlighted in the following examples.

In addition to the cited articles throughout this section, we refer the reader to the surveys [35] and [37] concerning the blow-analytic equivalence.

Definition 1.1. Let $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be two real analytic function germs. We say that f and g are *C^k-equivalent* if there exists $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ a C^k-diffeomorphism such that $f = gh$ (Here gh denotes the composition of g and h).

Example 1.2. Let $f(x, y) = x^3 - y^2$, whose zero set is a cusp. Then f is C^0-equivalent to $g(x, y) = x$ which is non-singular.

Example 1.3 ([73, Example 13.1])**.** We consider the Whitney family

$$f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0),\ t \in (0, 1)$$

defined by

$$f_t(x, y) = xy(y - x)(y - tx)$$

Then f_t and $f_{t'}$ are C^1-equivalent if and only if $t = t'$.

Let $F : \mathbb{R}^2 \times I \to \mathbb{R}$ be defined by $F(x, y, t) = xy(y - x)(y - tx)$ where $I = (0, 1)$. We denote by $\pi : M \to \mathbb{R}^2$ the blowing-up of $\mathbb{R}^2$ at the origin, and by $\Pi : M \times I \to \mathbb{R}^2 \times I$ the induced projection, $\Pi(p, t) = (\pi(p), t)$. Suppose that $\Phi : U_1 \to U_2$ is a real analytic isomorphism between two open neighborhoods of $\pi^{-1}(0) \times I$ in $M \times I$. Suppose, moreover, that $\pi^{-1}(0) \times I$ is invariant by Φ and that, for all $(p, t) \in U_1$, $\mathrm{pr}_I\, \Phi(p, t) = t$. Then Φ induces a homeomorphism $\varphi : V_1 \to V_2$ between two open neighborhoods $V_i = \Pi(U_i)$, $i = 1, 2$, of $\{0\} \times I$ in $\mathbb{R}^2 \times I$ that satisfies $\mathrm{pr}_I\, \varphi(p, t) = t$, for all $(p, t) \in V_1$, and $\varphi(0, t) = (0, t)$, for all $t \in I$.

T.-C. Kuo noticed in [52] that there exists such a Φ with the additional property that is $F(\varphi(p, t))$ is independent of t.

The concept of blow-analytic equivalence generalizes the previous observation. But, before defining it, we need to introduce the following notions.

Definition 1.4 ([74])**.** A *complexification* of an n-dimensional real analytic manifold M is given by an n-dimensional complex analytic manifold $M^{\mathbb{C}}$ and a real analytic isomorphism φ from M to a real analytic submanifold of $M^{\mathbb{C}}$ such that, for every $x \in M^{\mathbb{C}}$, there is an open neighborhood $U^{\mathbb{C}}$ of x, an open subset U' of $\mathbb{C}^n$ and a complex analytic isomorphism $\psi : U^{\mathbb{C}} \to U'$ mapping $\varphi(M) \cap U^{\mathbb{C}}$ to $\mathbb{R}^n \cap U'$.

Proposition 1.5 ([74, Proposition 1]). *A real analytic manifold M admits a complexification. Moreover, if $(M_1^{\mathbb{C}}, \varphi_1)$ and $(M_2^{\mathbb{C}}, \varphi_2)$ are two complexifications of M, then there exist an open neighborhood U_1 of $\varphi_1(M)$, an open neighborhood U_2 of $\varphi_2(M)$ and a complex analytic isomorphism $\psi : U_1 \to U_2$ such that $\psi(x) = \varphi_2\varphi_1^{-1}(x)$ for every $x \in \varphi_1(M)$.*

Let $\mu : M \to N$ be a proper real analytic map between two real analytic manifolds. Then μ extends to a complex analytic map $\mu^{\mathbb{C}} : U(M) \to U(N)$ where $U(M)$ (resp. $U(N)$) is an open neighborhood in a complexification of M (resp. N). From the previous proposition, we deduce that the complexification $\mu^{\mathbb{C}}$ of μ is unique as a germ at M up to a complex analytic isomorphism.

Definition 1.6 ([53]). Let $\mu : M \to N$ be a proper and surjective real analytic map between two real analytic manifolds. We say that μ is a *real modification* if there exists a complexification $\mu^{\mathbb{C}} : U(M) \to U(N)$ that is proper and bimeromorphic.

Remark 1.7. Some authors (e.g. [37]) don't assume that $\mu^{\mathbb{C}}$ is proper in the previous definition. We add this assumption, as in [53], in order to work with global blowings-up (instead of local ones) in Lemma 1.11.

Example 1.8. Notice that $\mu(x) = x^3$ is not a real modification.

Definition 1.9 ([53]). A map $f : M \to N$ between real analytic manifolds is *blow-analytic** if there exists a real modification $\mu : \tilde{M} \to M$ such that $f\mu : \tilde{M} \to N$ is real analytic.

We can define the blowing-up of a real analytic manifold along a closed submanifold using local coordinates. It consists in locally straightening the submanifold in order to use the classical definition of the blowing-up of the Euclidean space at the origin.

Let M be a real analytic manifold and C a closed submanifold of M. Up to real analytic isomorphism over M, there exists a unique proper map $\pi : \tilde{M} \to M$ with $\tilde{M}$ a real analytic manifold and such that :

1. The restriction $\pi_{|M \setminus C} : \tilde{M} \setminus \pi^{-1}(C) \to M \setminus C$ is an isomorphism.
2. Let U be a local coordinate chart given by $\varphi : U \to V \times W$, where $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^{d-m}$ are open neighborhoods of their respective origin such that $\varphi(C \cap U) = \{0\} \times W$. Let $\pi_0 : \tilde{V} \to V$ be the blowing-up of V at the origin. Then there is an isomorphism $\tilde{\varphi} : \pi^{-1}(U) \to \tilde{V} \times W$ such that the following diagram commutes

$\star$ The origin of the name seems to come from the fact that the modifications originally considered in [52] were just blowings-up.

$$\begin{CD} \pi^{-1}(U) @>{\tilde{\varphi}}>> \tilde{V} \times W \\ @V{\pi}VV @VV{(\pi_0, \mathrm{id}_W)}V \\ U @>>{\varphi}> V \times W \end{CD} \quad .$$

Definition 1.10. We say that $\pi : \tilde{M} \to M$ is the blowing-up of M along C.

Lemma 1.11 (A real version of Chow-Hironaka lemma [45, Corollary 2])**.** *Let $\mu : M \to N$ be a real modification. Then there exists a real analytic manifold $\tilde{M}$ and a real analytic map $\Psi : \tilde{M} \to M$ such that $\psi = \mu\Psi : \tilde{M} \to N$ is a locally finite sequence of blowings-up along non-singular centers.*

The following proposition is a direct consequence of the previous lemma.

Proposition 1.12 ([53])**.** *A map $f : M \to N$ between two real analytic manifolds is blow-analytic if and only if there is a locally finite sequence of blowings-up with non-singular centers $\sigma : \tilde{M} \to M$ such that $f\sigma : \tilde{M} \to N$ is real analytic.*

Example 1.13 ([53])**.** The function defined by $f(x,y) = \frac{x^2 y}{x^2+y^2}$ and $f(0,0) = 0$ is continuous but no C^1. It is blow-analytic via the blowing-up of $\mathbb{R}^2$ at the origin.

Example 1.14. The function defined on $\mathbb{R}^2 \setminus \{0\}$ by $f(x,y) = \frac{xy}{x^2+y^2}$ doesn't continuously extend through the origin. However it extends to a real analytic map after being composed with the blowing-up of $\mathbb{R}^2$ at the origin.

We can now give the definition of the blow-analytic equivalence.

Definition 1.15 ([53])**.** Two real analytic function germs $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are blow-analytic equivalent if there exists a homeomorphism $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ such that h and h^{-1} are blow-analytic and satisfies $f = gh$.

Remark 1.16. Some authors call blow-analytic homeomorphism a homeomorphism h such that h and h^{-1} are blow-analytic. We prefer to avoid this name since there is a confusion with a homeomorphism h such that only h is blow-analytic.

In order to prove that it is an equivalence relation, T.-C. Kuo gave the following characterization.

Proposition 1.17 ([53, Proposition 2])**.** *Let $\varphi : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ be a homeomorphism. Then φ and φ^{-1} are blow-analytic if and only if there are two real modifications μ_1, μ_2 and a real analytic isomorphism Φ such that the following diagram commutes*

$$\begin{CD} (M_1, \mu_1^{-1}(0)) @>{\Phi}>> (M_2, \mu_2^{-1}(0)) \\ @V{\mu_1}VV @VV{\mu_2}V \\ (\mathbb{R}^d, 0) @>>{\varphi}> (\mathbb{R}^d, 0) \end{CD} \quad .$$

Corollary 1.18. *The blow-analytic equivalence for real function germs* $(\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ *is an equivalence relation.*

Two questions arise naturally:

1. How to construct homeomorphisms satisfying the conditions in Definition 1.15 in order to prove that two germs are in the same blow-analytic equivalence class?
   The main way consists in putting the germs in a family and then to trivialize it by integrating a well chosen analytic vector field on an (equi)resolution space as for the results presented later in this section.

2. How to construct blow-analytic invariants in order to distinguish germs not in the same blow-analytic class?
   That will be the main subject of this article.

The blow-analytic equivalence has no continuous moduli for isolated singularities as shown in the following theorem.

Theorem 1.19 ([53, Theorem 1])**.** *A family of isolated singularities has only a finite number of blow-analytic classes, i.e. let* $F : (\mathbb{R}^d \times I, \{0\} \times I) \to (\mathbb{R}, 0)$ *be a real analytic function germ such that for every* $t \in I$, $f_t(x) = F(t, x) : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ *has an isolated singularity at the origin then the germs* $(f_t)_{t \in I}$ *define a finite number of blow-analytic classes.*

The results stated below show that with additional conditions a family of isolated singularities may define only one blow-analytic class.

Definition 1.20 (Blow-analytic triviality)**.** Let I be an interval and let $F : \mathbb{R}^d \times I \to \mathbb{R}$ be a family of real analytic function germs such that $f_t(0) = F(0, t) = 0$ for all $t \in I$.

Let $\mu : M \to \mathbb{R}^d$ be a real modification. We say that the family $(f_t)_{t \in I}$ is *blow-analytically trivial via* μ if there exist a real analytic isomorphism

$$\Phi : (M \times I, \mu^{-1}(0) \times I) \to (M \times I, \mu^{-1}(0) \times I)$$

and a homeomorphism

$$\varphi : (\mathbb{R}^d \times I, \{0\} \times I) \to (\mathbb{R}^d \times I, \{0\} \times I)$$

such that the following diagram commutes

$$\begin{tikzcd}
M \times I \arrow[rr, "\Phi"] \arrow[dr] \arrow[dd, "\mu\times\mathrm{id}_I"'] & & M \times I \arrow[dl] \arrow[dd, "\mu\times\mathrm{id}_I"] \\
& I & \\
\mathbb{R}^d \times I \arrow[rr, "\varphi"'] \arrow[ur] \arrow[dr, "F_0"'] & & \mathbb{R}^d \times I \arrow[ul] \arrow[dl, "F"] \\
& \mathbb{R} &
\end{tikzcd}$$

where there is $t_0 \in I$ such that for all $t \in I$, $F_0(x,t) = F(x,t_0)$.

In particular $F\varphi(x,t)$ doesn't depend on t and the maps Φ and φ preserve the t-levels.

A family of real analytic function germs is *blow-analytically trivial* if it is via some real modification μ.

Remark 1.21. If a family is blow-analytically trivial then its members are pairwise blow-analytically equivalent.

The strategy to prove the following results consists in integrating a well chosen vector field in order to construct the isomorphism of Definition 1.20.

Theorem 1.22 ([52])**.** *Let $F(x,t) : (\mathbb{R}^d,0) \times I \to (\mathbb{R},0)$ be real analytic. We use the following expansion $F(x,t) = \sum_\nu c_\nu(t)x^\nu$ and we set $H_j(x,t) = \sum_{|\nu|=j} c_\nu(t)x^\nu$. Let $k = \min\{j,\ H_j \not\equiv 0\}$.*
If for every $t \in I$, the origin is the only singular point of $H_k(\cdot,t)$, then the family defined by F is blow-analytically trivial via the blowing-up of $\mathbb{R}^d$ at the origin.

Example 1.23. The Whitney family is blow-analytically trivial.

Theorem 1.24 ([38, Theorem B])**.** *Let $f_t : (\mathbb{R}^d,0) \to (\mathbb{R},0), t \in I$, be a family of real analytic function germs which are non-degenerate with respect to their Newton polyhedra*. Assume furthermore that the Newton polyhedron of f_t doesn't depend on t. If, moreover, for each non-compact face γ which is not a coordinate face, the polynomial† $f_{t\gamma}$ doesn't depend on t, then the family (f_t) is blow-analytically trivial via a toric resolution* induced by the Newton polyhedron of f_t.*

Remark 1.25 ([38, Corollary 6.1])**.** If $d = 2$ then the assumption "for each non-compact non-coordinate face γ, $f_{t\gamma}$ doesn't depend on t" is superfluous.

If we assume that f_t is convenient‡ for all $t \in I$, then every non-compact

* See [4, §8].
† $f_{t\gamma}$ is the polynomial composed by the monomials on the face γ.
‡ i.e. for each i, some power of x_i appears in the expansion of f_t.

face is a coordinate face. Thus, we get the following corollary.

Corollary 1.26 ([38, Corollary 6.2])**.** *Let $f_t : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0), t \in I$, be a family of real analytic function germs which are non-degenerate with respect to their Newton polyhedra. If the Newton polyhedron of f_t doesn't depend on t and intersects coordinate axes then the family (f_t) is blow-analytically trivial via a toric resolution induced by the Newton polyhedron of f_t.*

Theorem 1.27 ([36, Theorem (0.2)])**.** *Let $f_t : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0), t \in I$, be a family of real analytic function germs. Set $F(x,t) = f_t(x) = \sum_\nu c_\nu(t)x^\nu$. Fix the weight system $w = (w_1, \dots, w_d) \in (\mathbb{N}_{>0})^d$. Let*

$$k = \min\left\{ j,\ H_j(x,t) = \sum_{\nu, \nu\cdot w = j} c_\nu(t)x^\nu \not\equiv 0 \right\}.$$

If, for every $t \in I$, the origin is the only singular point of $H_k(\cdot, t)$ then the family f_t is blow-analytically trivial via a toric resolution.

Example 1.28 ([33, Example 1])**.** The Briançon–Speder family

$$f_t(x,y,z) = z^5 + ty^6z + xy^7 + x^{15}$$

is blow-analytically trivial as soon as I doesn't contain $t_0 = -\frac{15^{\frac{1}{7}}(\frac{7}{2})^{\frac{4}{5}}}{3}$. Indeed, f_t is weighted homogeneous for the weights $(1,2,3)$ and defines an isolated singularity at the origin when $t \neq t_0$.

O. M. Abderrahmane [1, Théorème 3.3.1. & §3.2.] proved that if the leading term of $F(x,t)$ with respect to some convenient Newton polyhedron satisfies a special uniform Łojasiewicz condition then the family is blow-analytically trivial via a toric resolution. This generalizes Theorems 1.22, 1.26 and 1.27.

2. Arc-analytic maps

Proposition 2.1. *If $\mu : M \to N$ is a real modification and if $\gamma : (-\varepsilon, \varepsilon) \to N$ is a real analytic arc, then there exists a real analytic arc $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to M$ such that $\gamma = \mu\tilde{\gamma}$.*

Proof. Let $f^{\mathbb{C}} : M^{\mathbb{C}} \to N^{\mathbb{C}}$ be a complexification of f. It suffices to prove that there is a complexification of γ, $\gamma^{\mathbb{C}} : D^{\mathbb{C}} \to N^{\mathbb{C}}$, where $D^{\mathbb{C}}$ is a neighborhood of $(-\varepsilon, \varepsilon)$ in $\mathbb{C}$, that admits a holomorphic lift by $f^{\mathbb{C}}$.

Recall that a meromorphic map (in the sense of Remmert) of complex manifolds $F : X \to Y$ is a multivalued map such that its graph Γ_F is an analytic subset of $X \times Y$, which is mapped properly on X by the projection on the first coordinate, and off a nowhere dense analytic subset $P \subset Y$ (called the polar set or indeterminacy locus) this projection is a biholomorphic map, see for instance [27]. If $G : Z \to X$ is another meromorphic map such that $G^{-1}(P)$ is nowhere dense in Z then the composition map FG is meromorphic. We use this description for the inverse of $f^{\mathbb{C}}$.

Suppose first that $(\gamma^{\mathbb{C}})^{-1}(P) = \{0\}$. Then $\eta^{\mathbb{C}} = (f^{\mathbb{C}})^{-1}\gamma^{\mathbb{C}} : D^{\mathbb{C}} \to M^{\mathbb{C}}$, is a meromorphic arc. Its graph $\Gamma_{\eta^{\mathbb{C}}}$, the closure of $\{(t, \tilde{\gamma}^{\mathbb{C}}(t)); t \in D^{\mathbb{C}} \setminus \{0\}\}$, is an irreducible analytic curve in $D^{\mathbb{C}} \times M^{\mathbb{C}}$. Thus $\eta^{\mathbb{C}}(t)$ admits a unique accumulation point $p \in M^{\mathbb{C}}$ as $t \to 0$, and by Riemann removable singularity theorem, is holomorphic in $D^{\mathbb{C}}$ as required. A similar argument works if $(\gamma^{\mathbb{C}})^{-1}(P)$ is discrete or empty.

Let γ be a real analytic arc whose complexification is entirely included in P. Since $P \cap N$ is nowhere dense in N, there exists a real analytic map

$$\varphi(t,u) : V \to N,$$

where V is a neighborhood of $(-\varepsilon, \varepsilon) \times \{0\}$ in $\mathbb{R}^2$, such that $\gamma(t) = \varphi(t, 0)$ and such that the image of φ is not entirely included in P. Let $\varphi^{\mathbb{C}}(t,u) : V^{\mathbb{C}} \to N^{\mathbb{C}}$ be a complexification of φ. Then $\Phi^{\mathbb{C}} = (f^{\mathbb{C}})^{-1}\varphi^{\mathbb{C}}$ is meromorphic and, by Riemann removable singularity theorem, holomorphic except at a discrete subset of $V^{\mathbb{C}}$. Thus $\Gamma_{\Phi^{\mathbb{C}}}$ over $D = V^{\mathbb{C}} \cap \{u = 0\}$ is an analytic curve in $V^{\mathbb{C}} \times M^{\mathbb{C}}$. Its irreducible component that projects submersively onto D is the graph of a holomorphic lift of $\gamma^{\mathbb{C}}$ as required. ■

Remark 2.2. Using Lemma 1.11, we may assume that μ is a locally finite sequence of blowings-up and then use the universal property of blowings-up to prove that a real analytic arc non-entirely included in the center may be lifted.

Let $f : M \to N$ be a blow-analytic map then there is a real modification μ such that $\tilde{\mu} = f\mu$ is real analytic. Let $\gamma : (-\varepsilon, \varepsilon) \to N$ be a real analytic arc, then there exists a real analytic arc $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to M$ such that the following diagram commutes

$$\begin{tikzcd}
 & & \tilde{M} \arrow[dl, "\mu"'] \arrow[dr, "\tilde{\mu}"] & \\
(-\varepsilon,\varepsilon) \arrow[r, "\gamma"'] \arrow[urr, bend left, "\tilde{\gamma}"] & M \arrow[rr, "f"'] & & N
\end{tikzcd} .$$

Hence $f\gamma = \tilde{\mu}\tilde{\gamma}$. Thus a blow-analytic map sends real analytic arcs to real analytic arcs by composition. Such maps were first studied by K. Kurdyka [55].

Definition 2.3 ([55, Définition 4.1])**.** A map $f : M \to N$ between two real analytic varieties is *arc-analytic* if for every real analytic arc $\gamma : (-\varepsilon, \varepsilon) \to M$, the composition $f\gamma$ is also real analytic.

We proved the following proposition.

Proposition 2.4. *A blow-analytic map is arc-analytic.*

Thanks to the following remark, the previous proposition can be derived from the fact that a real analytic arc germ admits a lifting by a real modification. Such a property is weaker than Proposition 2.1 and is shown in [37, §5].

Remark 2.5. A map $f : M \to N$ is arc-analytic if and only if for every real analytic arc germ $\gamma : (\mathbb{R}, 0) \to M$, the composition germ $f\gamma$ is also real analytic.

Example 2.6 ([55, Exemple 6.1])**.** The function defined by $f(x,y) = \frac{x^3}{x^2+y^2}$ and $f(0,0) = 0$ is arc-analytic whereas it is not C^1.

We may notice that the Zariski closure of the graph of f is the Cartan umbrella defined by $x^3 = z(x^2+y^2)$ [55, Exemples 1.2].

It is natural to ask whether the converse is true: is an arc-analytic map blow-analytic? E. Bierstone, P. D. Milman and A. Parusiński [11] gave a counter-example with no additional assumption.

K. Kurdyka conjectured that a map is blow-analytic if and only if it is arc-analytic and subanalytic. It is still open, but the following results of E. Bierstone and P. D. Milman and of A. Parusiński are very close to the expected result.

Theorem 2.7 ([10, Theorem 1.4], [64, Theorem 3.1])**.** *A function $f : M \to \mathbb{R}$ defined on a non-singular real analytic variety is arc-analytic and subanalytic if and only if there exists a locally finite sequence $\sigma : \tilde{M} \to M$ of local blowings-up along non-singular centers such that $f\sigma$ is real analytic.*

Theorem 2.8 ([10, Theorem 1.1] Analytic version)**.** *Let $f : M \to \mathbb{R}$ be an arc-analytic function defined on a non-singular real analytic variety. Assume there exists*

$$G(x,y) = \sum_{i=0}^{p} G_i(x) y^i$$

a non-zero polynomial whose coefficients are real analytic functions on M such that

$$G(x, f(x)) = 0,\ x \in M.$$

Then f is blow-analytic.

Theorem 2.9 ([10, Theorem 1.1] Algebraic version)**.** *Let $f : M \to \mathbb{R}$ be an arc-analytic function defined on a non-singular real algebraic variety. Assume there exists*

$$G(x,y) = \sum_{i=0}^{p} G_i(x) y^i$$

a non-zero polynomial whose coefficients are regular functions on M such that

$$G(x, f(x)) = 0,\ x \in M$$

Then f is blow-analytic via a finite sequence of algebraic blowings-up with non-singular centers*.

The two previous results allow one to construct examples of blow-analytic functions.

Example 2.10 ([55, Exemple 6.1], [10, Examples 1.2])**.** The function defined by $f(x,y) = \frac{x^3}{x^2+y^2}$ and $f(0,0) = 0$ is arc-analytic and satisfies $G(x, y, f(x,y)) = 0$ where

$$G(x, y, z) = x^3 - z(x^2 + y^2)$$

Thus f is blow-analytic.

Example 2.11 ([10, Examples 1.2])**.** The function defined by $f(x,y) = \sqrt{x^4 + y^4}$ is arc-analytic and satisfies $G(x, y, f(x,y)) = 0$ where

$$G(x, y, z) = x^4 + y^4 - z^2$$

Thus f is blow-analytic.

It is known that a semialgebraic function† defined on a semialgebraic set satisfies a non-trivial polynomial equation [14, Lemma 2.5.2]. Following this fact and the statement of Theorem 2.9, it is natural to introduce the following notion.

Definition 2.12. A semialgebraic function defined on a real algebraic set is said to be *blow-Nash* if there exists a finite sequence of algebraic blowings-up with non-singular centers such that the composition is Nash‡.

Then, we may deduce the following result from Theorem 2.9.

Theorem 2.13. *A semialgebraic map defined on a non-singular real algebraic set is blow-Nash if and only if it is arc-analytic.*

The previous theorem admits a generalization to possibly singular real algebraic sets.

Theorem 2.14 ([17, Proposition 2.27])**.** *A continuous*§ *semialgebraic map defined on a real algebraic set is blow-Nash if and only if it is generically arc-analytic (i.e., there exists a nowhere dense algebraic subset such that every real analytic arc non entirely included in this subset is mapped to a real analytic arc by composition).*

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* We will see below that f is actually blow-Nash.

† i.e. a function whose graph is semialgebraic.

‡ i.e. real analytic and semialgebraic, see Definition 6.1.

§ Actually continuous on the closure of the non-singular locus.

Remark 2.15. We will see in Proposition 5.9 that a semialgebraic arc-analytic map defined on a real algebraic set is continuous. This is actually true for a subanalytic arc-analytic map [10, Remark 1.5.(2) & Lemma 6.8.]. The function given in [11] is arc-analytic but not continuous.

As pointed out in this section, blow-analyticity and arc-analyticity are very close notions. The constructions of the known invariants of the blow-analytic equivalence all rely on this fact. The next statement introduces the Fukui invariant and in Section 4 we will present the Koike–Parusiński zeta functions which are other invariants.

Theorem 2.16 ([34, Theorem 3.1])**.** *To a real analytic function germ $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$, we associate the set of orders of f restricted to real analytic arcs germs at the origin. This is the* Fukui invariant *whose formal definition is below:*

$$\mathcal{A}(f) = \left\{\operatorname{ord}_t f\gamma,\ \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0) \text{ real analytic}\right\}.$$

It is an invariant of the blow-analytic equivalence
i.e., if $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent then $\mathcal{A}(f) = \mathcal{A}(g)$.

Proof. Let $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be two blow-analytically equivalent real analytic function germs. Hence there exists $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ a homeomorphism with h and h^{-1} blow-analytic such that $f = gh$.

Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0)$ be a real analytic arc. By Proposition 2.4, $\eta = h\gamma$ is a real analytic arc. Since $g\eta = f\gamma$, we get $A(f) \subset A(g)$. The proof of the converse inclusion is exactly the same. ■

The following corollary is a direct consequence of the previous theorem.

Corollary 2.17. *If f and g are two real analytic function germs which are blow-analytic equivalent, then* $\operatorname{mult} f = \operatorname{mult} g$.

Remark 2.18 ([46, §7])**.** S. Izumi, S. Koike and T.-C. Kuo formulated versions with signs of the Fukui invariant which are also blow-analytic invariants:

$$\mathcal{A}^+(f) = \left\{\operatorname{ord}_t f\gamma,\ \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0),\ \exists\delta > 0,\ \forall t \in [0, \delta),\ f\gamma(t) \geq 0\right\},$$

$$\mathcal{A}^-(f) = \left\{\operatorname{ord}_t f\gamma,\ \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0),\ \exists\delta > 0,\ \forall t \in [0, \delta),\ f\gamma(t) \leq 0\right\}.$$

3. Classical motivic integration

For more complete surveys concerning classical motivic integration, we refer the reader to [24], [58] and [57]. The surveys [18], [72] and [13] introduce the machinery of motivic integration in the non-singular case (or with only “nice” singularities for the first one).

In this section, an algebraic variety is a reduced and separated scheme of finite type over a characteristic zero field k. This is the only section of this article which uses the language of schemes.

3.1 A brief overview of the theory in the non-singular case

Using tools coming from p-adic integration, V. V. Batyrev [8] proved that two birational Calabi–Yau varieties share the same Betti numbers. During a talk in Orsay in 1995, M. Kontsevich [50] introduced a "motivic" analogue of the p-adic integration which allows him to prove that two birational Calabi–Yau varieties share the same Hodge numbers. That's the beginning of motivic integration.

Then we got a deeper comprehension of this new theory, in particular it was generalized to possibly singular varieties, thanks to the works of J. Denef and F. Loeser [22], [26], V. V. Batyrev [7], [9] and E. Looijenga [58].

Definition 3.1. We denote by $K_0(\mathrm{Var}_k)$ the Grothendieck group of algebraic varieties over k. It is the free abelian group spanned by isomorphism classes $[X]$ of algebraic varieties over k modulo the following additivity relation:

(i) If Y is a closed subvariety of X then $[X] = [X \setminus Y] + [Y]$.

Moreover the fiber product over $\operatorname{Spec} k$ induces a ring structure:

(ii) $[X][Y] = [X \times_k Y]$.

Notations 3.2. Let fix some notations:

- We denote by $0 = [\varnothing]$ the class of the empty variety, it is the unit of the addition.
- We denote by $1 = [\mathrm{pt}] = [\operatorname{Spec} k]$ the class of the point, it is the unit of the product.
- We denote by $\mathbb{L} = [\mathbb{A}^1_k] = [\operatorname{Spec} k[x]]$ the class of the affine line.
- We denote by $\mathcal{M}_k = K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ the localization of $K_0(\mathrm{Var}_k)$ in $\{1, \mathbb{L}, \mathbb{L}^2, \ldots\}$ so that we can divide by the class of the affine line.

Remark 3.3. It is known that $K_0(\mathrm{Var}_k)$ is not an integral domain for a zero characteristic field k [67].

When k is a characteristic zero field, L. A. Borisov [15] recently proved that the class of the affine line $\mathbb{L}$ is a zero divisor, in particular the morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is not injective (see also [59]).

Remark 3.4. A Zariski-constructible set induces a well-defined class in $K_0(\mathrm{Var}_k)$.

Remark 3.5. If $p : E \to B$ is a Zariski piecewise trivial fibration* with fiber F

* i.e. we may split $B = \sqcup B_i$ as a finite disjoint union of locally closed sets for the Zariski topology such that $p^{-1}(B_i)$ is locally closed and isomorphic to $B_i \times F$.

then $[E] = [B][F]$ in $K_0(\mathrm{Var}_k)$.

We refer the reader to [39] and [40] for the basic properties of jet spaces and arc spaces that we are going to briefly introduce. Notice also that the study of singularities via the arc space was first studied by J. Nash [63] in 1968.

Definition 3.6. For X an algebraic variety and $n \in \mathbb{N}$, we denote by $\mathcal{L}_n(X)$ the *set of n-jets on* X, it is the $k[t]/t^{n+1}$-rational points of X, i.e.,

$$\mathcal{L}_n(X) = X\left(k[t]/t^{n+1}\right) = \mathrm{Hom}_{k-\mathrm{sch}}(\mathrm{Spec}\ k[t]/t^{n+1}\ , X).$$

These sets have a natural structure of schemes. Indeed, up to working locally, we may assume that $X = \mathrm{Spec}\left(k[x_1, \dots, x_d]/I\right)$ where $I = (f_1, \dots, f_m)$. Then an n-jet is just a morphism $\gamma : k[x_1, \dots, x_d]/I \to k[t]/t^{n+1}$. By setting $u_i = \gamma(x_i)$ and by interpreting it as a vector of k^{n+1}, we notice that $\mathcal{L}_n(X)$ is the subvariety of $k^{d(n+1)}$ given by the equations $f_i(u_1, \dots, u_d) = 0 \in k[t]/t^{n+1}$ modulo the previous identification. We may check that this construction doesn't depend on the choice of the generators of I. ⋆

The following example highlights the natural scheme structure of the set of n-jets.

Example 3.7. Consider the cusp

$$X = \mathrm{Spec}\left(k[x,y]/y^2 - x^3\right)$$

then

$$\begin{aligned}
&\mathcal{L}_1(X)(k)\\
&\quad = \left\{(a_0 + a_1 t, b_0 + b_1 t) \in \left(k[t]/t^2\right)^2,\ (b_0 + b_1 t)^2 - (a_0 + a_1 t)^3 \equiv 0 \mod t^2\right\}\\
&\quad = \left\{(a_0 + a_1 t, b_0 + b_1 t) \in \left(k[t]/t^2\right)^2,\ a_0^3 = b_0^2,\ 3a_1 a_0^2 = 2b_0 b_1\right\}.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}_2(X)(k)\\
&\quad = \left\{\begin{array}{l}(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \in \left(k[t]/t^3\right)^2,\\ \qquad\qquad (b_0 + b_1 t + b_2 t^2)^2 - (a_0 + a_1 t + a_2 t^2)^3 \equiv 0 \mod t^3\end{array}\right\}\\
&\quad = \left\{(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \in \left(k[t]/t^3\right)^2, \begin{array}{l} a_0^3 = b_0^2,\\ 3a_1 a_0^2 = 2b_0 b_1,\\ 3a_0^2 a_2 + 3a_0 a_1^2 = 2b_0 b_2 + b_1^2\end{array}\right\}.
\end{aligned}$$

Definition 3.8. Let $m, n \in \mathbb{N}$ with $m \geq n$, then the morphism

⋆ There is a more functorial construction in the beginning of [22].

$$k[t]/t^{m+1} \to k[t]/t^{n+1}$$

induces a truncation morphism

$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X).$$

Definition 3.9. The *scheme of (formal) arcs on* X is, by definition, the projective limit

$$\mathcal{L}(X) = \varprojlim \mathcal{L}_n(X).$$

Hence, for $n \in \mathbb{N}$, we have a natural truncation morphism

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X).$$

Remark 3.10. The functorial construction allows one to check that the k-rational points of $\mathcal{L}(X)$ are the $k[\![t]\!]$-rational points of X.

Remark 3.11. The space of n-jets, for a fixed n, is an algebraic variety. However, the space of arcs is a reduced and separated scheme but which is not of finite type, hence it can be seen as a "infinite dimensional algebraic variety".

Remark 3.12. Given an algebraic variety X, a theorem of Greenberg [41] ensures there exists $c > 0$ such that for all $n \in \mathbb{N}$ we have $\pi_n(\mathcal{L}(X)) = \pi_n^{cn}(\mathcal{L}_{cn}(X))$.

Then, using Chevalley theorem for schemes [42, Chapitre IV,Théorème 1.8.4], we may deduce that $\pi_n(\mathcal{L}(X))$ is Zariski-constructible as the image of a variety by a morphism.

The main concern of motivic integration consists in defining a measure on the arc space $\mathcal{L}(X)$ which maps a "measurable set" to an element in $K_0(\mathrm{Var}_k)$. We are now going to focus on the construction of these "measurable subsets" of $\mathcal{L}(X)$. The main idea consists in using the truncation morphisms⋆ in order to work with subsets of the jet spaces which are easier to use than the infinite dimensional scheme of arc.

Definition 3.13. Let X be a d-dimensional algebraic variety. We say that a subset A of $\mathcal{L}(X)$ is *stable* at the level $k \in \mathbb{N}$ if it satisfies the following conditions for $n \geq k$:

1. $\pi_n(A)$ is Zariski-constructible,
2. $A = \pi_n^{-1}\pi_n(A)$,
3. $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$ is a piecewise trivial fibration with fiber $\mathbb{A}_k^d$.

A stable set is a set which is stable at some level.

Proposition 3.14. *If a subset* A *of* $\mathcal{L}(X)$ *is stable then*

⋆ Truncation to order n is the motivic analogue of reduction modulo p^n in p-adic integration.

$$\tilde{\mu}(A) = [\pi_n(A)]\,\mathbb{L}^{-(n+1)d} \in \mathcal{M}_k$$

doesn't depend on n for n big enough.

This allows one to define a first family of measurable sets.

Definition 3.15. If $A \subset \mathcal{L}(X)$ is stable then we define its measure by $\tilde{\mu}(A)$.

The first part of the following result relies on Hensel lemma.

Lemma 3.16. *If X is a d-dimensional non-singular algebraic variety then the morphisms π_n^m and π_n are surjective. Moreover*

$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

is a piecewise trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$.

Remark 3.17. The previous result doesn't hold in the singular case* . Indeed, using the notations of Example 3.7, the preimage of the jet $(0,t) \in \mathcal{L}_1(X)$ by $\pi_1^2 : \mathcal{L}_2(X) \to \mathcal{L}_1(X)$ is empty.

Definition 3.18. A subset $A \subset \mathcal{L}(X)$ is a cylinder if $A = \pi_k^{-1}(C)$ for some $k \in \mathbb{N}$ and some Zariski-constructible subset C of $\mathcal{L}_k(X)$.

Corollary 3.19. *If X is a d-dimensional non-singular algebraic variety then a cylinder is stable. Moreover if $A = \pi_k^{-1}(C)$, with C a Zariski-constructible subset of $\mathcal{L}_k(X)$, then*

$$\tilde{\mu}(A) = [C]\mathbb{L}^{-(k+1)d}.$$

Example 3.20. If X is a d-dimensional non-singular algebraic variety then $\mathcal{L}(X)$ is stable and

$$\tilde{\mu}(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}.$$

In order to define the motivic integral, we are going to work with a completion of the ring $\mathcal{M}_k$. This will give us a notion of convergence.

Definition 3.21. For $m \in \mathbb{Z}$, we denote by $\mathcal{F}^m\mathcal{M}_k$ the subgroup of $\mathcal{M}_k$ spanned by the elements of the form $[S]\mathbb{L}^{-i}$ where $i - \dim S \geq m$. It defines a filtration† and we denote by $\hat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ with respect to this filtration‡, i.e.,

$$\hat{\mathcal{M}}_k = \varprojlim \mathcal{M}_k / \mathcal{F}^m\mathcal{M}_k \ .$$

* i.e. in this case, it is necessary to avoid confusion between the set $\mathcal{L}_n(X)$ of n-jets and the set $\pi_n(\mathcal{L}(X))$ of truncated arcs at order n.

† i.e. $\mathcal{F}^{m+1}\mathcal{M}_k \subset \mathcal{F}^m\mathcal{M}_k$ and $\mathcal{F}^m\mathcal{M}_k\mathcal{F}^n\mathcal{M}_k \subset \mathcal{F}^{m+n}\mathcal{M}_k$.

‡ E. Looijenga [58] calls it the completion with respect to the *virtual dimension.* V. V. Batyrev [7] [9] gives an equivalent description using a “non-archimedean norm”.

Remark 3.22. It is not known whether the morphism $\mathcal{M}_k \to \hat{\mathcal{M}}_k$ is injective, however the Euler characteristic and the Hodge polynomial factorise through the image of $\mathcal{M}_k$ in $\hat{\mathcal{M}}_k$ (see [22, (6.1)]), which is enough to prove the Kontsevich theorem concerning birational Calabi–Yau varieties.

Example 3.23. A sequence in $\mathcal{M}_k$ converges to 0 in $\hat{\mathcal{M}}_k$ if and only if the virtual dimension* of its elements converges to $-\infty$.

Example 3.24. A series $\sum a_n$ converges in $\hat{\mathcal{M}}_k$ if and only if a_n converges to 0 in $\hat{\mathcal{M}}_k$.

Example 3.25. In $\hat{\mathcal{M}}_k$, $\sum_{i\geq 0} \mathbb{L}^{-ki}$ converges and its limit is the multiplicative inverse of $1-\mathbb{L}^{-k}$.

Definition 3.26. Let $A \subset \mathcal{L}(X)$ be a stable set and $\alpha : A \to \mathbb{N}\cup\{\infty\}$ be a function whose each fiber is stable and such that $\tilde{\mu}(\alpha^{-1}(\infty)) = 0$. We say that $\mathbb{L}^{-\alpha}$ is integrable if the following sequence converges in $\hat{\mathcal{M}}_k$:

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\tilde{\mu} = \sum_n \tilde{\mu}(\alpha^{-1}(n))\mathbb{L}^{-n}.$$

Example 3.27. Let $\mathcal{I}$ be an ideal sheaf then $\mathbb{L}^{-\operatorname{ord}_{\mathcal{I}}}$ is integrable† .

Theorem 3.28 (Kontsevich transformation rule)**.** *Let $A \subset \mathcal{L}(X)$ be a stable set, $h : Y \to X$ be a proper birational map between two non-singular algebraic varieties and $\alpha : \mathcal{L}(X) \to \mathbb{N}\cup\{\infty\}$ be such that $\mathbb{L}^{-\alpha}$ is integrable. Then*

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\tilde{\mu} = \int_{h_*^{-1}(A)} \mathbb{L}^{-(\alpha h_* + \operatorname{ord}_{\mathrm{Jac}_h})} \mathrm{d}\tilde{\mu}$$

where $h_* : \mathcal{L}(Y) \to \mathcal{L}(X)$ is induced by h and where Jac_h is the ideal sheaf locally spanned by the jacobian determinant of h.

We are now able to prove Kontsevich theorem.

Corollary 3.29 ([50])**.** *Two birational Calabi–Yau varieties share the same Hodge numbers.*

Proof. Let X and Y be two birational Calabi–Yau varieties. Since they are birational, there exists a non-singular algebraic variety Z and two proper birational maps such that the following diagram commutes

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* We define the virtual dimension of $\alpha \in \mathcal{M}_k$ as the only integer m such that $\alpha \in \mathcal{F}^{-m}\mathcal{M}_k \setminus \mathcal{F}^{-m+1}\mathcal{M}_k$.

† $\operatorname{ord}_{\mathcal{I}}$ is the contact order along $\mathcal{I}$: an arc induces a morphism $\gamma : \mathcal{O}_X \to k[\![t]\!]$ and then we denote $\operatorname{ord}_{\mathcal{I}}(\gamma) = \sup\{e,\, \gamma(\mathcal{I}) \subset (t^e)\}$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[rr] & & Y \end{tikzcd}.$$

Then

$$\begin{aligned}\mathbb{L}^{-d}[X] &= \int_{\mathcal{L}(X)} \mathbb{L}^{-0}\mathrm{d}\tilde{\mu} = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\operatorname{ord}_{\mathrm{Jac}_{\pi_X}}}\mathrm{d}\tilde{\mu} = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\operatorname{ord}_{\mathrm{Jac}_{\pi_Y}}}\mathrm{d}\tilde{\mu} \\ &= \int_{\mathcal{L}(Y)} \mathbb{L}^{-0}\mathrm{d}\tilde{\mu} = \mathbb{L}^{-d}[Y].\end{aligned}$$

Indeed $\operatorname{div}(\mathrm{Jac}_{\pi_X}) = K_{Z/X} = K_Z - \pi_X^* K_X = K_Z$ since X is Calabi–Yau. We get exactly the same equality for Y.

Hence $[X] = [Y]$ in $\hat{\mathcal{M}}_k$.

We conclude thanks to Remark 3.22. ∎

Whereas stable sets are enough for our purpose (and usually they suffice in the non-singular case), in general a bigger family of measurable sets is necessary. Indeed, for example, $\mathcal{L}(X)$ is not stable if X is singular. It is possible to define such a larger set of measurable sets. A first attempt appears in [22, Definition-Proposition 3.2]. A second and even larger notion of measurable sets is defined in [7, Definition 2.16], [9], [58, Proposition 2.2] and [25, Appendix].

3.2 Denef–Loeser motivic zeta functions

3.2.1 The naive motivic zeta function

Definition 3.30. Let $f : X \to \mathbb{A}_k^1$ be a non-constant morphism where X is a d-dimensional non-singular algebraic variety. For $n \in \mathbb{N}_{>0}$, we set

$$\mathfrak{X}_n(f) = \left\{\gamma \in \mathcal{L}_n(X),\, \operatorname{ord}_t f_n\gamma = n\right\},$$

where $f_n : \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}_k^1)$ is induced by f and where

$$\operatorname{ord}_t : \mathcal{L}_n(\mathbb{A}_k^1) \to \{0, \dots, n, \infty\}$$

is defined as follows. To an n-jet $\gamma \in \mathcal{L}_n(\mathbb{A}_k^1)$ there is a natural associated morphism $\tilde{\gamma} : k[x] \to k[t]/t^{n+1}$ and we denote $\operatorname{ord}_t \gamma = \sup\{e,\, t^e | \tilde{\gamma}(x)(t)\}$.

Then we define the *naive motivic zeta function* of f by

$$Z_f^{\mathrm{naive}}(T) = \sum_{n \geq 1} [\mathfrak{X}_n(f)]\, \mathbb{L}^{-nd} T^n \in \mathcal{M}_k[\![T]\!].$$

Remark 3.31. In $\hat{\mathcal{M}}_k[\![T]\!]$, the naive motivic zeta function may be defined as a motivic integral as follows.

Let s be a formal variable and $T = \mathbb{L}^{-s}$. For $A \subset \mathcal{L}(X)$ a stable set and $\alpha, \beta : A \to \mathbb{N} \cup \{\infty\}$ two functions whose each fiber is stable and such that β is integrable when restricted to each fiber of α, we set

$$\begin{aligned}\int_A \mathbb{L}^{-(\alpha \cdot s+\beta)} \mathrm{d}\tilde{\mu} &:= \sum_{i,j} \tilde{\mu}\left(\alpha^{-1}(i) \cap \beta^{-1}(j)\right) \mathbb{L}^{-(is+j)} \\ &= \sum_i \left(\int_{\alpha^{-1}(i)} \mathbb{L}^{-\beta_{|\alpha^{-1}(i)}} \mathrm{d}\tilde{\mu} \right) \mathbb{L}^{-is} \in \hat{\mathcal{M}}_k[\![T]\!].\end{aligned}$$

Define $\operatorname{ord}_t : \mathcal{L}(\mathbb{A}^1_k) \to \{0, 1, \ldots, \infty\}$ using the morphism $\tilde{\gamma} : k[x] \to k[\![t]\!]$ naturally associated to an arc $\gamma \in \mathcal{L}(\mathbb{A}^1_k)$.

Set

$$Z^{\mathrm{int}}_f(T) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_t \overline{f} \cdot s} \mathrm{d}\tilde{\mu} \in \hat{\mathcal{M}}_k[\![T]\!]$$

where $\overline{f} : \mathcal{L}(X) \to \mathcal{L}(\mathbb{A}^1_k)$ is induced by f.

Then $Z^{\mathrm{naive}}_f(T) = \mathbb{L}^d Z^{\mathrm{int}}_f(T) - \left[X \setminus f^{-1}(0)\right] \in \hat{\mathcal{M}}_k[\![T]\!]$.

This construction points out the analogy with the Igusa (p-adic) zeta function.

The following rationality formula is a direct consequence of Denef–Loeser Key Lemma [22, Lemma 3.4] used to generalize the motivic change of variables formula to the singular case.

Theorem 3.32 ([21, Theorem 2.2.1]). *Let $f : X \to \mathbb{A}^1_k$ be as in the previous definition. Let $h : Y \to X$ be a resolution of $(X, X_0(f) = f^{-1}(0))$. Denote by $(E_i)_{i \in J}$ the irreducible components of $h^{-1}(X_0(f))$. For $i \in J$, we denote by N_i the multiplicity of E_i along the divisor associated to fh and by $\nu_i - 1$ the multiplicity of E_i along the divisor associated to Jac_h. For $I \subset J$, we set $E^\bullet_I = \cap_{i \in I} E_i \setminus \cup_{j \notin I} E_j$. Then*

$$Z^{\mathrm{naive}}_f(T) = \sum_{\varnothing \neq I \subset J} [E^\bullet_I](\mathbb{L}-1)^{|I|} \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

Remark 3.33.

$$Z^{\mathrm{int}}_f(T) = \mathbb{L}^{-d} \sum_{I \subset J} [E^\bullet_I](\mathbb{L}-1)^{|I|} \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

3.2.2 The equivariant motivic zeta function following Guibert–Loeser–Merle

We denote by $\mathbf{G}_{\mathrm{m}} = \operatorname{Spec}\left(k[x,y]/xy - 1\right)$ the multiplicative algebraic group over k. It may be identified with the affine line without the origin.

The following Grothendieck ring is defined in [44].

Definition 3.34. Let S be an algebraic variety over a field k. Let $n \in \mathbb{N}_{>0}$. We define $K_0(\mathrm{Var}^n_{S\times\mathbf{G}_\mathrm{m}})$ as the free abelian group spanned by equivariant isomorphism classes of varieties of the form $Y \to S \times \mathbf{G}_\mathrm{m}$ over $S \times \mathbf{G}_\mathrm{m}$ where Y is equipped with a good action* of $\mathbf{G}_\mathrm{m}$ such that each fiber of $\pi_1 : Y \to S$ is invariant by the action and such that $\pi_2 : Y \to \mathbf{G}_\mathrm{m}$ satisfies $\pi_2(\lambda \cdot x) = \lambda^n \pi_2(x)$ modulo the following relations:

(i) If Z is a closed $\mathbf{G}_\mathrm{m}$-invariant subvariety of Y then $[Y \setminus Z] + [Z] = [Y]$,

(ii) Let $f : Y \times \mathbb{A}^m_k \to Y \to S \times \mathbf{G}_\mathrm{m}$ equipped with two actions σ and σ' which are liftings† of the same action τ on Y then‡

$$[f : Y \times \mathbb{A}^m_k \to S \times \mathbf{G}_\mathrm{m}, \sigma] = [f : Y \times \mathbb{A}^m_k \to S \times \mathbf{G}_\mathrm{m}, \sigma'].$$

The fiber product over $S \times \mathbf{G}_\mathrm{m}$ induces a ring structure:

(iii) $[X][Y] = [X \times_{S\times\mathbf{G}_\mathrm{m}} Y]$ where the $\mathbf{G}_\mathrm{m}$-action on $X \times_{S\times\mathbf{G}_\mathrm{m}} Y$ is diagonal. The product unit is given by the identity map $1 = [S \times \mathbf{G}_\mathrm{m} \to S \times \mathbf{G}_\mathrm{m}]$ where the $\mathbf{G}_\mathrm{m}$-action is trivial on S and given by $\lambda \cdot x = \lambda^n x$ on the first $\mathbf{G}_\mathrm{m}$.

Finally, the cartesian product induces a structure of $K_0(\mathrm{Var}_k)$-module.

(iv) If $[X] \in K_0(\mathrm{Var}_k)$ and $[Y \to S \times \mathbf{G}_\mathrm{m}] \in K_0(\mathrm{Var}^n_{S\times\mathbf{G}_\mathrm{m}})$ then

$$[X][Y] = [X \times Y \to Y \to S \times \mathbf{G}_\mathrm{m}]$$

where the $\mathbf{G}_\mathrm{m}$-action is trivial on X.

Notation 3.35. The class $\mathbb{L} \in K_0(\mathrm{Var}_k)$ induces, via the scalar product, a class

$$\mathbb{L} = \mathbb{L} \cdot 1 = [\mathbb{A}^1_k \times S \times \mathbf{G}_\mathrm{m} \to S \times \mathbf{G}_\mathrm{m}]$$

where $\lambda \cdot (x, s, r) = (x, s, \lambda^n r)$.

Then we set

$$\mathcal{M}^n_{S\times\mathbf{G}_\mathrm{m}} = K_0(\mathrm{Var}^n_{S\times\mathbf{G}_\mathrm{m}}) \left[\mathbb{L}^{-1}\right].$$

Notation 3.36. For $n = km$ with $k \in \mathbb{N}_{>0}$ we define the morphism $\theta_{mn} : \mathrm{Var}^m_{S\times\mathbf{G}_\mathrm{m}} \to \mathrm{Var}^n_{S\times\mathbf{G}_\mathrm{m}}$ which only changes the action by $\lambda \cdot x = \lambda^k \cdot x$. This morphism is compatible with the Grothendieck module and with the localization by $\mathbb{L}$ so that we can construct $K_0(\mathrm{Var}_{S\times\mathbf{G}_\mathrm{m}})$ and $\mathcal{M}_{S\times\mathbf{G}_\mathrm{m}}$ using inductive limits.

Definition 3.37. Let $f : X \to \mathbb{A}^1_k$ be a non-constant morphism where X is a non-singular algebraic variety. For $n \in \mathbb{N}_{>0}$, we set

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* i.e. the orbits are included in open affine parts in order to be able to work locally.

† i.e. $\mathrm{pr}_1(\lambda \cdot_\sigma (y, x)) = \mathrm{pr}_1(\lambda \cdot_{\sigma'} (y, x)) = \lambda \cdot_\tau y$.

‡ This relation is purely technical, it allows one to factorise terms of the zeta function whose actions are not exactly the same.

$$\mathfrak{X}_n(f) = \{\gamma \in \mathcal{L}_n(X),\, \mathrm{ord}_t\, f_n\gamma = n\}\,.$$

We define a $\mathbf{G}_{\mathrm{m}}$-action on $\mathfrak{X}_n(f)$ by $\lambda \cdot \gamma(t) = \gamma(\lambda t)$.

We define the *angular component* morphism $\mathrm{ac}_{f,n} : \mathfrak{X}_n(f) \to \mathbf{G}_{\mathrm{m}}$ by $\mathrm{ac}_{f,n}(\gamma) = \mathrm{ac}(f\gamma)$ where $\mathrm{ac} : \mathcal{L}_n(\mathbb{A}^1_k) \to \mathbb{A}^1_k$ associates to γ the first non-zero coefficient of $\tilde{\gamma}(x)(t)$ (or 0 if $\tilde{\gamma}(x) = 0$) where $\tilde{\gamma} : k[x] \to k[\![t]\!]$ is induced by γ.

Finally we define $\tilde{f}_n : \mathfrak{X}_n(f) \to X_0(f)$ which maps γ to its origin.

Hence the class

$$\left[(\tilde{f}_n, \mathrm{ac}_{f,n}) : \mathfrak{X}_n(f) \to X_0(f) \times \mathbf{G}_{\mathrm{m}}\right] \in K_0(\mathrm{Var}^n_{X_0(f)\times\mathbf{G}_{\mathrm{m}}})$$

is well-defined.

Then the *equivariant motivic zeta function* of f is defined by

$$Z_f(T) = \sum_{n\geq 1} [\mathfrak{X}_n(f)]\mathbb{L}^{-nd}T^n \in \mathcal{M}_{X_0(f)\times\mathbf{G}_{\mathrm{m}}}[\![T]\!].$$

Similarly to the naive case, we have a rationality formula in terms of resolution. We refer the reader to [44, §3.6] for the construction of $[U_I]$.

Theorem 3.38 ([25, §3.3], [44, §3.6])**.**

$$Z_f(T) = \sum_{\varnothing\neq I\subset J} [U_I] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}}.$$

Definition 3.39 (Motivic Milnor fiber)**.** We denote by

$$\mathscr{S}_f = -\lim_{T\infty} Z_f(T) \in \mathcal{M}_{X_0(f)\times\mathbf{G}_{\mathrm{m}}}$$

the motivic Milnor fiber of f.

Theorem 3.40 ([44, Theorem 5.18])**.** *We have the following relation*

$$\mathscr{S}_{f_1\oplus f_2} = -\mathscr{S}_{f_1} * \mathscr{S}_{f_2} + \mathscr{S}_{f_1} + \mathscr{S}_{f_2}$$

for some product

$$* : \mathcal{M}_{X_0(f)\times\mathbf{G}_{\mathrm{m}}} \times \mathcal{M}_{X_0(f)\times\mathbf{G}_{\mathrm{m}}} \to \mathcal{M}_{X_0(f)\times\mathbf{G}_{\mathrm{m}}}.$$

We may notice that in [23, Definition 4.2.2], J. Denef and F. Loeser work with slightly different settings. They not only consider arcs such that $\mathrm{ord}_t\, f\gamma = n$ but also arcs with $\mathrm{ord}_t\, f\gamma > n$ (with no action in this last case). This allows them to get a convolution formula for this "modified equivariant zeta function". Particularly, this allows one to recover Theorem 3.40.

This is the strategy used in what follows when we introduce some modified zeta function $\tilde{Z}_f(T)$ satisfying

$$\tilde{Z}_{f_1 \oplus f_2}(T) = \tilde{Z}_{f_1}(T) \circledast \tilde{Z}_{f_2}(T)$$

where $\circledast$ consists in applying coefficientwise some convolution product.

4. Koike–Parusiński zeta functions

As noted before, blow-analyticity and arc-analyticity are closely related. And since motivic integration is an "integration theory" on arc spaces, it is natural to try using it in order to study the blow-analytic equivalence. Such a strategy was first initiated by S. Koike and A. Parusiński [47].

They define a naive zeta function and two zeta functions with signs (which play the role of the motivic equivariant zeta function) in a way similar to the ones of Denef–Loeser but they realized the motivic measure through the Euler characteristic with compact support.

Notice that we are going to work with real analytic arcs (and not formal arcs).

Definition 4.1. We define the space of arcs mapping the origin to the origin

$$\mathcal{L}(\mathbb{R}^d, 0) = \left\{ \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0),\ \gamma \text{ real analytic} \right\}$$

and the space of n-jets mapping the origin to the origin⋆

$$\mathcal{L}_n(\mathbb{R}^d, 0) = \left\{ \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \cdots + \mathbf{a}_n t^n,\ \mathbf{a}_i \in \mathbb{R}^d \right\}.$$

Definition 4.2 ([47, §1.1])**.** Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a real analytic function. For $n \in \mathbb{N}_{>0}$, we set

$$\mathfrak{X}_n(f) = \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = ct^n + \cdots,\ c \neq 0 \right\},$$

$$\mathfrak{X}_n^{>}(f) = \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = ct^n + \cdots,\ c > 0 \right\},$$

$$\mathfrak{X}_n^{<}(f) = \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = ct^n + \cdots,\ c < 0 \right\}$$

and we define Koike–Parusiński zeta functions by

$$Z_f^{\chi_c, \text{naive}}(T) = \sum_{n \geq 1} \chi_c\left(\mathfrak{X}_n(f)\right)(-1)^{nd} T^n \in \mathbb{Z}[\![T]\!],$$

⋆ Since $\mathbb{R}^d$ is non-singular, we may assume that n-jets are just truncated arcs. We may first use Hensel lemma, as in the algebraic classical motivic case, to lift an n-jet to a formal arc, and then apply Artin approximation theorem [5] to find a real analytic arc whose expansion coincides with the previous formal arc up to degree n. This allows one to prove that an n-jet may be lifted to a real analytic arc.

$$Z_f^{\chi_c,>}(T) = \sum_{n\geq 1} \chi_c\left(\mathfrak{X}_n^{>}(f)\right)(-1)^{nd}T^n \in \mathbb{Z}[\![T]\!],$$

$$Z_f^{\chi_c,<}(T) = \sum_{n\geq 1} \chi_c\left(\mathfrak{X}_n^{<}(f)\right)(-1)^{nd}T^n \in \mathbb{Z}[\![T]\!].$$

Remark 4.3 ([47, §1.1])**.** $Z_f^{\chi_c,\mathrm{naive}}(T) = Z_f^{\chi_c,>}(T) + Z_f^{\chi_c,<}(T)$.

Remark 4.4. Contrary to Denef–Loeser motivic zeta functions, these zeta functions are local since we restrict to arcs mapping the origin to the origin.

S. Koike and A. Parusiński [47, Lemma 4.2] proved a first adaptation to the real case of Denef–Loeser key lemma for the motivic change of variables formula. This allows them to get the following rationality formulas.

Theorem 4.5 ([47, §1.2])**.** *Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a real analytic function germ. There exists $\sigma : (M, \sigma^{-1}(0)) \to (\mathbb{R}^d, 0)$ a locally finite sequence of blowings-up with non-singular centers such that $f\sigma$ and the jacobian determinant $\operatorname{Jac}\sigma$ simultaneously have only normal crossings and such that σ is an isomorphism outside the zero set of f. We denote by $(E_i)_{i\in J}$ the irreducible components of $(f\sigma)^{-1}(0)$ and, for $i \in J$,*

$$N_i = \operatorname{mult}_{E_i} f\sigma \qquad \text{and} \qquad \nu_i - 1 = \operatorname{mult}_{E_i} \operatorname{Jac}\sigma.$$

We consider the following natural stratification of M given, for $I \subset J$, by

$$E_I^\bullet = \bigcap_{i\in I} E_i \setminus \bigcup_{j\notin I} E_j.$$

Then

$$Z_f^{\chi_c,\mathrm{naive}}(T) = \sum_{\varnothing\neq I\subset J} (-2)^{|I|}\chi_c\left(E_I^\bullet \cap \sigma^{-1}(0)\right) \prod_{i\in I} \frac{(-1)^{\nu_i}T^{N_i}}{1-(-1)^{\nu_i}T^{N_i}}$$

and, for $\varepsilon \in \{<,>\}$,

$$Z_f^{\chi_c,\varepsilon}(T) = \sum_{\varnothing\neq I\subset J} (-1)^{|I|} K_I^\varepsilon \prod_{i\in I} \frac{(-1)^{\nu_i}T^{N_i}}{1-(-1)^{\nu_i}T^{N_i}}.$$

We refer the reader to [47, §1.2] for the definition of K_I^ε.

The previous rationality formulas allow one to prove that the Koike–Parusiński zeta functions are invariants of the blow-analytic equivalence.

Theorem 4.6 ([47, Theorem 4.5])**.** *If $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are two blow-analytically equivalent real analytic function germs then*

$$Z_f^{\chi_c,\mathrm{naive}} = Z_g^{\chi_c,\mathrm{naive}}, \qquad Z_f^{\chi_c,>} = Z_g^{\chi_c,>}, \qquad Z_f^{\chi_c,<} = Z_g^{\chi_c,<}.$$

Definition 4.7 ([47, §2])**.** For $\varepsilon \in \{<,>\}$, we define the modified Koike–Parusiński zeta function with sign ε by

$$\tilde{Z}_f^{\chi_c,\varepsilon}(T) = \frac{1 - Z_f^{\chi_c,\mathrm{naive}}(T)}{1-T} - 1 + Z_f^{\chi_c,\varepsilon}(T).$$

Theorem 4.8 ([47, Theorem 2.3])**.** *For $\varepsilon \in \{<,>\}$, the following formula holds*

$$\tilde{Z}_{f_1 \oplus f_2}^{\chi_c,\varepsilon}(T) = \tilde{Z}_{f_1}^{\chi_c,\varepsilon}(T) \odot \tilde{Z}_{f_2}^{\chi_c,\varepsilon}(T)$$

where the product $\odot$ is the Hadamard product which consists in applying coefficientwise the classical product of $\mathbb{Z}$.

Remark 4.9. The couples $(Z_f^{\chi_c,>}(T), Z_f^{\chi_c,<}(T))$ and $(\tilde{Z}_f^{\chi_c,>}(T), \tilde{Z}_f^{\chi_c,<}(T))$ are equivalent. Indeed, we have the following formula [47, (2.5)]:

$$Z_f^{\chi_c,\varepsilon}(T) = \frac{1 + \tilde{Z}_f^{\chi_c,>}(T) + \tilde{Z}_f^{\chi_c,<}(T)}{1+T} + 1 + \tilde{Z}_f^{\chi_c,\varepsilon}(T).$$

Using Koike–Parusiński zeta functions and Fukui invariants, S. Koike and A. Parusiński [47, Theorem 6.1] classified entirely Brieskorn polynomials⋆ in two variables up to the blow-analytic equivalence.

However, Koike–Parusiński zeta functions alone are not enough to distinguish some blow-analytic equivalence classes of such Brieskorn polynomials, see [47, Remark 6.3].

Using these invariants, O. M. Abderrahmane [2] proved that the blow-analytic type of a singular weigthed homogeneous polynomial in two variables non-degenerate with respect to its Newton polyhedron determines its weights.

5. The virtual Poincaré polynomial of $\mathcal{AS}$-sets

Most of the material of this section is covered by the survey [56].

5.1 Arc-symmetric sets

In the article [55], where K. Kurdyka introduces arc-analytic maps, he also introduces arc-symmetric sets. An arc-symmetric set is a semialgebraic subset of $\mathbb{R}^d$ such that given a real analytic arc on $\mathbb{R}^d$, either this arc is entirely included in the subset or it meets it at isolated points only.

Definition 5.1 ([55, Définition 1.1])**.** A semialgebraic subset S of $\mathbb{R}^d$ is *arc-symmetric* if it satisfies the following equivalent conditions:

⋆ i.e. polynomials of the form $\pm x^p \pm y^q$.

(i) For $\gamma : (-1,1) \to \mathbb{R}^d$ a real analytic arc, if $\mathrm{Int}(\gamma^{-1}(S)) \neq \varnothing$ then $\gamma^{-1}(S) = (-1,1)$,
(ii) For $\gamma : (-1,1) \to \mathbb{R}^d$ a real analytic arc, if $\gamma((-1,0)) \subset S$ then $\gamma((-1,1)) \subset S$,
(iii) For $\gamma : (-1,1) \to \mathbb{R}^d$ an injective real analytic arc, if $\gamma((-1,0)) \subset S$ then $\gamma((-1,1)) \subset S$,
(iv) For $\gamma : (-1,1) \to \mathbb{R}^d$ a Nash arc* , if $\gamma((-1,0)) \subset S$ then $\gamma((-1,1)) \subset S$.

Example 5.2. The unicity of the analytic continuation ensures that real algebraic sets (resp. real analytic and semialgebraic sets) are arc-symmetric.

The following example highlights the usefulness of arc-symmetric sets.

Example 5.3 ([55, Exemple 1.2])**.** Let $V = \{z(x^2+y^2) = x^3\} \subset \mathbb{R}^3$ be the Cartan umbrella. Then V is irreducible as an algebraic (or analytic) set but can be decomposed non-trivially into the union of two arc-symmetric sets: the handle, that is the z-axis, and the canopy, that is the (Euclidean) closure of $V \setminus \{(0,0,z),\ z \neq 0\}$.

Hence arc-symmetric sets are finer than analytic components of a real algebraic set.

The following remark is a direct consequence of the curve selection lemma [14, Proposition 8.1.13].

Remark 5.4 ([55, Remarque 1.3])**.** An arc-symmetric set is closed for the Euclidean topology.

Theorem 5.5 ([55, Théorème 1.4])**.** *Arc-symmetric subsets of $\mathbb{R}^d$ are exactly the closed sets of a noetherian topology on $\mathbb{R}^d$ that we will denote by $\mathcal{AR}$.*

Definition 5.6 ([55, Définition 2.10])**.** The $\mathcal{AR}$-topology being noetherian, we may associate to X its Krull dimension $\dim_{\mathcal{AR}} X$ defined as the length of the longest strictly increasing sequence of $\mathcal{AR}$-irreducible subsets ending at X.

Proposition 5.7 ([55, Proposition 2.11])**.** *Let S be a semialgebraic subset, then the following dimensions coincide:*
- *The $\mathcal{AR}$-dimension* $\dim_{\mathcal{AR}} \overline{S}^{\mathcal{AR}}$,
- *The Zariski dimension* $\dim_{\mathrm{Zar}} \overline{S}^{\mathrm{Zar}}$,
- *The geometric dimension* $\dim S$ *defined as the greatest dimension of real analytic submanifolds included in S.*

5.2 More properties of arc-analytic maps

The notion of arc-analytic map can be extended naturally to maps between

* i.e. a real analytic arc whose graph is semialgebraic, see Definition 6.1.

arc-symmetric sets.

Definition 5.8 ([55, Définition 4.1])**.** A map $f : X \to Y$ between two arc-symmetric sets is *arc-analytic* if it maps by composition real analytic arcs on X to real analytic arcs on Y.

Proposition 5.9 ([55, Théorème 4.2, Proposition 5.1])**.** *Let $f : X \to \mathbb{R}^n$ be an arc-analytic and semialgebraic map defined on X an arc-symmetric subset of $\mathbb{R}^d$, then*
- *The graph of f is an arc-symmetric subset of $\mathbb{R}^d \times \mathbb{R}^n$,*
- *The map f is continuous for the Euclidean topology,*
- *The map f is continuous for the $\mathcal{AR}$-topology (particularly, the inverse image by f of an arc-symmetric set is arc-symmetric),*
- *If moreover f is injective and proper then its image is arc-symmetric.*

Note that in general the image of a proper injective polynomial map defined on a real algebraic set is not algebraic. Thus arc-symmetric sets appear naturally as the images of such maps.

The following theorem states that a semialgebraic and arc-analytic map is real analytic outside some nowhere dense set.

Theorem 5.10 ([55, Théorème 5.2])**.** *Let $f : X \to \mathbb{R}^n$ be a semialgebraic and arc-analytic map defined on an arc-symmetric set. Then*

$$\dim \operatorname{Sing} f \leq \dim X - 2$$

where $\operatorname{Sing} f$ is the set of points of X where f is not real analytic.

The following result is a direct consequence of Proposition 5.7. It is some kind of "unicity of the arc-analytic continuation".

Theorem 5.11 ([55, Proposition 5.3])**.** *Let $f, g : X \to \mathbb{R}^n$ be two semialgebraic and arc-analytic maps defined on an irreducible arc-symmetric set X. If $f = g$ on an open semialgebraic subset U satisfying $\dim U = \dim X$ then $f = g$ on X.*

Finally, there is a Nullstellensatz theorem for semialgebraic and arc-analytic functions.

Theorem 5.12 ([55, Proposition 6.5])**.** *Let $f, g : X \to \mathbb{R}$ be two semialgebraic and arc-analytic functions defined on an arc-symmetric set. If $f^{-1}(0) \subset g^{-1}(0)$, then there exists a semialgebraic and arc-analytic function $h : X \to \mathbb{R}$ together with an integer $k \in \mathbb{N}$ such that $g^k = f \cdot h$.*

5.3 Constructible categories

Definition 5.13 ([65, Definition 2.4])**.** Let $\mathcal{C}$ be a collection of semialgebraic subsets. We say that $\mathcal{C}$ is a *constructible category* if it satisfies the following axioms:

A1. $\mathcal{C}$ contains real algebraic sets,

A2. $\mathcal{C}$ is stable by union, intersection and taking the complement,

A3. a. The inverse image of a $\mathcal{C}$-set by a map between two $\mathcal{C}$-sets whose graph is in $\mathcal{C}$ is a $\mathcal{C}$-set,

b. The image of a $\mathcal{C}$-set by a injective map between two $\mathcal{C}$-sets whose graph is in $\mathcal{C}$ is a $\mathcal{C}$-set,

A4. A locally compact $\mathcal{C}$-set is Euler in codimension 1, i.e. if X is a locally compact $\mathcal{C}$-set, there exists a semialgebraic subset Y of X satisfying $\dim Y \leq \dim X - 2$ such that $X \setminus Y$ is Euler* .

Example 5.14 ([65, Theorem 2.8])**.** Zariski-constructible sets (i.e. sets of the boolean algebra spanned by real algebraic sets) form a constructible category denoted by $\mathcal{AC}$. It is the smallest constructible category since it is included in any other constructible category.

A constructible category admits a closure which is compatible with the dimension.

Theorem 5.15 ([65, Theorem 2.5])**.** *Let $\mathcal{C}$ be a constructible category and X be a locally closed $\mathcal{C}$-set. Then for every subset A of X there is a smallest closed $\mathcal{C}$-subset of X which contains A. It is denoted by $\overline{A}^{\mathcal{C}}$. Thus any other closed $\mathcal{C}$-subset of X which contains A must contain $\overline{A}^{\mathcal{C}}$.*

Remark 5.16 ([65, Remark 2.7])**.** If A is semialgebraic then $\dim A = \dim \overline{A}^{\mathcal{C}}$. Moreover, if A is a $\mathcal{C}$-set then $\overline{A}^{\mathcal{C}} = A \cup \overline{(\overline{A} \setminus A)}^{\mathcal{C}}$, in particular $\dim(\overline{A}^{\mathcal{C}} \setminus A) < \dim A$.

Remark 5.17 ([65, p361])**.** Notice that the closure of the constructible category $\mathcal{AC}$ is not the Zariski closure. More precisely, if A is a semialgebraic subset of a real algebraic set, then $\overline{A} \subset \overline{A}^{\mathcal{AC}} \subset \overline{A}^{\mathrm{Zar}}$ but these inclusions may be strict.

For example if A is the non-singular part of maximal dimension of the Whitney umbrella $\{zx^2 = y^2\} \subset \mathbb{R}^3$ then $\overline{A} \subsetneq \overline{A}^{\mathcal{AC}} = \overline{A}^{\mathrm{Zar}}$.

For example if A is the non-singular part of maximal dimension of the Cartan umbrella $\{z(x^2 + y^2) = x^3\} \subset \mathbb{R}^3$ then $\overline{A} = \overline{A}^{\mathcal{AC}} \subsetneq \overline{A}^{\mathrm{Zar}}$.

5.4 The $\mathcal{AS}$ collection

We are going to work with a slightly different framework from arc-symmetric

* i.e. X is Euler if for all $x \in X$ the Euler-Poincaré characteristic of X in x, $\chi(X, X \setminus x) = \sum(-1)^i \dim H_i(X, X \setminus x; \mathbb{Z}_2)$, is odd. We may prove ([65, Corollary 4.7]) that a locally compact $\mathcal{C}$-set is actually Euler. The interest of this axiom resides in the fact that a locally compact semialgebraic set is Euler in codimension 1 if and only if it admits a fundamental class for its Borel–Moore homology with $\mathbb{Z}_2$ coefficients ([65, Remark 1.5]). Hence the sets of a constructible category share topological properties with real algebraic sets, see [70].

sets. It is the collection $\mathcal{AS}$ which was defined by A. Parusiński [65].

Definition 5.18. The $\mathcal{AS}$ collection is the boolean algebra spanned by arc-symmetric subsets of a real projective space* $\mathbb{P}^n_{\mathbb{R}}$.

We get the following characterization.

Proposition 5.19 ([65, §4.2])**.** *A subset S of $\mathbb{P}^n_{\mathbb{R}}$ is an $\mathcal{AS}$-set if and only if it is semialgebraic and satisfies the following condition: for every real analytic arc $\gamma : (-1,1) \to \mathbb{P}^n_{\mathbb{R}}$ satisfying $\gamma((-1,0)) \subset S$, there exists $\varepsilon > 0$ such that $\gamma((0,\varepsilon)) \subset S$.*

Remark 5.20. Closed† $\mathcal{AS}$-subsets of $\mathbb{P}^n_{\mathbb{R}}$ are exactly the arc-symmetric subsets of $\mathbb{P}^n_{\mathbb{R}}$.

The following example shows that $\mathcal{AS}$-sets have a better behavior at infinity than arc-symmetric sets.

Example 5.21 ([65, §4.2])**.** The hyperbola branch $\{(x,y) \in \mathbb{R}^2,\, xy = 1, x > 0\}$ is arc-symmetric but is not $\mathcal{AS}$ since if we embed the branch in $\mathbb{P}^2_{\mathbb{R}}$, we may find a real analytic arc going from a branch to the other one.

The following result is an $\mathcal{AS}$ version of Theorem 5.5. The proof is quite similar as we can see in the proof of [65, Theorem 2.5].

Theorem 5.22. *The $\mathcal{AS}$-closed‡ subsets of $\mathbb{P}^n_{\mathbb{R}}$ are exactly the closed sets of a noetherian topology on $\mathbb{P}^n_{\mathbb{R}}$.*

Another advantage of the $\mathcal{AS}$-collection is that it is a constructible category.

Theorem 5.23 ([65, Theorem 4.5 & Proposition 4.6])**.**
The $\mathcal{AS}$ collection is a constructible category. Moreover it is the biggest one since every other constructible category is contained in $\mathcal{AS}$. This is also the only constructible category which contains the connected components of compact real algebraic sets.

In particular, the image of an $\mathcal{AS}$-set by an injective map whose graph is $\mathcal{AS}$ is an $\mathcal{AS}$-set. Contrary to the arc-symmetric case (See Proposition 5.9), it is not necessary to assume that the map is proper. The following example shows that the properness is not a superfluous condition.

Example 5.24 ([56, Remark 3.6])**.** The image of the hyperbola branch $\{(x,y) \in \mathbb{R}^2,\, xy = 1, x > 0\}$ by the projection on the first coordinate is not arc-symmetric.

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* By [14, Theorem 3.4.4], $\mathbb{P}^n_{\mathbb{R}}$ is biregularly isomorphic to an algebraic subset of an Euclidean space $\mathbb{R}^N$.

† for the Euclidean topology.

‡ for the Euclidean topology.

The following result may be easily deduced from [65, Theorem 4.3]. We may see this result as an analogue of the Chevalley theorem which states that, over an algebraically closed field, the image of a Zariski-constructible set by a regular map is Zariski-constructible.

Theorem 5.25. *The image of an $\mathcal{AS}$-set by a regular map whose each fiber has an odd Euler characteristic with compact support is an $\mathcal{AS}$-set.*

5.5 The virtual Poincaré polynomial

C. McCrory and A. Parusiński [60] proved there exists a unique additive invariant of real algebraic varieties which coincides with the Poincaré polynomial with $\mathbb{Z}_2$ coefficients for compact non-singular algebraic varieties. It is called the *virtual Poincaré polynomial*.

This invariant was extended to $\mathcal{AS}$-sets by G. Fichou [28] who also showed that it is invariant under Nash-isomorphisms*.

These constructions rely on the weak factorization theorem [75, 3] in a way similar to what has been done by F. Bittner [12] to prove that $K_0(\mathrm{Var}_k)$ is spanned by the isomorphism classes of projective non-singular algebraic varieties with the following relations: $0 = [\varnothing]$ and $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$ where $\mathrm{Bl}_Y X$ is the blowing-up of X along a non-singular closed subvariety Y with exceptional divisor E.

Using a new construction, C. McCrory and A. Parusiński proved that the virtual Poincaré polynomial for $\mathcal{AS}$-sets is in fact an invariant up to bijections whose graph is $\mathcal{AS}$ (See the proof of [61, Theorem 4.6] together with [61, Proposition 4.3]).

Theorem 5.26 ([60], [28], [61])**.** *There is a unique map*† $\beta : \mathcal{AS} \to \mathbb{Z}[u]$, *called the* virtual Poincaré polynomial, *such that*

- $\beta(X \sqcup Y) = \beta(X) + \beta(Y)$,
- $\beta(X \times Y) = \beta(X)\beta(Y)$,
- *If* $X \neq \varnothing$ *then* $\deg \beta(X) = \dim X$ *and the leading coefficient of* $\beta(X)$ *is positive*‡,
- *If* X *is compact and non-singular then* $\beta(X) = \sum_i \dim H_i(X, \mathbb{Z}_2)u^i$,
- *If* X *and* Y *are two* $\mathcal{AS}$*-sets such that there is a bijection with* $\mathcal{AS}$*-graph between them then* $\beta(X) = \beta(Y)$.

Remark 5.27. Notice that if we work with semialgebraic sets with no additional assumption then every additive invariant of semialgebraic sets up to semialgebraic

* Two $\mathcal{AS}$-sets A and B are Nash-isomorphic if there exist two compact Nash manifolds M and N such that $A \subset M$ and $B \subset N$ and a Nash-isomorphism $\varphi : M \to N$ such that $\varphi(A) = B$, see [28, Definition 2.15 & Theorem 2.16].

† $\mathcal{AS}$ is a set since each $\mathcal{AS}$-set is embedded in a projective space.

‡ $\beta(\varnothing) = 0$

homeomorphisms factorises through the Euler characteristic with compact support*.

This highlights that the virtual Poincaré polynomial, which encodes the dimension, is a very powerful additive invariant.

Notice also that we recover the Euler characteristic with compact support by evaluating the virtual Poincaré polynomial in $u=-1$.

Example 5.28. Since $\mathbb{P}^1_{\mathbb{R}}$ is compact and non-singular we have $\beta(\mathbb{P}^1_{\mathbb{R}})=u+1$ and so, by additivity, $\beta(\mathbb{R})=\beta(\mathbb{P}^1_{\mathbb{R}})-\beta(\mathrm{pt})=u$.

The following example shows that the virtual Poincaré polynomial is not a topological invariant.

Example 5.29 ([60])**.** The strict transform of the blowing-up of the eight curve $H=\{y^2=x^2-x^4\}$ at the origin is a circle and the inverse image of the origin consists in two points. Hence $\beta(\mathbb{S}^1)-2\beta(\mathrm{pt})=\beta(H)-\beta(\mathrm{pt})$ and so $\beta(H)=u$.

If we blow-up the following union of two circles tangent at the origin $T=\{((x+1)^2+y^2-1)((x-1)^2+y^2-1)=0\}$, then the strict transform consists in two circles and the inverse image of the origin consists in two points. Hence $2\beta(\mathbb{S}^1)-2\beta(\mathrm{pt})=\beta(T)-\beta(\mathrm{pt})$ and so $\beta(T)=2u+1$.

6. The blow-Nash equivalence

6.1 Nash maps

Nash maps and Nash manifolds were first studied by J. Nash [62] where he considered real analytic functions satisfying non trivial polynomial equations. Thanks to this notion, in the same article, he proves that a connected and compact C^∞-manifold is diffeomorphic to a non-singular connected component of a real algebraic variety. This result was improved by A. Tognoli [71]: a compact C^∞-manifold is diffeomorphic to a non-singular real algebraic set.

M. Artin and B. Mazur [6] gave a characterization of Nash functions in order to define an abstract notion of Nash manifold.

Nash functions are powerful since they share good algebraic properties with polynomial maps and good geometric properties with real analytic geometry such as an implicit function theorem [14, Proposition 2.9.7 & Corollary 2.9.8]. We refer the reader to [14, §8] and [69] for more details.

* See [68]. More precisely the Grothendieck ring of semialgebraic sets up to semialgebraic homeomorphisms is isomorphic to $\mathbb{Z}$ via the Euler characteristic with compact support. This is due to the following cell decomposition property of semialgebraic sets: a semialgebraic set may be splitted as a disjoint union of semialgebraic sets which are semialgebraically homeomorphic to $(0,1)^d$. We may conclude by noticing that we can cover $(0,1)^d$ with two semialgebraic sets that are semialgebraically isomorphic to $(0,1)^d$ and such that the intersection is semialgebraically isomorphic to $(0,1)^{d-1}$.

Definition 6.1 ([62], [14, §8])**.** Let $U \subset \mathbb{R}^d$ be an open semialgebraic set. A function $f : U \to \mathbb{R}$ is *Nash* if it satisfies the following equivalent conditions:

1. f is semialgebraic and C^∞,
2. f is real analytic and satisfies a non trivial polynomial equation.

Definition 6.2. A map $f : U \to \mathbb{R}^n$ defined on an open semialgebraic set is *Nash* if so are its component functions.

Definition 6.3. A d-dimensional *Nash manifold* is a semialgebraic set M of $\mathbb{R}^p$ such that every $x \in M$ admits a semialgebraic open neighborhood which straightens M, i.e. there exists U a semialgebraic open neighborhood of the origin in $\mathbb{R}^p$, V a semialgebraic open neighborhood of x in $\mathbb{R}^p$ and $\varphi : V \to U$ a Nash diffeomorphism such that $\varphi(x) = 0$ and $\varphi(M \cap V) = \mathbb{R}^d \times \{0\}$.

Example 6.4. We may derive from the Jacobian criterion and the Nash implicit function theorem that a non-singular real algebraic set has a natural structure of Nash manifold.

M. Artin and B. Mazur gave a description of Nash maps as maps which lift to polynomial maps on irreducible non-singular real algebraic sets.

Theorem 6.5 (Artin-Mazur description [6] [14, Theorem 8.4.4])**.** *Let $M \subset \mathbb{R}^p$ be an n-dimensional Nash manifold which is semialgebraically connected. Let $f : M \to \mathbb{R}^k$ be a Nash map. Then there exists an n-dimensional irreducible non-singular real algebraic set $V \subset \mathbb{R}^{p+q}$, an open semialgebraic subset M' of V, a Nash diffeomorphism $\sigma : M \to M'$ and a polynomial map $g : V \to \mathbb{R}^k$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R}^{p+q} \arrow[dd, "\Pi"'] & V \arrow[l, hook'] \arrow[dr, "g"] & \\
& M' \arrow[u, hook] & \mathbb{R}^k \\
\mathbb{R}^p & M \arrow[l, hook'] \arrow[u, "\sigma"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where Π is the projection on the first p coordinates.

Moreover M' is a connected component of $\Pi^{-1}(M) \cap V$.

6.2 The blow-Nash equivalence

G. Fichou [28] introduces zeta functions similar to the ones of S. Koike and A. Parusiński but realized through the virtual Poincaré polynomial which is a richer invariant than the Euler characteristic with compact support.

Whereas his zeta functions are well defined for real analytic function germs, he has to restrict to Nash function germs in order to ensure that the object involved in the rationality formula are $\mathcal{AS}$.

For this technical reason, he introduces an algebraic version of the blow-analytic equivalence for Nash function germs called the blow-Nash equivalence. It is not known whether the first definition of the blow-Nash equivalence is an equivalence relation. For this reason, the definition evolves in [29] to get an equivalence relation. But with this last definition the Fichou zeta functions are not invariants of the blow-Nash equivalence (but we may still derive from them few weaker invariants [29, Proposition 2.6, Proposition 3.2 & Theorem 3.4]). Finally the definition of the blow-Nash equivalence stabilized as a notion which is midway from the previous notions [30, 31, 32]. From now on, the term "blow-Nash equivalence" will refer to this last notion.

Fichou zeta functions are invariants of the blow-Nash equivalence but it was still not obvious originally that this notion is an equivalence relation.

Definition 6.6 ([28, Definition 4.1])**.** Two Nash function germs $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are *blow-Nash equivalent* if there exist two Nash modifications*

$$\mu_f : (M_f, \mu_f^{-1}(0)) \to (\mathbb{R}^d, 0) \quad \text{and} \quad \mu_g : (M_g, \mu_g^{-1}(0)) \to (\mathbb{R}^d, 0)$$

such that $f\mu_f$, Jac_{μ_f}, $g\mu_g$ and Jac_{μ_g} have normal crossings only, a Nash diffeomorphism

$$\Phi : (M_f, \mu_f^{-1}(0)) \to (M_g, \mu_g^{-1}(0))$$

which preserves the multiplicies of Jac_{μ_f} and Jac_{μ_g} along the irreducible components of the exceptional loci and which induces a semialgebraic homeomorphism $\varphi : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ such that the following diagram commutes

$$\begin{tikzcd}
(M_f, \mu_f^{-1}(0)) \arrow[r, "\Phi"] \arrow[d, "\mu_f"'] & (M_g, \mu_g^{-1}(0)) \arrow[d, "\mu_g"] \\
(\mathbb{R}^d, 0) \arrow[r, "\varphi"] \arrow[dr, "f"'] & (\mathbb{R}^d, 0) \arrow[d, "g"] \\
 & (\mathbb{R}, 0)
\end{tikzcd} .$$

The blow-Nash equivalence has properties similar to the ones of the blow-analytic equivalence concerning the absence of continuous moduli for isolated singularities.

6.3 Fichou zeta functions

Definition 6.7 ([28, §3.1])**.** Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a real analytic function germ. For $n \in \mathbb{N}_{>0}$, we set

* A Nash modification is a proper surjective Nash map whose complexification is proper and bimeromorphic.

$$\mathfrak{X}_n(f) = \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = ct^n + \cdots,\ c \neq 0\right\},$$

$$\mathfrak{X}_n^+(f) = \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = t^n + \cdots\right\},$$

$$\mathfrak{X}_n^-(f) = \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = -t^n + \cdots\right\}$$

and we define the Fichou zeta functions by

$$Z_f^{\beta,\mathrm{naive}}(T) = \sum_{n\geq 1} \beta\left(\mathfrak{X}_n(f)\right) u^{-nd} T^n \in \mathbb{Z}\left[u, u^{-1}\right] [\![T]\!],$$

$$Z_f^{\beta,+}(T) = \sum_{n\geq 1} \beta\left(\mathfrak{X}_n^+(f)\right) u^{-nd} T^n \in \mathbb{Z}\left[u, u^{-1}\right] [\![T]\!],$$

$$Z_f^{\beta,-}(T) = \sum_{n\geq 1} \beta\left(\mathfrak{X}_n^-(f)\right) u^{-nd} T^n \in \mathbb{Z}\left[u, u^{-1}\right] [\![T]\!].$$

Remark 6.8. Similarly to the Koike–Parusiński zeta functions, the Fichou zeta functions are local.

We may derive the following rationality formulas from an adaptation of Denef–Loeser key lemma for the motivic change of variables formula.

Theorem 6.9 ([28, Proposition 3.2 & Proposition 3.5])**.** *Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash function germ. There exists $\sigma : (M, \sigma^{-1}(0)) \to (\mathbb{R}^d, 0)$ a finite sequence of algebraic blowings-up with non-singular centers such that $f\sigma$ and $\operatorname{Jac}\sigma$ simultaneously have normal crossings only and such that σ is an isomorphism outside the zero set of f. We denote by $(E_i)_{i\in J}$ the irreducible components of $(f\sigma)^{-1}(0)$ and, for $i \in J$,*

$$N_i = \operatorname{mult}_{E_i} f\sigma \qquad \text{and} \qquad \nu_i - 1 = \operatorname{mult}_{E_i} \operatorname{Jac}\sigma.$$

Then

$$Z_f^{\beta,\mathrm{naive}}(T) = \sum_{\varnothing\neq I\subset J} (u-1)^{|I|} \beta\left(E_I^\bullet \cap \sigma^{-1}(0)\right) \prod_{i\in I} \frac{u^{-\nu_i} T^{N_i}}{1 - u^{-\nu_i} T^{N_i}}$$

and, for $\varepsilon \in \{-, +\}$,

$$Z_f^{\beta,\varepsilon}(T) = \sum_{\varnothing\neq I\subset J} (u-1)^{|I|-1} \beta\left(\widetilde{E_I^{\bullet}}^{\varepsilon} \cap \sigma^{-1}(0)\right) \prod_{i\in I} \frac{u^{-\nu_i} T^{N_i}}{1 - u^{-\nu_i} T^{N_i}}.$$

We refer the reader to [28, Lemma 3.12] for the definition of $\widetilde{E_I^{\bullet}}^{\varepsilon}$.

Remark 6.10 ([28, Remark 3.3])**.** In Theorem 6.9, it is necessary to assume that

f is Nash in order to ensure that $E_I^\bullet$ is $\mathcal{AS}$ so that $\beta(E_I^\bullet)$ is well defined.

The next theorem is a direct consequence of the previous rationality formulas.

Theorem 6.11 ([28, Theorem 4.8])**.** *If $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are two blow-Nash equivalent Nash function germs then*

$$Z_f^{\beta,\text{naive}} = Z_g^{\beta,\text{naive}}, \qquad Z_f^{\beta,+} = Z_g^{\beta,+}, \qquad Z_f^{\beta,-} = Z_g^{\beta,-}.$$

Remark 6.12 ([28, p. 678])**.** In the blow-Nash setting, Fichou zeta functions generalize the Fukui invariants since the virtual Poincaré polynomial is non-zero for a non-empty $\mathcal{AS}$-set.

Fichou zeta functions allow one to obtain a complete classification of Brieskorn polynomials up to three variables for the blow-Nash equivalence [28]. More recently they were used by G. Fichou and T. Fukui [32] to prove that the blow-Nash class of a three variables non-degenerate (with respect to its Newton polyhedron) convenient weighted homogeneous polynomial determines its weight system.

No convolution formula is known for Fichou zeta functions.

7. The arc-analytic equivalence

7.1 Definition and first properties

In [16], a characterization of the blow-Nash equivalence is given in terms of arc-analytic maps. This relation is called the arc-analytic equivalence and allows one to prove that it is an equivalence relation as expected. Moreover it avoids using Nash modifications.

Definition 7.1 ([16, Definition 7.5])**.** Let $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be two Nash function germs. We say that f and g are arc-analytic equivalent if there exists a semialgebraic homeomorphism $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ such that

(i) $f = gh$,
(ii) h is arc-analytic,
(iii) there exists $c > 0$ such that $|\det \mathrm{d}h| > c$ where $\mathrm{d}h$ is defined.

Theorem 7.2 ([16, Proposition 7.7])**.** *The arc-analytic equivalence is an equivalence relation.*

Theorem 7.3 ([16, Proposition 7.9])**.** *The arc-analytic equivalence coincides with the blow-Nash equivalence, i.e. two Nash function germs are arc-analytic equivalent if and only if they are blow-Nash equivalent.*

Corollary 7.4 ([16, Corollary 7.10])**.** *The blow-Nash equivalence is an equivalence relation.*

A recent preprint by A. Parusiński and L. Paunescu [66] announces that the arc-analytic equivalence admits no continuous moduli even for families of non-isolated singularities. The following result can be deduced from [66, Theorem 8.5] and the proof of [66, Theorem 3.3], see also the formula [66, (3.9)].

Theorem 7.5. *Let* $F : (\mathbb{R}^d \times I, \{0\} \times I) \to (\mathbb{R}, 0)$ *be a Nash germ. Then the germs* $f_t(x) = F(t,x) : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$*,* $t \in I$*, define a finite number of arc-analytic classes.*

7.2 A new zeta function

7.2.1 A Grothendieck ring

In order to obtain a zeta function with a convolution formula, we are going to work with an adaptation of Guibert–Loeser–Merle Grothendieck group to $\mathbb{R}^*$-equivariant $\mathcal{AS}$-sets up to $\mathcal{AS}$-bijections following [16, §3].

Definition 7.6 ([16, Definition 3.1])**.** Let $K_0(\mathcal{AS})$ be the free abelian group spanned by symbols* $[X]$ with $X \in \mathcal{AS}$ modulo the following relations

(1) Let $X, Y \in \mathcal{AS}$. If there is a bijection $X \to Y$ whose graph is $\mathcal{AS}$ then

$$[X] = [Y],$$

(2) For $X \in \mathcal{AS}$ and $Y \subset X$ a closed $\mathcal{AS}$-subset we set

$$[X \setminus Y] + [Y] = [X],$$

The cartesian product induces a ring structure:

(3) $[X][Y] = [X \times Y]$.

Remark 7.7. The unit of the addition is the class of the empty set denoted by

$$0 = [\varnothing].$$

The one of the multiplication is the class of the point denoted by

$$1 = [\mathrm{pt}].$$

We denote by $\mathbb{L}_{\mathcal{AS}} = [\mathbb{R}]$ the class of the affine line and we set

$$\mathcal{M}_{\mathcal{AS}} = K_0(\mathcal{AS}) \left[\mathbb{L}_{\mathcal{AS}}^{-1}\right].$$

Remark 7.8. For $A, B \in \mathcal{AS}$ we have $[A \sqcup B] = [A] + [B]$.

Remark 7.9. The virtual Poincaré polynomial factorises through a ring morphism

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* It is well defined since $\mathcal{AS}$ is a set.

$$\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$$

which extends to a ring morphism

$$\beta : \mathcal{M}_{\mathcal{AS}} \to \mathbb{Z}[u, u^{-1}].$$

Definition 7.10. For $n \in \mathbb{N}_{>0}$, we denote by $\mathcal{AS}^n_{\mathrm{mon}}$ the category whose objects are of the form

$$\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*$$

where $X \in \mathcal{AS}$, the graph $\Gamma_{\varphi_X} \in \mathcal{AS}$, the graph of the action $\Gamma_{\mathbb{R}^* \times X \to X} \in \mathcal{AS}$ and finally for all $(\lambda, x) \in \mathbb{R}^* \times X$, $\varphi_X(\lambda \cdot x) = \lambda^n \varphi(x)$ and whose morphisms are equivariant maps⋆ with $\mathcal{AS}$-graph over $\mathbb{R}^*$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & \mathbb{R}^* & \end{tikzcd}.$$

Remark 7.11. One may notice that an isomorphism in $\mathcal{AS}^n_{\mathrm{mon}}$ is just an equivariant bijection with $\mathcal{AS}$-graph over $\mathbb{R}^*$.

Definition 7.12 ([16, Definition 3.4])**.** For $n \in \mathbb{N}_{>0}$, we denote by $K_0(\mathcal{AS}^n_{\mathrm{mon}})$ the free abelian group spanned by symbols

$$[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*]$$

where $\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^* \in \mathcal{AS}^n_{\mathrm{mon}}$ modulo the relations:

(1) If $\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*$ and $\varphi_Y : \mathbb{R}^* \circlearrowright Y \to \mathbb{R}^*$ are isomorphic in $\mathcal{AS}^n_{\mathrm{mon}}$ then we set

$$[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*] = [\varphi_Y : \mathbb{R}^* \circlearrowright Y \to \mathbb{R}^*],$$

(2) If Y is a $\mathbb{R}^*$-invariant closed $\mathcal{AS}$-subset of X then

$$[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*] = \left[\varphi_{X|Y} : \mathbb{R}^* \circlearrowright Y \to \mathbb{R}^*\right] + \left[\varphi_{X|X \setminus Y} : \mathbb{R}^* \circlearrowright X \setminus Y \to \mathbb{R}^*\right],$$

(3) Let $\varphi_Y : \mathbb{R}^* \circlearrowright_\tau Y \to \mathbb{R}^* \in \mathcal{AS}^n_{\mathrm{mon}}$ and $\psi = \varphi_Y \operatorname{pr}_Y : Y \times \mathbb{R}^m \to \mathbb{R}^*$. Let σ and σ' be two actions of $\mathbb{R}^*$ on $Y \times \mathbb{R}^m$ which are two liftings† of τ then $\psi : \mathbb{R}^* \circlearrowright_\sigma (Y \times \mathbb{R}^m) \to \mathbb{R}^*$ and $\psi : \mathbb{R}^* \circlearrowright_{\sigma'} (Y \times \mathbb{R}^m) \to \mathbb{R}^*$ are in $\mathcal{AS}^n_{\mathrm{mon}}$ and we add the relation

$$[\psi : \mathbb{R}^* \circlearrowright_\sigma (Y \times \mathbb{R}^m) \to \mathbb{R}^*] = [\psi : \mathbb{R}^* \circlearrowright_{\sigma'} (Y \times \mathbb{R}^m) \to \mathbb{R}^*].$$

⋆ i.e. $f(\lambda \cdot x) = \lambda \cdot f(x)$.

† i.e. $\operatorname{pr}_Y(\lambda \cdot_\sigma x) = \lambda \cdot_\tau \operatorname{pr}_Y(x)$.

The fiber product over $\mathbb{R}^*$ induces a ring structure:

(4) We add the relation

$$[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*]\,[\varphi_Y : \mathbb{R}^* \circlearrowright Y \to \mathbb{R}^*] = [X \times_{\mathbb{R}^*} Y \to \mathbb{R}^*]$$

where the action of $\mathbb{R}^*$ on $X \times_{\mathbb{R}^*} Y$ is diagonal from the previous ones.

The cartesian product induces a structure of $K_0(\mathcal{AS})$-algebra* :

(5) Let $[A] \in K_0(\mathcal{AS})$ and $[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*] \in K_0(\mathcal{AS}^n_{\mathrm{mon}})$ then we set

$$[A] \cdot [\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*] = [\varphi_X\, \mathrm{pr}_X : A \times X \to \mathbb{R}^*]$$

where the action is trivial on A.

Remark 7.13. The unit of the addition is the class of the empty set

$$0 = [\emptyset]$$

and the unit of the product is

$$\mathbb{1}_n = [\mathrm{id} : \mathbb{R}^* \to \mathbb{R}^*]$$

where the action on the first $\mathbb{R}^*$ is $\lambda \cdot r = \lambda^n r$.

The class of the affine line $\mathbb{L}_{\mathcal{AS}} \in K_0(\mathcal{AS})$ induces, by the scalar product, a class

$$\mathbb{L}_n = \mathbb{L}_{\mathcal{AS}} \cdot \mathbb{1}_n = [\mathrm{pr}_{\mathbb{R}^*} : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^*]$$

where the action is $\lambda \cdot (r, x) = (r, \lambda^n x)$.

We set $\mathcal{M}^n = K_0(\mathcal{AS}^n_{\mathrm{mon}})\left[\mathbb{L}_n^{-1}\right]$ which has a natural structure of $\mathcal{M}_{\mathcal{AS}}$-algebra.

Definition 7.14. We define a directed partial order on $\mathbb{N}_{>0}$ by

$$n \prec m \Leftrightarrow \exists k \in \mathbb{N}_{>0},\, n = km$$

and for $n \prec m$ we define the morphism

$$\theta_n^m : \mathcal{AS}^m_{\mathrm{mon}} \to \mathcal{AS}^n_{\mathrm{mon}}$$

which only changes the action by $\lambda \cdot_n x = \lambda^k \cdot_m x$.

Then we set $\mathcal{AS}_{\mathrm{mon}} = \varinjlim \mathcal{AS}^n_{\mathrm{mon}}$, $K_0(\mathcal{AS}_{\mathrm{mon}}) = \varinjlim K_0(\mathcal{AS}^n_{\mathrm{mon}})$ and $\mathcal{M} = \varinjlim \mathcal{M}^n$.

Notation 7.15. • $\mathbb{1} = \varinjlim \mathbb{1}_n$

* Using the notation introduced below, the algebra structure is given by the structural morphism $K_0(\mathcal{AS}) \to K_0(\mathcal{AS}^n_{\mathrm{mon}})$ defined by $[A] \to [A] \cdot \mathbb{1}_n = [\mathrm{pr}_{\mathbb{R}^*} : A \times \mathbb{R}^* \to \mathbb{R}^*]$ where the $\mathbb{R}^*$-action is $\lambda \cdot (a, r) = (a, \lambda^n r)$.

- $\mathbb{L} = \varinjlim \mathbb{L}_n$

Remark 7.16. Notice that $K_0(\mathcal{AS}_{\mathrm{mon}})$ has a natural structure of $K_0(\mathcal{AS})$-algebra and that $\mathcal{M}$ has a natural structure of $\mathcal{M}_{\mathcal{AS}}$-algebra.

Notice also that $\mathbb{L} = \mathbb{L}_{\mathcal{AS}} \cdot \mathbb{1}$ and that, since the localization commutes with inductive limit,

$$\mathcal{M} = K_0(\mathcal{AS}_{\mathrm{mon}}) \left[\mathbb{L}^{-1}\right]$$

Proposition 7.17 ([16, End of §3])**.** *The map $\mathcal{AS}^n_{\mathrm{mon}} \to \mathcal{AS}$ defined by*

$$(\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*) \mapsto X$$

induces the following forgetful morphisms:
- *Morphisms of $K_0(\mathcal{AS})$-modules $\overline{\ \cdot\ } : K_0(\mathcal{AS}^n_{\mathrm{mon}}) \to K_0(\mathcal{AS})$,*
- *A morphism of $K_0(\mathcal{AS})$-modules $\overline{\ \cdot\ } : K_0(\mathcal{AS}_{\mathrm{mon}}) \to K_0(\mathcal{AS})$,*
- *Morphisms of $\mathcal{M}_{\mathcal{AS}}$-modules $\overline{\ \cdot\ } : \mathcal{M}^n \to \mathcal{M}_{\mathcal{AS}}$,*
- *A morphism of $\mathcal{M}_{\mathcal{AS}}$-modules $\overline{\ \cdot\ } : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}$.*

Proposition 7.18 ([16, §4.2.1])**.** *Let $\varepsilon \in \{<, >\}$. The map $\mathcal{AS}^n_{\mathrm{mon}} \to \mathcal{AS}$ defined by*

$$(\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*) \mapsto \varphi_X^{-1}(\mathbb{R}_{\varepsilon 0})$$

induces the following forgetful morphisms
- *Morphisms of $K_0(\mathcal{AS})$-modules $F^\varepsilon : K_0(\mathcal{AS}^n_{\mathrm{mon}}) \to K_0(SA)$,*
- *A morphism of $K_0(\mathcal{AS})$-modules $F^\varepsilon : K_0(\mathcal{AS}_{\mathrm{mon}}) \to K_0(SA)$,*
- *Morphisms of $\mathcal{M}_{\mathcal{AS}}$-modules $F^\varepsilon : \mathcal{M}^n \to \mathcal{M}_{SA}$,*
- *A morphism of $\mathcal{M}_{\mathcal{AS}}$-modules $F^\varepsilon : \mathcal{M} \to \mathcal{M}_{SA}$,*

where $K_0(SA) \simeq \mathbb{Z}$ is the Grothendieck ring of semialgebraic sets up to semialgebraic homeomorphisms and $\mathcal{M}_{SA} \simeq \mathbb{Z}$ the localization by the class of the affine real line.

Remark 7.19. Previous morphisms are morphisms of modules (and not of algebras) since the multiplication comes from the cartesian product in $K_0(\mathcal{AS})$ and from the fiber product in $K_0(\mathcal{AS}_{\mathrm{mon}})$. It is highlighted in the following example.

Example 7.20. • $\beta(\overline{\mathbb{1}}) = \beta(\mathbb{R}^*) = u - 1 \neq 1 = \beta(\mathrm{pt})$.
- $\chi_c(F^{>}\mathbb{1}) = \chi_c(\mathbb{R}_{>0}) = -1 \neq 1 = \chi_c(\mathrm{pt})$.

Proposition 7.21 ([16, §4.2.2])**.** *Let $\varepsilon \in \{+, -\}$. The map $\mathcal{AS}^n_{\mathrm{mon}} \to \mathcal{AS}$ defined by*

$$(\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*) \mapsto \varphi_X^{-1}(\varepsilon 1)$$

induces the following forgetful morphisms:
- *Morphisms of $K_0(\mathcal{AS})$-algebras $F^\varepsilon : K_0(\mathcal{AS}^n_{\mathrm{mon}}) \to K_0(\mathcal{AS})$,*

- *A morphism of $K_0(\mathcal{AS})$-algebras $F^\varepsilon : K_0(\mathcal{AS}_{\mathrm{mon}}) \to K_0(\mathcal{AS})$,*
- *Morphisms of $\mathcal{M}_{\mathcal{AS}}$-algebras $F^\varepsilon : \mathcal{M}^n \to \mathcal{M}_{\mathcal{AS}}$,*
- *A morphism of $\mathcal{M}_{\mathcal{AS}}$-algebras $F^\varepsilon : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}$.*

Remark 7.22. Previous morphisms are morphisms of algebras since the fiber product over a point coincides with the cartesian product.

7.2.2 Definition and properties of the zeta function

Definition 7.23 ([16, Definition 4.2])**.** Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash function germ. For $n \in \mathbb{N}_{>0}$, we set

$$\mathfrak{X}_n(f) = \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\ f\gamma(t) = ct^n + \cdots,\ c \neq 0\right\}$$

together with the angular component morphism $\mathrm{ac}_f^n : \mathfrak{X}_n(f) \to \mathbb{R}^*$ defined by $\mathrm{ac}_f^n(\gamma) = \mathrm{ac}(f\gamma) = c$ and with the $\mathbb{R}^*$-action defined by $\lambda \cdot \gamma(t) = \gamma(\lambda t)$. In this way the class

$$[\mathfrak{X}_n(f)] := \left[\mathrm{ac}_f^n : \mathbb{R}^* \circlearrowright \mathfrak{X}_n(f) \to \mathbb{R}^*\right]$$

is well defined in $K_0(\mathcal{AS}^n_{\mathrm{mon}})$.

The zeta function of [16] is defined by

$$Z_f(T) = \sum_{n \geq 1} [\mathfrak{X}_n(f)]\, \mathbb{L}^{-nd} T^n \in \mathcal{M}[\![T]\!].$$

This zeta function encodes the previous ones of Koike–Parusiński and Fichou.

Proposition 7.24 ([16, §4.2])**.** *We recover Koike–Parusiński zeta functions applying coefficientwise the morphisms $\chi_c(\overline{\ \cdot\ })$, $\chi_c F^>$ or $\chi_c F^<$ to $Z_f(T)$.*

Proposition 7.25 ([16, §4.2])**.** *We recover Fichou zeta functions applying coefficientwise the morphisms $\beta(\overline{\ \cdot\ })$, βF^+ or βF^- to $Z_f(T)$.*

This zeta function admits a rationality formula (from which we may derive the rationality formulas of Koike–Parusiński zeta functions and Fichou zeta functions).

Theorem 7.26 ([16, Theorem 4.22])**.** *Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash function germ. There exists $\sigma : (M, \sigma^{-1}(0)) \to (\mathbb{R}^d, 0)$ a finite sequence of algebraic blowings-up with non-singular centers such that $f\sigma$ and $\mathrm{Jac}\,\sigma$ simultaneously have normal crossings only and such that σ is an isomorphism outside the zero set of f. We denote by $(E_i)_{i\in J}$ the irreducible components of $(f\sigma)^{-1}(0)$ and, for $i \in J$,*

$$N_i = \mathrm{mult}_{E_i} f\sigma \qquad \text{and} \qquad \nu_i - 1 = \mathrm{mult}_{E_i} \mathrm{Jac}\,\sigma.$$

Then

$$Z_f = \sum_{\varnothing \neq I \subset J} \left[U_I \cap (\sigma p_I)^{-1}(0)\right] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

We refer the reader to [16, §4.3.2] for the definition of U_I and p_I.

We derive from the previous rationality formula that the zeta function considered in this section is an invariant of the arc-analytic equivalence.

Theorem 7.27 ([16, Theorem 7.11])**.** *If $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are two arc-analytic equivalent Nash function germs then $Z_f = Z_g$.*

7.2.3 A convolution formula

Definition 7.28 ([16, Notation 6.1])**.** We define the convolution product

$$* : K_0(\mathcal{AS}^m_{\mathrm{mon}}) \times K_0(\mathcal{AS}^n_{\mathrm{mon}}) \to K_0(\mathcal{AS}^{mn}_{\mathrm{mon}})$$

as the unique $K_0(\mathcal{AS})$-bilinear map satisfying

$$\begin{aligned}
&[\varphi_1 : \mathbb{R}^* \circlearrowleft_{\sigma_1} X_1 \to \mathbb{R}^*] * [\varphi_2 : \mathbb{R}^* \circlearrowleft_{\sigma_2} X_2 \to \mathbb{R}^*] \\
&= -\left[\varphi_1 + \varphi_2 : \mathbb{R}^* \circlearrowleft_{\tau_1} ((X_1 \times X_2) \setminus (\varphi_1 + \varphi_2)^{-1}(0)) \to \mathbb{R}^*\right] \\
&\quad + \left[\mathrm{pr}_{\mathbb{R}^*} : \mathbb{R}^* \circlearrowleft_{\tau_2} ((\varphi_1 + \varphi_2)^{-1}(0) \times \mathbb{R}^*) \to \mathbb{R}^*\right],
\end{aligned}$$

where $\lambda \cdot_{\tau_1} (x_1, x_2) = (\lambda^n \cdot_{\sigma_1} x_1, \lambda^m \cdot_{\sigma_2} x_2)$ and $\lambda \cdot_{\tau_2} (x_1, x_2, r) = (\lambda^n \cdot_{\sigma_1} x_1, \lambda^m \cdot_{\sigma_2} x_2, \lambda^{mn} r)$.

Remark 7.29. This induces a $K_0(\mathcal{AS})$-bilinear map $* : K_0(\mathcal{AS}_{\mathrm{mon}}) \times K_0(\mathcal{AS}_{\mathrm{mon}}) \to K_0(\mathcal{AS}_{\mathrm{mon}})$ (resp. $\mathcal{M}_{\mathcal{AS}}$-bilinear map $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$) which is associative, commutative and whose unit is $\mathbb{1}$.

Definition 7.30 ([16, §6])**.** We define the modified zeta function of f by

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

where $Z_f^{\mathrm{naive}}(T)$ is obtained by applying coefficientwise $\alpha \mapsto \overline{\alpha} \cdot \mathbb{1}$ to $Z_f(T)$.

Remark 7.31 ([16, Remark 6.9])**.** Applying coefficientwise the forgetful morphism (resp. $F^>$, resp. $F^<$) then the Euler characteristic with compact support we recover the modified zeta functions of Koike–Parusiński.

Remark 7.32 ([16, Corollary 6.14])**.** $Z_f(T)$ and $\tilde{Z}_f(T)$ encode the same information. Indeed

$$Z_f(T) = \tilde{Z}_f(T) + \frac{\mathbb{L} - \tilde{Z}_f^{\mathrm{naive}}(T)}{\mathbb{L} - T} - \mathbb{1}.$$

We may recover Remark 4.9 from both previous remarks.

Theorem 7.33 ([16, Theorem 6.15])**.** *For $i \in \{1,2\}$, let $f_i : (\mathbb{R}^{d_i}, 0) \to (\mathbb{R}, 0)$ be a Nash function germ. We define $f_1 \oplus f_2 : (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, 0) \to (\mathbb{R}, 0)$ by $f_1 \oplus f_2(x_1, x_2) = f_1(x_1) + f_2(x_2)$. Then*

$$\tilde{Z}_{f_1 \oplus f_2}(T) = -\tilde{Z}_{f_1}(T) \circledast \tilde{Z}_{f_2}(T)$$

where $\circledast$ is the Hadamard product which consists in applying $*$ coefficientwise.

We may recover Theorem 4.8 from the last theorem.

The convolution formula of this section allows one to get a real version of a result of Yoshinaga–Suzuki [76]: the arc-analytic type of a Brieskorn polynomial determines its exponents (See [16, Corollary 8.4]).

Notice there exists an effective formula to compute the zeta function of a polynomial non-degenerate with respect to its Newton polyhedron [16, Theorem 5.15]. Such a formula was already known in the topological case [20, §5], the p-adic case [19], the classical motivic integration case [43, §2.1] and for Fichou zeta functions [32].

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Department of Mathematics, Faculty of Science,
Saitama University,
255 Shimo-Okubo, Sakura-ku, Saitama 338-8570, Japan.
e-mail: jbcampesato@mail.saitama-u.ac.jp