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Local differential geometry of singular curves with finite multiplicities

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Abstract

We introduce the notion of multiplicity for a singular C^∞ (resp. analytic) curve. When the multiplicity is m, we show (Theorem 1.1) that the m-th root of the arc length parameter is a C^∞ (resp. analytic) parameter. We show a closed formula (Theorem 2.1) for curvatures in arbitrary parameterizations, which enable us to investigate the asymptotic behavior of curvatures at a singular point. As an application, we obtain a version of the fundamental theorem of curves at a singular point.

As pointed out at the beginning of [2], the singular points of analytic space curves have been investigated by various authors. An abstract of [2] was published in 1916 (Bull. Amer. Math. Soc. 22 (1916), page 268), and this research topic has a history of more than 100 years. Talking about regular space curves, the fundamental theorem of curves says that curvatures of curves decide their shapes. This was generalized by T. Sasai ([3, 4]) for analytic space curves with singularities. We show a version of the fundamental theorem of C^∞-curves at a singular point (Theorem 2.8) for curves with multiplicity m, as a byproduct of our approach. Since the author is not able to find any literature which claims Theorem 2.8, it is probably new. But the author is not able to distinguish the known things (including forgotten things and folklore) and new things in the results of this article. The author feels that it is impossible to be aware about all important results in this field. Since the author is not an expert on history, he decides to prepare the paper in this form.

The paper is organized as follows: In the first section, we define the notion of multiplicity for space curves and, when the multiplicity is m, we show (Theorem 1.1) that the m-th root of the arc length parameter is a C^∞ (resp. analytic) parameter. We next show a closed formula (Theorem 2.1) for curvatures in arbitrary parameterizations, and show a version of the fundamental theorem on C^∞ space curves with a singularity of multiplicity m. We present a proof of Theorem 2.1 in the third section.

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Dedicated to the memory of Professor Takao Sasai who told the author the problem to extend the fundamental theorem of curves to a singular point

1. Multiplicity and arc length parameter

Let $f : [0,\varepsilon) \to \mathbb{R}^n$ be a C^∞-map with $f(0) = 0$. We say that f is of **multiplicity** m **at** $t = 0$ if there is a C^∞-map $\tilde{f} : [0,\varepsilon) \to \mathbb{R}^n$ with the following property:

$$f(t) = \frac{t^m}{m}\tilde{f}(t), \qquad \tilde{f}(0) \neq 0.$$

We denote this number m by $\operatorname{ord} f$.

We say that f is **of finite multiplicity** at $t = 0$, if f is of multiplicity m at $t = 0$ for some positive integer m. We say that f is **flat** at $t = 0$ if there are no such positive integer m.

We consider a non-constant C^∞-map $\gamma : [0,\varepsilon) \to \mathbb{R}^n$, $\gamma(0) = 0$, as a curve in the Euclidean space $\mathbb{R}^n$. We denote elements of $\mathbb{R}^n$ as vertical vectors. The following lemma implies that the multiplicity m is an intrinsic invariant.

Theorem 1.1. *If γ is of multiplicity m at $t = 0$, then there is a C^∞-parameter $u = u(t)$ so that u^m/m is an arc length parameter. Moreover, we have $\left|\frac{d\gamma}{du}\right| = u^{m-1}$.*

Proof. We have $d\gamma/dt = t^{m-1}\boldsymbol{T}(t)$ where $\boldsymbol{T}(t) = \tilde{\gamma} + (t/m)\tilde{\gamma}_t$. We remark that $\boldsymbol{T}(0) \neq 0$. The arc length parameter s satisfies that

$$\frac{ds}{dt} = \left|\frac{d\gamma}{dt}\right| = t^{m-1}|\boldsymbol{T}(t)|.$$

Define a parameter u by $u^m/m = s$, that is,

$$\frac{u^m}{m} = s = \int_0^t \tau^{m-1}|\boldsymbol{T}(\tau)|d\tau. \tag{1.1}$$

We remark that when we express the right hand side by $\varphi(t)$, $d\varphi/dt$ has order $m-1$ in t. This implies that $\varphi(t) = t^m\phi(t)$ for some C^∞-function $\phi(t)$ with $\phi(0) > 0$. So we take a parameter u by $u = t(m\phi(t))^{1/m}$. Remark that u is a C^∞-function in t. We also remark that

$$\left|\frac{d\gamma}{du}\right| = \left|\frac{ds}{du}\right|\left|\frac{d\gamma}{ds}\right| = \left|\frac{ds}{du}\right| = u^{m-1}. \qquad \square$$

We remark that u is an analytic parameter when γ is analytic.

In the notation in the proof above, we have

$$\frac{du}{dt} = \left(\frac{t}{u}\right)^{m-1}|\boldsymbol{T}(t)|, \tag{1.2}$$

$$u^{m-1}du = t^{m-1}|\boldsymbol{T}(t)|dt, \tag{1.3}$$

$$u^{m-1}T_i(t)|\boldsymbol{T}(t)|^{-1}du = t^{m-1}T_i(t)dt, \tag{1.4}$$

$$\int_0^u v^{m-1} T_i(t)|\boldsymbol{T}(t)|^{-1} dv = \int_0^t \tau^{m-1} T_i(\tau) d\tau = x_i \tag{1.5}$$

where $\boldsymbol{T}(t) = {}^t(T_1(t), \ldots, T_n(t))$. Here ${}^t\boldsymbol{a}$ denotes the transpose of $\boldsymbol{a}$. The $(m+k)$-jet of γ determines $(k-1)$-jet of $\boldsymbol{T}(t)$. If $u = u(t)$ denote the function of t with (1.2), then the equation (1.3) implies $(k-1)$-jet of $\boldsymbol{T}(t)$ determines k-jet of $u(t)$. In fact, setting $u = \sum_{i \geq 1} u_i t^i$, we have

$$\sum_{j_1, \ldots, j_m \geq 1} j_1 u_{j_1} \cdots u_{j_m} t^{j_1 + \cdots + j_m} = t^m |\boldsymbol{T}(t)|$$

and $(k-1)$-jet of $\boldsymbol{T}(t)$ determines $u_1, \ldots, u_k$.

For a C^∞-map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, we say that γ is of **multiplicity** m (resp. **of finite multiplicity**, **flat**) at $t = 0$ if the following two maps $[0, \varepsilon) \to \mathbb{R}^n$ defined by $t \to \gamma(t)$ and $t \mapsto \gamma(-t)$ are of multiplicity m (resp. of finite multiplicity, flat) at $t = 0$.

Example 1.2. *Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}^n$ be two analytic maps. Then the the image of C^∞-map $\gamma : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto \alpha(g(t)) + \beta(g(-t))$, is*

$$\{\alpha(t) : t \geq 0\} \cup \{\beta(t) : t \geq 0\}$$

where $g(t) = e^{-1/t}$, $t > 0$; 0, $t \leq 0$. We remark that γ is flat at $t = 0$.

Example 1.3. *For a C^∞-curve defined by $s \mapsto \gamma(s) = \frac{s}{\sqrt{2}}\binom{\cos\log s/\sqrt{2}}{\sin\log s/\sqrt{2}}$, $s > 0$, s is an arc length parameter. Setting $s = e^{-1/t}$, this curve is expressed by $t \mapsto e^{-1/t}\binom{\cos 1/t}{-\sin 1/t}$, which extends to a C^∞-function at $t = 0$. This is flat at $t = 0$.*

Consider a C^∞-map

$$\mu : [0, \varepsilon) \to \mathbb{R}^n,\ t \mapsto \frac{1}{m}\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{pmatrix}, \qquad \begin{aligned} &\phi_1(t) = t^m, \\ &\lim_{t \to 0} \frac{\phi_{i+1}(t)}{\phi_i(t)} = 0, \quad i = 1, 2, \ldots, n-1. \end{aligned}$$

Here ε is a small positive number. The image of a C^∞-map $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ with multiplicity m is represented by the image of the map in this form, up to suitable rotation. The functions $\phi_2(t), \ldots, \phi_n(t)$ are differential geometric invariants. We call this expression **Monge normal form**, since this is classical Monge normal form when $(n, m) = (2, 1)$.

For arbitrary C^∞-map $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ of multiplicity m, there is a diffeomorphism germ $\tau : [0, \varepsilon) \to [0, \varepsilon)$ at 0 so that $\gamma(\tau(t))$ is in Monge normal form, that is,

$$\gamma(\tau(t)) = \frac{1}{m}\begin{pmatrix} t^m \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{pmatrix}.$$

We remark that $(m+j)$-jet of γ determines $(m+j)$-jets of $\phi_2(t), \ldots, \phi_m(t)$.

If we take a parameter u so that t^m/m is an arc length parameter, we can express the curve by

$$\text{(1.6)} \quad \gamma(u) = \begin{pmatrix} \varphi_1(u) \\ \vdots \\ \varphi_n(t) \end{pmatrix}, \quad m = \operatorname{ord}\varphi_1(u), \ \lim_{t\to 0}\frac{\varphi_{i+1}(u)}{\varphi_i(u)} = 0, i = 1, \ldots, n-1.$$

composing a suitable rotation, if necessary. This is also a normal form of a curve with multiplicity m. We remark that $(m+j)$-jet of Monge normal form determine the $(m+j)$-jet of the normal form (1.6) and vice versa.

Remark 1.4. *S. Shiba and M. Umehara ([5]) has analyzed* $3/2$*-cusp in the plane* $\mathbb{R}^2$ *using the square root of an arc length parameter as a parameter (they call it the half-arclength parameter).*

2. Curvatures

Let us consider a curve $\gamma : [0, \varepsilon) \to \mathbb{R}^n$, $t \mapsto \gamma(t)$, with multiplicity m at $t = 0$. We have a parameter u so that u^m/m is an arc length parameter. Set $\boldsymbol{w}_k = d^k\gamma/dt^k$, $k = 1, \ldots, n$. We consider an orthogonal frame $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ so that

$$\langle \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\rangle_{\mathbb{R}} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\rangle_{\mathbb{R}}, \quad k = 1, \ldots, n-1, \text{ for } 0 < t < \varepsilon.$$

Define V_k, $k = 1, \ldots, n-1$, by

$$V_k = \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \ldots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \ldots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle \end{vmatrix}^{1/2}, \quad \text{and} \quad V_n = |\boldsymbol{w}_1 \ \cdots \ \boldsymbol{w}_n|.$$

Recall that the curvatures κ_i, $i = 1, \ldots, n-1$, are defined by the formula

$$\frac{d}{ds}\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ddots & \vdots \\ 0 & -\kappa_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \kappa_{n-1} \\ 0 & \ldots & 0 & -\kappa_{n-1} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$$

where s is an arc length parameter. The curvatures κ_i in arbitrary parametarization are given by the following

Theorem 2.1. $\kappa_i = \dfrac{V_{i-1}V_{i+1}}{V_1 V_i^2}$, $i = 1, \dots, n-1$.

This formula enables us to analyse the asymptotic behaviour of the curvatures at a singular point. When the author first showed this formula, he did not know any such a closed formula for curvatures in terms of Gram determinants. After he showed this, he has found the paper [1], which showed the formula for curvatures using Gram-Schmidt process. Before Theorem 4.2 on page 702 loc. cite., H. Gluck said "Looking carefully at the chain rule formulas for the derivatives of F in terms of those of F^* leads one to the following conclusions", and stated the formulas for curvatures for arbitrary parameterization. So some details were not presented. We present a proof of Theorem 2.1 in the next section, since our proof gives an explicit expression of the orthonormal projection, and looks different to that in [1].

Let us assume that there are C^∞-functions $g(t)$ and $\boldsymbol{T} = \boldsymbol{T}(t) \in \mathbb{R}^n$ with

$$\frac{d\gamma}{dt} = g(t)\boldsymbol{T}(t), \quad \boldsymbol{T}(0) \neq 0.$$

We consider a C^∞-map

$$\hat{\gamma} : [0, \varepsilon) \to \mathbb{R}^n \quad \text{with } \frac{d\hat{\gamma}}{dt} = \boldsymbol{T}(t).$$

Since $\boldsymbol{T}(0) \neq 0$, $\hat{\gamma}$ is regular at $t = 0$. It is clear that t is an arc length parameter of $\hat{\gamma}$, i.e., $|\boldsymbol{T}| = 1$, if and only if $\int_0^t g(\tau)d\tau$ is an arc length parameter of γ.

Let $\hat{\kappa}_k$ $(k = 1, \dots, n-1)$ denote the curvatures of $\hat{\gamma}$.

Theorem 2.2. $|g(t)|\kappa_k = \hat{\kappa}_k$ *for* $k = 1, \dots, n-1$.

Corollary 2.3. *If* t^m/m *is an arc length parameter of* γ*, then* $t^{m-1}\kappa_k = \hat{\kappa}_k$ *for* $k = 1, \dots, n-1$.

Proof. Assume that $|\boldsymbol{T}| = 1$ and set $g(t) = t^{m-1}$ in the previous Theorem. $\square$

Definition 2.4. *Define* $\widehat{V}_k$ $(k = 1, \dots, n-1)$ *by*

$$\widehat{V}_k = \begin{vmatrix} \langle \boldsymbol{T}, \boldsymbol{T} \rangle & \langle \boldsymbol{T}, \boldsymbol{T}_t \rangle & \dots & \langle \boldsymbol{T}, \boldsymbol{T}^{(k-1)} \rangle \\ \langle \boldsymbol{T}_t, \boldsymbol{T} \rangle & \langle \boldsymbol{T}_t, \boldsymbol{T}_t \rangle & \dots & \langle \boldsymbol{T}_t, \boldsymbol{T}^{(k-1)} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{T}^{(k-1)}, \boldsymbol{T} \rangle & \langle \boldsymbol{T}^{(k-1)}, \boldsymbol{T}_t \rangle & \dots & \langle \boldsymbol{T}^{(k-1)}, \boldsymbol{T}^{(k-1)} \rangle \end{vmatrix}^{1/2},$$

and $\widehat{V}_n = |\boldsymbol{T} \ \boldsymbol{T}_t \ \cdots \ \boldsymbol{T}^{(n-1)}|$. *By convention, we set* $\widehat{V}_0 = 1$.

By Theorem 2.1, we have $\hat{\kappa}_k = \widehat{V}_{k-1}\widehat{V}_{k+1}/(\widehat{V}_1\widehat{V}_k^2)$. So Theorem 2.2 is a consequence of the following

Theorem 2.5. $\kappa_k = \dfrac{\widehat{V}_{k-1}\widehat{V}_{k+1}}{|g(t)|\widehat{V}_1\widehat{V}_k^2}, \quad k = 1, 2, \dots, n-1.$

Remark 2.6. *When t is an arc length parameter of $\hat{\gamma}$, or equivalently, $\int_0^t g(\tau)d\tau$ is an arc length parameter of γ, i.e., $\int_0^t |d\gamma/dt(\tau)|d\tau$, we have*

$$\langle \boldsymbol{T}, \boldsymbol{T}\rangle = 1, \ \langle \boldsymbol{T}, \boldsymbol{T}_t\rangle = 0, \ \langle \boldsymbol{T}_t, \boldsymbol{T}_t\rangle + \langle \boldsymbol{T}, \boldsymbol{T}_{tt}\rangle = 0, \ \dots,$$

$$\widehat{V}_1 = 1, \ \widehat{V}_2 = |\boldsymbol{T}_t|, \ \widehat{V}_3 = \sqrt{|\boldsymbol{T}_t|^2|\boldsymbol{T}_{tt}|^2 - |\boldsymbol{T}_t|^6 - \langle \boldsymbol{T}_t, \boldsymbol{T}_{tt}\rangle^2}, \quad \textit{and so on.}$$

Set $\boldsymbol{T}(t) = (T_1(t), T_2(t), \dots, T_n(t))$. After composing a suitable rotation, we can assume that $\lim_{t\to 0} T_{k+1}/T_k = 0$ for $k = 1, 2, \dots, n-1$. If we assume that $\operatorname{ord} T_n < \infty$, then there are non-negative integers $e_1, \dots, e_{n-1}$ so that

$$\begin{aligned}
\operatorname{ord} T_2 =& 1 + e_1,\\
\operatorname{ord} T_3 =& 2 + e_1 + e_2,\\
&\cdots\\
\operatorname{ord} T_n =& n - 1 + e_1 + e_2 + \cdots + e_{n-1}.
\end{aligned}$$

Corollary 2.7. *We have* $\operatorname{ord}\kappa_k = e_k - m + 1$, $k = 1, \dots, n-1$. *In particular, κ_k is bounded if $e_k \geq m - 1$.*

Proof. Since $\widehat{V}_k^2$ is the sum of squares of $k \times k$ minors of the matrix

$$(\boldsymbol{T}\ \boldsymbol{T}_t\ \boldsymbol{T}_{tt}\ \dots\ \boldsymbol{T}^{(k-1)}),$$

we have that $\operatorname{ord} T_k - (k-1) = e_1 + \cdots + e_k$ and

$$\begin{aligned}
\operatorname{ord}\widehat{V}_k =& e_1 + (e_1 + e_2) + \cdots + (e_1 + \cdots + e_{k-1})\\
=& (k-1)e_1 + (k-2)e_2 + \cdots + e_{k-1}.
\end{aligned}$$

By Theorem 2.5, we conclude that

$$m - 1 + \operatorname{ord}\kappa_k = \operatorname{ord}\widehat{V}_{k-1} + \operatorname{ord}\widehat{V}_{k+1} - 2\operatorname{ord}\widehat{V}_k = e_k. \qquad \square$$

Setting $\widehat{V}_k = t^{(k-1)e_1 + (k-2)e_2 + \cdots + e_{k-1}}\widehat{w}_k$, we have

$$t^{m-e_k-1}\kappa_k = \frac{\widehat{w}_{k-1}\widehat{w}_{k+1}}{\widehat{w}_1\widehat{w}_k^2}.$$

The j-jet of $\boldsymbol{T}$ determines $(j-k+1)$-jet of $\boldsymbol{T}^{(k-1)}$, $(j-k+1)$-jet of $\widehat{V}_k$ and thus $(j-k+1-[(k-1)e_1 + (k-2)e_2 + \cdots + e_{k-1}])$-jet of $\widehat{w}_k$. Since

$$\begin{aligned}&j-k+1-[(k-1)e_1+(k-2)e_2+\cdots+e_{k-1}]\\ >&j-(k+1)+1-[ke_1+(k-1)e_2+\cdots+e_k],\end{aligned}$$

the j-jet of $\boldsymbol{T}$ determines $(j-k-[ke_1+(k-1)e_2+\cdots+e_k])$-jet of $\widehat{w}_{k-1}\widehat{w}_{k+1}/\widehat{w}_1\widehat{w}_k^2$.

Theorem 2.8. *Let $k_1(t)$, $\dots$, $k_{n-1}(t)$ be C^∞-functions defined on $[0,\varepsilon)$. Assume that $k_1(t)>0$, $\dots$, $k_{n-2}(t)>0$ for $t\neq 0$. Then there exists a curve $\gamma(t)$ of multiplicity m so that t^m/m is an arc length parameter and the curvatures are given by $\kappa_1(t)=k_1(t)/t^{m-1}$, $\dots$, $\kappa_{n-1}(t)=k_{n-1}(t)/t^{m-1}$. Such a curve is unique up to a motion of $\mathbb{R}^n$.*

Proof. The fundamental theorem of space curves shows the existence of a regular curve $\hat\gamma$ in $\mathbb{R}^n$ so that t is an arc length parameter of $\hat\gamma$ and that the curvatures of $\hat\gamma$ are $k_1(t)$, $\dots$, $k_{n-1}(t)$. This shows the existence of a C^∞-map $\boldsymbol{T}(t)$ so that $k_k=\widehat{V}_{k-1}\widehat{V}_{k+1}/(\widehat{V}_1\widehat{V}_k^2)$, $k=1,\dots,n-1$, where $\widehat{V}_k$ are defined in Definition 2.4. Remark that $\widehat{V}_1=1$ and $d\hat\gamma/dt=\boldsymbol{T}(t)$. Solving the differential equation $d\gamma/dt=t^{m-1}\boldsymbol{T}(t)$, we conclude the existence of a curve $\gamma(t)$ of multiplicity m so that t^m/m is an arc length parameter and the curvatures are given by $\kappa_1(t)=k_1(t)/t^{m-1}$, $\dots$, $\kappa_{n-1}(t)=k_{n-1}(t)/t^{m-1}$, as a consequence of Theorem 2.2. Such a curve is unique up to a motion of $\mathbb{R}^n$. ☐

Remark 2.9. *We show a flat version of the previous theorem is possible as follows: Let $k_1(t)$, $\dots$, $k_{n-1}(t)$ be C^∞-functions defined on $(0,\varepsilon)$ with $k_1(t)>0$, $\dots$, $k_{n-2}(t)>0$. Then there is a regular curve $\hat\gamma(t)$, $t\in(0,\varepsilon)$, in $\mathbb{R}^n$, whose curvatures are given by*

$$\hat\kappa_1(t)=k_1(t),\ \dots,\ \hat\kappa_{n-1}(t)=k_{n-1}(t),\qquad \text{for } t\in(0,\varepsilon)$$

and t is an arc length parameter of $\hat\gamma$. Let $s(t)$ is a C^∞-function on $(0,\varepsilon)$ which is increasing with $s(0)=0$. We assume that $s(t)$ is flat at $t=0$. Since $\boldsymbol{T}(t)=d\hat\gamma/dt$ is unit vectors, $s_t(t)\boldsymbol{T}(t)$ extends to $t=0$ as a C^∞-function. Then the integration of $s_t(t)\boldsymbol{T}(t)$ provides a curve $\gamma(t)$ whose curvatures $\kappa_i(t)$ are given by $\kappa_i(t)=\hat\kappa_i(t)/s_t(t)$, $i=1,\dots,n-2$, as a consequence of Theorem 2.2.

3. Proof of Theorem 2.1

Let $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n$ denote the frame defined at the beginning of the previous section.

Lemma 3.1. *We have*

$$\boldsymbol{a}_k=\frac{\tilde{\boldsymbol{a}}_k}{V_{k-1}V_k},\quad \tilde{\boldsymbol{a}}_k=\begin{vmatrix}\langle\boldsymbol{w}_1,\boldsymbol{w}_1\rangle & \dots & \langle\boldsymbol{w}_1,\boldsymbol{w}_{k-1}\rangle & \boldsymbol{w}_1\\ \vdots & \ddots & \vdots & \vdots\\ \langle\boldsymbol{w}_k,\boldsymbol{w}_1\rangle & \dots & \langle\boldsymbol{w}_k,\boldsymbol{w}_{k-1}\rangle & \boldsymbol{w}_k\end{vmatrix}\quad (k=1,\dots,n-1)$$

and $\boldsymbol{a}_n = \dfrac{\tilde{\boldsymbol{a}}_n}{|\tilde{\boldsymbol{a}}_n|}$, $\quad \tilde{\boldsymbol{a}}_n = \begin{vmatrix} \langle \boldsymbol{e}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{e}_1, \boldsymbol{w}_{n-1}\rangle & \boldsymbol{e}_1 \\ \vdots & & \vdots & \vdots \\ \langle \boldsymbol{e}_n, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{e}_n, \boldsymbol{w}_{n-1}\rangle & \boldsymbol{e}_n \end{vmatrix}$ *where* $\boldsymbol{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ *i-th row.*

Proof. Let $\pi_k : \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection to the normal space of the linear span of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$. Then

$$\boldsymbol{a}_1 = \frac{\boldsymbol{w}_1}{|\boldsymbol{w}_1|}, \quad \boldsymbol{a}_2 = \frac{\pi_1(\boldsymbol{w}_2)}{|\pi_1(\boldsymbol{w}_2)|}, \ \dots, \ \boldsymbol{a}_{k+1} = \frac{\pi_k(\boldsymbol{w}_{k+1})}{|\pi_k(\boldsymbol{w}_{k+1})|}, \ k = 2, \dots, n-2.$$

We remark that π_k is given by

$$\pi_k(\boldsymbol{v}) = \frac{1}{V_k^2} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle & \boldsymbol{w}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle & \boldsymbol{w}_k \\ \langle \boldsymbol{v}, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{v}, \boldsymbol{w}_k\rangle & \boldsymbol{v} \end{vmatrix}.$$

In fact, if $\boldsymbol{v}$ is normal to the linear span of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$, then $\pi_k(\boldsymbol{v}) = \boldsymbol{v}$; and if $\boldsymbol{v}$ is a linear combination of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$, then $\pi_k(\boldsymbol{v}) = 0$.

$$\begin{aligned} \langle \pi_k(\boldsymbol{w}_{k+1}), \pi_k(\boldsymbol{w}_{k+1})\rangle &= \frac{1}{V_k^2} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_1, \pi_k(\boldsymbol{w}_{k+1})\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_k, \pi_k(\boldsymbol{w}_{k+1})\rangle \\ \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_{k+1}, \pi_k(\boldsymbol{w}_{k+1})\rangle \end{vmatrix} \\ &= \frac{1}{V_k^2} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle & 0 \\ \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_{k+1}, \pi_k(\boldsymbol{w}_{k+1})\rangle \end{vmatrix} \\ &= \langle \boldsymbol{w}_{k+1}, \pi_k(\boldsymbol{w}_{k+1})\rangle \\ &= \frac{1}{V_k^2} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_1, \boldsymbol{w}_{k+1}\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_k, \boldsymbol{w}_{k+1}\rangle \\ \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_k\rangle & \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_{k+1}\rangle \end{vmatrix} = \frac{V_{k+1}^2}{V_k^2}. \end{aligned}$$

We then obtain that

$$\frac{\pi_k(\boldsymbol{w}_{k+1})}{|\pi_k(\boldsymbol{w}_{k+1})|} = \frac{1}{V_k V_{k+1}} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_k\rangle & \boldsymbol{w}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_k, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_k, \boldsymbol{w}_k\rangle & \boldsymbol{w}_k \\ \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_1\rangle & \dots & \langle \boldsymbol{w}_{k+1}, \boldsymbol{w}_k\rangle & \boldsymbol{w}_{k+1} \end{vmatrix}.$$

It is clear that $\langle \boldsymbol{a}_i, \tilde{\boldsymbol{a}}_n \rangle = 0$, $i = 1, \dots, n-1$. It is enough to show that $\langle \tilde{\boldsymbol{a}}_n, \tilde{\boldsymbol{a}}_n \rangle = V_{n-1}^2$.

$$\begin{aligned}
\langle \tilde{\boldsymbol{a}}_n, \tilde{\boldsymbol{a}}_n \rangle &= \begin{vmatrix} \langle \boldsymbol{e}_1, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{e}_1, \boldsymbol{w}_{n-1} \rangle & \langle \boldsymbol{e}_1, \tilde{\boldsymbol{a}}_n \rangle \\ \vdots & & \vdots & \vdots \\ \langle \boldsymbol{e}_n, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{e}_n, \boldsymbol{w}_{n-1} \rangle & \langle \boldsymbol{e}_n, \tilde{\boldsymbol{a}}_n \rangle \end{vmatrix} \\
&= \sum_{i=1}^{n} (-1)^{n+i} M_i \langle \boldsymbol{e}_i, \tilde{\boldsymbol{a}}_n \rangle = \sum_{i=1}^{n} M_i^2 = V_{n-1}^2
\end{aligned}$$

where $M_i = |\langle \boldsymbol{e}_j, \boldsymbol{w}_1 \rangle \ \dots \ \langle \boldsymbol{e}_j, \boldsymbol{w}_{n-1} \rangle|_{j=1,\dots,i-1,i+1,\dots,n}$. □

Proof of Theorem 2.1. For $i = 1, \dots, n-2$, we have

$$\begin{aligned}
\kappa_i =& \Big\langle \frac{d}{ds} \boldsymbol{a}_i, \boldsymbol{a}_{i+1} \Big\rangle = \Big\langle \frac{d}{ds} \frac{\tilde{\boldsymbol{a}}_i}{V_{i-1} V_i}, \ \frac{\tilde{\boldsymbol{a}}_{i+1}}{V_i V_{i+1}} \Big\rangle \\
=& \Big\langle \Big(\frac{d}{ds} \frac{1}{V_{i-1} V_i} \Big) \tilde{\boldsymbol{a}}_i, \ \frac{\tilde{\boldsymbol{a}}_{i+1}}{V_i V_{i+1}} \Big\rangle + \Big\langle \frac{1}{V_{i-1} V_i} \frac{d}{ds} \tilde{\boldsymbol{a}}_i, \ \frac{\tilde{\boldsymbol{a}}_{i+1}}{V_i V_{i+1}} \Big\rangle \\
=& \frac{1}{V_{i-1} {V_i}^2 V_{i+1}} \frac{dt}{ds} \Big\langle \frac{d}{dt} \tilde{\boldsymbol{a}}_i, \ \tilde{\boldsymbol{a}}_{i+1} \Big\rangle \quad (\text{since } \langle \tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{a}}_{i+1} \rangle = 0) \\
=& \frac{1}{V_{i-1} {V_i}^2 V_{i+1}} \frac{1}{\left| \frac{d\gamma}{dt} \right|} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_{i-1} \rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_{i-1}, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{w}_{i-1}, \boldsymbol{w}_{i-1} \rangle & 0 \\ \frac{d}{dt} \langle \boldsymbol{w}_i, \boldsymbol{w}_1 \rangle & \dots & \frac{d}{dt} \langle \boldsymbol{w}_i, \boldsymbol{w}_{i-1} \rangle & \langle \boldsymbol{w}_{i+1}, \tilde{\boldsymbol{a}}_{i+1} \rangle \end{vmatrix} \\
& \quad (\text{since } \langle \boldsymbol{w}_j, \tilde{\boldsymbol{a}}_{i+1} \rangle = 0, \quad j = 1, 2, \dots, i-1) \\
=& \frac{1}{V_1 V_{i-1} {V_i}^2 V_{i+1}} {V_{i-1}}^2 {V_{i+1}}^2 = \frac{V_{i-1} V_{i+1}}{V_1 {V_i}^2}.
\end{aligned}$$

We also have that

$$\begin{aligned}
\kappa_{n-1} &= \Big\langle \frac{d}{ds} \boldsymbol{a}_{n-1}, \boldsymbol{a}_n \Big\rangle = \Big\langle \frac{d}{ds} \frac{\tilde{\boldsymbol{a}}_{n-1}}{V_{n-2} V_{n-1}}, \ \frac{\tilde{\boldsymbol{a}}_n}{V_{n-1}} \Big\rangle = \Big\langle \frac{1}{V_{n-2} V_{n-1}} \frac{d}{ds} \tilde{\boldsymbol{a}}_{n-1}, \ \frac{\tilde{\boldsymbol{a}}_n}{V_{n-1}} \Big\rangle \\
&= \frac{1}{V_{n-2} {V_{n-1}}^2} \frac{1}{\left| \frac{d\gamma}{dt} \right|} \begin{vmatrix} \langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{w}_1, \boldsymbol{w}_{n-2} \rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \boldsymbol{w}_{n-2}, \boldsymbol{w}_1 \rangle & \dots & \langle \boldsymbol{w}_{n-2}, \boldsymbol{w}_{n-2} \rangle & 0 \\ \frac{d}{dt} \langle \boldsymbol{w}_{n-1}, \boldsymbol{w}_1 \rangle & \dots & \frac{d}{dt} \langle \boldsymbol{w}_{n-1}, \boldsymbol{w}_{n-1} \rangle & \langle \boldsymbol{w}_n, \tilde{\boldsymbol{a}}_n \rangle \end{vmatrix} = \frac{V_{n-2} V_n}{V_1 {V_{n-1}}^2}. \ \square
\end{aligned}$$

Proof of Theorem 2.5. Since $d\gamma/dt = g(t)\boldsymbol{T}$, we have

$$\begin{aligned}
\frac{d^2\gamma}{dt^2} =& (g(t)\boldsymbol{T})_t = g_t(t)\boldsymbol{T} + g(t)\boldsymbol{T}_t, \\
\frac{d^3\gamma}{dt^3} =& g_{tt}(t)\boldsymbol{T} + 2g_t(t)\boldsymbol{T}_t + g(t)\boldsymbol{T}_{tt},
\end{aligned}$$

$$\cdots$$

$$\frac{d^k\gamma}{dt^k} = \sum_{i=0}^{k} \binom{k}{i} g^{(k-i)}(t)\boldsymbol{T}^{(i)}.$$

These imply that

$$\left(\frac{d\gamma}{dt}\ \cdots \frac{d^k\gamma}{dt^k}\right) = \left(\boldsymbol{T}\ \boldsymbol{T}_t\ \cdots\ \boldsymbol{T}^{(k-1)}\right)P, \quad P = \begin{pmatrix} g(t) & \cdots & * \\ & \ddots & \vdots \\ \text{\huge 0} & & g(t) \end{pmatrix}$$

and we conclude that

$$V_k^2 = \det\left({}^tP\,{}^t\left(\boldsymbol{T}\ \boldsymbol{T}_t\ \cdots\ \boldsymbol{T}^{(k-1)}\right)\left(\boldsymbol{T}\ \boldsymbol{T}_t\ \cdots\ \boldsymbol{T}^{(k-1)}\right)P\right) = g(t)^{2k}\widehat{V}_k^2.$$

We complete the proof by Theorem 2.1. □

References

[1] H. Gluck, Higher curvatures of curves in Euclidean space, Amer. Math. Monthly, 73 (1966), 699–704.

[2] A. Ranum, The singular point of analytic space curve, Trans. Amer. Math. Soc., 31 (1929), 145–163.

[3] T. Sasai, The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations, Tohoku Math. J., 36 (1984), 17–24.

[4] T. Sasai, Geometry of analytic space curves with singularities and regular singularities of differential equations, Funkcial. Ekvac., 30 (1987), 283–303.

[5] S. Shiba and M. Umehara, The behavior of curvature functions at cusps and inflection points, Differential Geom. Appl. 30 (2012), 285–299.

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