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On the image of the associated form morphism

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Abstract

Let $\mathbb{C}[x_1, \ldots, x_n]_{d+1}$ be the vector space of homogeneous forms of degree $d+1$ on $\mathbb{C}^n$, with $n, d \geq 2$. In earlier articles by J. Alper, M. Eastwood and the author, we introduced a morphism, called A, that assigns to every nondegenerate form the so-called associated form lying in the space $\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$. One of the reasons for our interest in A is the conjecture—motivated by the well-known Mather-Yau theorem on complex isolated hypersurface singularities—asserting that all regular GL_n-invariant functions on the affine open subvariety $\mathbb{C}[x_1, \ldots, x_n]_{d+1,\Delta}$ of forms with nonvanishing discriminant can be obtained as the pull-backs by means of A of the rational GL_n-invariant functions on $\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$ defined on $\mathrm{im}(A)$. The morphism A factors as $A = \mathbf{A} \circ \mathrm{grad}$, where grad is the gradient morphism and $\mathbf{A}$ assigns to every n-tuple of forms of degree d with nonvanishing resultant a form in $\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$ defined analogously to $A(f)$ for a nondegenerate f. In order to establish the conjecture, it is important to study the image of $\mathbf{A}$. In the present paper, we show that $\mathrm{im}(\mathbf{A})$ is an open subset of an irreducible component of each of the so-called catalecticant varieties V, $\mathrm{Gor}(T)$ and describe the closed complement to $\mathrm{im}(\mathbf{A})$, at the same time clarifying and extending known results on these varieties. Furthermore, for $n = 3$, $d = 2$ we give a description of the complement to $\mathrm{im}(\mathbf{A})$ via the zero locus of the Aronhold invariant of degree 4, which establishes an analogy with the case $n = 2$ where this complement is known to be the vanishing locus of the catalecticant for any $d \geq 2$.

1. Introduction

This paper is motivated by a new construction in classical invariant theory that originated in article [EI] and was further explored in [AI1], [AI2], [AIK]. Fix integers $n \geq 2$ and $d \geq 2$ and let $\mathbb{C}[x_1, \ldots, x_n]_{d+1,\Delta}$ be the complex affine open subvariety of the space $\mathbb{C}[x_1, \ldots, x_n]_{d+1}$ of homogeneous forms of degree $d+1$ in n variables where the discriminant Δ does not vanish. Consider the Milnor algebra $M(f) := \mathbb{C}[x_1, \ldots, x_n]/(f_{x_1}, \ldots, f_{x_n})$ of the isolated singularity at the origin of the hypersurface in $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, \ldots, x_n]_{d+1,\Delta}$ and let $\mathfrak{m} \subset M(f)$ be the maximal ideal. One can then introduce a form on the n-dimensional quotient $\mathfrak{m}/\mathfrak{m}^2$ with values in the one-dimensional socle $\mathrm{Soc}(M(f))$ of $M(f)$ as follows:

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$$\mathfrak{m}/\mathfrak{m}^2 \to \mathrm{Soc}(M(f)), \quad z \mapsto \widehat{z}^{\,n(d-1)},$$

where $\widehat{z}$ is any element of $\mathfrak{m}$ that projects to $z \in \mathfrak{m}/\mathfrak{m}^2$. There is a canonical isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}^n$ and, since the Hessian of f generates the socle, there is also a canonical isomorphism $\mathrm{Soc}(M(f)) \cong \mathbb{C}$. Hence, one obtains a form $A(f)$ of degree $n(d-1)$ on $\mathbb{C}^n$, which is called the *associated form of* f. This form is very natural; in particular, it is a Macaulay inverse system for the Milnor algebra $M(f)$.

The main object of our study in [AI1], [AI2], [AIK] was the morphism

$$A\colon \mathbb{C}[x_1,\dots,x_n]_{d+1,\Delta} \to \mathbb{C}[y_1,\dots,y_n]_{n(d-1)}, \quad f \mapsto A(f)$$

of affine varieties. As first observed in [EI], for certain values of n and d one can recover all GL_n-invariant rational functions on forms of degree $d+1$ from those on forms of degree $n(d-1)$ by evaluating the latter on associated forms, i.e., by composing them with A. Motivated by the above fact, in [AI1] we proposed a conjecture asserting that an analogous statement holds for all n and d (cf. [EI, Conjecture 3.2]):

Conjecture 1.1. *For any regular* GL_n*-invariant function* J *on* $\mathbb{C}[x_1,\dots,x_n]_{d+1,\Delta}$ *there exists a rational* GL_n*-invariant function* $\widetilde{J}$ *on* $\mathbb{C}[y_1,\dots,y_n]_{n(d-1)}$ *defined on the image of* A *such that* $J = \widetilde{J}\circ A$.

In other words, the conjecture asserts that the invariant theory of forms of degree $d+1$ can be extracted, in a canonical way, from that of forms of degree $n(d-1)$ at least at the level of rational invariant functions. While this statement is quite intriguing from the purely invariant-theoretic viewpoint, it was originally motivated—as explained in [EI], [AI1]—by complex singularity theory, specifically, by the well-known Mather-Yau theorem (see [MY] and also [Be], [Sh], [GLS, Theorem 2.26]). In [AI2], Conjecture 1.1 was shown to hold for binary forms of any degree, and in [AI1] its weaker variant was established for arbitrary n, d.

In this paper, we obtain results towards settling the conjecture in full generality, which are at the same time of interest in a broader algebraic context. The morphism A factors as $A = \mathbf{A}\circ\mathrm{grad}$, where $\mathrm{grad} : \mathbb{C}[x_1,\dots,x_n]_{d+1} \to \mathbb{C}[x_1,\dots,x_n]_d^{\oplus n}$ is the gradient morphism and $\mathbf{A} : (\mathbb{C}[x_1,\dots,x_n]_d^{\oplus n})_{\mathrm{Res}} \to \mathbb{C}[y_1,\dots,y_n]_{n(d-1)}$ assigns to every n-tuple $\mathbf{f} = (f_1,\dots,f_n)$ of forms of degree d with nonvanishing resultant the *associated form of* $\mathbf{f}$ defined analogously to $A(f)$, with the partial derivative f_{x_j} replaced by f_j for all j. Note that for every $\mathbf{f}$ the form $\mathbf{A}(\mathbf{f})$ is a Macaulay inverse system for the zero-dimensional complete intersection algebra $M(\mathbf{f}) := \mathbb{C}[x_1,\dots,x_n]/(f_1,\dots,f_n)$. As explained in [AI2, Section 3], in order to establish Conjecture 1.1 for all n, d, it is important to study the image of $\mathbf{A}$. In this paper we show that $\mathrm{im}(\mathbf{A}) \subset \mathbb{C}[y_1,\dots,y_n]_{n(d-1)}$ is an open subset of an irreducible component of each of the *catalecticant varieties* V and $\mathrm{Gor}(T)$ (see Section 3 for the definitions) and give a description of the closed complement to $\mathrm{im}(\mathbf{A})$. We note that a number of other properties of the morphism $\mathbf{A}$ (as well as the gradient morphism) essential for confirming Conjecture 1.1 were obtained in the recent paper [F].

The irreducible components of catalecticant varieties are of general interest and have been studied regardless of Conjecture 1.1 (see [IK, Chapter 4] and references therein for details). In particular, in [IK, Theorem 4.17] it was shown that $\mathrm{Gor}(T)$ has an irreducible component containing $\mathrm{im}(\mathbf{A})$ as a dense subset and the dimension of this component was found. On the other hand, an analogous fact for V (which is the catalecticant variety most relevant to our study of the morphism $\mathbf{A}$) appears to be only known in the cases (i) $n = 3$, $d \geq 3$, (ii) $n = 4$, $d = 2, 3$, (iii) $n = 5$, $d = 2$ (see [IK, Theorem 4.19 and Corollary 4.18]), and one of our aims is to bring the results on V in line with those on $\mathrm{Gor}(T)$.

In this paper, we, first of all, refine and extend Theorems 4.17 and 4.19 of [IK]. Namely, in Section 3 we show that the set $\mathrm{im}(\mathbf{A})$ is open (not just dense) in an irreducible component of each of V, $\mathrm{Gor}(T)$ for all n, d and explicitly describe the closed complement to $\mathrm{im}(\mathbf{A})$ (see Theorem 3.3). Note that finding a suitable characterization of this complement is important for resolving Conjecture 1.1 (see Remark 3.5). As the proof of Theorem 4.17 in [IK] is quite brief, we also provide an alternative derivation—with full details—of the dimension formula for $\mathrm{im}(\mathbf{A})$. Note that, although we assume the base field to be $\mathbb{C}$, our arguments work for any algebraically closed field k of characteristic zero and even apply to the case $\mathrm{char}(k) > n(d-1)$, with $n(d-1)$ being the socle degree of $M(\mathbf{f})$ for all $\mathbf{f} = (f_1, \dots, f_n) \in (k[x_1, \dots, x_n]_d^{\oplus n})_{\mathrm{Res}}$. We also stress that our clarifications and extensions of results of [IK] only apply in the case of zero-dimensional complete intersections with homogeneous ideal generators of equal degrees.

In fact, ideally, one would like to have a better description of the complement to $\mathrm{im}(\mathbf{A})$ than the one provided by Theorem 3.3. Namely, it would be desirable to represent it as the intersection of the relevant irreducible component of V with the zero locus of an SL_n-invariant form on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$. This is indeed possible for $n = 2$, in which case the SL_2-invariant in question is the catalecticant (see [AI2, Proposition 4.3]). In Section 4 we show that such a representation is also valid for $n = 3$, $d = 2$, with the corresponding SL_3-invariant being the Aronhold invariant of degree 4 (see Proposition 4.1).

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2. Preliminaries on associated forms and the morphism A

In this section we introduce the main object of our study. What follows is an abridged version of the exposition given in [AI2, Section 2].

Fix $n \geq 2$ and for any nonnegative integer j define $\mathbb{C}[x_1, \ldots, x_n]_j$ to be the vector space of homogeneous forms of degree j in $x_1, \ldots, x_n$ over $\mathbb{C}$. Clearly, one has $\mathbb{C}[x_1, \ldots, x_n] = \oplus_{j=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_j$. Next, fix $d \geq 2$ and consider the vector space $\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n}$ of n-tuples $\mathbf{f} = (f_1, \ldots, f_n)$ of forms of degree d. Recall that the resultant Res on the space $\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n}$ is a form with the property that $\operatorname{Res}(\mathbf{f}) \neq 0$ if and only if $f_1, \ldots, f_n$ have no common zeroes away from the origin (see, e.g., [GKZ, Chapter 13]).

For $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n}$, we now introduce the algebra

$$M(\mathbf{f}) := \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

and recall a well-known lemma (see, e.g., [AI2, Lemma 2.4] and [SS, p. 187]):

Lemma 2.1. *The following statements are equivalent:*

(1) *the resultant* $\operatorname{Res}(\mathbf{f})$ *is nonzero;*

(2) *the algebra* $M(\mathbf{f})$ *has finite vector space dimension;*

(3) *the morphism* $\mathbf{f}\colon \mathbb{A}^n(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ *is finite;*

(4) *the* n*-tuple* $\mathbf{f}$ *is a homogeneous system of parameters of* $\mathbb{C}[x_1, \ldots, x_n]$*, i.e., the Krull dimension of* $M(\mathbf{f})$ *is* 0.

If the above conditions are satisfied, then $M(\mathbf{f})$ *is a local standard graded complete intersection algebra whose socle* $\operatorname{Soc}(M(\mathbf{f}))$ *is generated in degree* $n(d-1)$ *by the image* $\overline{\operatorname{jac}(\mathbf{f})} \in M(\mathbf{f})$ *of the Jacobian* $\operatorname{jac}(\mathbf{f}) := \det \operatorname{Jac}(\mathbf{f})$*, where* $\operatorname{Jac}(\mathbf{f})$ *is the Jacobian matrix* $\big(\partial f_i/\partial x_j\big)_{i,j}$.

Remark 2.2. As we pointed out in Lemma 2.1, the algebra $M(\mathbf{f})$ has a natural standard grading: $M(\mathbf{f}) = \bigoplus_{i=0}^{\infty} M(\mathbf{f})_i$. It is well-known (see, e.g., [St, Corollary 3.3]) that the corresponding Hilbert function $H(x) := \sum_{i=0}^{\infty} t_i\, x^i$, with $t_i := \dim_{\mathbb{C}} M(\mathbf{f})_i$, is given by

$$H(x) = (x^{d-1} + \cdots + x + 1)^n. \tag{2.1}$$

Next, we let $(\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\operatorname{Res}}$ be the affine open subvariety of $\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n}$ that consists of all n-tuples of forms with nonzero resultant. We now define the *associated form* $\mathbf{A}(\mathbf{f}) \in \mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$ of $\mathbf{f} = (f_1, \ldots, f_n) \in (\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\operatorname{Res}}$ by the formula

$$(y_1\overline{x}_1 + y_2\overline{x}_2 + \cdots + y_n\overline{x}_n)^{n(d-1)} = \mathbf{A}(\mathbf{f})(y_1, \ldots, y_n) \cdot \overline{\operatorname{jac}(\mathbf{f})} \in M(\mathbf{f}),$$

where $\overline{x}_i \in M(\mathbf{f})$ is the image of x_i. It is not hard to see that the induced map

$$\mathbf{A}\colon (\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\operatorname{Res}} \to \mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}, \quad \mathbf{f} \mapsto \mathbf{A}(\mathbf{f})$$

is a morphism of affine varieties. This morphism is quite natural; in particular, it has an important equivariance property (see [AI2, Lemma 2.7]). In article [AI2] we studied $\mathbf{A}$ in relation to Conjecture 1.1 stated in the introduction.

We will now interpret $\mathbf{A}$ in different terms. Recall that the algebra $\mathbb{C}[y_1,\dots,y_n]$ is a $\mathbb{C}[x_1,\dots,x_n]$-module via differentiation:

$$(h \diamond F)(y_1,\dots,y_n) := h\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right) F(y_1,\dots,y_n), \tag{2.2}$$

where $h \in \mathbb{C}[x_1,\dots,x_n]$ and $F \in \mathbb{C}[y_1,\dots,y_n]$. For a positive integer j, differentiation induces a perfect pairing

$$\mathbb{C}[x_1,\dots,x_n]_j \times \mathbb{C}[y_1,\dots,y_n]_j \to \mathbb{C}, \quad (h,F) \mapsto h \diamond F;$$

it is often referred to as the *polar pairing*. For $F \in \mathbb{C}[y_1,\dots,y_n]_j$, we now introduce the homogenous ideal, called the *annihilator* of F,

$$F^{\perp} := \{h \in \mathbb{C}[x_1,\dots,x_n] \mid h \diamond F = 0\} \subset \mathbb{C}[x_1,\dots,x_n],$$

which is clearly independent of scaling and thus is well-defined for F in the projective space $\mathbb{P}(\mathbb{C}[y_1,\dots,y_n]_j)$. It is well-known that the quotient $\mathbb{C}[x_1,\dots,x_n]/F^{\perp}$ is a standard graded local Artinian Gorenstein algebra of socle degree j and the following holds (cf. [IK, Lemma 2.12]):

Proposition 2.3. *The correspondence $F \mapsto \mathbb{C}[x_1,\dots,x_n]/F^{\perp}$ induces a bijection*

$$\mathbb{P}(\mathbb{C}[y_1,\dots,y_n]_j) \to \left\{ \begin{array}{l} \textit{local Artinian Gorenstein algebras } \mathbb{C}[x_1,\dots,x_n]/I \\ \textit{of socle degree } j\textit{, where the ideal } I \textit{ is homogeneous} \end{array} \right\}.$$

Remark 2.4. Given a homogenous ideal $I \subset \mathbb{C}[x_1,\dots,x_n]$ such that $\mathbb{C}[x_1,\dots,x_n]/I$ is a local Artinian Gorenstein algebra of socle degree j, Proposition 2.3 implies that there is a form $F \in \mathbb{C}[y_1,\dots,y_n]_j$, unique up to scaling, such that $I = F^{\perp}$. In fact, the uniqueness part of this statement can be strengthened: if $I \subset F^{\perp}$, then $I = F^{\perp}$ and all forms with this property are mutually proportional. Indeed, $I \subset F^{\perp}$ implies $I_j \subset F^{\perp}$, where $I_j := I \cap \mathbb{C}[x_1,\dots,x_n]_j$, and the claim follows from the fact that I_j has codimension 1 in $\mathbb{C}[x_1,\dots,x_n]_j$. Any such form F is called *a* (*homogeneous*) *Macaulay inverse system for* $\mathbb{C}[x_1,\dots,x_n]/I$ and its image in $\mathbb{P}(\mathbb{C}[y_1,\dots,y_n]_j)$ is called *the* (*homogeneous*) *Macaulay inverse system for* $\mathbb{C}[x_1,\dots,x_n]/I$.

We have (see [AI2, Proposition 2.11]):

Proposition 2.5. *For any $\mathbf{f} \in (\mathbb{C}[x_1,\dots,x_n]_d^{\oplus n})_{\mathrm{Res}}$, the form $\mathbf{A}(\mathbf{f})$ is a Macaulay inverse system for the algebra $M(\mathbf{f})$.*

By Proposition 2.5, the morphism $\mathbf{A}$ can be thought of as a map assigning to every element $\mathbf{f} \in (\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\mathrm{Res}}$ a particular Macaulay inverse system for the algebra $M(\mathbf{f})$.

We now let $U_{\mathrm{Res}} \subset \mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$ be the locus of forms F such that the subspace $F^{\perp} \cap \mathbb{C}[x_1, \ldots, x_n]_d$ is n-dimensional and has a basis with nonvanishing resultant. It is easy to see that U_{Res} is locally closed in $\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}$, hence is a variety (see, e.g., Proposition 3.2 below for details). By Proposition 2.5, the image of $\mathbf{A}$ is contained in U_{Res}. Moreover, if $F \in U_{\mathrm{Res}}$, then for the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ generated by $F^{\perp} \cap \mathbb{C}[x_1, \ldots, x_n]_d$, we have the inclusion $I \subset F^{\perp}$. By Remark 2.4, the form F is the inverse system for $\mathbb{C}[x_1, \ldots, x_n]/I$, and therefore $F = \mathbf{A}(\mathbf{f})$ for some basis $\mathbf{f} = (f_1, \ldots, f_n)$ of $F^{\perp} \cap \mathbb{C}[x_1, \ldots, x_n]_d$. Thus, we have proved:

Proposition 2.6. $\mathrm{im}(\mathbf{A}) = U_{\mathrm{Res}}$.

The constructions of the morphism $\mathbf{A}$ can be projectivized. Indeed, denote by $\mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)$ the Grassmannian of n-dimensional subspaces of the space $\mathbb{C}[x_1, \ldots, x_n]_d$. The resultant Res on $\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n}$ descends to a section, also denoted by Res, of a power of the very ample generator of the Picard group of $\mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)$. Let $\mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)_{\mathrm{Res}}$ be the affine open subvariety where Res does not vanish; it consists of all n-dimensional subspaces of $\mathbb{C}[x_1, \ldots, x_n]_d$ having a basis with nonzero resultant. Consider the morphism

$$(\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\mathrm{Res}} \to \mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)_{\mathrm{Res}}, \quad \mathbf{f} = (f_1, \ldots, f_n) \mapsto \langle f_1, \ldots, f_n \rangle,$$

where $\langle \cdot \rangle$ denotes linear span. Then, by the equivariance property (see [AI2, Lemma 2.7]), the morphism $\mathbf{A}$ composed with the projection $\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)} \setminus \{0\} \to \mathbb{P}(\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)})$ factors as

$$(\mathbb{C}[x_1, \ldots, x_n]_d^{\oplus n})_{\mathrm{Res}} \to \mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)_{\mathrm{Res}} \xrightarrow{\widehat{\mathbf{A}}} \mathbb{P}(\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)}).$$

By Proposition 2.5, the morphism $\widehat{\mathbf{A}}$ can be thought of as a map assigning to every subspace $W \in \mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)_{\mathrm{Res}}$ *the* Macaulay inverse system for the algebra $M(\mathbf{f})$, where $\mathbf{f} = (f_1, \ldots, f_n)$ is any basis of W.

Proposition 2.6 implies

Proposition 2.7. $\mathrm{im}(\widehat{\mathbf{A}}) = \mathbb{P}(U_{\mathrm{Res}})$, *where* $\mathbb{P}(U_{\mathrm{Res}})$ *is the image of* U_{Res} *in the projective space* $\mathbb{P}(\mathbb{C}[y_1, \ldots, y_n]_{n(d-1)})$.

It turns out that $\widehat{\mathbf{A}} : \mathrm{Gr}(n, \mathbb{C}[x_1, \ldots, x_n]_d)_{\mathrm{Res}} \to \mathbb{P}(U_{\mathrm{Res}})$ is in fact an isomorphism (see [AI2, Proposition 2.13]). This last result will be utilized in our considerations of the relevant catalecticant varieties in the next section.

3. The catalecticant varieties and a description of im(A)

Let

$$K := \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]_d = \binom{d+n-1}{n-1}.$$

Consider the quasiaffine variety

$$U := U_{K-n}(n(d-1)-d, d; n) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$$

and the affine subvariety

$$V := V_{K-n}(n(d-1)-d, d; n) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$$

as defined in [IK, p. 5]. Specifically, set

$$L := \dim_{\mathbb{C}} \mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d} = \binom{n(d-1)-d+n-1}{n-1}$$

and let $\{\mathtt{m}_1, \dots, \mathtt{m}_K\}$, $\{\mathbf{m}_1, \dots, \mathbf{m}_L\}$ be the standard monomial bases in the spaces $\mathbb{C}[x_1, \dots, x_n]_d$ and $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d}$, respectively, with the monomials numbered in accordance with some orders, which we will fix from now on. For a form $F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ let $F_j := \mathtt{m}_j \diamond F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d}$, $j = 1, \dots, K$, where $\diamond$ is defined in (2.2). Expanding $F_1, \dots, F_K$ with respect to $\{\mathbf{m}_1, \dots, \mathbf{m}_L\}$, we obtain an $L \times K$-matrix $D(F)$ called the *catalecticant matrix*. Then the varieties U and V are described as

$$U = \{F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)} \mid \operatorname{rank} D(F) = K - n\},$$
$$V = \{F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)} \mid \operatorname{rank} D(F) \le K - n\}.$$

Note that U is a dense open subset of V (see [IK, Lemma 3.5]).

Clearly, $V \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ is the affine subvariety given by the condition of the vanishing of all $(K-n+1)$-minors of $D(F)$. Observe that for $n = 2$ one has $K = d+1$, $L = d-1$, and therefore the matrix $D(F)$ has no $(K-1)$-minors, hence $V = \mathbb{C}[y_1, y_2]_{2(d-1)}$. Similarly, for $n = 3$, $d = 2$, we have $K = 6$, $L = 3$, therefore $D(F)$ has no $(K-2)$-minors, hence $V = \mathbb{C}[y_1, y_2, y_3]_3$. Notice that in all other cases one has $L \ge K$, and therefore V is a proper affine subvariety of $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ unless $n = 2$ or $n = 3$, $d = 2$.

Next, let $T := (t_0, t_1, \dots, t_{n(d-1)}) = (1, n, \dots, n, 1)$ be the Gorenstein sequence from the Hilbert function (2.1), which is symmetric about $n(d-1)/2$. Consider the quasiaffine variety $\mathrm{Gor}(T)$ that consists of all forms $F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ such that the Hilbert function of the standard graded local Artinian Gorenstein algebra $\mathbb{C}[x_1, \dots, x_n]/F^{\perp}$ is T. Clearly, $\mathrm{Gor}(T)$ is an open subset of the affine subvariety $\mathrm{Gor}_{\le}(T) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ consisting of all forms F for which the Gorenstein sequence of $\mathbb{C}[x_1, \dots, x_n]/F^{\perp}$ does not exceed

T. Analogously to V, the variety $\mathrm{Gor}_{\leq}(T)$ is defined by the vanishing of all (t_i+1)-minors of the corresponding matrices constructed analogously to $D(F)$, for $i=1,\dots,n(d-1)-1$. Following [IK], we call V and $\mathrm{Gor}(T)$ the *catalecticant varieties.*

Remark 3.1. We note that [IK] introduces more general catalecticant varieties (and even schemes), but V and $\mathrm{Gor}(T)$ are the ones most relevant to our study of the morphism $\mathbf{A}$, thus in the present paper only these two varieties are considered.

We have the obvious inclusions

$$U_{\mathrm{Res}} \subset \mathrm{Gor}(T) \subset U \subset V, \tag{3.1}$$

where $U_{\mathrm{Res}} \subset \mathbb{C}[y_1,\dots,y_n]_{n(d-1)}$ was defined in Section 2. To better understand the relationship between U_{Res}, $\mathrm{Gor}(T)$, U and V, we will now introduce a certain closed subset of U.

Cover U by open subsets U_α, each of which is given by the condition of the nonvanishing of a particular $(K-n)$-minor of the catalecticant matrix $D(F)$. In what follows, on each U_α we will define a regular function R_α. Let, for instance, U_{α_0} be the subset of U described by the nonvanishing of the principal $(K-n)$-minor of $D(F)$. For $F \in \mathrm{U}_{\alpha_0}$ we will now find a canonical basis of the solution set $\mathcal{S}(F)$ of the homogeneous system $D(F)\gamma = 0$, where γ is a column-vector in $\mathbb{C}^K$. Since $\operatorname{rank} D(F) = K-n$, one has $\dim_{\mathbb{C}} \mathcal{S}(F) = n$. Split $D(F)$ into blocks as follows:

$$D(F) = \left(\begin{array}{cc} \boxed{A(F)} & \boxed{B(F)} \\ \multicolumn{2}{c}{\boxed{\quad C(F) \quad}} \end{array}\right),$$

where $A(F)$ has size $(K-n)\times(K-n)$ (recall that $\det A(F) \neq 0$), $B(F)$ has size $(K-n)\times n$, and $C(F)$ has size $(L-K+n)\times K$. We also split the column-vector γ as $\gamma = \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix}$, where γ is in $\mathbb{C}^{K-n}$ and γ'' is in $\mathbb{C}^n$. Then $\mathcal{S}(F)$ is given by the condition $\gamma' = -A(F)^{-1}B(F)\gamma''$. Therefore, the vectors

$$\gamma_j(F) := \begin{pmatrix} -A(F)^{-1}B(F)\mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}, \quad j = 1,\dots,n,$$

form a basis of $\mathcal{S}(F)$ for every $F \in \mathrm{U}_{\alpha_0}$, where $\mathbf{e}_j$ is the jth standard basis vector in $\mathbb{C}^n$.

Clearly, the components $\gamma_j^1,\dots,\gamma_j^K$ of γ_j are regular functions on U_{α_0} for each j, and we define $r_{j,\alpha_0} := \sum_{i=1}^K \gamma_j^i\,\mathtt{m}_i$, $j=1,\dots,n$, where, as before, $\{\mathtt{m}_1,\dots,\mathtt{m}_K\}$ is the standard monomial basis in $\mathbb{C}[x_1,\dots,x_n]_d$. Then the d-forms $r_{1,\alpha_0}(F),\dots,r_{n,\alpha_0}(F)$ constitute a basis of the intersection $F^\perp \cap \mathbb{C}[x_1,\dots,x_n]_d$ for every $F \in \mathrm{U}_{\alpha_0}$. Set $R_{\alpha_0} := \mathrm{Res}(r_{1,\alpha_0},\dots,r_{n,\alpha_0})$. Clearly, R_{α_0} is a regular function on U_{α_0}, and we define Z_{α_0} to be its zero locus.

Arguing as above for every U_α, we introduce a regular function R_α on U_α and its zero locus Z_α. Notice that if for some α, α' the intersection $\mathrm{U}_{\alpha,\alpha'} := \mathrm{U}_\alpha \cap \mathrm{U}_{\alpha'}$ is nonempty, then $Z_\alpha \cap \mathrm{U}_{\alpha,\alpha'} = Z_{\alpha'} \cap \mathrm{U}_{\alpha,\alpha'}$. Thus, the loci Z_α glue together into a closed subset Z of U. If U' is an irreducible component of U, then the intersection $Z \cap U'$ is either a hypersurface in U', or all of U', or empty. Notice also that Z is GL_n-invariant, which follows from the general formula

$$(CF)^\perp \cap \mathbb{C}[x_1,\ldots,x_n]_j = C^{-t}\,(F^\perp \cap \mathbb{C}[x_1,\ldots,x_n]_j), \quad j = 0,\ldots,n(d-1),$$

for all $C \in \mathrm{GL}_n$, $F \in \mathbb{C}[y_1,\ldots,y_n]_{n(d-1)}$.

We will now establish:

Proposition 3.2. *One has* $U_{\mathrm{Res}} = \mathrm{Gor}(T) \setminus Z = U \setminus Z = V \setminus \overline{Z}$.

Proof. It is clear that $U_{\mathrm{Res}} = U \setminus Z$, thus inclusions (3.1) imply $U_{\mathrm{Res}} = \mathrm{Gor}(T) \setminus Z = U \setminus Z$. Further, to see that $U \setminus Z = V \setminus \overline{Z}$, we need to prove that $V \setminus U \subset \overline{Z}$. As shown in the proof of [IK, Lemma 3.5], in every neighborhood of every form $F \in V \setminus U$ there exists $\widehat{F} \in U$ such that all elements of $\widehat{F}^\perp \cap \mathbb{C}[x_1,\ldots,x_n]_d$ have a common zero away from the origin. Thus, $F \in \overline{Z}$ as required. ☐

Next, by Proposition 2.7, the morphism $\widehat{\mathbf{A}} : \mathrm{Gr}(n, \mathbb{C}[x_1,\ldots,x_n]_d)_{\mathrm{Res}} \to \mathbb{P}(U_{\mathrm{Res}})$ is surjective. In fact, by [AI2, Proposition 2.13], the map $\widehat{\mathbf{A}}$ is an isomorphism, therefore we have

$$\dim_{\mathbb{C}} \mathbb{P}(U_{\mathrm{Res}}) = \dim_{\mathbb{C}} \mathrm{Gr}(n, \mathbb{C}[x_1,\ldots,x_n]_d) = Kn - n^2,$$

which implies

$$\dim_{\mathbb{C}} U_{\mathrm{Res}} = Kn - n^2 + 1 =: N. \tag{3.2}$$

As U_{Res} is irreducible, we obtain the following result:

THEOREM 3.3. *There exist irreducible components* $\mathrm{Gor}(T)^\circ$, U°, V° *of the varieties* $\mathrm{Gor}(T)$, U, V, *respectively, such that* $U_{\mathrm{Res}} = \mathrm{Gor}(T)^\circ \setminus Z = U^\circ \setminus Z = V^\circ \setminus \overline{Z}$, *with* $\dim_{\mathbb{C}} \mathrm{Gor}(T)^\circ = \dim_{\mathbb{C}} U^\circ = \dim_{\mathbb{C}} V^\circ = N$, *where* N *is defined in* (3.2).

As by Proposition 2.6 we have $\mathrm{im}(\mathbf{A}) = U_{\mathrm{Res}}$, Theorem 3.3 yields a description of the image of the morphism $\mathbf{A}$ in terms of $\mathrm{Gor}(T)$, U, V and Z.

Remark 3.4. Theorem 4.17 of [IK] shows that $\mathrm{Gor}(T)$ has an irreducible component containing U_{Res} as a dense subset and the dimension of this component is equal to N. The proof given in [IK] does not explicitly utilize the morphism $\mathbf{A}$ and is somewhat brief overall. Also, Theorem 4.19 of [IK] (cf. Corollary 4.18

therein) yields that U_{Res} is dense in an irreducible component of V in the following cases: (i) $n = 3$, $d \geq 3$, (ii) $n = 4$, $d = 2, 3$, (iii) $n = 5$, $d = 2$. In comparison with these results, Theorem 3.3 stated above is more precise because:

- it treats both $\mathrm{Gor}(T)$ and V simultaneously for all n, d;
- it shows that U_{Res} is in fact open (not just dense) in an irreducible component of each of $\mathrm{Gor}(T)$ and V and explicitly describes the closed complement to U_{Res} in terms of the subset Z;
- its proof gives a complete argument for the formula for $\dim_{\mathbb{C}} U_{\mathrm{Res}}$.

Remark 3.5. Describing the complement to $\mathrm{im}(\mathbf{A}) = U_{\mathrm{Res}}$ in V° is of particular importance for settling Conjecture 1.1. Theorem 3.3 offers a description in terms of the set Z, but, ideally, one would like to show that there exists an SL_n-invariant form on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ whose zero locus intersects V° in $V^\circ \setminus \mathrm{im}(\mathbf{A})$. This indeed holds for $n = 2$, in which case $V^\circ = V = \mathbb{C}[y_1, y_2]_{2(d-1)}$ and $\mathbb{C}[y_1, y_2]_{2(d-1)} \setminus \mathrm{im}(\mathbf{A})$ is the zero locus of the catalecticant (see [AI2, Proposition 4.3]). The above fact was instrumental for establishing Conjecture 1.1 in the binary case in [AI2]. In the next section we will show that an analogous statement is also valid for $n = 3$, $d = 2$. Notice that, by [EI], the conjecture holds in this situation as well.

4. The case $n = 3$, $d = 2$

In this section we set $n = 3$, $d = 2$. Notice that the associated form of any element of $(\mathbb{C}[x_1, x_2, x_3]_2^{\oplus 3})_{\mathrm{Res}}$ is a ternary cubic and that $V^\circ = V = \mathbb{C}[y_1, y_2, y_3]_3$. Let S be the degree four Aronhold invariant. An explicit formula for S can be found, for example, in [DK, p. 250]. Namely, for a ternary cubic

$$\begin{aligned} c(y_1, y_2, y_3) = ay_1^3 + by_2^3 + cy_3^3 + 3dy_1^2y_2 + 3ey_1^2y_3 + 3fy_1y_2^2 + \\ 3gy_2^2y_3 + 3hy_1y_3^2 + 3iy_2y_3^2 + 6jy_1y_2y_3 \end{aligned}$$

one has

$$\begin{aligned} S(c) = {} & abcj - bcde - cafg - abhi - j(agi + bhe + cdf) + \\ & afi^2 + ahg^2 + bdh^2 + bie^2 + cgd^2 + cef^2 - j^4 + \\ & 2j^2(fh + id + eg) - 3j(dgh + efi) - f^2h^2 - i^2d^2 - \\ & e^2g^2 + ideg + egfh + fhid. \end{aligned} \tag{4.1}$$

We will now state the result of this section, which for $n = 3$, $d = 2$ provides a more explicit description of the complement $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \mathrm{im}(\mathbf{A})$ than the one given by Theorem 3.3.

Proposition 4.1. *One has* $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \mathrm{im}(\mathbf{A}) = \{S = 0\}$.

Proof. We utilize canonical forms of ternary cubics. Namely, every nonzero ternary cubic is linearly equivalent to one of the following:

$$\begin{aligned}
&c_{1,t} := y_1^3 + y_2^3 + y_3^3 + t y_1 y_2 y_3, && t^3 \neq -27,\\
&c_2 := y_1^3 + y_2^2 y_3 && \text{(cuspidal cubic)},\\
&c_3 := y_1^3 + y_1^2 y_3 + y_2^2 y_3 && \text{(nodal cubic)},\\
&c_4 := y_1^2 y_3 + y_2 y_3^2,\\
&c_5 := y_1^3 + y_1 y_2 y_3,\\
&c_6 := y_1 y_2 y_3,\\
&c_7 := y_1^2 y_2 + y_1 y_2^2,\\
&c_8 := y_1^2 y_2,\\
&c_9 := y_1^3
\end{aligned}$$

(see, e.g., [K, p. 44]). Using formula (4.1) it is now easy to deduce

$$\{S = 0\} = \{0\} \cup O(c_{1,0}) \cup O(c_2) \cup O(c_4) \cup O(c_7) \cup O(c_8) \cup O(c_9),$$

where for a ternary cubic c we denote by $O(c)$ its GL_3-orbit. In particular, we have $\{S = 0\} = \overline{O(c_{1,0})}$, which is the closure of the locus of ternary forms representable as the sum of the cubes of three linear forms (cf. [Ba, Theorems 2.1, 2.2] and [DK, Proposition 5.13.2]).

To see that $\mathrm{im}(\mathbf{A})$ does not intersect the zero locus of S, we find the degree two component of the annihilator of each of the cubics $c_{1,0}, c_2, c_4, c_7, c_8, c_9$:

$$\begin{aligned}
&c_{1,0}^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1 x_2, x_1 x_3, x_2 x_3 \rangle,\\
&c_2^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1 x_2, x_1 x_3, x_3^2 \rangle,\\
&c_4^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1^2 - x_2 x_3, x_1 x_2, x_2^2 \rangle,\\
&c_7^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1^2 + x_2^2 - x_1 x_2, x_1 x_3, x_2 x_3, x_3^2 \rangle,\\
&c_8^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1 x_3, x_2^2, x_2 x_3, x_3^2 \rangle,\\
&c_9^{\perp} \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2 \rangle.
\end{aligned}$$

We thus see that for the cubics c_7, c_8, c_9 the corresponding annihilator components have dimension greater than 3 and that in the remaining situations they have zeroes away from the origin. It then follows that

$$\mathrm{im}(\mathbf{A}) \subset \mathbb{C}[y_1, y_2, y_3]_3 \setminus \{S = 0\}.$$

In order to show that $\mathbf{A}$ maps $(\mathbb{C}[x_1, x_2, x_3]_2^{\oplus 3})_{\mathrm{Res}}$ onto $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \{S = 0\}$, we need to prove that each of the cubics $c_{1,t}$, c_3, c_5, c_6 lies in $\mathrm{im}(\mathbf{A})$, where $t \neq 0$, $t^3 \neq 216$ (notice that $c_{1,0}$ and $c_{1,\tau}$ with $\tau^3 = 216$ are linearly equivalent—see, e.g., [AIK, p. 603]). First of all, $c_{1,t}$, with $t \neq 0$, $t^3 \neq 216$, is proportional to the associated form of the nondegenerate cubic $c_{1,-18/t}$ and c_6 to the associated form of the nondegenerate cubic $c_{1,0}$ (see, e.g., [AIK, Section 2.2]). Next, we calculate the degree two component of the annihilator of each of the cubics c_3, c_5:

$$c_3^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1^2 - x_2^2 - 3x_1x_3, x_1x_2, x_3^2 \rangle,$$
$$c_5^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 = \langle x_1^2 - 6x_2x_3, x_2^2, x_3^2 \rangle.$$

This shows that c_3, c_5 lie in U_{Res} hence in $\mathrm{im}(\mathbf{A})$.

The proof is now complete. □

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