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On blow-analytic equivalence of plane curves

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Abstract

Discrete topological invariants are presented to classify real plane curve singularities or isolated surface singularities by blow-analytic equivalence. A characterization of basic singularities like nodes are given. An 'exotic' blowing up is defined to factor a contraction of curves in a surface.

1. Introduction

Two germs of embedded real plane curve singularities are *blow-analytically equivalent* if there is a homeomorphism between them which can be realized by some simple blowings up and down. Here, a *simple blowing up* is a blowing up of a smooth point in a real surface. This equivalence is first studied in [1] for the case there is only one analytic branch. They are, suprisingly, shown to be all equivalent.

As an example we reprise here the so-called 'Kobayashi-Kuo example'. Take a cusp $(\{y^2 - x^3 = 0\}, 0)$. By usual three blowings up one reaches a good resolution. We perform one more blowing up at a smooth point of the exceptional curve with even self-intersection number. In the complex case, one has two (-3)-curves, which are, in the real case, isomorphic to (-1)-curves, thus contractible

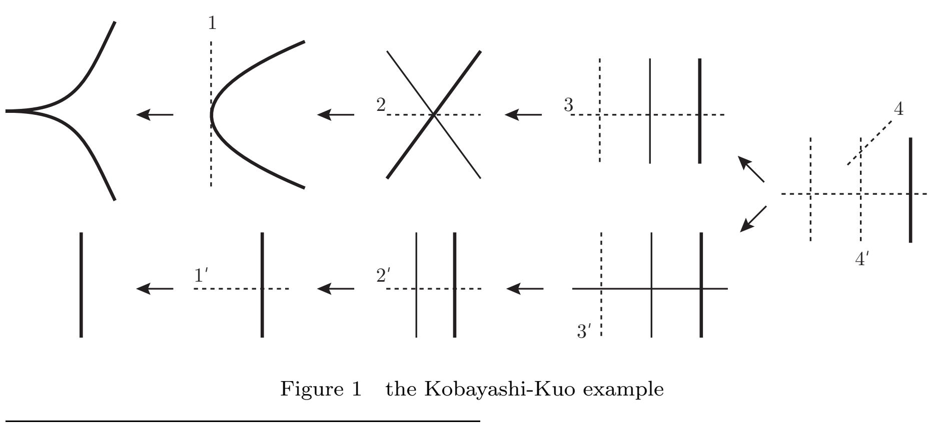

Figure 1 the Kobayashi-Kuo example

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to smooth points. The third and the fourth exceptional curves can be contracted and one gets a smooth line in a plane.

Each compact smooth real analytic curve is diffeomorphic to S^1, and has its *parity* if it is embedded in a smooth surface. The parity is *even* or *odd* according to the tubular neighborhood is oriented or not, respectively. In Figure 1, the bold lines are strict transforms of the original curves, dotted lines are odd and real lines are even. The number n designates that the curve is the exceptional curve of n-th simple blowing up.

We present in this paper a basic framework to classify germs of real plane curves. We add some useful invariants to [1] and give a rough classification of bibranched case. After completing this work, Valle [2] gave a finiteness result fixing the discrete invariant μ' here and the number of components. She also gave a complete classification of tribranched curves with $\mu' \leq 2$.

2. An invariant μ and surface singularities

Definition 2.1. Let X be a surface which is a tubular neighborhood of a sum of compact smooth curves $\{E_i\}$ $(1 \leq i \leq n)$ intersecting normally whose dual graph is a tree. For $(X, \{E_i\})$, or for the dual graph whose vertices are weighted by the parity of the curve, we define μ to be the corank of the intersection matrix A, which is independent of the numbering of E_i's. Note that the intersection numbers take value in the field $\boldsymbol{Z}/2\boldsymbol{Z}$.

The surface X is contractible by a deformation retract to a union of S^1's. Assume furthermore that X is connected for simplicity. Let us denote the number of connected component of the boundary ∂X by m.

Proposition 2.2. *The corank μ is one less than the number of the connected component m.*

Proof. We write a $(\boldsymbol{Z}/2\boldsymbol{Z})$-vector space $H_1(X, \boldsymbol{Z}/2\boldsymbol{Z})$ simply as H, which is spanned by the classes of E_i's. The matrix A determines a symmetric bilinear form on H and the number μ is equal to $\dim\ker A$.

Consider a standard exact sequence of $(\boldsymbol{Z}/2\boldsymbol{Z})$-vector spaces:

$$\begin{array}{ccccccc} & & & 0 & \to & H_2(X, \partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \\ \to & H_1(\partial X, \boldsymbol{Z}/2\boldsymbol{Z}) & \to & H_1(X, \boldsymbol{Z}/2\boldsymbol{Z}) & \stackrel{\alpha}{\to} & H_1(X, \partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \\ \to & H_0(\partial X, \boldsymbol{Z}/2\boldsymbol{Z}) & \to & H_0(X, \boldsymbol{Z}/2\boldsymbol{Z}) & \to & 0 \end{array}$$

where

(1) $H_0(X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong \boldsymbol{Z}/2\boldsymbol{Z}$, $H_1(X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus n}$,

(2) $H_0(\partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus m}$, $H_1(\partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus m}$ and

(3) $H_2(X, \partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong \boldsymbol{Z}/2\boldsymbol{Z}$.

Hence $H_1(X, \partial X, \boldsymbol{Z}/2\boldsymbol{Z}) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus n}$. Since $H_1(X, \partial X, \boldsymbol{Z}/2\boldsymbol{Z}) = H_1(X, \partial X, \boldsymbol{Z}) \otimes \boldsymbol{Z}/2\boldsymbol{Z}$ is isomorphic to $H^1(X, \boldsymbol{Z}) \otimes \boldsymbol{Z}/2\boldsymbol{Z}$ by intersection pairing, for $v, w \in H_1(X, \boldsymbol{Z}/2\boldsymbol{Z})$ we have $\alpha(v)(w) = {}^t vAw$. Hence $\mu = \dim \ker \alpha$ equals to $(m-1)$. □

Lemma 2.3. *The number μ is invariant under simple blowing up and down, provided that E_i's remain to be normal crossing.*

Proof. The corank μ is independent of the choice of the basis of H, and the new exceptional curve is odd and orthogonal to all total transforms. □

Corollary 2.4. *The number μ is an invariant for real analytic class of isolated real surface singularities.*

Remark 2.5. Since $\mu = 0$ if and only if the determinant of A is one, we can rewrite the assertions in [1] using μ. For instance, if all the curves are odd, the condition $\mu = 0$ is equivalent to the existence of a composition of simple blowings up $\beta : (X, \cup E_i) \to (\boldsymbol{R}^2, \{0\})$.

Since the classes of E_i's form a basis of H, there exists an orthogonal decomposition $H = \ker A \oplus H'$ where H' is a subspace spanned by basis $\{E_{i_1}, \ldots, E_{i_{n-\mu}}\}$ for a suitable choice of $i_1, \ldots, i_{n-\mu} \subset \{1, \ldots, n\}$. The matrix associated to the bilinear form on H' has determinant one.

Proposition 2.6. *Suppose that $\mu = 1$. Then $(X, \{E_i\})$ can be transformed by simple blowings up and down to a neighborhood of a single even curve.*

Proof. Since $\ker A$ is spanned by a single vector, the space H' is spanned by all but one curve, say E_j. Let Γ' be the full subgraph which does not contain E_j. Since Γ' has determinant one and all the connected component of Γ' has at most one intersection with E_j, one can contract Γ' leaving E_j smooth by Proposition 8 in [1]. The remaining E_j becomes even since the determinant is zero. □

Corollary 2.7. *An isolated real surface singularity with $\mu = 1$ is blow-analytically equivalent to an ordinary double point $(\{x^2 + y^2 = z^2\}, 0)$.*

3. An invariant μ' of branched plane curves

Definition 3.1. We call a germ of analytic curve in $\boldsymbol{R}^2$ which has an isolated singularity with several branches at the origin a *branched plane curve*. We only consider the real support of the curve, and do not take account in multiplicities of components, embedded primes, imaginary components. Let $(C, 0)$ be a branched curve and $\beta : X \to \boldsymbol{R}^2$ be a *simple good resolution*, in the sense that:

(1) ("simple") the map β is a composition of simple blowings up β_i $(1 \le i \le n)$ and each center of β_i is above the origin of $\boldsymbol{R}^2$,

(2) ("good") the set $\beta^{-1}(C)$ is a union of smooth curves, no three components meet at a point and any two components intersect at most at a point transversally.

A simple good resolution exists e.g. by Hironaka's theorem.

Lemma 3.2. *Each analytic branch of a plane curve at a point P intersects a small neibourhood of P at two points. Especially, an analytic branch does not have an end point.*

Proof. Since the germ of a real curve at a smooth point is real analytically a line, the strict transform of each branch of the plane curve in a simple good resolution intersects with the boundary of a small tubular neighborhood of the exceptional set at two points. Since the contraction map is homeomorphic outside the exceptional set, the original branch has also two intersection points at the boundary of a tubular neighborhood of P. □

Fix a simple good resolution of C. Since the dual graph of the exceptional curves is a tree, there exists the minimal subtree c of exceptional curves which combines the components of the strict transforms of C. When C is bibranched, c is a chain. Let V' be the set of exceptional curves which do not participate in c. Let Γ' be the full subgraph of the total dual graph of exceptional curves, whose vertex set is V'. We define μ' to be the corank of Γ', and Δ' to be the determinant of the intersection matrix of Γ'.

Lemma 3.3. *The numbers μ' and Δ' are independent of the choice of simple good resolution.*

Proof. The center of simple blowing up is one of the following:

(1) an intersection of two curves which are either a curve in c or a component of the strict transform of C,

(2) a simple point on an exceptional curve in c or

(3) a point on an exceptional curve in V'.

In the case (1), the tree c gets longer and Γ' does not change at all. The case (2) creates a new cell which belongs to V'; since this is a launch extension for Γ', the number μ' does not change. The case (3) extends Γ' and this does not change μ'.

Blowing down of a branch point in the union of the exceptional divisors and the strict transform destroys the property of 'goodness', thus simple blowing down in a category of a simple good resolution does not harm c. □

Thus μ' and Δ' are well-defined for a germ of branched curve in $\boldsymbol{R}^2$, since any branched curve has a simple good resolution and two such resolutions always have a common simple good resolution.

Example 3.4. One can construct an example with arbitrary big μ' as follows.

Start from a single odd curve v_0 and perform a tick extension on it. A tick extension has two branches and take the shorter branch consisting of a single vertex, which we call v_1. Performing again a tick extension on v_1 and get v_2. Repeat tick extensions and you get v_n. Let two noncompact smooth curves intersect general single points on v_0 and v_n, respectively, and add arbitrary number of noncompact curves on the chain between v_0 and v_n. The configuration has $\mu = n$ and the image C is at least bibranched.

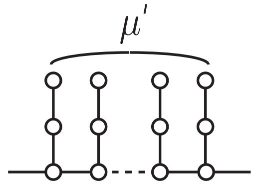

Figure 2 Curves with arbitrary μ'

Proposition 3.5. *A germ of bibranched curve* $(C, 0)$ *in* $\boldsymbol{R}^2$ *is blow-analytically equivalent to a germ* $(\{xy = 0\}, 0)$ *if and only if a good resolution of* $(C, 0)$ *has* $\Delta' = 1$.

Proof. The germ $(\{xy = 0\}, 0)$ in $\boldsymbol{R}^2$ itself is a simple good resolution and has $\Delta' = 1$. Since one can always have a common simple good resolution, 'only if' part is proven by the previous lemma.

Suppose that some simple good resolution of $(C, 0)$ has $\Delta' = 1$. Performing additional simple blowings up if necessary, one can assume that all the exceptional curves in V' are odd. Since the determinant of a graph is the product of the determinants of all the connected components, each connected component of Γ' has determinant one.

For each connected component, there exists a unique vertex which is connected to the chain c. Regarding that vertex to be 'c' in the Proposition 8 of [Kobayashi-Kuo1], one can contract all the vertices in V' by a sequence of simple blowings down leaving the curves in the chain smooth.

The remaining chain may be a mixture of odd and even curves. You can contract any odd curves; after the contraction, the graph remains to be a chain with two components of the strict transforms of C at both ends. When there exists a pair of adjoining two even curves, blowing up the intersection of those two curves creates three linear odd curves, which can be contracted by three simple blowings down. Thus we can decrease the length of the chain. Since the determinant of the chain is one, eventually one reaches the stage where there remains only one odd curve left; one can contract that last exceptional curve and get two smooth curves intersecting transversally in $\boldsymbol{R}^2$. □

Example 3.6. Two germs $(\{y(y^2-x^3)=0\},0)$, $(\{y(y-x^3)=0\},0)$ have $\mu'=0$, hence blow-analytically equivalent to an ordinary double point.

We say two plane curve germs are blow-analytically equivalent *in the level of the graph* if the germs have a common resolution graph (including the strict transform).

Theorem 3.7. *Two bibranched curves in* $\boldsymbol{R}^2$ *are blow-analytically equivalent in the level of the graph if and only if they have a common* μ'.

Proof. Let X be a simple good resolution of a bibranched curve. We will show that X can be transformed to a standard configuration by simple blowings up and down. One can assume that all the curves are odd as before.

Similarly, as in the proof of Proposition 8 in [1], one can contract the vertices whose distance with the chain c is greater than two. Performing twin contractions if necessary, one can assume that the connected components of Γ' are either (A) a single odd vertex or (B) two adjoining odd vertices. The number of branches of type (B) is equal to μ'. Let us denote the branches by γ_j's $(1 \leq j \leq \mu')$.

Contract all the branches of type (A); Now the chain c may be a mixture of odd or even curves. Each branch γ_j is joined with c at a unique vertex v_j in c, and if $j_1 \neq j_2$, the vertices v_{j_1} and v_{j_2} are different by Corollary 10 in [1]. We can assume that v_j is ordered (increasingly or decreasingly) according to the order in the chain.

Next contract vertices in c other than v_j's; we can assume that each segment separated by v_j's has at most one vertex, which is even.

Now the Theorem follows from the two facts below. □

Lemma 3.8. *Even if the parity of* $v=v_j$ *is reversed, the equivalence class of the configuration does not change.*

Proof. The graphs $v-o-o$ and $o-v-o-o$, where v and o's are odd curves, are equivalent, since they have a common resolution $o-v-o-o=8$. □

Claim 3.9. *There are no even curves left in the segments.*

Proof. This comes from the condition that the total determinant is one, so here we forget the noncompact curves. We will show the Claim by induction on μ'.

If μ' is zero, there is only one room for those even curves, and if there really is, the determinant is zero. Next suppose that $\mu'>0$ and that there is an even curve outside of v_1, that is, at one edge. The curve, blown up at a general point, becomes two adjoining odd curves. Those curves together with the closest γ_1 constitute an impossible subgraph as usual. Thus, there are no even curves at either edges. By the previous Lemma, one can assume that v_1 is odd, hence one can perform a rifle contraction to v_1 and γ_1. Then the number μ' has been decreased by one. $\square$

Proof of Theorem, continued. Thus we get to a configuration where the chain c consists of a linear μ' odd curves and at each curve in c a branch of type (B) is clinging. The Theorem is proven. $\square$

Example 3.10. A germ of bibranched curve with $\mu' = 1$ is blow-analytically equivalent to a germ $(\{x(x-y^2)=0\},0)$. In fact, it is easy to see that the dual graph determines completely the tubular neibourhood in this case. It is also easy to see that a germ of bibranched curve with $\mu' = 2$ is blow-analytically equivalent to a germ $(\{(y^2-x^5)(y-x^2)=0\},0)$.

When C has more branches, you can choose any subset of branches and consider similar invariants.

Definition 3.11. Let X be a simple good resolution of a germ of curve $(C,0)$ in $\boldsymbol{R}^2$, which is n-branched. We denote the components of strict transform of C in X by $C_1,\dots,C_n$.

Let k be any integer between 0 and n. Choose arbitrary k components among $C_1,\dots,C_n$, say, $\{C_{i_1},\dots,C_{i_k}\}$. Let $c_{i_1,\dots,i_k}$ be the minimal subtree of exceptional curves which combines them, and $\mathit{\Gamma}_{i_1,\dots,i_k}$ be the full complementary subgraph. We define μ_k to be an unordered list (i.e. a set admitting repetition) $\{\operatorname{corank}\mathit{\Gamma}_{i_1,\dots,i_k} | 1 \le i_1 < \cdots < i_k \le n\}$.

Remark 3.12. (1) We have $\mu_0 = \{\mu\}$ and $\mu_n = \{\mu'\}$. We need not assume that X comes from $\boldsymbol{R}^2$ by simple blowings up for the definition of μ_k.

(2) The numbers μ_k $(k > 1)$ are invariant under simple blowing up and down, but μ_1 is not.

The invariance of μ_k follows by classifying the center of a simple blowing up. If the center is in $c_{i_1,\dots,i_k}$, the blowing up adds at most one odd curve in the new $\mathit{\Gamma}_{i_1,\dots,i_k}$. If the center is outside $c_{i_1,\dots,i_k}$, it is a simple blowing up of $\mathit{\Gamma}_{i_1,\dots,i_k}$. In both cases $\operatorname{corank}\mathit{\Gamma}_{i_1,\dots,i_k}$ does not change.

Proposition 3.13. *A germ of tribranched curve in $\boldsymbol{R}^2$ with $\mu' = 0$ is blow-analytically equivalent either to*

(1) *an ordinary triple point $(\{xy(x+y)=0\},0)$, or to*

(2) *a germ $(\{xy(x-y^2)=0\},0)$,*

according to $\mu_2 = \{1,1,1\}$ or $\{0,1,1\}$.

Proof. Since $\mu' = 0$, one can contract all the curves in V' leaving the components of c smooth as in the bibranched case. The resulting configuration including the component of the strict transform is a star. We denote by v the central vertex of the star. As in the bibranched case, one can contract each segment between v and the strict transform, performing some simple blowings up if necessary, and leaves possibly one even curve on each segment. Since $\mu = 0$, that can happen on at most one segment and, in either case, the graph determines a blow-analytically equivalent class.

(1) If all the segments vanish, v is now an odd curve. In this case, one has a triple point.

(2) Otherwise, one segment has an even curve v'. This curve v may be odd or even; but in either case, one can blow up a point on v' and reach the configuration $o - o - o$, where all three curves are odd, one end curve intersects one noncompact component and the other end curve intersects two noncompact components. This is nothing but a simple good resolution of (2).

One can easily calculate μ_2 for each case. □

4. An exotic blowing up

Proposition 4.1. *Let f be a following map:*

$$f : \boldsymbol{R}P^1 \times \boldsymbol{R}P^1 \setminus \{(\infty, \infty)\} \to \boldsymbol{R}^2$$

sending (u, v) to

$$(x, y) = \left(\frac{uv^2}{1 + u^2 + v^2}, \frac{u^2 v}{1 + u^2 + v^2} \right).$$

Then the following holds:

(1) *f is real analytic everywhere,*

(2) *f sends the axes $\{0\} \times \boldsymbol{R}P^1$ and $\boldsymbol{R}P^1 \times \{0\}$ to the origin,*

(3) *$f(u, \infty) = (u, 0)$ and $f(\infty, v) = (0, v)$ and*

(4) *f induces a real analytic isomorphism between $\{(u, v) | u \neq 0$ and $v \neq 0\}$ and $\boldsymbol{R}^2 \setminus (0, 0)$.*

Proof. (1) Let us take a local coordinate (ξ, η) around a point $(u, v) = (a, b)$. If both a and b are finite, we can take $\xi = u - a$, $\eta = v - b$. Then $x = (\xi + a)(\eta + b)^2/(1 + a^2 + b^2 + 2a\xi + 2b\eta + \xi^2 + \eta^2)$ is obviously a real analytic function of (ξ, η). So is y.

If $a = \infty$, we take $\xi = 1/u$ and $\eta = v - b$. Note that only one of a and b can be the infinity. In this case, the map

$$(x, y) = \frac{1}{1 + \xi^2(\eta + b)^2} \left(\xi(\eta + b)^2, \eta + b \right)$$

are again real analytic. The other cases are similar.

(2) Obvious from the definition.

(3) Obvious from the equation above when $(\xi, \eta) \to (0, 0)$.

(4) Assume first that u, v are finite and $uv \neq 0$. In this case $xy \neq 0$. The Jacobian matrix $\partial(x, y)/\partial(u, v)$ is equal to

$$\frac{1}{(u^2+v^2+1)^2}\begin{pmatrix} v^2(v^2-u^2+1) & 2uv(u^2+1) \\ 2uv(v^2+1) & u^2(u^2-v^2+1) \end{pmatrix},$$

whose determinant is

$$-\frac{u^2v^2(3+4u^2+4v^2+2u^2v^2+u^4+v^4)}{(u^2+v^2+1)^4} < 0.$$

Thus f is locally one-to-one by the inverse function theorem.

On the other hand, u satisfies the following equation:

$$u^3 - \frac{x^2+y^2}{x}u^2 - \frac{y^2}{x} = 0.$$

One can easily check that this equation has exactly one real root. Thus f is one-to-one in this case.

Next Assume that $u = \infty$ and $v \neq 0$. Let $s = 1/u$ and then we have

$$(x, y) = \frac{(sv^2, v)}{1+s^2+s^2v^2}.$$

The Jacobian matrix is

$$\frac{\partial(x, y)}{\partial(s, v)} = \frac{1}{(1+s^2+s^2v^2)^2}\begin{pmatrix} 1-s^2-s^2v^2 & 2sv(1+s^2) \\ -2sv(1+v^2) & 1+s^2-s^2v^2 \end{pmatrix},$$

which is the identity matrix when $s = 0$. The other case is similar. □

Let X be an analytic neighborhood of two even rational smooth curves which intersect normally at one point. The surface X can be blown down to a smooth surface so that the two curves are sent to a smooth point. We call this map an *exotic blowing down*. More generally,

Definition 4.2. An *exotic blowing down* is a surjective real analytic map between two dimensional real analytic manifolds $g : M \to N$ such that

(1) M has an open subset U which is real analytically isomorphic to $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$,

(2) $g_{|U}$ is real analytically equivalent to f in Proposition 4.1 and

(3) g is real analytic isomorphism outside U.

We have proved that the determinant of the intersection matrix is an invariant under simple blowing up and down. The exotic blowing up is represented by three simple blowings up and one simple blowing down, thus the determinant is also invariant under exotic blowings up and down.

The following asserts that the converse is true in the category of real analytic blowings up.

Theorem 4.3. *Let X be a tree of smooth rational curves whose determinant is one. Then X can be contracted to $\boldsymbol{R}^2$ by a composition of simple and exotic blowings down.*

Moreover, if there is a noncompact curve normally crossing at only one curve, then one can choose the contraction map whose restriction to the noncompact curve is isomorphism to the image in $\boldsymbol{R}^2$.

Remark 4.4. The neighborhood in the definition of exotic blowing down above cannot be contracted by a composition of smooth blowings down only, since there are no odd curves.

Proof. Let c be any fixed vertex and $d(v)$ be the distance from c to v. Assume the maximum of d is greater than one. First contract all odd vertices which attain the maximum of d.

Suppose there exists an even vertex v which attains the maximum of d. The vertex v has a unique adjacent vertex w. If there are other vertices which is adjacent to w and has the maximum d, the total determinant is zero using the lemma. Thus there are exactly two adjacent vertices to w, that is, v and say x. If w is odd, contract w and then v; if w is even, apply exotic blow down for w and v. In either case, the number of vertices decreases by two. By repeating the above process, one eventually get to the case where d is at most one.

Contract all the odd vertices which has $d = 1$ leaves only the chain c and the even adjacent vertices. Again, the number of even adjacent vertices is at most one. If c is alone, c is odd by determinantal reason. One can apply a simple blowing down. Otherwise, apply an exotic blowing down or two simple blowings down beginning from c according to the parity of c.

Thus we can contract the orignal surface to $\boldsymbol{R}^2$; and if there is a noncompact component crossing normally to the curve corresponding to c, it is mapped isomorphically to the image in $\boldsymbol{R}^2$. □

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