Saitama Math. J. 115
Vol. 31 (2017), 115–136

Comprehensive Gröbner systems approach to b-functions of μ-constant deformations

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(*Received 10 July, 2016; Accepted 26 October, 2016*)

Abstract

A method for computing b-functions associated with semi-quasihomogeneous isolated singularities is considered in the context of symbolic computation. A new method of computing b-functions and relevant holonomic D-modules associated with μ-constant deformations is described. The key of the resulting algorithm is the use of the notions of comprehensive Gröbner systems of a special class of Poincaré-Birkhoff-Witt algebra and that of Weyl algebra. Several b-functions of μ-constant deformations of bimodal singularities are given as the result of the computation.

1. Introduction

The b-function, or Bernstein-Sato polynomial, is an important complex analytic invariant of hypersurface singularities. Many researchers of singularity theory have studied b-functions and relations between b-functions and singularities [1, 4, 5, 6, 9, 14, 17, 18, 19, 20, 22, 23, 25, 26, 27, 28, 29, 33, 36, 39, 40, 42, 43].

Let b_f be the b-function of a semi-quasihomogeneous polynomial f with parameters. Then, b_f may change with the values of parameters. T. Yano in [43] studied the b-function of the μ-constant deformation of $x^5 + y^5$ and M. Kato computed b-functions of the μ-constant deformations of $x^7 + y^5$ and $x^9 + y^4$ in [15, 16]. Moreover, P. Cassou-Nogués computed b-functions of μ-constant deformations of $x^5 + y^4$ and $x^7 + y^6$ in [5, 6]. B-functions of μ-constant deformations have been studied by many researchers. See [3, 5, 7, 10, 11, 31, 32].

There exist mainly two different kinds of approaches for computing b-functions [3, 27, 29, 39]. The first approach requires an annihilating ideal of f^s in rings of partial differential operators to compute the b-function b_f where s is an indeterminate. The second approach computes b-functions without computing the annihilating ideal of f^s. We follow the first approach to study b-functions of μ-constant deformations.

In [21], we have presented algorithms for computing comprehensive Gröbner systems in rings of partial differential operators and a special class of Pincaré-

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2010 Mathematics Subject Classification. 14F10, 32S25, 13P10

Key words and phrases. semi-quasihomogeneous singularities, μ-constant deformation, Gröbner bases.

Birkhoff-Witt algebras. We also have implemented the algorithms in the computer algebra system Risa/Asir [24].

In this paper, we show that an algorithm for computing parametric b-functions can be constructed by using comprehensive Gröbner systems. We propose a comprehensive Gröbner systems approach for studying b-function of μ-constant deformations. We provide a method that utilizing holonomic D-modules to compute b-function of μ-constant deformations.

This paper is organized as follows. In section 2, we see comprehensive Gröbner systems in rings of partial differential operators. In section 3, we review comprehensive Gröbner systems in Poincaré-Birkhoff-Witt algebra and give a method for computing parametric b-functions. In section 4, we describe structures of holonomic D-module associated with roots of b-functions. In section 5, we give b-functions of μ-constant deformations of several non-unimodal singularities. In section 6, we introduce an idea for avoiding heavy computation of b-functions.

2. Comprehensive Gröbner systems

Here we recall the notations of comprehensive Gröbner systems in rings of partial differential operators. For details, we refer the reader to [21].

Let K be a field of characteristic zero, $\bar{K}$ an algebraic closure of K. The set of natural numbers $\mathbb{N}$ includes zero, $\mathbb{C}$ is the field of complex numbers and $\mathbb{Q}$ is the field of rational numbers.

Let $K\langle x, \partial_x\rangle$ denote the Weyl algebra, the ring of linear partial differential operators with coefficients in K, where $x = (x_1, \ldots, x_n)$, $\partial_x = (\partial_1, \ldots, \partial_n)$, $\partial_i = \frac{\partial}{\partial x_i}$ with relations

$$x_i x_j = x_j x_i,\ \partial_i\partial_j = \partial_j\partial_i,\ \partial_j x_i = x_i\partial_j\ (i \neq j) \text{ and } \partial_i x_i = x_i\partial_i + 1.$$

Let $u = (u_1, \ldots, u_m)$ be variables such that $u \cap x = \emptyset$, $K[u]\langle x, \partial_x\rangle$ a ring of partial differential operators with coefficients in a polynomial ring $K[u]$. The symbol $\mathrm{pp}(x, \partial_x)$ is the set of power products of $x \cup \partial_x$.

Throughout the paper we assume that a partial differential operator in $K\langle x, \partial_x\rangle$ (or $K[u]\langle x, \partial_x\rangle$), is always represented in the *canonical form* that is each power product of a partial differential operator is written as $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$ where $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{N}$.

We have the following natural K-vector space isomorphism $\Psi : K\langle x, \partial_x\rangle \to K[x, \xi]$ $(x^\beta \partial_x^\gamma \to x^\beta \xi^\gamma)$ where $K[x, \xi]$ is a commutative polynomial ring, $\xi = (\xi_1, \ldots, \xi_n)$ corresponds to $\partial_x = (\partial_1, \partial_2, \ldots, \partial_n)$. For example, let $p = 3x_1^2 x_2 \partial_1 + x_2\partial_2 \in \mathbb{C}\langle x_1, x_2, \partial_1, \partial_2\rangle$. Then $\Psi(p) = 3x_1^2 x_2 \xi_1 + x_2\xi_2 \in \mathbb{C}[x_1, x_2, \xi_1, \xi_2]$. For all $i \in \{1, \ldots, n\}$, the inverse map Ψ^{-1} is defined as changing symbols ξ_i into ∂_i.

Fix a term ordering $\succ$ on $\mathrm{pp}(x, \partial_x)$ and let $p \in K\langle x, \partial_x\rangle$. Then, $\mathrm{lpp}(p)$, $\mathrm{lm}(p)$ and $\mathrm{lc}(p)$ denote as the leading power product, leading monominal and leading coefficient of $\Psi(p)$ in $K[x,\xi]$. Furthermore, for a subset P in $K\langle x,\partial_x\rangle$, we define $\mathrm{lpp}(P) := \{\mathrm{lpp}(p)|p\in P\}$, $\mathrm{lm}(P) := \{\mathrm{lm}(p)|\ p\in P\}$ and $\mathrm{lc}(P) := \{\mathrm{lc}(p)|p\in P\}$.

Let $p_1,\dots,p_r \in K\langle u,x,\partial_x\rangle$ (or $K[u]\langle x,\partial_x\rangle$). Then, the left ideal generated by $p_1,\dots,p_r$ is written as $Id(p_1,\dots,p_r)$.

Definition 1. *Fix a term ordering on* $\mathrm{pp}(x,\partial_x)$. *Let* $p_1,\dots,p_r\in K\langle x,\partial_x\rangle$ *and* $G=\{g_1,\dots,g_r\}\subset Id(p_1,\dots,p_r)\subset K\langle u,x,\partial_x\rangle$. *Then,* G *is a Gröbner basis of* $Id(p_1,\dots,p_r)$ *if* G *satisfies* $Id(\mathrm{lm}(I)) = Id(\mathrm{lm}(g_1),\dots,\mathrm{lm}(g_r))$.

There exist algorithms and implementations to compute Gröbner bases of ideals in $K\langle u,x,\partial_x\rangle$.

For every $\bar{a}\in\bar{K}^m$, we define the canonical specialization homomorphism $\sigma_{\bar{a}}: K[u]\langle x,\partial_x\rangle \to \bar{K}\langle x,\partial_x\rangle$ as a map that substitutes u by $\bar{a}$ in a partial differential operator $p(u,x,\partial_x)\in K[u]\langle x,\partial_x\rangle$. The image under $\sigma_{\bar{a}}$ of an ideal $I\subset K[u]\langle x,\partial_x\rangle$ is denoted by $\sigma_{\bar{a}}(I) := \{\sigma_{\bar{a}}(p)|p\in I\}\subseteq \bar{K}\langle x,\partial_x\rangle$.

For instance, let $p = 3u_1u_2x_1^3\partial_1^2 + u_2x_1\partial_1 + x_1$ in $\mathbb{C}[u_1,u_2]\langle x_1,\partial_1\rangle$ and $(-3,1),(0,-\frac{2}{3})\in\mathbb{C}^2$. Then, by substituting values $(-3,1),(0,-\frac{2}{3})$ into (u_1,u_2), we get $\sigma_{(-3,1)}(p) = -9x_1^3\partial_1^2 + x_1\partial_1 + x_1$ and $\sigma_{(0,-\frac{2}{3})}(p) = -\frac{2}{3}x_1\partial_1 + x_1$.

For $g_1,\dots,g_r\in K[u]$, $\mathbb{V}(g_1,\dots,\ g_r)\subseteq\bar{K}^m$ denotes the affine variety of $g_1,\dots,g_r$, i.e., $\mathbb{V}(g_1,\ \dots,g_r) := \{\bar{a}\in\bar{K}^m|\ g_1(\bar{a}) = \cdots = g_r(\bar{a}) = 0\}$, $\mathbb{V}(0)=\bar{K}^m$ and $\mathbb{V}(1)=\emptyset$. We call an algebraically constructible set $\mathbb{V}(g_1,\dots,g_r)\setminus\mathbb{V}(g'_1,\dots,g'_{r'})\subseteq\bar{K}^m$ with $g_1,\dots,g_r,g'_1,\dots,g'_{r'}\in K[u]$, a *stratum.* (Notation $\mathbb{A}_1,\mathbb{A}_2,\dots,\mathbb{A}_\ell$ are used to represent strata.)

The definition of comprehensive Gröbner systems is the key ingredient of this paper.

Definition 2 (CGS)**.** *Fix a term ordering on* $\mathrm{pp}(x,\partial_x)$. *Let* P *be a subset of* $K[u]\langle x,\partial_x\rangle$, $\mathbb{A}_1,\dots,\mathbb{A}_\ell$ *strata in* $\bar{K}^m$ *and let* $G_1,\dots,G_\ell$ *be subsets in* $K[u]\langle x,\partial_x\rangle$. *A finite set* $\mathcal{G} = \{(\mathbb{A}_1,G_1),\dots,(\mathbb{A}_\ell,G_\ell)\}$ *of pairs is called a comprehensive Gröbner system CGS on* $\mathbb{A}_1\cup\cdots\cup\mathbb{A}_\ell$ *for* P *if for all* $\bar{a}\in\mathbb{A}_i$, $\sigma_{\bar{a}}(G_i)$ *is a Gröbner basis of* $Id(\sigma_{\bar{a}}(P))$ *in* $\bar{K}\langle x,\partial_x\rangle$ *for each* $i=1,\dots,\ell$. *We call a pair* $(\mathbb{A}_i,G_i)$ *segment of* $\mathcal{G}$. *We simply say* $\mathcal{G}$ *is a comprehensive Gröbner system for* P *if* $\mathbb{A}_1\cup\cdots\cup\mathbb{A}_\ell=\bar{K}^m$.

There exist algorithms for computing comprehensive Gröbner systems. We have adapted the algorithm [21] for computing CGSs and implemented it in the computer algebra system Risa/Asir.

To the best of our knowledge, our implementation is currently, in the rings of partial differential operators, only one implementation for computing CGSs.

Example 3. *Let* $F=\{x_1\partial_1^2\partial_2^3 + ax_1\partial_1^3, \partial_1^2 + bx_2\partial_1\partial_2, x_1\partial_1^2 + 3x_2\partial_2^2 + bx_1\partial_1^2\}\subset\mathbb{C}[a,b]\langle x_1,x_2,\partial_1,\partial_2\rangle$ *and* $\succ$ *the total degree lexicographic term ordering s.t.* $x_1\succ$

$x_2 \succ \partial_1 \succ \partial_2$ where $\partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2}$. Then, a CGS of F w.r.t. $\succ$ is the following.

$$\begin{aligned}
&\{(\mathbb{C}^2 \setminus \mathbb{V}(ab(b+1)), \{b^4\partial_1\partial_2 + b^3\partial_1\partial_2, ab\partial_1^2, abx_2\partial_2^2\}),\\
&(\mathbb{V}(a)\setminus \mathbb{V}(b(b+1),a), \{b^4\partial_1\partial_2 + b^3\partial_1\partial_2, b^3\partial_1^2 + b^2\partial_1^2, b^2x_2\partial_2^2\}),\\
&(\mathbb{V}(a,b+1), \{\partial_1\partial_2^2, \partial_1^2\partial_2 - \partial_1\partial_2, \partial_1^3 - \partial_1^2, x_2\partial_2^2.x_2\partial_1\partial_2 - \partial_1^2\}),\\
&(\mathbb{V}(a,b), \{\partial_1^2, x_2\partial_2^2\}),\\
&(\mathbb{V}(b+1)\setminus \mathbb{V}(a,b+1), \{a\partial_1\partial_2, a\partial_1^2, x_2\partial_2^2\}),\\
&(\mathbb{V}(b)\setminus \mathbb{V}(a,b), \{\partial_1^2, x_2\partial_2^2\})\ \}.
\end{aligned}$$

3. The Poincaré-Birkhoff-Witt algebra and b-functions

Let f be a non-constant polynomial in $\mathbb{C}[x]$. Then, the annihilating ideal of f^s is

$$\mathrm{Ann}(f^s) := \{p \in \mathbb{C}\langle s, x, \partial_x\rangle \mid pf^s = 0\}$$

where s is an indeterminate.

The global b-function or the Bernstein-Sato polynomial of f is defined as the monic generator $b_f(s)$ of

$$(\mathrm{Ann}(f^s) + Id(f)) \cap \mathbb{C}[s].$$

It is known that the b-function of f always has $s+1$ as a factor and has a form $(s+1)\tilde{b}_f(s)$, where $\tilde{b}_f(s) \in \mathbb{C}[s]$. The polynomial $\tilde{b}_f(s)$ is called the **reduced** b-function of f.

Here first, we recall the approach of Briançon-Maisonobe [3] for computing a basis of $\mathrm{Ann}(f^s)$. Second, we review a computation method of parametric b-functions.

Consider $\mathbb{C}\langle \partial t, s\rangle$ with the relation

$$\partial t \cdot s = s\partial t - \partial t$$

and let $\mathbb{C}\langle x, \partial_x, [\partial t, s]\rangle$ denote the Poincaré-Birkhoff-Witt algebra $\mathbb{C}\langle x, \partial_x\rangle \otimes_{\mathbb{C}} \mathbb{C}\langle \partial t, s\rangle$ with relations

$$x_is = sx_i, \partial_is = s\partial_i, x_i\partial t = \partial tx_i, \partial_i\partial t = \partial t\partial_i, \partial t \cdot s = s\partial t - \partial t,$$
$$x_ix_j = x_jx_i, \partial_i\partial_j = \partial_j\partial_i, \partial_jx_i = x_i\partial_j (i \neq j) \text{ and } \partial_ix_i = x_i\partial_i + 1.$$

Moreover, consider the following left ideal in $\mathbb{C}\langle x, \partial_x, [\partial t, s]\rangle$:

$$I = Id\left(f\partial t + s, \frac{\partial}{\partial x_1} + \partial t\frac{\partial f}{\partial x_1}, \frac{\partial}{\partial x_2} + \partial t\frac{\partial f}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} + \partial t\frac{\partial f}{\partial x_n}\right).$$

Briançon and Maisonobe proved in [3] that $\mathrm{Ann}(f^s) = I \cap \mathbb{C}\langle s, x, \partial_x\rangle$ and hence the latter can be computed via the Gröbner basis in $\mathbb{C}\langle x, \partial_x, [\partial t, s]\rangle$, w.r.t. an elimination ordering for $\{\partial t\}$.

Definition 4. *Fix a term ordering on* $\mathrm{pp}(x, \partial_x, \partial t, s)$. *Let* $p_1, \ldots, p_r \in K\langle x, \partial_x, \partial t, s\rangle$ *and* $G = \{g_1, \ldots, g_r\} \subset Id(p_1, \ldots, p_r) \subset K\langle x, \partial_x, [\partial t, s]\rangle$. *Then,* G *is a Gröbner basis of* $Id(p_1, \ldots, p_r)$ *if* G *satisfies* $Id(\mathrm{lm}(I)) = Id(\mathrm{lm}(g_1), \ldots, \mathrm{lm}(g_r))$.

There exists an algorithm for computing Gröbner bases in $K\langle x, \partial_x, [\partial t, s]\rangle$ ([21]). Actually, we have implemented the algorithm in the computer algebra system Risa/Asir. Hence, we can obtain a Gröbner basis of the ideal $\mathrm{Ann}(f^s)$ by utilizing Briançon-Maisonobe's method.

We turn to parametric cases. We can define and compute comprehensive Gröbner bases in $K[u]\langle x, \partial_x, [\partial t, s]\rangle$ in the same way where u are variables (parameters) s.t. $u \cap x = \emptyset$. Thus, we are able to obtain a basis of the parametric ideal $\mathrm{Ann}(f^s)$ where $f \in K[u][x]$.

Algorithm 1 `ParaAnn`

Specification: `ParaAnn`(f)
Computing a parametric basis of $\mathrm{Ann}(f^s)$.
Input: $f \in K[u][x]$.
Output: $\mathcal{B} = \{(\mathbb{A}_1, B_1), (\mathbb{A}_2, B_2), \ldots, (\mathbb{A}_\ell, B_\ell)\}$: For all $\bar{a} \in \mathbb{A}_i$, $\sigma_{\bar{a}}(B_i)$ is a basis of $\mathrm{Ann}(\sigma_{\bar{a}}(f)^s)$, for each $i \in \{1, \ldots, \ell\}$.
BEGIN
$\mathcal{B} \leftarrow \emptyset$;
$I \leftarrow \{f\partial t + s, \frac{\partial}{\partial x_1} + \partial t\frac{\partial f}{\partial x_1}, \frac{\partial}{\partial x_2} + \partial t\frac{\partial f}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} + \partial t\frac{\partial f}{\partial x_n}\}$;
$\succ_{\partial t} \leftarrow$ an elimination ordering for $\{\partial t\}$;
$\mathcal{G} \leftarrow$ compute a CGS for I w.r.t. $\succ_{\partial t}$ in $K[u]\langle x, \partial_x, [\partial t, s]\rangle$ ([21]);
while $\mathcal{G} \neq \emptyset$ **do**
  select $(\mathbb{A}, G)$ from $\mathcal{G}$; $\mathcal{G} \leftarrow \mathcal{G}\backslash\{(\mathbb{A}, G)\}$;
  $\mathcal{B} \leftarrow \mathcal{B} \cup \{(\mathbb{A}, G \cap K\langle s, x, \partial_x\rangle)\}$;
end-while
return $\mathcal{B}$;
END

We have implemented the algorithm `ParaAnn` in the computer algebra system Risa/Asir.

Example 5. *Let* $f = x_1^3 + ax_1x_2^2 + bx_2^2 \in \mathbb{C}[a, b][x_1, x_2]$ *where* a, b *are parameters.*

Then, our implementation outputs the following as parametric bases of $\mathrm{Ann}(f^s)$ *in* $\mathbb{C}[a,b]\langle s, x, \partial_x\rangle$.

1. *If parameters* (a,b) *belong to* $\mathbb{C}^2 \setminus \mathbb{V}(ab)$, *then a basis of* $\mathrm{Ann}(f^s)$ *is*

$$\begin{aligned} B_1 = \{ & a x_1^2 x_2 \partial_2 - 2b x_1 \partial_1 - 3b x_1 x_2 \partial_2 + 6bs x_1 + 2a^2 x_2^3 \partial_2 - 2a^2 s x_2^2, \\ & -2a x_1^2 \partial_1 - 2a x_1 x_2 \partial_2 + 6as - 2b x_1 \partial_1 - 3b x_2 \partial_2 + 6bs, \\ & -3x_1^2 \partial_2 + 2a x_1 x_2 \partial_1 - a x_2^2 \partial_2 + 2b x_2 \partial_1 \}. \end{aligned}$$

2. *If parameters* (a,b) *belong to* $\mathbb{V}(a) \setminus \mathbb{V}(a,b)$, *then a basis of* $\mathrm{Ann}(f^s)$ *is*

$$B_2 = \{2x_1\partial_1 + 3x_2\partial_2 - 6s, -3x_1\partial_2 + 2bx_2\partial_1\}.$$

3. *If parameters* (a,b) *belong to* $\mathbb{V}(b) \setminus \mathbb{V}(a,b)$, *then a basis of* $\mathrm{Ann}(f^s)$ *is*

$$B_3 = \{x_1\partial_1 + x_2\partial_2 - 3s, -3x_1^2\partial_2 + 2ax_1x_2\partial_1 - ax_2^2\partial_2\}.$$

4. *If parameters* (a,b) *belong to* $\mathbb{V}(a,b)$, *then a basis of* $\mathrm{Ann}(f^s)$ *is*

$$B_4 = \{x_1\partial_1 - 3s, \partial_2\}.$$

Note that the sets B_1, B_2, B_3 *and* B_4 *will be used in Example 6, again.*

As the monic generator of $(\mathrm{Ann}(f^s) + Id(f)) \cap \mathbb{C}[s]$ is the b-function of f, we are able to construct an algorithm for computing b-functions of the parametric polynomial f as follows.

Algorithm 2 `ParaBF`

Specification: `ParaBF`(f)
Computing b-functions of a parametric polynomial f.
Input: $f \in \mathbb{C}[u][x]$. $\succ$: a block term ordering s.t. $\{x, \partial_x\} \gg s$
Output: $\mathcal{P} = \{(\mathbb{A}_1, b_1(s)), (\mathbb{A}_2, b_2(s)), \ldots, (\mathbb{A}_\ell, b_\ell(s))\}$: If parameters u belong to $\mathbb{A}_i$, then $b_i(s)$ is the b-function of f where $i \in \{1, \ldots, \ell\}$.
BEGIN
$\mathcal{P} \leftarrow \emptyset$; $\mathcal{B} \leftarrow$`ParaAnn`(f);
while $\mathcal{B} \neq \emptyset$ **do**
  select $(\mathbb{A}, B)$ from $\mathcal{B}$; $\mathcal{B} \leftarrow \mathcal{B} \setminus \{(\mathbb{A}, B)\}$;
  $\mathcal{G} \leftarrow$ compute a CGS for $B \cup \{f\}$ w.r.t. $\succ$ on $\mathbb{A}$ in $\mathbb{C}\langle s, x, \partial_x\rangle$;
  while $\mathcal{G} \neq \emptyset$ **do**
    select (A', G) from $\mathcal{G}$; $\mathcal{G} \leftarrow \mathcal{G} \setminus \{(A', G)\}$;
    $b(s) \leftarrow$ the smallest element of $G \cap \mathbb{C}[s]$ w.r.t. $\succ$;
    $\mathcal{P} \leftarrow \mathcal{P} \cup \{(\mathbb{A}', b(s))\}$;

end-while
end-while
return $\mathcal{P}$;
END

We illustrate the algorithm with the following example.

Example 6. *Let $f = x_1^3 + ax_1x_2^2 + bx_2^2$ be a polynomial in $\mathbb{C}[a,b][x_1,x_2]$, $\succ$ the total degree lexicographic term ordering s.t. $\partial_1 \succ \partial_2 \succ x_1 \succ x_2$ on $\mathrm{pp}(x_1,x_2,\partial_1,\partial_2)$ and $\succ_s$ the block term ordering s.t. $\{x,\partial_x\} \gg s$ with $\succ$ where a,b are parameters.*

0. *Compute parametric bases of $Ann(f^s)$, which is already given in Example 5.*
1. *Compute a CGS for $B_1 \cup \{f\}$ w.r.t. $\succ_s$ on $\mathbb{C}^2 \setminus \mathbb{V}(ab)$ where B_1 is from Example 5. Then,*

   $\{(\mathbb{C}^2 \setminus \mathbb{V}(ab), G_1 = \{s^3 + 3s^2 + 107/36s + 35/36, (s+1)x_2, (-6s^2 - 13s - 7)x_1, (-3s-3)x_1^2 + (-as-a)x_2^2, x_1^3 + ax_2^2x_1 + bx_2^2, (-a^3x_2^2\partial_2 - 3b^2\partial_2)x_1^2 + a^2bx_2^2x_1\partial_2 - a^4x_2^4\partial_2 - 2a^4x_2^3 - 4ab^2x_2^2\partial_2 + -6ab^2x_2 + 2b^3x_2\partial_1, a^2x_1^2x_2\partial_2 - abx_1x_2\partial_2 + 2b^2x_1\partial_1 + a^3x_2^3\partial_2 - 2a^3sx_2^2 + 3b^2x_2\partial_2 - 6b^2s\})\}$

   is the CGS. Hence, $G_1 \cap \mathbb{C}[s] = \{s^3 + 3s^2 + 107/36 + 35/36\}$. Therefore, if parameters (a,b) belong to $\mathbb{C}^2 \setminus \mathbb{V}(ab)$, then the b-function of f is

   $$s^3 + 3s^2 + \frac{107}{36}s + \frac{35}{36} = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right).$$

2. *A CGS for $B_2 \cup \{f\}$ w.r.t. $\succ_s$ on $\mathbb{V}(a)\setminus \mathbb{V}(a,b)$ is*

   $\{(\mathbb{V}(a)\setminus \mathbb{V}(a,b), \{s^3+3s^2+107/36s+35/36, (s+1)x_2, 6s^2x_1+13sx_1+7x_1, (s+1)x_1^2, x_1^3 + bx_2^2, -3x_1^2\partial_2 + 2bx_2\partial_1, 2x_1\partial_1 + 3x_2\partial_2 - 6s\})\}.$

   Therefore, if parameters (a,b) belong to $\mathbb{V}(a)\setminus \mathbb{V}(a,b)$, then the b-function of f is

   $$s^3 + 3s^2 + \frac{107}{36}s + \frac{35}{36} = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right).$$

3. *A CGS for $B_3 \cup \{f\}$ w.r.t. $\succ_s$ on $\mathbb{V}(b)\setminus \mathbb{V}(a,b)$ is*

   $\{(\mathbb{V}(b)\setminus \mathbb{V}(a,b)), \{s^4 + 4s^3 + 53/9s^2 + 34/9s + 8/9, (3s^3 + 10s^2 + 11s + 4)x_2, (3s^3+10s^2+11s+4)x_1, (3s^2+7s+4)x_2^2, (s+1)x_1x_2, (3s+3)x_1^2 + (as+a)x_2^2, (s+1)x_2^3, x_1^3 + ax_2^2x_1, sx_2^2\partial_2 + x_2^2\partial_2 - 3s^2x_2 - 4sx_2 - x_2, x_1^2\partial_2 + ax_2^2\partial_2 - 2asx_2, 3s^2x_1\partial_2 + 6sx_1\partial_2 + 3x_1\partial_2 + as^2x_2\partial_1 + 2asx_2\partial_1 + a\partial_1, x_1\partial_1 + x_2\partial_2 - 3s, 3sx_1x_2\partial_2 + 3x_2x_2\partial_2 - 9s^2x_1 - 15sx_1 - 6x_1 - asx_2^2\partial_1 - ax_2^2\partial_1, -x_1x_2\partial_2^2 + 3sx_1\partial_2 + x_1\partial_2 + ax_2^2\partial_1\partial_2 - 2asx_2\partial_1\})\}.$

Therefore, if parameters (a, b) *belong to* $\mathbb{V}(a) \setminus \mathbb{V}(a, b)$*, then the b-function of* f *is*

$$s^4 + 4s^3 + 53/9s^2 + 34/9s + 8/9 = (s+1)^2 \left(s + \frac{2}{3}\right)\left(s + \frac{4}{3}\right).$$

4. A CGS for $B_4 \cup \{f\}$ *w.r.t.* $\succ_s$ *on* $\mathbb{V}(a, b)$ *is*

$\{(\mathbb{V}(a,b), \{s^3 + 2s^2 + 11/9s + 2/9, 3s^2x_1 + 5sx_1 + 2x_1, (s+1)x_1^2, x_1^3, \partial_2, x_1\partial_1 - 3s\})\}$

Therefore, if parameters (a, b) *belong to* $\mathbb{V}(a, b)$*, then the b-function of* f *is*

$$s^3 + 2s^2 + \frac{11}{9}s + \frac{2}{9} = (s+1)\left(s + \frac{1}{3}\right)\left(s + \frac{2}{3}\right).$$

We have implemented the algorithm `ParaBF` in the computer algebra system Risa/Asir.

4. CGS and supports of D-modules

Let f be a non-constant polynomial in $\mathbb{C}[x]$. Let us regard s as a "parameter" and compute a CGS of $\mathrm{Ann}(f^s) \cup \{f\}$ w.r.t. a block term ordering s.t. $\partial \succ x$ in $\mathbb{C}[s]\langle x, \partial_x\rangle$. Then, the CGS may give us the supports of roots of b_f.

Let us consider $f = x_1x_3^2 + x_2^3 \in \mathbb{C}[x_1, x_2, x_3]$. In this example, let $\succ_x$ be the total degree lexicographic term ordering s.t. $x_1 \succ x_2 \succ x_3$ on $\mathrm{pp}(x_1, x_2, x_3)$ and $\succ_\partial$ be the total degree lexicographic term ordering s.t. $\partial_1 \succ \partial_2 \succ \partial_3$ on $\mathrm{pp}(\partial_1, \partial_2, \partial_3)$. A CGS of $\mathrm{Ann}(f^s) \cup \{f\}$ w.r.t. the block term ordering $\succ$ s.t. $\{\partial_1, \partial_2, \partial_3\} \gg \{x_1, x_2, x_3\}$ (with $\succ_x$ and $\succ_\partial$) in $\mathbb{C}[s]\langle x_1, x_2, x_3, \partial_1, \partial_2, \partial_3\rangle$, is the following.

- If $s = -1$, then $G_1 = \{2x_2\partial_2 + 3x_3\partial_3 + 6, -2x_1\partial_1 + x_3\partial_3, 3x_2^2\partial_1 - x_3^2\partial_2, -2x_1x_3\partial_2 + 3x_2\partial_3, x_1x_3^2 + x_2^3, 9x_2x_3\partial_1\partial_3 + 6x_2\partial_2^2 + 2x_3^2\partial_2^2, 27x_3^2\partial_1\partial_3^2 - 4x_3^2\partial_2^3 + 81x_3\partial_1\partial_3 + 24\partial_1\}$ is a Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$ w.r.t. $\succ$ in $\mathbb{C}\langle x_1, x_2, x_3, \partial_1, \partial_2, \partial_3\rangle$.
- If $s = -\frac{4}{3}$, then $G_2 = \{x_1, x_2, x_3\partial_3 + 2, x_3^2\}$ is a Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$.
- If $s = -\frac{5}{3}$, then $G_3 = \{x_1, x_3\partial_3 + 2, x_3^2, x_2\partial_2 + 2, x_2^2\}$ is a Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$.
- If $s = -\frac{5}{6}$, then $G_4 = \{x_3, x_2, 2x_1\partial_1 + 1\}$ is a Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$.
- If $s = -\frac{7}{6}$, then $G_5 = \{x_3, x_2\partial_2 + 2, x_2^2, 2x_1\partial_1 + 1\}$ is a Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$.

· If $(s+1)(s+\frac{4}{3})(s+\frac{5}{3})(s+\frac{5}{6})(s+\frac{7}{6}) \neq 0$, then $\{1\}$ is the Gröbner basis of $\mathrm{Ann}(f^s) \cup \{f\}$.

Note that the b-function of f is $(s+1)(s+\frac{4}{3})(s+\frac{5}{3})(s+\frac{5}{6})(s+\frac{7}{6})$. That is, the b-function appears in the output as the last stratum and the roots of $b_f(s)$ also appear in the other strata above.

We have the following theorem.

Theorem 7 ([21]). *An algorithm for computing CGSs [21], always outputs $\sqrt{b_f(s)}$ and all roots of the $b_f(s) = 0$ as strata where $\sqrt{b_f(s)}$ is the squarefree polynomial of $b_f(s)$.*

We borrow from the paper [43] the following theorem.

Theorem 8. *Let $\gamma \in \mathbb{Q}$ and $J_f = Id\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. Set*

$$M_{(\gamma,f)} = \mathbb{C}\langle s, x, \partial_x \rangle / (\mathrm{Ann}(f^s) + Id(f) + Id(s-\gamma)).$$

Then, if $\tilde{b}_f(\gamma) \neq 0$, then $M_{(\gamma,f)} = \{0\}$, and if $\tilde{b}_f(\gamma) = 0$, then $M_{(\gamma,f)}$ is a holonomic D-module and supp$(M_{(\gamma,f)}) \subseteq$ Sing(S) where supp$(M_{(\gamma,f)})$ is the support of $M_{(\gamma,f)}$ and Sing(S) is the singular locus of the hypersurface $S = \{x \in \mathbb{C}^n \,|\, f(x) = 0\}$, i.e., Sing$(S) = \mathbb{V}\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

Let us consider $f = x_1x_3^2 + x_2^3 \in \mathbb{C}[x_1, x_2, x_3]$, again. Focus on each system of partial differential equations of the CGS of $\mathrm{Ann}(f^s) \cup \{f\}$. Then, we can compute supports of each holonomic D-module $M_{(\gamma,f)}$ from the CGS, namely, $\mathrm{supp}(M_{(-\frac{4}{3},f)}) = \mathbb{V}(G_2 \cap \mathbb{C}[x_1,x_2,x_3]) = \mathbb{V}(x_1,x_2,x_3^2)$, $\mathrm{supp}(M_{(-\frac{5}{3},f)}) = \mathbb{V}(G_3 \cap \mathbb{C}[x_1,x_2,x_3]) = \mathbb{V}(x_1,x_2^2,x_3^2)$, $\mathrm{supp}(M_{(-\frac{5}{6},f)}) = \mathbb{V}(G_4 \cap \mathbb{C}[x_1,x_2,x_3]) = \mathbb{V}(x_2,x_3)$ and $\mathrm{supp}(M_{(-\frac{7}{6},f)}) = \mathbb{V}(G_5 \cap \mathbb{C}[x_1,x_2,x_3]) = \mathbb{V}(x_2^2,x_3)$. Thus, we see that, the support of the holonomic D-module associated with $s = -\frac{4}{3}, -\frac{5}{3}$ is the origin and the support of the holonomic D-module associated with $s = -\frac{5}{6}, -\frac{7}{6}$ is the complex line $y = z = 0$. Note that the characteristic varieties of the holonomic D-modules above can also easily determined from the outputs above.

5. B-functions of μ-constant deformations

We have obtained lists of b-functions of typical μ-constant deformations, that are the main results of this paper.

First, we see a simple example which is the unimodal singularity E_{12} in section 5.1, to facilitate our results. In fact, if f is a unimodal singularity, then our implementation return b-functions of the μ-constant deformation f within a few seconds. Thus, we mainly consider bimodal singularities. We show b-functions of eight bimodal singularities, that are obtained by our implementation within one month.

All results of b-functions in this paper have been computed on a PC with [OS: Windows 7 (64 bit), CPU: Intel Core i-7-5930K CPU @ 3.50 GHz 3.50 GHz, RAM: 64 GB].

5.1 $x^3 + y^7$ (E_{12} singularity)

The Milnor number μ of the singularity $x^3 + y^7 = 0$ is 12, and the μ-constant deformation is given by

$$f = x^3 + y^7 + axy^5$$

where a is a parameter.

In order to compute b-functions of the μ-constant deformation, first of all, we need to compute a parametric basis of $\mathrm{Ann}(f^s)$ that is the following.

- If the parameter belong to $\mathbb{C} \setminus \mathbb{V}(a)$, then $B_1 = \{125a^3x^2y(\frac{\partial}{\partial x})^2 + 630x^2(\frac{\partial}{\partial x})^2 + 100a^3xy^2\frac{\partial}{\partial y}\frac{\partial}{\partial x} + 550a^3xy\frac{\partial}{\partial x} + 543xy\frac{\partial}{\partial y}\frac{\partial}{\partial x} + 315sx\frac{\partial}{\partial x} - 12ax(\frac{\partial}{\partial y})^2 + 2688x\frac{\partial}{\partial x} - 65ay^5(\frac{\partial}{\partial x})^2 + 30a^2y^4\frac{\partial}{\partial y}\frac{\partial}{\partial x} + 20a^3y^3(\frac{\partial}{\partial y})^2 + 80a^2y^3\frac{\partial}{\partial x} + (190a^3y^2\frac{\partial}{\partial y} + 117y^2(\frac{\partial}{\partial y})^2 - 1125a^3s^2y - 1275a^3sy + 126sy\frac{\partial}{\partial y} + 999y\frac{\partial}{\partial y} - 6615s^2 - 6174s, -5ax^2\frac{\partial}{\partial x} - 7xy^2\frac{\partial}{\partial x} - 2axy\frac{\partial}{\partial y} + 15asx - 3y^3\frac{\partial}{\partial y} + 21sy^2, -125a^4x^2\frac{\partial}{\partial x} - 50a^4xy\frac{\partial}{\partial y} + 1029xy\frac{\partial}{\partial x} + 375a^4sx - 21ax\frac{\partial}{\partial y} + 35a^2y^4\frac{\partial}{\partial x} - 5a^3y^3\frac{\partial}{\partial y} + 441y^2\frac{\partial}{\partial y} - 3087sy, 3x^2\frac{\partial}{\partial y} - 5axy^4\frac{\partial}{\partial x} - 7y^6\frac{\partial}{\partial x} + ay^5\frac{\partial}{\partial y}\}$ is a basis of $\mathrm{Ann}(f^s)$.
- If the parameter belong to $\mathbb{V}(a)$, then $B_2 = \{7x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 21s, 3x^2\frac{\partial}{\partial y} - 7y^6\frac{\partial}{\partial x}\}$ is a basis of $\mathrm{Ann}(f^s)$.

Next, we compute a CGS $\{(\mathbb{A}_1, G_1), (\mathbb{A}_2, G_2), \ldots, (\mathbb{A}_\ell, G_\ell)\}$ of $B_i \cup \{f\}$ w.r.t. a block term order s. t. $\{x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \gg s$. After that for each $i \in \{1, \ldots, \ell\}$, we select a generator $b_f(s)$ of $G_i \cap \mathbb{C}[s]$. Then, we obtain the b-functions as Table 1.

Table 1 b-functions of $x^3 + y^7 + axy^5$

stratum	global b-function
$\mathbb{C} \setminus \mathbb{V}(a)$	$(s+1)(s+\frac{10}{21})\boxed{(s+\frac{11}{21})}(s+\frac{13}{21})(s+\frac{16}{21})(s+\frac{17}{21})$ $(s+\frac{19}{21})(s+\frac{20}{21})(s+\frac{22}{21})(s+\frac{23}{21})(s+\frac{25}{21})(s+\frac{26}{21})(s+\frac{29}{21})$
$\mathbb{V}(a)$	$(s+1)(s+\frac{10}{21})(s+\frac{13}{21})(s+\frac{16}{21})(s+\frac{17}{21})$ $(s+\frac{19}{21})(s+\frac{20}{21})(s+\frac{22}{21})(s+\frac{23}{21})(s+\frac{25}{21})(s+\frac{26}{21})(s+\frac{29}{21})\boxed{(s+\frac{32}{21})}$

Furthermore, by using a CGS, we can compute a holonomic D-module associated with $s = \gamma$ for each root γ of the b-function, namely, we can obtain a set $G_{(\gamma,f)} \subset \mathbb{C}\langle x, \partial_x\rangle$ that satisfies $M_{(\gamma,f)} = \mathbb{C}\langle x, \partial_x\rangle / \mathit{Id}(G_{(\gamma,f)})$.

If the parameter a belongs to $\mathbb{C} \setminus \mathbb{V}(a)$, then

$$
\begin{aligned}
G_{(-\frac{10}{21},f)} &= \{x, y\},\\
G_{(-\frac{11}{21},f)} &= \{x, y\},\\
G_{(-\frac{13}{21},f)} &= \{y^2, x, a^3y + 49y\tfrac{\partial}{\partial y} + 98\},\\
G_{(-\frac{16}{21},f)} &= \Big\{x, 37a^6y^2 - 245a^3y - 7203y\tfrac{\partial}{\partial y} - 21609, 136ya^6 + 1470a^3y\tfrac{\partial}{\partial y}\\
&\qquad +21609y\left(\tfrac{\partial}{\partial y}\right)^2 + 2940a^3 + 86436\tfrac{\partial}{\partial y}\Big\},\\
G_{(-\frac{17}{21},f)} &= \{ax + 7y^2, 10a^4x + 147ax\tfrac{\partial}{\partial y} - 1029y, xy, x^2, \ldots\}\\
&\ \vdots\\
G_{(-\frac{29}{21},f)} &= \{-1273a^6x^2y + 9261a^3x^2 + 64827x^2\tfrac{\partial}{\partial y} + 30870a^2xy^2, x^3, \ldots\}.
\end{aligned}
$$

If the parameter a belongs to $\mathbb{V}(a)$, then

$$
\begin{aligned}
G_{(-\frac{10}{21},f)} &= \{x, y\},\\
G_{(-\frac{13}{21},f)} &= \{x, y\tfrac{\partial}{\partial y} + 2, y^2\},\\
G_{(-\frac{16}{21},f)} &= \{x, y\tfrac{\partial}{\partial y} + 3, y^3\},\\
G_{(-\frac{17}{21},f)} &= \{y, y\tfrac{\partial}{\partial y} + 2, x^2\},\\
&\ \vdots\\
G_{(-\frac{32}{21},f)} &= \{y\tfrac{\partial}{\partial y} + 6, x\tfrac{\partial}{\partial x} + 2, x^2, y^6\}.
\end{aligned}
$$

5.2 $x^3 + y^{10}$ (E_{18} singularity)

The Milnor number μ of the singularity $x^3+y^{10} = 0$ is 18, and the μ-constant deformation is given by

$$f = x^3 + y^{10} + axy^7 + bxy^8$$

where a, b are parameter. The algorithm `ParaBF` outputs Table 2 as the parametric b-function of f.

Let us consider $(\mathbb{V}(a^4 - 64b)\setminus \mathbb{V}(a, b), b_f^{(2)})$. If parameters (a, b) belong to $\mathbb{V}(a^4 - 64b)\setminus \mathbb{V}(a, b)$, then the reduced b-function of f is $\tilde{b}_f = b_f^{(2)}/(s + 1)$. One can check $\mathrm{supp}(M_{(\gamma,f)}) \subseteq \mathrm{Sing}(S)$ where γ is a root of $\tilde{b}_f = 0$ and $S = \{(x, y)|f(x, y) = 0\}$. Then, all roots of $\tilde{b}_0 = \tilde{b}_f/(s + 1) = 0$ are on the origin and the root of $\tilde{b}_f/\tilde{b}_0 = s + 1 = 0$ is on another isolated singular point $(x, y) = \left(\dfrac{a^2}{32b^3}, \dfrac{-a}{4b}\right)$. In fact, the Gröbner basis of $\mathrm{Ann}(f^s) \cap \{f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\}$ for $s = -1$ in $\mathbb{C}\langle x, y\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$ is $\{4aby + a^2, 8x - a^2y^4\}$.

Set

$$
\begin{aligned}
B(s) = {}& (s + \tfrac{5}{6})(s + \tfrac{7}{6})(s + \tfrac{8}{15})(s + \tfrac{11}{15})(s + \tfrac{13}{15})(s + \tfrac{14}{15})(s + \tfrac{16}{15})(s + \tfrac{17}{15})(s + \tfrac{19}{15})\\
&\times(s + \tfrac{13}{30})(s + \tfrac{19}{30})(s + \tfrac{23}{30})(s + \tfrac{29}{30})(s + \tfrac{31}{30})(s + \tfrac{37}{30})(s + \tfrac{41}{30})
\end{aligned}
$$

Table 2 global b-functions of $x^3 + y^{10} + axy^7 + bxy^8$

strata	global b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a(a^4 - 64b))$	$b_f^{(1)} = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{7}{15})(s+\frac{8}{15})$ $\times(s+\frac{11}{15})(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{19}{15})(s+\frac{13}{30})$ $\times(s+\frac{17}{30})(s+\frac{19}{30})(s+\frac{23}{30})(s+\frac{29}{30})(s+\frac{31}{30})(s+\frac{37}{30})(s+\frac{41}{30})$
$\mathbb{V}(a^4 - 64b)\setminus \mathbb{V}(a,b)$	$b_f^{(2)} = (s+1)^2(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{7}{15})(s+\frac{8}{15})$ $\times(s+\frac{11}{15})(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{19}{15})(s+\frac{13}{30})$ $\times(s+\frac{17}{30})(s+\frac{19}{30})(s+\frac{23}{30})(s+\frac{29}{30})(s+\frac{31}{30})(s+\frac{37}{30})(s+\frac{41}{30})$
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$b_f^{(3)} = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{8}{15})(s+\frac{11}{15})$ $\times(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{19}{15})(s+\frac{22}{15})(s+\frac{13}{30})$ $\times(s+\frac{17}{30})(s+\frac{19}{30})(s+\frac{23}{30})(s+\frac{29}{30})(s+\frac{31}{30})(s+\frac{37}{30})(s+\frac{41}{30})$
$\mathbb{V}(a,b)$	$b_f^{(4)} = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{8}{15})(s+\frac{11}{15})$ $\times(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{19}{15})(s+\frac{22}{15})(s+\frac{13}{30})$ $\times(s+\frac{19}{30})(s+\frac{23}{30})(s+\frac{29}{30})(s+\frac{31}{30})(s+\frac{37}{30})(s+\frac{41}{30})(s+\frac{47}{30})$

that is the common factor of $b_f^{(1)}/(s+1), b_f^{(2)}/(s+1), b_f^{(3)}/(s+1)$ and $b_f^{(4)}/(s+1)$.

As we are considering μ-constant deformations, b-functions of the μ-constant deformation f are given in Table 3, by collecting roots of the b-functions on the origin.

Table 3 b-functions of $x^3 + y^{10} + axy^7 + bxy^8$ on the origin

strata	b-function on the origin	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{7}{15})(s+\frac{17}{30})$	18
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{22}{15})(s+\frac{17}{30})$	18
$\mathbb{V}(a,b)$	$B(s)(s+\frac{22}{15})(s+\frac{47}{30})$	18

5.3 $x^3 + y^{11}$ (E_{20} singularity)

The Milnor number μ of the singularity $x^3+y^{11} = 0$ is 20, and the μ-constant deformation is given by

$$f = x^3 + y^{11} + axy^8 + bxy^9$$

where a, b are parameters. The algorithm `ParaBF` outputs Table 4 as the parametric b-function of f where

$$B(s) = (s+\tfrac{14}{33})(s+\tfrac{17}{33})(s+\tfrac{20}{33})(s+\tfrac{23}{33})(s+\tfrac{25}{33})(s+\tfrac{26}{33})(s+\tfrac{28}{33})(s+\tfrac{29}{33})(s+\tfrac{31}{33})$$
$$\times(s+\tfrac{32}{33})(s+\tfrac{34}{33})(s+\tfrac{35}{33})(s+\tfrac{37}{33})(s+\tfrac{38}{33})(s+\tfrac{40}{33})(s+\tfrac{41}{33})(s+\tfrac{43}{33})(s+\tfrac{46}{33}).$$

Let us consider $(\mathbb{V}(16a^5 + 3125b^2)\setminus \mathbb{V}(a,b), b_f^{(2)})$. If parameters (a, b) belong

Table 4 global b-functions of $x^3 + y^{11} + axy^8 + bxy^9$

strata	global b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a(16a^5 + 3125b^2))$	$b_f^{(1)} = B(s)(s+1)(s+\frac{16}{33})(s+\frac{19}{33})$
$\mathbb{V}(16a^5 + 3125b^2) \setminus \mathbb{V}(a,b)$	$b_f^{(2)} = B(s)(s+1)^2(s+\frac{16}{33})(s+\frac{19}{33})$
$\mathbb{V}(a) \setminus \mathbb{V}(a,b)$	$b_f^{(3)} = B(s)(s+1)(s+\frac{19}{33})(s+\frac{49}{33})$
$\mathbb{V}(a,b)$	$b_f^{(4)} = B(s)(s+1)(s+\frac{49}{33})(s+\frac{52}{33})$

to $\mathbb{V}(16a^5+3125b^2)\setminus \mathbb{V}(a,b)$, then the reduced b-function of f is $\tilde{b}_f = b_f^{(2)}/(s+1)$. One can check $\mathrm{supp}(M_{(\gamma,f)}) \subseteq \mathrm{Sing}(S)$ where γ is a root of $\tilde{b}_f = 0$ and $S = \{(x,y)|f(x,y) = 0\}$. Then, all roots of $\tilde{b}_0 = \tilde{b}_f/(s+1) = 0$ are on the origin and $\tilde{b}_f/\tilde{b}_0 = 0$ $(s = -1)$ is on another isolated singular point $(x,y) = \left(\dfrac{4a^2}{25b^3}, \dfrac{-2a}{5b}\right)$. The b-functions of the μ-constant deformation f are given in Table 5.

Table 5 b-functions of $x^3 + y^{11} + axy^8 + bxy^9$ on the origin

strata	b-function on the origin	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{16}{33})(s+\frac{19}{33})$	20
$\mathbb{V}(a) \setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{19}{33})(s+\frac{49}{33})$	20
$\mathbb{V}(a,b)$	$B(s)(s+\frac{49}{33})(s+\frac{52}{33})$	20

5.4 $x^4 + y^7$ (W_{18} singularity)

The Milnor number μ of the singularity $x^4+y^7 = 0$ is 18, and the μ-constant deformation is given by

$$f = x^4 + y^7 + ax^2y^4 + bx^2y^5$$

where a, b are parameters. The algorithm `ParaBF` outputs Table 6 where

$$B(s) = (s+\tfrac{9}{14})(s+\tfrac{11}{14})(s+\tfrac{13}{14})(s+\tfrac{15}{14})(s+\tfrac{17}{14})(s+\tfrac{19}{14})(s+\tfrac{11}{28})(s+\tfrac{15}{28})(s+\tfrac{19}{28}) \times(s+\tfrac{23}{28})(s+\tfrac{25}{28})(s+\tfrac{27}{28})(s+\tfrac{29}{28})(s+\tfrac{31}{28})(s+\tfrac{33}{28})(s+\tfrac{37}{28}).$$

strata	global b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a(a^3+27b))$	$b_f^{(1)} = B(s)(s+1)(s+\frac{13}{28})(s+\frac{17}{28})$
$\mathbb{V}(a^3+27b)\setminus \mathbb{V}(a,b)$	$b_f^{(2)} = B(s)(s+1)^2(s+\frac{13}{28})(s+\frac{17}{28})$
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$b_f^{(3)} = B(s)(s+1)(s+\frac{17}{28})(s+\frac{41}{28})$
$\mathbb{V}(a,b)$	$b_f^{(4)} = B(s)(s+1)(s+\frac{41}{28})(s+\frac{45}{28})$

Table 6 global b-functions of $x^4+y^7+ax^2y^4+bx^2y^5$

Let us consider $(\mathbb{V}(a^3+27b)\setminus \mathbb{V}(a,b), b_f^{(2)})$. If parameters (a,b) belong to $\mathbb{V}(a^3+27b)\setminus \mathbb{V}(a,b)$, then the reduced b-function of f is $\tilde{b}_f = b_f^{(2)}/(s+1)$. One can check $\mathrm{supp}(M_{(\gamma,f)}) \subseteq \mathrm{Sing}(S)$ where γ is a root of $\tilde{b}_f = 0$ and $S = \{(x,y)|f(x,y)=0\}$. Then, all roots of $\tilde{b}_0 = \tilde{b}_f/(s+1) = 0$ are on the origin and $\tilde{b}_f/\tilde{b}_0 = 0$ $(s=-1)$ is on two isolated singular points $(x,y) = \left(\pm\frac{a}{3b}, \frac{-a}{3b}\right)$. The b-functions of the μ-constant deformation f are given in Table 7.

Table 7 b-functions of $x^4+y^7+ax^2y^4+bx^2y^5$ on the origin

strata	b-function on the origin	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{13}{28})(s+\frac{17}{28})$	18
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{17}{28})(s+\frac{41}{28})$	18
$\mathbb{V}(a,b)$	$B(s)(s+\frac{41}{28})(s+\frac{45}{28})$	18

5.5 x^3y+y^8 (Z_{17} singularity)

The Milnor number μ of the singularity $x^3y+y^8=0$ is 17, and the μ-constant deformation is given by

$$f = x^3y+y^8+axy^6+bxy^7$$

where a,b are parameters. By executing the algorithm `ParaBF`, we can obtain Table 8 as the parametric reduced b-function of f where

$$B(s) = (s+1)(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{5}{12})(s+\tfrac{11}{12})(s+\tfrac{13}{12}) \\ \times(s+\tfrac{13}{24})(s+\tfrac{17}{24})(s+\tfrac{19}{24})(s+\tfrac{23}{24})(s+\tfrac{25}{24})(s+\tfrac{29}{24})(s+\tfrac{31}{24}).$$

In all the cases, all roots of reduced b-functions are on the origin and the degree of all the reduced b-functions is 17.

5.6 $x^3+yz^2+y^7$ (Q_{16} singularity)

The Milnor number μ of the singularity $x^3+yz^2+y^7=0$ is 16, and the μ-constant deformation is given by

Table 8 reduced b-functions of $x^3y + y^8 + axy^6 + bxy^7$

strata	reduced b-function	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{7}{12})(s+\frac{11}{24})$	17
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{7}{12})(s+\frac{35}{24})$	17
$\mathbb{V}(a,b)$	$B(s)(s+\frac{19}{12})(s+\frac{35}{24})$	17

$$f = x^3 + yz^2 + y^7 + axy^5 + bxz^2$$

where a, b are parameters. The algorithm `ParaBF` outputs Table 9 where

$$B(s) = (s+1)(s+\tfrac{4}{3})(s+\tfrac{5}{3})(s+\tfrac{19}{21})(s+\tfrac{22}{21})(s+\tfrac{25}{21})(s+\tfrac{26}{21}) \\ \times(s+\tfrac{29}{21})(s+\tfrac{31}{21})(s+\tfrac{32}{21})(s+\tfrac{34}{21})(s+\tfrac{37}{21})(s+\tfrac{38}{21}).$$

Table 9 global b-functions of $x^3 + yz^2 + y^7 + axy^5 + bxz^2$

strata	global b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a(27a^4+256b))$	$b_f^{(1)} = B(s)(s+1)(s+\frac{20}{21})(s+\frac{23}{21})$
$\mathbb{V}(27a^4+256b)\setminus \mathbb{V}(a,b)$	$b_f^{(2)} = B(s)(s+1)(s+\frac{3}{2})(s+\frac{20}{21})(s+\frac{23}{21})$
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$b_f^{(3)} = B(s)(s+1)(s+\frac{23}{21})(s+\frac{41}{21})$
$\mathbb{V}(a,b)$	$b_f^{(4)} = B(s)(s+1)(s+\frac{41}{21})(s+\frac{44}{21})$

Let us consider $(\mathbb{V}(27a^4 + 256b)\setminus \mathbb{V}(a,b), b_f^{(2)})$. If parameters (a, b) belong to $\mathbb{V}(27a^4 + 256b)\setminus \mathbb{V}(a,b)$, then the reduced b-function of f is $\tilde{b}_f = b_f^{(2)}/(s+1)$. One can check $\mathrm{supp}(M_{(\gamma,f)}) \subseteq \mathrm{Sing}(S)$ where γ is a root of $\tilde{b}_f$ and $S = \{(x,y,z)|f(x,y,z) = 0\}$. Then, all roots of $\tilde{b}_0 = \tilde{b}_f/(s+\frac{3}{2}) = 0$ are on the origin and $\tilde{b}_f/\tilde{b}_0 = s+\frac{3}{2} = 0$ is on two isolated singular points $(x,y,z) = \left(\dfrac{3a}{4b^2}, \dfrac{3a}{4b}, \pm\dfrac{3a}{4b^2\sqrt{b}}\right)$. The b-functions of the μ-constant deformation f on the origin are given in Table 10.

Table 10 b-functions of $x^3 + yz^2 + y^7 + axy^5 + bxz^2$ on the origin

strata	b-function on the origin	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(ab)$	$B(s)(s+\frac{20}{21})(s+\frac{23}{21})$	14
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{23}{21})(s+\frac{41}{21})$	14
$\mathbb{V}(a,b)$	$B(s)(s+\frac{41}{21})(s+\frac{44}{21})$	14

Note that the Milnor number 16 does not coincide with the degree of the b-function 14. If $a = b = 0$, then the multiplicity of the holonomic system $M_{(-\frac{4}{3},f)}$

defined by $Id(G_{(-\frac{4}{3},f)})$ is equal to 2 where

$$G_{(-\frac{4}{3},f)} = \left\{x, y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} + 7, z^2, yz, y^4\right\}.$$

Actually, the algebraic local cohomology solutions of $M_{(-\frac{4}{3},f)}$ is $\mathrm{Span}_{\mathbb{C}}\left(\begin{bmatrix}1\\xyz^2\end{bmatrix},\begin{bmatrix}1\\xy^4z\end{bmatrix}\right)$. (See [37, 38, 42, 43].) Similarly, the multiplicity of the holonomic system $M_{(-\frac{5}{3},f)}$ defined by $Id(G_{(-\frac{5}{3},f)})$ is equal to 2 where

$$G_{(-\frac{5}{3},f)} = \left\{y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} + 7, x\frac{\partial}{\partial x} + 2, z^2, yz, x^2, y^4\right\},$$

and the holonomic system $M_{(\gamma,f)}$ is simple for every roots γ ($\neq -\frac{4}{3}, -\frac{5}{3}$) of the reduced b-function. The sum of the multiplicities is equal to 16.

The same fact can be verified for the other strata $\mathbb{C}\setminus\mathbb{V}(a)$ and $\mathbb{V}(a)\setminus\mathbb{V}(a,b)$ by analyzing the structure of CGSs in the same way.

5.7 $x^3 + yz^2 + xy^5$ (Q_{17} singularity)

The Milnor number μ of the singularity $x^3 + yz^2 + xy^5 = 0$ is 17, and the μ-constant deformation is given by

$$f = x^3 + yz^2 + xy^5 + ay^8 + by^9$$

where a, b are parameters. By executing the algorithm ParaBF, we can obtain Table 11 as the parametric reduced b-function of f where

$$B(s) = (s+\tfrac{3}{2})(s+\tfrac{4}{3})(s+\tfrac{5}{3})(s+\tfrac{7}{6})(s+\tfrac{11}{6})(s+\tfrac{9}{10})(s+\tfrac{13}{10})(s+\tfrac{17}{10})$$
$$\times(s+\tfrac{31}{30})(s+\tfrac{37}{30})(s+\tfrac{41}{30})(s+\tfrac{43}{30})(s+\tfrac{47}{30})(s+\tfrac{49}{30})(s+\tfrac{53}{30}).$$

Table 11 reduced b-functions of $x^3 + yz^2 + xy^5 + ay^8 + by^9$

strata	reduced b-function	degree of the b-function
$\mathbb{C}^2\setminus\mathbb{V}(a)$	$B(s)(s+\frac{11}{10})(s+\frac{29}{30})$	17
$\mathbb{V}(a)\setminus\mathbb{V}(a,b)$	$B(s)(s+\frac{11}{10})(s+\frac{59}{30})$	17
$\mathbb{V}(a,b)$	$B(s)(s+\frac{21}{10})(s+\frac{59}{30})$	17

In all the cases, all roots of reduced b-functions are on the origin.

5.8 $x^2z + yz^2 + y^6$ (S_{17} singularity)

The Milnor number μ of the singularity $x^2z + yz^2 + y^6 = 0$ is 17, and the μ-constant deformation is given by

$$f = x^2z + yz^2 + y^6 + ay^4z + bz^3$$

where a, b are parameters. By executing the algorithm ParaBF, we can obtain Table 12 as the parametric reduced b-function of f where

$$B(s) = (s+\tfrac{3}{2})(s+\tfrac{4}{3})(s+\tfrac{5}{3})(s+\tfrac{7}{6})(s+\tfrac{11}{6})(s+\tfrac{7}{8})(s+\tfrac{11}{8})(s+\tfrac{13}{8}) \times(s+\tfrac{25}{24})(s+\tfrac{29}{24})(s+\tfrac{31}{24})(s+\tfrac{35}{24})(s+\tfrac{37}{24})(s+\tfrac{41}{24})(s+\tfrac{43}{24}).$$

Table 12 reduced b-functions of $x^2z + yz^2 + y^6 + ay^4z + bz^3$

strata	reduced b-function	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{9}{8})(s+\frac{23}{24})$	17
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{9}{8})(s+\frac{47}{24})$	17
$\mathbb{V}(a,b)$	$B(s)(s+\frac{17}{8})(s+\frac{47}{24})$	17

In all the cases, all roots of reduced b-functions are on the origin.

5.9 $x^3 + xz^2 + y^5$ (U_{16} singularity)

The Milnor number μ of the singularity $x^3 + xz^2 + y^5 = 0$ is 16, and the μ-constant deformation is given by

$$f = x^3 + xz^2 + y^5 + ay^2z^2 + by^3z^2$$

where a, b are parameter. The algorithm ParaBF outputs Table 13 where

$$B(s) = (s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{8}{5})(s+\tfrac{9}{5})(s+\tfrac{13}{15})(s+\tfrac{16}{15})(s+\tfrac{19}{15})(s+\tfrac{22}{15})(s+\tfrac{23}{15})(s+\tfrac{26}{15}).$$

Table 13 global b-functions of $x^3 + xz^2 + y^5 + ay^2z^2 + by^3z^2$

strata	global b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a(27a^4 + 256b))$	$b_f^{(1)} = B(s)(s+1)(s+\frac{14}{15})(s+\frac{17}{15})$
$\mathbb{V}(27a^4 + 256b)\setminus \mathbb{V}(a,b)$	$b_f^{(2)} = B(s)(s+1)(s+\frac{3}{2})(s+\frac{14}{15})(s+\frac{17}{15})$
$\mathbb{V}(a)\setminus \mathbb{V}(a,b)$	$b_f^{(3)} = B(s)(s+1)(s+\frac{17}{15})(s+\frac{29}{15})$
$\mathbb{V}(a,b)$	$b_f^{(4)} = B(s)(s+1)(s+\frac{29}{15})(s+\frac{32}{15})$

Let us consider $(\mathbb{V}(27a^4 + 256b)\setminus \mathbb{V}(a,b), b_f^{(2)})$. If parameters (a, b) belong to $\mathbb{V}(27a^4 + 256b)\setminus \mathbb{V}(a,b)$, then the reduced b-function of f is $\tilde{b}_f = b_f^{(2)}/(s+1)$. One can check $\mathrm{supp}(M_{(\gamma,f)}) \subseteq \mathrm{Sing}(S)$ where γ is a root of $\tilde{b}_f = 0$ and $S = \{(x,y,z)|f(x,y,z) = 0\}$. Then, all roots of $\tilde{b}_0 = \tilde{b}_f/(s+\frac{3}{2}) = 0$ are on the origin and $\tilde{b}_f/\tilde{b}_0 = s + \frac{3}{2} = 0$ is on two isolated singular points

Table 14 b-functions of $x^3 + xz^2 + y^5 + ay^2z^2 + by^3z^2$ on the origin

strata	b-function on the origin	degree of the b-function
$\mathbb{C}^2 \setminus \mathbb{V}(a)$	$B(s)(s+\frac{9}{8})(s+\frac{23}{24})$	12
$\mathbb{V}(a) \setminus \mathbb{V}(a,b)$	$B(s)(s+\frac{9}{8})(s+\frac{47}{24})$	12
$\mathbb{V}(a,b)$	$B(s)(s+\frac{17}{8})(s+\frac{47}{24})$	12

$(x,y) = \left(\frac{-3a^2}{64b^2}, \frac{-a}{4b}, \pm\frac{a}{4b\sqrt{b}}\right)$. The b-functions of the μ-constant deformation f on the origin, are given in Table 14.

Note that the Milnor number 16 does not coincide with the degree of the b-function 12. One can check that the multiplicity of the holonomic system $M_{(-\frac{6}{5},f)}$, $M_{(-\frac{7}{5},f)}$, $M_{(-\frac{8}{5},f)}$ and $M_{(-\frac{9}{5},f)}$ is equal to 2.

5.10 Concluding remarks

We have tried to compute more than 40 μ-constant deformations of non-unimodal singularities by our implementation of Algorithm 2, directly. However, we have obtained only 8 examples of the μ-constant deformations that it took less than "one month" to compute. Other examples need more RAM and time to get b-functions. In general, the computation complexity of algorithms for computing b-functions is quite big. Thus, the computation complexity of parametric b-functions is quite big, too. Anyway, our implementation of Algorithm 2 could return 8 new b-functions of μ-constant deformations of non-unimodal singularities.

In order to avoid the heavy computation, V. Levandovskyy and J. Martín have introduced a smart idea in [17]. We will adopt the idea to the parametric case in the next section.

6. Checking roots of b-functions

Let $f = f_0 + g \in \mathbb{C}[u][x] = \mathbb{C}[u_1,\dots,u_m][x_1,\dots,x_n]$ be a semi-quasihomogeneous polynomial where f_0 is the quasihomogeneous part and g is a linear combinations of upper monomials with parameters u. Then, f can be regard as a μ-constant deformation of f_0 with an isolated singularity at the origin. The following is the classical result by M. Kashiwara [14]. The upper bound statement is due to [10, 11, 33, 34, 41].

Theorem 9. *Let $E_{f_0} = \{\alpha | b_{f_0}(\alpha) = 0\}$ where b_{f_0} is the b-function of f_0 on the origin. Then, for $e \in \mathbb{C}^m$, the set of roots of b-function of $\sigma_e(f)$, on the origin,*

$$E_{\sigma_e(f)} = \{\alpha | b_{\sigma_e(f)}(\alpha) = 0\}$$

becomes a subset of $E = \{\alpha + k | \alpha \in E_{f_0}, k \in \mathbb{Z}, -n < \alpha + k < 0\}$ where $\mathbb{Z}$ is the *set of integers. That is, $E_{\sigma_e(f)} \subset E$, for $e \in \mathbb{C}^m$.*

Empirically, the computational complexity of a Gröbner basis of $B \cup \{f\}$ is bigger than the computational complexity of an annihilating ideal of f^s where B is a basis of the annihilating ideal of f^s. Thus, in many cases, our implementation can return a basis of the annihilating ideal of f^s, but it takes more than "one week" to return the Gröbner basis of $B \cup \{f\}$.

In fact, the closed formula of b_{f_0} is known, thus, the set E of the estimated roots of b_f can be computed by Theorem 9. Hence, in order to avoid the big computation, we can decide roots of b_f and holonomic D-modules by computing a Gröbner basis of $Id(B \cup \{f, s-\nu\})$ where $\nu \in E$. If the reduced Gröbner basis is $\{1\}$, then ν is not a root of b_f, otherwise, ν is a root of b_f. This idea is due to V. Levandovskyy and J. Martín [17]. By the following algorithm, one can check whether ν is a root of the b-function of f or not. Moreover, if ν is a root of the b-function of f, then one can obtain the holonomic D-module associated with ν, too.

Algorithm 3 `CheckingRoot`

Specification: `CheckingRoot`$(f, s-\nu, \succ)$
Checking whether ν is a root of the b-function of f or not.
Input: $f \in \mathbb{C}[u][x]$, $s-\nu \in \mathbb{Q}[s]$, $\succ$: a term order.
Output: $\mathcal{R} \subset \bar{K}^m$: For $(A, G) \in \mathcal{R}$, if the parameters u belong to $\mathbb{A}$, then ν is a root of the b-function of f and G is a basis of the holonomic D-module associated with ν, otherwise, ν is not a root of the b-function of f.
BEGIN
$\mathcal{B} \leftarrow$ `ParaAnn`(f) by Algorithm 1; $\mathcal{R} \leftarrow \emptyset$;
while $\mathcal{B} \neq \emptyset$ **do**
  select an element $(\mathbb{A}, B)$ from $\mathcal{B}$; $\mathcal{B} \leftarrow \mathcal{B} \backslash \{(\mathbb{A}, B)\}$;
  $\mathcal{G} \leftarrow$compute a CGS of $Id(G \cup \{f, s-\nu\})$ w.r.t. $\succ$ on $\mathbb{A}$ in $\mathbb{C}\langle s, x, \partial_x\rangle$;
  while $\mathcal{G} \neq \emptyset$ **do**
  select an element $(\mathbb{A}', G')$ from $\mathcal{G}$; $\mathcal{G} \leftarrow \mathcal{G} \backslash \{(\mathbb{A}', G')\}$;
   if G' does not have a constant element **then**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(\mathbb{A}', G' \cap \mathbb{C}[u]\langle x, \partial_x\rangle)\}$;
   end-if
  end-while
end-while
return $\mathcal{R}$;
END

We give a simple example of Algorithm 3.

Let us return to section 5.1. Consider $s + \frac{3}{2}$ and compute a CGS of

$B_1 \cup \{f, s+\frac{2}{3}\}$ on $\mathbb{C}\setminus\mathbb{V}(a)$. Then, the CGS is $\{(\mathbb{C}\setminus\mathbb{V}(a), \{1\})\}$. Thus, $s=-\frac{2}{3}$ is not a root of $b_f(s)$ on $\mathbb{C}\setminus\mathbb{V}(a)$.

We turn to the case $s+\frac{11}{21}$. Our implementation returns $\{(\mathbb{C}\setminus\mathbb{V}(a), \{x, y, s+\frac{11}{21}\})\}$ as the CGS of $B_1 \cup \{f, s+\frac{11}{21}\}$ on $\mathbb{C}^2\setminus\mathbb{V}(a)$. Thus, $s=-\frac{11}{21}$ is a root of the b-function on $\mathbb{C}\setminus\mathbb{V}(a)$ and defines the ideal generated by $\{x, y\}$.

In general, the computational complexity of $Id(G\cup\{f, s-\nu\})$ is much smaller than the computational complexity of $Id(G\cup\{f\})$. That's why Algorithm 3 may work well, where the notations G and ν are from Algorithm 3.

Here we only give a sketch of ideas. We will report the computation results of Algorithm 3 elsewhere.

Acknowledgements

This work has been partly supported by JSPS Grant-in-Aid for Young Scientists (B) (No.15K17513) and Grant-in-Aid for Scientific Research (C) (No. 15K04891).

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