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A remark on metrically conical surface singularities of Brieskorn type

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Abstract

Birbrair, Fernandes, and Neumann proved that a two-dimensional isolated Brieskorn hypersurface singularity is metrically conical if and only if the two smallest weights are the same. We extend this result for two-dimensional isolated Brieskorn complete intersection singularities.

1. Introduction

Let $(V, o) \subset (\mathbb{C}^m, o)$ be a germ of a normal complex surface singularity and $f\colon X \to V$ a good resolution, i.e., a resolution with simple normal crossing exceptional divisor. Let E denote the exceptional divisor on X. Let $B_\epsilon \subset \mathbb{C}^m$ denote the ball of radius ϵ centered at the origin and S_ϵ its boundary. It is known that for sufficiently small ϵ, $V \cap B_\epsilon$ is homeomorphic to the cone $C(M)$ over the link $M := V \cap S_\epsilon$. The standard metric on $\mathbb{C}^m$ induces a metric on V given by arc-length within V. This metric is called the inner metric and independent of the choice of embedding $(V, o) \subset (\mathbb{C}^m, o)$, up to bi-Lipschitz equivalence. The germ (V, o) is said to be *metrically conical* if $V \cap B_\epsilon$ is bi-Lipschitz equivalent to the metric cone $C(M)$ with respect to the inner metric ([1], [2], [3]).

Birbrair and Fernandes [1] first provided weighted homogeneous surface singularities which are not metrically conical. Subsequently, Birbrair, Fernandes, and Neumann showed that non metrically conical singularities are common. In fact, they proved that if a weighted homogeneous surface singularity is metrically conical then the two smallest weights are the same ([2]), and that the converse holds for Brieskorn hypersurface singularities, namely, the germ $(\{x_1^a + x_2^b + x_3^b = 0\}, o) \subset (\mathbb{C}^3, o)$ with $2 \le a \le b$ is metrically conical ([3]). In this paper we show the following.

Theorem 1.1. *Suppose that $(V, o) \subset (\mathbb{C}^m, o)$ is a germ of an isolated complete intersection surface singularity of Brieskorn type defined by*

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$$V = \{\,(x_i) \in \mathbb{C}^m \mid q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0, \quad j = 3, \ldots, m\},$$

where a_i are integers with $2 \leq a_1 \leq \cdots \leq a_m$. Then the singularity (V, o) is metrically conical if and only if $a_{m-1} = a_m$.

Recently, Birbrair, Neumann, and Pichon [4] gave a classification of surface singularities up to bi-Lipschitz equivalence, in terms of the so-called thick-thin decomposition. They proved that the germ (V, o) is metrically conical if and only if (V, o) has no thin piece ([4, Corollary 1.8]), and gave a construction of the thick-thin decomposition using "sufficiently good" resolution of (V, o) ([4, §2]). Our theorem is obtained as an application of their results.

2. The proof of the theorem

A divisor on X supported in E is called a cycle. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{V,o}$. For a function $h \in \mathfrak{m} \setminus \{0\}$, let $(h)_E$ denote the exceptional part of the divisor $\mathrm{div}_X(h \circ f)$. Then the smallest one among the cycles $(h)_E$, $h \in \mathfrak{m} \setminus \{0\}$, is called the *maximal ideal cycle*. We denote by Z the maximal ideal cycle on X. Clearly, the intersection number ZE_i is non-positive for every irreducible component E_i of E. A component E_i is called an *$\mathcal{L}$-curve* ([4, §2]) if $ZE_i < 0$. We call $f\colon X \to V$ a *right resolution* if the following three conditions are satisfied.

(1) $\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X(-Z)$, i.e., f factors through the blowing up by $\mathfrak{m}$.

(2) No two $\mathcal{L}$-curves intersect.

(3) f is the minimum among good resolutions of (V, o) satisfying two conditions above.

Note that the right resolution can be obtained by blowing-ups from the minimal good resolution. An irreducible component E_i is called a *node* if E_i is not a rational curve or $(E - E_i)E_i \geq 3$. (In [4, §2], $\mathcal{L}$-curve is also called a node.) The exceptional set E is said to be *star-shaped* if it has just one node.

Now we assume that (V, o) is as in Theorem 1.1. Let $f\colon X \to V$ be the minimal good resolution. It is known that E is star-shaped except for the case $V = \{x_1^2 + x_2^2 + x_3^m = 0\}$ (E is a chain in this case). So we assume that E is star-shaped. By [4, Corollary 7.11], we obtain the following.

Proposition 2.1. *The germ (V, o) is metrically conical if and only if the node is a unique $\mathcal{L}$-curve on the right resolution.*

A concrete description of the maximal ideal cycle for a Brieskorn hypersurface singularity was obtained by Konno and Nagashima [5], and their result was extended to Brieskorn complete intersections in [6]. By [6, Theorem 6.1], the maximal ideal cycle Z coincides with $(x_m)_E$. Let

$$d = \mathrm{lcm}(a_1, \ldots, a_{m-1}), \quad \alpha_m = a_m / \gcd(a_m, d)$$

(see p.125 and p.128 of [6]). The $\mathcal{L}$-curves are given in [6, Theorem 4.4] (those are expressed as $E_{m,s_m,\xi}$). In fact, the node is the unique $\mathcal{L}$-curve if and only if $\alpha_m = 1$, namely, $a_m \mid d$. Assume that this condition is satisfied. If $\mathfrak{m}\mathcal{O}_X \neq \mathcal{O}_X(-Z)$, by resolving the base points of the linear system $H^0(\mathcal{O}_X(-Z))$, on the right resolution, we have a node other than $\mathcal{L}$-curves (see p. 135 for the structure of the base points). Therefore the node is a unique $\mathcal{L}$-curve on the right resolution if and only if $\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X(-Z)$; f is the right resolution in this case. By [6, Proposition 6.4], this condition is equivalent to that $d/a_{m-1} < d/a_m + 1$. Hence it follows from Proposition 2.1 that if (V, o) is metrically conical, then we obtain $a_{m-1} \geq a_m$; by assumption, $a_{m-1} = a_m$. Conversely, if $a_{m-1} = a_m$, then $d/a_{m-1} < d/a_m + 1$ and $\alpha_m = 1$. Hence f is the right resolution and the node is a unique $\mathcal{L}$-curve.

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