Classical formulae on projective surfaces and 3-folds with ordinary singularities, revisited

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Abstract

From a viewpoint of global singularity theory, we revisit classical formulae of Salmon-Cayley-Zeuthen on numerical characters of projective surfaces and analogous formulae of Severi-Segre-Roth on those of 3-folds. The theory of universal polynomials associated to local and multi-singularities of maps provides a unified way for reproducing those classical enumerative formulae.

1. Introduction

In this paper, from a viewpoint of global singularity theory, we revisit classical formulae of Salmon-Cayley-Zeuthen for projective surfaces in 3-space and analogous formulae of Severi-Segre-Roth for 3-folds in 4-space. For simplicity, we work over $\mathbb{C}$ throughout. In classical literature, numerical projective characters of a subvariety of projective space are defined by the degree of individual singularity loci, e.g. the number of triple points, and also by the degree of *polars*, i.e. the degree of the critical loci of linear projections. For a surface with *ordinary singularities* in $\mathbb{P}^3$ (see §3.1 for the definition), there are known five relations among nine numerical characters, which were originally given by Salmon [20, XVII] (cf. Baker [2, IV] and Semple-Roth [22, IX, §3]); those have rigorously been reformulated within modern algebraic geometry by Piene [18]. For a 3-fold in $\mathbb{P}^4$, called *primals* classically, there are analogous works of Roth [19] (and also Severi, B. Segre etc); recently, new formulae on Chern numbers have been shown by Tsuboi [23, 24] using modern intersection theory.

In contrast, our approach is based on singularity theory of holomorphic maps and complex cobordism theory. The aim of this paper is to demonstrate an effective way to reproduce and generalize those classical formulae by means of universal polynomials associated to local and multi-singularities of maps, called *Thom polynomials* (cf. Kleiman [9, 10], Colley [4], Fehér-Rimányi [5, 6], Kazarian [7, 8], Ohmoto [14, 16]). For a surface with ordinary singularities in $\mathbb{P}^3$, there are

essentially four numerical characters, which actually correspond to Chern mono-mials $1, c_1, c_1^2, c_2$ of the normalization of the surface. As an analogy, for a 3-fold in $\mathbb{P}^4$ with ordinary singularities (see §4.1), Roth [19] claimed that there are seven independent characters among more than 20 numerical characters. We show that all numerical characters can explicitly be computed by using universal polynomi-als, and especially, independent characters correspond to seven Chern monomials $c_1^i c_2^j c_3^k$ $(0 \leq i + j + k \leq 3)$ of its normalization (Theorem 4.1). In particular, our computation recovers exactly the same formulae as those of Tsuboi [24], although methods are completely different (Remark 4.5). It should be noted that we make use of an advanced version of universal polynomial for computing *weighted Euler characteristics* of singularity loci of prescribed type, which is based on equivari-ant Chern-Schwartz-MacPherson (CSM) classes established by the second author [14, 16]. Indeed, the CSM class satisfies a variant of Grothendieck-Riemann-Roch formula, so the additivity and the covariant functoriality of CSM classes are fit for interpreting or generalizing standard classical arguments for counting invariants, e.g. addition-deletion principle.

A generalization to higher dimensional case would be possible. In fact, for a projective variety $X = f(M) \subset \mathbb{P}^n$ given as the image variety of a locally stable map $f : M^m \to \mathbb{P}^n$ $(m < n)$, the composition of f with a generic linear projection onto lower dimensional projective space becomes a locally stable map, that has been proven by Mather [12] and Bruce-Kirk [3]. Therefore, in general, numerical characters of such a subvariety X would be expressed in terms of degrees of Chern monomials of M and the hyperplane class by using universal polynomials. For another kind of classical enumerative problems, e.g. counting lines which have a prescribed contact with a given surface (cf. Salmon [20]), universal polynomials for *unstable* singularities are effective, that has been discussed in our another paper [21].

2. Singularities of maps and Thom polynomials

2.1 Local and multi-singularities of maps

All maps are assumed to be holomorphic throughout. Let κ denote the rel-ative codimension $n - m$ for a map $f : M \to N$ from an m-fold to an n-fold. As local classification of maps, it is very natural to think of the equivalence relation of map-germs via local coordinate changes of the source and the tar-get: $f, g : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ are $\mathcal{A}$-*equivalent* if there are biholomorphic germs $\sigma : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ and $\tau : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $g = \tau \circ f \circ \sigma^{-1}$. Ob-viously, the equivalence is defined for any germs $f : (M, p) \to (N, f(p))$, and an equivalence class is called a local singularity type or $\mathcal{A}$-singularity type. We say that a map-germ f is *stable* if any deformation of f is trivial up to (parametric) $\mathcal{A}$-equivalence. Stable-germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ are characterized by its local algebra

$$Q_f := \mathcal{O}_{\mathbb{C}^m,0}/f^*\mathfrak{m}_n\mathcal{O}_{\mathbb{C}^m,0},$$

where $\mathcal{O}_{\mathbb{C}^m,0}$ is the ring of holomorphic function-germs at 0 and $\mathfrak{m}_n$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ – a theorem of J. Mather (cf. [13]) says that if stable-germs $f, g : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ have isomorphic local algebras, then they are $\mathcal{A}$-equivalent. For stable-germs, $\kappa = n - m$ is more essential rather than m, n.

Example 2.1. Let $\kappa \geq 0$. The A_μ-singularity is characterized by

$$Q_{A_\mu} = \mathbb{C}[x]/\langle x^{\mu+1}\rangle.$$

Normal forms of stable A_μ-singularity types in low dimensions are as follows:

$$\mathbb{C}^2 \to \mathbb{C}^2 \quad A_1 : (x^2, y), \quad A_2 : (x^3 + yx, y);$$
$$\mathbb{C}^2 \to \mathbb{C}^3 \quad A_1 : (x^2, xy, y);$$
$$\mathbb{C}^3 \to \mathbb{C}^3 \quad A_1 : (x^2, y, z),\ A_2 : (x^3 + yx, y, z),\ A_3 : (x^4 + yx^2 + zx, y, z);$$
$$\mathbb{C}^3 \to \mathbb{C}^4 \quad A_1 : (x^2, xy, y, z).$$

Those germs are classically known [20]; the image (or the critical value locus) is a typical singularity of singular surfaces and 3-folds, which will be discussed later (§3 and §4).

A *multi-singularity type* is an $\mathcal{A}$-equivalence class of germs (multi-germs) $f : (M, S) \to (N, q)$ of holomorphic maps at finite subsets $S = \{p_1, \cdots, p_k\} \subset f^{-1}(q)$ in M. A multi-singularity of map-germs is *stable* if it is stable under any deformation up to $\mathcal{A}$-equivalence. In particular, if $f : (M, S) \to (N, q)$ is a stable multi-singularty, then each mono-germ $f : (M, p_i) \to (N, q)$ is stable and the sum of their codimensions does not exceed the target dimension n. A multi-singularity type may be regarded just as a collection of $\mathcal{A}$-types η_i of mono-germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$; an *ordered* type is denoted by $\underline{\eta} := (\eta_1, \eta_2, \cdots, \eta_r)$, which is used below to distinguish the first entry η_1 from others.

A proper holomorphic map $f : M \to N$ between complex manifolds is *locally stable* if for any $y \in N$ and for any finite subset S of $f^{-1}(y)$, the germ $f : (M, S) \to (N, y)$ is stable. Let M and N be of dimension m and n, respectively, and f a locally stable map. Given an $\mathcal{A}$-type η of $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, we set

$$\eta(f) = \{\, p \in M \mid \text{the germ of } f \text{ at } p \text{ is of type } \eta \,\}$$

and call its closure the *η-singularity locus* of f, which becomes an analytic closed subset of M. For a stable multi-singularity type $\underline{\eta}$, we set

$$\underline{\eta}(f) := \left\{\, p_1 \in \eta_1(f) \,\left|\, \begin{array}{l} \exists \text{ distinct } p_2, \cdots, p_r \in f^{-1}f(p_1) - \{p_1\} \\ \text{s.t. } f \text{ at } p_i \text{ is of type } \eta_i \end{array} \right.\right\}$$

and call its closure $\overline{\underline{\eta}(f)} \subset M$ the *multi-singularity locus of type $\underline{\eta}$ in the source*.

In case of $m \leq n$, the restriction map

$$f : \overline{\underline{\eta}(f)} \to \overline{f(\underline{\eta}(f))}$$

is finite-to-one; let $\deg_1 \underline{\eta}$ denote the degree of this map, that is equal to the number of η_1-type appearing in the tuple $\underline{\eta}$. e.g. $\deg_1 A_0^3 = 3$.

Example 2.2. In case of $(m, n) = (3, 4)$, local stable singularity type is only A_1, which yields the critical locus $\overline{A_1(f)}$ of a stable map $f : M^3 \to N^4$ (it becomes a smooth curve). Stable multi-singularity types are the following types whose components are mutually transverse:

$A_0^2 := A_0 A_0$ (double point locus in the source space);
$A_0^3 := A_0 A_0 A_0$ (triple point curve);
$A_0^4 := A_0 A_0 A_0 A_0$ (quadruple points);
$A_0 A_1$ and $A_1 A_0$ (preimage of stationary points).

2.2 Thom polynomials

To each local singularity type η of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+\kappa}, 0$, one can assign a unique universal polynomial $tp(\eta)$ of quotient Chern classes $c_i = c_i(f)$ which expresses the η-singularity locus of appropriately generic maps $f : M \to N$ (cf. [5, 6, 7, 16]):

$$[\overline{\eta(f)}] = tp(\eta)(f) \in H^*(M).$$

Here $c_i(f)$ is defined by the i-th component of $c(f) = 1 + c_1(f) + c_2(f) + \cdots$ where

$$c(f) = c(f^* TN - TM) = \frac{1 + f^* c_1(TN) + \cdots}{1 + c_1(TM) + \cdots}.$$

This fact originates in René Thom (perhaps, his Strasbourg Lecture in 1957), thus $tp(\eta)$ is usually called the *Thom polynomial* of η. As simplest examples, $tp(A_1)$, $tp(A_2)$ and $tp(A_3)$ in cases of $\kappa = 0, 1$ are shown in Tables 1 and 2 below.

Furthermore, one may expect the existence of universal polynomials for stable multi-singularity types, which generalize the *multiple point formulae* studied by Kleiman [9, 10]. However the general problem is known to be very hard and a full conjecture has been proposed in Kazarian [7]. Nevertheless, there are some computational results in a special case of maps having only A_μ-singularities with $\kappa \geq 0$, called *curvilinear maps* [9, 10, 4, 7, 8]. An algorithm given by Colley [4] verifies the following claim in at least low codimensional case (cf. [8]); To each multi-singularity type $A_\bullet = (A_{\mu_1}, A_{\mu_2}, \cdots)$ (in other words, a type of *multiple stationary points*) of maps with codimension κ, one can associate a universal polynomial $tp(A_\bullet)$ of variables $c_i = c_i(f)$ and $s_I = f^* s_I(f)$ with rational coefficients such that it holds that

$$\left[\overline{A_\bullet(f)}\right] = tp(A_\bullet)(f) \in H^*(M; \mathbb{Q})$$

for proper stable curvilinear maps $f : M^m \to N^{m+\kappa}$. Here $s_I(f)$ denote the *Landweber-Novikov classes* of f multi-indexed by $I = (i_1 i_2 \cdots)$:

$$s_I = s_I(f) = f_*(c_1(f)^{i_1} c_2(f)^{i_2} \cdots) \in H^*(N).$$

In case of $\kappa = 0$ and 1, the universal polynomials $tp(A_\bullet)$ with low codimension are explicitly given as in Tables 1 and 2 [4, 8]. We often write $s_I(f)$ for its pullback $f^* s_I(f) \in H^*(M)$, i.e. drop the notation f^* for short.

Table 1 $\kappa = 0$

type	codim	tp
A_1	1	c_1
A_2	2	$c_1^2 + c_2$
A_1^2	2	$c_1 s_1 - 4c_1^2 - 2c_2$
A_3	3	$c_1^3 + 3c_1 c_2 + 2c_3$
A_1^3	3	$\frac{1}{2}\left(\begin{array}{c} c_1 s_1^2 - 4c_2 s_1 - 4c_1 s_2 - 2c_1 s_{01} - 8c_1^2 s_1 \\ +40c_1^3 + 56c_1 c_2 + 24c_3 \end{array}\right)$
$A_1 A_2$	3	$c_1 s_2 + c_1 s_{01} - 6c_1^3 - 12c_1 c_2 - 6c_3$
$A_2 A_1$	3	$c_1^2 s_1 + c_2 s_1 - 6c_1^3 - 12c_1 c_2 - 6c_3$

Table 2 $\kappa = 1$

type	codim	tp
A_0^2	1	$s_0 - c_1$
A_1	2	c_2
A_0^3	2	$\frac{1}{2}(s_0^2 - s_1 - 2s_0 c_1 + 2c_1^2 + 2c_2)$
$A_0 A_1$	3	$s_{01} - 2c_1 c_2 - 2c_3$
$A_1 A_0$	3	$s_0 c_2 - 2c_1 c_2 - 2c_3$
A_0^4	3	$\frac{1}{3!}\left(\begin{array}{c} s_0^3 - 3s_0 s_1 + 2s_2 + 2s_{01} - 3s_0^2 c_1 + 3s_1 c_1 \\ +6s_0 c_1^2 + 6s_0 c_2 - 6c_1^3 - 18c_1 c_2 - 12c_3 \end{array}\right)$

2.3 Euler characteristics

For a complex analytic variety X which admits singular points, i.e. $\mathcal{O}_{X,p} \not\cong \mathcal{O}_{\mathbb{C}^m,0}$, the Chern class $c(TX)$ no longer exists, because of the lack of the tangent bundle. However, there is a well-behaved alternative in homology, $c^{\mathrm{SM}}(X) \in H_*(X)$, called the *Chern-Schwartz-MacPherson class* (CSM class) of X [11]: the CSM class satisfies a nice functorial property and the normalization condition that $c^{\mathrm{SM}}(X) = c(TX) \frown [X]$ if X is non-singular. In particular, if X is irreducible, proper and possibly singular, the degree of the class is the (topological) Euler characteristic $\chi(X)$ and the top component is the fundamental class, i.e.

$$c^{\mathrm{SM}}(X) = \chi(X)[pt] + \cdots + [X] \in H_*(X).$$

For a closed subvariety X in an ambient manifold M, the *Segre*-SM class of

the embedding $\iota : X \to M$ is defined by

$$s^{\mathrm{SM}}(X, M) := \iota^* c(TM)^{-1} \frown c^{\mathrm{SM}}(X) \in H_*(X).$$

If X is non-singular, the SSM class is just the inverse normal Chern class $c(-\nu_{M,X})$ $(= c(TX)/\iota^* c(TM))$. We consider $\iota_* c^{\mathrm{SM}}(X)$, $\iota_* s^{\mathrm{SM}}(X, M) \in H^*(M)$ via the Poincaré dual, unless specifically mentioned.

In [14], an equivariant version of CSM classes has been established. As a direct corollary, it is proven in [16, Thm. 4.4] (also [14, Thm. 5.5], [15, 17]) that for a local singularity type η, there exists a unique universal polynomial $tp^{\mathrm{SM}}(\overline{\eta})$ in $c_i(f)$ so that

$$c^{\mathrm{SM}}(\overline{\eta(f)}) = c(TM) \cdot tp^{\mathrm{SM}}(\overline{\eta})(f) \in H^*(M)$$

for appropriately generic $f : M \to N$. We call $tp^{\mathrm{SM}}(\overline{\eta})$ the *universal SSM class* of $\overline{\eta}$, or roughly the *higher Thom polynomial* of $\overline{\eta}$ (cf. [1]). It follows from the above form of $c^{\mathrm{SM}}(X)$ that the leading term of the universal polynomial is just the Thom polynomial of η,

$$tp^{\mathrm{SM}}(\overline{\eta}) = tp(\eta) + h.o.t.,$$

and the degree gives $\chi(\overline{\eta(f)})$. Let us see the simplest examples. For

$$\begin{aligned} A_1 &: (x, \boldsymbol{z}) \mapsto (x^2, \boldsymbol{z}) & (\kappa = 0) \\ A_1 &: (x, y, \boldsymbol{z}) \mapsto (x^2, xy, y, \boldsymbol{z}) & (\kappa = 1) \end{aligned}$$

low degree terms of the universal SSM class are computed as follows (see [16, p.222]):

$$(1) \qquad tp^{\mathrm{SM}}(\overline{A_1}) = \begin{cases} c_1 - c_1^2 + c_1^3 - (c_1^4 + c_2^2 - c_1 c_3) + \cdots & (\kappa = 0) \\ c_2 - (c_1 c_2 + c_3) + \cdots & (\kappa = 1). \end{cases}$$

Then, for stable maps $f : M^m \to N^{m+\kappa}$ with low dimensions and M compact, the Euler characteristics of the locus $S(f) := \overline{A_1(f)} \subset M$ is computed by

$$\chi(S(f)) = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\overline{A_1})(f).$$

Next, for multi-singularity types in low dimensions, it has been stated in [16, §6.2] that there is a universal polynomial of $c_i = c_i(f)$ and $s_I = s_I(f)$ such that

$$c^{\mathrm{SM}}(\overline{A_\bullet(f)}) = c(TM) \cdot tp^{\mathrm{SM}}(\overline{A_\bullet})(f) \in H^*(M; \mathbb{Q}),$$

for proper stable curvilinear maps $f : M \to N$. If M is compact, we have

$$\chi(\overline{A_\bullet(f)}) = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\overline{A_\bullet})(f).$$

In [16, p.248], low degree terms of some universal polynomials $tp^{\mathrm{SM}}(A_\bullet)$ are explicitly computed, e.g.,

$$(2) \qquad tp^{\mathrm{SM}}(\overline{A_0^2}) = (s_0 - c_1) + \tfrac{1}{2}(2c_2 + 2c_1 s_0 - s_0^2 - s_1) + \cdots . \qquad (\kappa = 1)$$

Note that the leading term $s_0 - c_1$ is just $tp(\overline{A_0^2})$ (the double point formula). Moreover, it is also shown in Theorems 6.5 and 6.10 in [16] that for proper stable curvilinear maps $f : M^m \to N^{m+1}$, the Euler characteristics of the image hypersurface $X = f(M)$ and the (singular) double locus $D = \overline{f(A_0^2(f))}$ in N are expressed as

$$\chi(X) = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\alpha_{\mathrm{im}})(f)$$

and

$$\chi(D) = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\alpha_{\mathrm{im}}(2))(f),$$

where

$$\begin{aligned}
tp^{\mathrm{SM}}(\alpha_{\mathrm{im}}) &= 1 + \tfrac{1}{2}(c_1 - s_0) + \tfrac{1}{6}(s_0^2 + 2s_1 - 2c_1 s_0 - c_1^2 - c_2) \\
&\quad + \tfrac{1}{24}\left(\begin{array}{l} 2c_1^3 - 10c_1 c_2 + 2c_1^2 s_0 + 2c_2 s_0 + 3c_1 s_0^2 \\ -s_0^3 + 14s_{01} + 5c_1 s_1 - 5s_0 s_1 - 6s_2 \end{array} \right) \\
&\quad + \cdots . \\
tp^{\mathrm{SM}}(\alpha_{\mathrm{im}}(2)) &= \tfrac{1}{2}(s_0 - c_1) + \tfrac{1}{6}(-c_1^2 + 5c_2 + 4c_1 s_0 - 2s_0^2 - s_1) \\
&\quad + \tfrac{1}{24}\left(\begin{array}{l} 2c_1^3 + 38c_1 c_2 + 24c_3 + 2c_1^2 s_0 - 22c_2 s_0 - 9c_1 s_0^2 \\ +3s_0^3 - 14s_{01} - 7c_1 s_1 + 7s_0 s_1 + 2s_2 \end{array} \right) \\
&\quad + \cdots .
\end{aligned}$$

Those universal polynomials will be used in the following sections.

3. Classical formulae of surfaces in 3-space

A simple application of Thom polynomials recovers formulae of Salmon-Cayley-Zeuthen of surfaces in 3-space. This is a prototype for a further application to 3-folds which will be discussed in the next section.

3.1 Surfaces with ordinary singularities

Let $X \subset \mathbb{P}^3$ be a reduced surface of degree d having only *ordinary singularities.* That is, the singular locus of X consists of crosscap points and transverse double and triple points, which are locally defined, respectively, by equations up to local analytic coordinate changes (Figure 1):
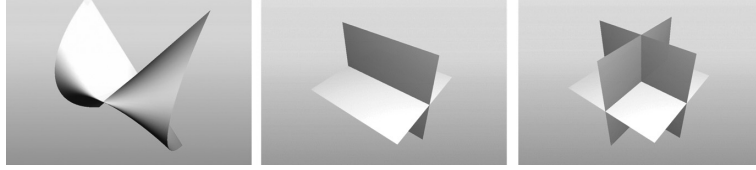
$$xy^2 - z^2 = 0, \quad xy = 0, \quad xyz = 0.$$

Figure 1   Surface with ordinary singularities in 3-space.

Note that the A_1-singularity $\mathbb{C}^2 \to \mathbb{C}^3$, $(u,v) \mapsto (x,y,z) = (v^2, u, uv)$, is a normalization of the crosscap. Hence, the normalization M of X becomes smooth, and the composition map of the projection and the inclusion into the ambient space, denoted by $f : M \to \mathbb{P}^3$ with $X = f(M)$, is locally stable (cf. [18, 13]). We do not need the condition that M is embedded in some higher dimensional projective space and f is realized by a linear projection. Let Γ be the double point locus in the source and D its image (the double curve):

$$\Gamma = \overline{A_0^2(f)} \subset M, \quad D = f(\Gamma) \subset \mathbb{P}^3.$$

Note that Γ is an immersed curve having nodes at $A_0^3(f)$ (the preimage of triple points) on M; D is an immersed space curve with triple points. The crosscap points are located on D, where $\Gamma \to D$ is doubly ramified.

From now on, we always use cohomology with rational coefficients: $H^* = H^*(-, \mathbb{Q})$. We denote the hyperplane class of $\mathbb{P}^3$ by

$$a = c_1(\mathcal{O}_{\mathbb{P}^3}(1)) \in H^2(\mathbb{P}^3).$$

Put

$$f_*(1) = da, \quad f_* c_1(TM) = \xi_1 a^2, \quad f_*(c_1(TM)^2) = \xi_2 a^3,$$

$$f_* c_2(TM) = \xi_{01} a^3 \in H^*(\mathbb{P}^3).$$

Then

$$c(f) = c(f^* T\mathbb{P}^3 - TM) = \frac{(1 + \tilde{a})^4}{1 + c_1(TM) + c_2(TM)}$$
$$= 1 + (4\tilde{a} - c_1(TM)) + (6\tilde{a}^2 - 4\tilde{a}c_1(TM) + c_1(TM)^2 - c_2(TM)) + \cdots$$

($\tilde{a} := f^* a \in H^2(M)$) and

$$s_0(f) = f_*(1) = da,$$
$$s_1(f) = f_*(c_1(f)) = (4d - \xi_1)a^2,$$
$$s_2(f) = f_*(c_1(f)^2) = (16d - 8\xi_1 + \xi_2)a^3,$$
$$s_{01}(f) = f_*(c_2(f)) = (6a - 4\xi_1 + \xi_2 - \xi_{01})a^3.$$

3.2 Numerical projective characters

Classically, nine numerical characters of X are introduced [2, 22]:

- $\mu_0 = d$: the degree of X
- μ_1: the rank of X (the first polar)
- μ_2: the class of X (the second polar)
- κ: the number of cusps in generic projection $X \to \mathbb{P}^2$
- ϵ_0: the degree of the double curve D
- ϵ_1: the rank of D
- ρ: the class of immersion of D in X
- C: the number of crosscaps
- T: the number of triple points.

We express the above numerical characters in terms of $d, \xi_1, \xi_2, \xi_{01}$ by using Thom polynomials. First, C, T, ϵ_0 immediately follows from Thom polynomials of A_1, A_0^2 and A_0^3 (Table 2) applied to $f : M \to \mathbb{P}^3$:

$$C = \int f_*(Tp(A_1)(f)) = 6d - 4\xi_1 + \xi_2 - \xi_{01},$$

$$T = \tfrac{1}{3} \int f_*(Tp(A_0^3)(f))$$

$$= \tfrac{1}{6}(44d - 12d^2 + d^3 + (3d - 24)\xi_1 + 4\xi_2 - 2\xi_{01}),$$

$$\epsilon_0 = \tfrac{1}{2} \int f_*(Tp(A_0^2)(f)) \cdot a = \tfrac{1}{2}(d^2 - 4d + \xi_1).$$

Conversely, we can express ξ_1, ξ_2, ξ_{01} in terms of ϵ_0, C, T:

Lemma 3.1. *It holds that*

$$\xi_1 = d(4 - d) + 2\epsilon_0,$$
$$\xi_2 = d(d - 4)^2 + (16 - 3d)\epsilon_0 + 3T - C,$$
$$\xi_{01} = d(d^2 - 4d + 6) + (8 - 3d)\epsilon_0 + 3T - 2C.$$

Remark 3.2. Lemma 3.1 is classical (see [18, Prop.1]). In old literature, the degree $\mu_0 = d$, the rank μ_1, the class μ_2 and the degree of pinch points C of X are more frequently used; they are called *elementary numerical characters*. A major classical interest was to express intrinsic invariants of projective varieties in terms of their extrinsic invariants (=numerical characters). For instance, two basic invariants are introduced for projective surfaces:

$$\omega := \mu_2 - 6\mu_1 + 9\mu_0 + C + 1, \quad I := \mu_2 - 2\mu_1 + 3\mu_0 - 4,$$

called the *Castelnuovo-Enriques invariant* and the *Zeuthen-Segre invariant*, respectively ([2, Chap.V], [22, pp.223–224]). Their sum is an absolute birational

invaraiant: $\omega + I = 12p_a + 9$, where p_a is the arithmetic genus of X. That is verified in our context as follows: by the computation of μ_1 and μ_2 below, we see

$$\omega = \xi_2 + 1, \quad I = \xi_{01} - 4,$$

thus by using the Hirzebruch-Riemann-Roch formula,

$$\omega + I = \int (c_1(TM)^2 + c_2(TM)) - 3 = 12\chi(M, \mathcal{O}_M) - 3 = 12p_a + 9.$$

To compute other five characters, we need to discuss generic projections of X and D. Consider a generic projection of X from a general point p in $\mathbb{P}^3$. Put

$$g = pr \circ f : M \to \mathbb{P}^2$$

to be the composition of f with the canonical projection $pr : \mathbb{P}^3 - p \to \mathbb{P}^2$. Obviously g has the degree d. Let $S(g)$ be the closure of $A_1(g) \subset M$, i.e. the critical curve of the map g. In this case, the curve is non-singular and closed.

The space $\mathbb{P}^3 - p$ is isomorphic to the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$, and the isomorphism $H^*(\mathbb{P}^2) = H^*(\mathbb{P}^3 - p)$ is given by the pullback of the projection. Denote by a' the hyperplane class of $\mathbb{P}^2$; then $\tilde{a} := g^*a' = f^*a \in H^2(M)$. The quotient Chern class for g is $c(g) = (1 + \tilde{a})^3 c(TM)^{-1}$ and $g_*(1) = g_*([M]) = d$. Substitute $c_i(g)$ for c_i in (higher) Thom polynomials, then one gets the degrees and the Euler characteristics of corresponding loci of g.

We are now ready to compute the remaining five numerical characters.

• The degree κ is the ramification divisor of $S(g) \to g(S(g)) \subset \mathbb{P}^2$, that is just the number of cusps of g. Thus

$$\kappa = \int f_*(tp(A_2)(g)) = 12d - 9\xi_1 + 2\xi_2 - \xi_{01}$$

$$= d(d-1)(d-2) + (6 - 3d)\epsilon_0 + 3T.$$

• The _rank_ μ_1 of X is defined as the degree of the first polar of X, i.e. the degree of the locus of points at which tangent planes contain a given general point p. That is equal to the degree of $S(g)$:

$$\mu_1 = \int f_*(tp(A_1)(g)) = 3d - \xi_1 = d(d-1) - 2\epsilon_0.$$

• The _class_ μ_2 of X is defined as the degree of the second polar of X, i.e. the degree of the locus of points at which tangent planes contain a given general line L. First, consider a generic projection g to $\mathbb{P}^2$ from p. The contour curve $g(S(g)) \subset \mathbb{P}^2$ has the degree μ_1 and μ_2 polar points from a general point $pt \in \mathbb{P}^2$; then L is given by the line joining p and pt. The composed map

$S(g) \to \mathbb{P}^2 - pt \to \mathbb{P}^1$ has $\mu_2 + \kappa$ critical points, thus by the Riemann-Hurwitz formula (i.e. $tp(A_1) = c_1$)

$$\mu_2 + \kappa = 2\mu_1 - \chi(S(g)).$$

The Euler characteristic of $S(g)$ is computed using (1) in §2.3:

$$\chi(S(g)) = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\overline{A_1})(g) = -9d + 9\xi_1 - 2\xi_2.$$

Hence,

$$\begin{aligned}
\mu_2 &= 3d - 2\xi_1 + \xi_{01} \\
&= d(d-1)^2 + (4-3d)\epsilon_0 + 3T - 2C.
\end{aligned}$$

• The *class ρ of immersion of D in X* is defined by the number of points in D at which one of the tangent plane of X contains a given point p. Crosscap points should be excluded because of the lack of the tangent plane. In our situation, curves Γ and $S(g)$ on M are transverse to each other; in particular the intersection contains crosscap points. Hence ρ is computed as follows:

$$\begin{aligned}
\rho &= \Gamma \cdot S(g) - C \\
&= -18d + 3d^2 + (11-d)\xi_1 - 2\xi_{01} + \xi_2 \\
&= (d-2)\epsilon_0 - 3T.
\end{aligned}$$

• The *rank ϵ_1 of D* is defined by the degree of its polar, i.e., the number of planes tangent to D which contain a given general line L. First we resolves (non-transverse) triple points of D to get smooth D'. Let $\pi : D' \to \mathbb{P}^1$ be the composed map with the projection from L. Then ϵ_0 is just the degree of π, and ϵ_1 is equal to the number of critical points of π. Hence, by $tp(A_1) = c_1$, we have

$$\epsilon_1 = \epsilon_0 \chi(\mathbb{P}^1) - \chi(D').$$

Obviously, $\chi(D') = \chi(D) + 2T$. On the other hand, we see that

$$\begin{aligned}
\chi(D) &= \int_M c(TM) \cdot tp^{\mathrm{SM}}(\alpha_{\mathrm{im}}(2)))(f) \\
&= \tfrac{1}{3}(7d + 6d^2 - d^3 - \tfrac{5}{2}\xi_{01} - 12\xi_1 + \tfrac{7}{2}\xi_2) \\
&= (4-d)\epsilon_0 + T + \tfrac{1}{2}C.
\end{aligned}$$

Combining all of the above,

$$\begin{aligned}
\epsilon_1 &= 3d^2 - 21d + (13-d)\xi_1 + \tfrac{3}{2}\xi_{01} - \tfrac{5}{2}\xi_2 \\
&= (d-2)\epsilon_0 - 3T - \tfrac{1}{2}C.
\end{aligned}$$

As seen above, all numerical characters are expressed by $d, \xi_1, \xi_2, \xi_{01}$. These expressions immediately imply the following five basic relations among nine characters in [2, Chapter IV] (cf. [18, Thm.2]):

Proposition 3.3 ([2]). *The above nine characters of a projective surface in $\mathbb{P}^3$ with ordinary singularities satisfy*

$$
\begin{array}{llll}
(i) & d(d-1) = \mu_1 + 2\epsilon_0, & (ii) & \mu_1(d-2) = \kappa + \rho, \\
(iii) & \epsilon_0(d-2) = \rho + 3T, & (iv) & 2\rho - 2\epsilon_1 = C, \\
(v) & \mu_2 + 2C = \mu_1 + \kappa.
\end{array}
$$

4. Classical formulae of 3-folds in 4-space

4.1 3-folds with ordinary singularities

Let $X \subset \mathbb{P}^4$ be a reduced hypersurface of degree d with *ordinary singularities*, i.e., singularities are locally given by the following equations:

crosscaps	(A_1)	$xy^2 = z^2$
double point locus	(A_0^2)	$xy = 0$
triple point curve	(A_0^3)	$xyz = 0$
quadruple points	(A_0^4)	$xyzw = 0$
stationary points	$(A_0 A_1)$	$(xy^2 - z^2)w = 0.$

There are many examples obtained by generic linear projections of smooth 3-folds sitting in higher dimensional projective space. In fact, it is shown that almost all projections of smooth 3-folds into $\mathbb{P}^4$ are locally stable (Mather [12]).

As with the case of surfaces in $\mathbb{P}^3$, the A_1-singularity $\mathbb{C}^3 \to \mathbb{C}^4$, $(u, v, t) \mapsto (x, y, z, w) = (v^2, u, uv, t)$, locally gives the normalization of the crosscap locus (a curve on X). Therefore, a normalization M of X becomes to be smooth and we have a locally stable map $f : M \to \mathbb{P}^4$ with $X = f(M)$ (cf. [23, 24, 13]). Note that it is not needed that f is realized by some linear projection. The double point locus $\Gamma = \overline{A_0^2(f)}$ in the source M is a closed surface with ordinary singularities; the double curve, triple points and crosscaps of Γ are just $\overline{A_0^3(f)}$, $A_0^4(f)$ and $A_0 A_1(f)$, respectively. The critical locus $\overline{A_1(f)}$ in M is an immersed curve lying on Γ, and denote by C its image in $\mathbb{P}^4$. The loci $A_1 A_0(f)$ and $A_0 A_1(f)$ are just the preimage of the stationary point locus denoted by St. The image of Γ, called the *double surface* $D := f(\Gamma)$, is a projective surface which has singularities along the target triple point curve $T = f(\overline{A_0^3(f)})$. Note that $f|_\Gamma : \Gamma \to D$ is ramified along C, and the source triple point locus $\overline{A_0^3(f)}$ is ramified on T at St.

4.2 Numerical projective characters

As a natural generalization, several attempts to find invariants of 3-folds (primals) were intensively made by algebraic geometers in that age, e.g. B. Segre, Severi, Todd, Eger and Roth; there are (at least) twenty numerical characters of

a 3-fold X in $\mathbb{P}^4$:

- d: the degree of X;
- μ_0: the degree of the double surface $D = f(\overline{A_0^2}) \subset X \subset \mathbb{P}^4$;
- t: the degree of the triple point curve $T = f(\overline{A_0^3})$;
- q: the number of quadruple points $Q = f(A_0^4)$;
- s_t: the number of stationary points $St = f(A_0 A_1) = f(A_1 A_0)$;
- γ: the degree of the critical curve $C = f(\overline{A_1})$;
- characters associated to projections of X, D, T and C, i.e.,
  - rank m_1, first class m_2 and class m_3 of X;
  - rank μ_1 and class μ_2 of D;
  - rank of T and C;
  - class of immersions of T and C in X;
  - class of immersions of T and C in D;
  - apparent characters of D: crosscaps, triple points and the double curve for the image of a generic projection $D \to \mathbb{P}^3$.

Roth [19] claimed that there are seven independent characters among them. However, it is not easy to follow his arguments because several intermediate characters are newly introduced and a number of relations among many those quantities are discussed together. Therefore, instead of tracking Roth's proofs, we show directly that there are seven independent characters corresponding to Chern monomials of M, and all other characters can be expressed by those ones. Put

$$f_*(1) = da, \quad f_*(c_1(TM)) = \xi_1 a^2, \quad \cdots, \quad f_*(c_3(TM)) = \xi_{001} a^4$$

in $H^*(\mathbb{P}^4; \mathbb{Q}) = \mathbb{Q}[a]/\langle a^5 \rangle$. The rank of C is written by its degree γ and $\chi_C := \int_C c_1(TC)$, so we use χ_C, instead.

Theorem 4.1. *Any numerical projective characters of a 3-fold in 4-space with ordinary singularities are expressed in terms of $d, \xi_1, \cdots, \xi_{001}$. In particular, all numerical characters are generated by $d, \mu_0, t, q, s_t, \gamma$ and χ_C.*

Theorem 4.1 follows from Examples 4.2, 4.3 and 4.4 below. We conjecture that the assertion would be a general phenomenon in any dimension.

Example 4.2. Seven quantities $d, \mu_0, t, q, s_t, \gamma, \chi_C$ are expressed in terms of $d, \xi_1, \cdots, \xi_{001}$, and vice versa, as in Table 3 and Table 4. In fact, they are immediately computed by Thom polynomials in Table 2 ($\kappa = 1$) and $tp^{SM}(\overline{A_1})$ as in (1) in §2.3 applied to our stable map $f : M^3 \to \mathbb{P}^4$, and conversely, the degree of Chern monomials of M are solved.

Example 4.3. We compute explicitly elementary characters of X (degree d, rank m_1, first class m_2 and class m_3) in a similar way as in §3.2. Let $m_0 = d$, the

$$\mu_0 = \frac{1}{2}\int f_*(tp(A_0^2)(f)) \cdot a^2 = \frac{1}{2}(-5d + d^2 + \xi_1),$$

$$t = \frac{1}{3}\int f_*(tp(A_0^3)(f)) \cdot a$$

$$= \frac{1}{3}(35d - \frac{15}{2}d^2 + \frac{1}{2}d^3 - \xi_{01} - 15\xi_1 + \frac{3}{2}d\xi_1 + 2\xi_2),$$

$$\gamma = \int f_*(tp(A_1)(f)) \cdot a = 10d - \xi_{01} - 5\xi_1 + \xi_2,$$

$$q = \frac{1}{4}\int f_*(tp(A_0^4)(f))$$

$$= \frac{1}{4}(-295d + \frac{355}{6}d^2 - 5d^3 + \frac{1}{6}d^4 + 2\xi_{001} + (25 - \frac{4}{3}d)\xi_{01}$$

$$+(200 - 25d + d^2)\xi_1 + \frac{1}{2}\xi_1^2 - 7\xi_{11} + (-55 + \frac{8}{3}d)\xi_2 + 6\xi_3),$$

$$s_t = \int f_*(tp(A_0A_1)(f))$$

$$= -120d + 10d^2 + 2\xi_{001} + (20 - d)\xi_{01} + (90 - 5d)\xi_1 - 6\xi_{11}$$

$$+(d - 30)\xi_2 + 4\xi_3,$$

$$\chi_C = \int_M c(TM) \cdot tp^{\mathrm{SM}}(\overline{A_1})(f)$$

$$= -60d + \xi_{001} + 10\xi_{01} + 55\xi_1 - 4\xi_{11} - 20\xi_2 + 3\xi_3.$$

<div align="center">Table 3   Thom polynomials applied to $f : M \to \mathbb{P}^4$</div>

degree of X.

• The *rank m_1 of X* is defined by the degree of the locus S_1 consisting of points at which tangent planes contain a given general point p. Let $g : M \to \mathbb{P}^3$ be the composition of the normalization map f with generic projection from p. Then g is stable and $S_1 = \overline{A_1(g)}$ is a smooth surface; denote by $i : S_1 \hookrightarrow M$ the inclusion. For $i_*(1) = tp(A_1)(g) = c_1(g)$, we have

$$m_1 = \int f_*(tp(A_1)(g)) = 4d - \xi_1 = d(d-1) - 2\mu_0.$$

This is called the Cayley formula in [19]. Furthermore, using $tp^{\mathrm{SM}}(A_1)(g) = c_1(g) - c_1(g)^2 + \cdots$ in (1) or using an embedded resolution of S_1 (for computing $c_1(TS_1)^2$), we can see that

$$i_*(c_1(TS_1)) = -c_1(g)^2 + c_1(g)c_1(TM),$$
$$i_*(c_2(TS_1)) = c_1(g)^3 - c_1(g)^2c_1(TM) + c_1(g)c_2(TM),$$
$$i_*(c_1(TS_1)^2) = c_1(g)^3 - 2c_1(g)^2c_1(TM) + c_1(g)c_1(TM)^2.$$

• The *first class m_2 of X* is defined by the degree of the locus S_2 consisting of points at which tangent planes contain a given general line L. Assume $p \in L$. Project the image singular surface $g(S_1) \subset \mathbb{P}^3$ from $pt = L \cap \mathbb{P}^3$, and denote the composed map by $h : S_1 \to \mathbb{P}^2$; we can assume that h is stable. Then

$$\xi_0 = \int f_*(1) \cdot a^3 = d,$$

$$\xi_1 = \int f_*(c_1(TM)) \cdot a^2 = 5d - d^2 + 2\mu_0,$$

$$\xi_2 = \int f_*(c_1(TM)^2) \cdot a$$
$$= 25d - 10d^2 + d^3 + (20 - 3d)\mu_0 + 3t - \gamma,$$

$$\xi_{01} = \int f_*(c_2(TM)) \cdot a$$
$$= 10d - 5d^2 + d^3 + (10 - 3d)\mu_0 + 3t - 2\gamma,$$

$$\xi_3 = \int_M c_1(TM)^3$$
$$= 125d - 75d^2 + 15d^3 - d^4 + (150 - 45d + 4d^2 - 2\mu_0)\mu_0$$
$$+ 4q - \tfrac{1}{2}s_t + (45 - 4d)t + (-10 + \tfrac{1}{2}d)\gamma - \chi_C,$$

$$\xi_{11} = \int_M c_1(TM)c_2(TM)$$
$$= 50d - 35d^2 + 10d^3 - d^4 + (70 - 30d + 4d^2 - 2\mu_0)\mu_0$$
$$+ 4q + (30 - 4d)t - 5\gamma - 2\chi_C,$$

$$\xi_{001} = \int_M c_3(TM)$$
$$= 10d - 10d^2 + 5d^3 - d^4 + (20 - 15d + 4d^2 - 2\mu_0)\mu_0 + 4q$$
$$+ \tfrac{3}{2}s_t + (15 - 4d)t + (10 - \tfrac{3}{2}d)\gamma - 4\chi_C.$$

Table 4   Chern numbers of M and other degrees.

$S(h) := \overline{A_1(h)}$ is a smooth curve in S_1; denote by $j : S(h) \hookrightarrow S_1$ the inclusion. By using $tp^{\mathrm{SM}}(A_1)(h)$, it is easy to see that

$$j_*(1) = c_1(h), \quad j_*(c_1(TS(h))) = c_1(h)c_1(TS_1) - c_1(h)^2 \quad \in H^*(S_1).$$

Notice that the critical locus $S(h)$ contains the cuspidal points of g; in fact,

$$S(h) = S_2 \sqcup \overline{A_2(g)} \quad \text{(disjoint)}.$$

Hence, the degree m_2 of S_2 is given by using $tp(A_1) = c_1$ and $tp(A_2) = c_1^2 + c_2$:

$$m_2 = \int f_* i_*(tp(A_1)(h)) - \int f_*(tp(A_2)(g))$$
$$= 6d - 3\xi_1 - \xi_{01}$$
$$= d(d-1)^2 + (4 - 3d)\mu_0 + 3t - 2\gamma.$$

• The *class m_3 of X* is defined by the number of points at which tangent planes contain a given general plane Π. Assume $L \subset \Pi$. Now the image plane curve $h(S(h)) \subset \mathbb{P}^2$ admits only cusps and nodes as its singularities. Project it from

$pt_2 = \Pi \cap \mathbb{P}^2$ to $\mathbb{P}^1$; denote the resulting map by $h' : S(h) \to \mathbb{P}^1$. Also denote the projection of the cuspidal locus of g by

$$h'' := h'|_{\overline{A_2(g)}} : \overline{A_2(g)} \to \mathbb{P}^1.$$

Then the number of critical points of h' is the sum of four quantities;

- $A = m_3 = \#$ polar points of $h(S_2)$ from pt_2,
- $B = \#$ cuspidal points of $h(S_2)$,
- $C = \#$ polar points of $h(A_2(g))$ from pt_2,
- $D = \#$ cuspidal points of $h(A_2(g)) = \#$ swallowtail $A_3(g)$ of g.

Also the number of critical points of h'' is just $C + D$, and the number of cusps of h is $B + D$. By using Thom polynomials

- $A + B + C + D = \int tp(A_1)(h') = \int c_1(h')$;
- $C + D = \int tp(A_1)(h'') = \int c_1(h'')$;
- $B + D = \int tp(A_2)(h) = \int c_1(h)^2 + c_2(h)$;
- $D = \int tp(A_3)(g) = \int c_1(g)^3 + 3c_1(g)c_2(g) + 2c_3(g)$.

Computing the degrees in the right hand sides of above equalities (i.e. computing Gysin images via i_*, j_* and f_*), we see that

$$
\begin{aligned}
m_3 &= 4d - \xi_{001} + 2\xi_{01} - 3\xi_1 \\
&= d(d-1)^3 + (-6 + 9d - 4d^2 + 2\mu_0)\mu_0 - 4q \\
&\quad - \tfrac{3}{2}s_t + (4d - 9)t + (\tfrac{3}{2}d - 14)\gamma + 4\chi_C
\end{aligned}
$$

Example 4.4. We explain how to express in terms of $d, \cdots, \xi_{001}$ numerical characters associated to D, e.g. rank μ_1 and class μ_2 (=rank and class of the image surface $\pi(D) \subset \mathbb{P}^3$ under a generic projection). Recall that the source double point locus $\Gamma (\subset M)$ is a surface with ordinary singularities. Take a resolution Γ' of Γ; the composed map $\varphi : \Gamma' \to M$ into the ambient 3-fold is a stable map so that

$$(3) \qquad \varphi(A_0^k(\varphi)) = A_0^{k+1}(f) \quad (k = 1, 2, 3), \quad \varphi(A_1(\varphi)) = A_0 A_1(f).$$

Let $\mathcal{R}$ be the $\mathbb{Q}$-subalgebra of $H^*(M; \mathbb{Q})$ generated by Chern classes $c_i(TM)$, the pushforward of all Chern monomials $f^* f_*(c_I(TM))$ and $\tilde{a} := f^* a$. Notice that the degree of the image via f_* of an element of $\mathcal{R}$ is a polynomial of $d, \xi_1, \cdots, \xi_{001}$. Obviously, $c_i(f)$ and $f^* s_I(f) = f^* f_*(c_I(f))$ belong to $\mathcal{R}$, thus universal polynomials tp and tp^{SM} applied to f are all in $\mathcal{R}$. It follows from (3) that

$$\tfrac{1}{k}\varphi_* tp(A_0^k)(\varphi) = tp(A_0^{k+1})(f), \quad \varphi_* tp(A_1)(\varphi) = tp(A_0 A_1)(f),$$

thus those classes are in $\mathcal{R}$. Recall that

$$\varphi_* tp(A_0)(\varphi) = \varphi_*(1) = s_0(\varphi),$$
$$\varphi_* tp(A_0^2)(\varphi) = s_0(\varphi)^2 - s_1(\varphi),$$
$$\varphi_* tp(A_0^3)(\varphi) = \tfrac{1}{2}(s_0(\varphi)^3 - 3s_0(\varphi)s_1(\varphi) + 2s_2(\varphi) + 2s_{01}(\varphi)),$$
$$\varphi_* tp(A_1)(\varphi) = s_{01}(\varphi),$$

and thus

$$s_0(\varphi) = \varphi_*(1),$$
$$s_1(\varphi) = c_1(M)\varphi_*(1) - \varphi_*(c_1(\Gamma')),$$
$$s_2(\varphi) = c_1(M)^2\varphi_*(1) - 2c_1(M)\varphi_*(c_1(\Gamma')) + \varphi_*(c_1(\Gamma')^2),$$
$$s_{01}(\varphi) = c_2(M)\varphi_*(1) - c_1(M)\varphi_*(c_1(\Gamma')) + \varphi_*(c_1(\Gamma')^2) - \varphi_*(c_2(\Gamma'))$$

are in $\mathcal{R}$. Hence

$$\varphi_*(1), \;\; \varphi_*(c_1(T\Gamma')), \;\; \varphi_*(c_1(T\Gamma')^2), \;\; \varphi_*(c_2(T\Gamma')) \in \mathcal{R}.$$

In particular, their degrees are written in terms of $d, \cdots, \xi_{001}$. To compute numerical characters associated to D, we take composed maps $\pi \circ f \circ \varphi$ with generic linear projections π from a point and a line. Notice that $\Gamma \to D$ is doubly ramified along smooth C. In a quite similar way as in the computations of the rank and the class of a surface in §3.2, μ_1 and μ_2 of D can be interpreted in terms of degrees of the critical loci of $\pi \circ f \circ \varphi$, its cuspidal locus and C. Then we see that they are expressed by the degree μ_0 of D, $\int c_1(T\Gamma')$ and $\int c_2(T\Gamma')$, thus by $d, \cdots, \xi_{001}$. Apparent characters of D (characters of singular loci of projections of D) are also computed in the same way.

Remark 4.5. As an analogy to Lemma 3.1, Chern numbers $\xi_3, \xi_{11}, \xi_{001}$ of the normalization M of a projective 3-fold X with ordinary singularities have been studied by Tsuboi [23, 24] in a completely different method using the excess intersection formula and Piene's formulae of polar classes. Together with seven characters $d, \mu_0, \cdots, \chi_C$, his result involves the intersection product of the canonical divisor of M with $S = \overline{A_1(f)}\,(\simeq C = f(S))$; the number is computed by our method as

$$K_M \cdot S = -c_1(TM) \cdot tp(A_1)$$
$$= -10\xi_1 + \xi_{11} + 5\xi_2 - \xi_3$$
$$= \tfrac{1}{2}s_t - \tfrac{1}{2}d\gamma - \chi_C$$

(also note that $\chi(C, \mathcal{O}_C) = \tfrac{1}{2}\chi_C$ for the smooth curve C of crosscap points). Then it turns out that Tsuboi's formulae in [24] and ours in Table 4 completely coincide. We also confirm the class m_3 of X computed in [24] (cf. Example 4.3 above).

Example 4.6. (Example 3.3 in [24]) Let $\iota : M = \mathbb{P}^3 \to \mathbb{P}^9$ be the quadratic Veronese embedding:

$$\iota[z_0; z_1; z_2; z_3] = [z_0^2; z_1^2; z_2^2; z_3^2; z_0 z_1; z_0 z_2; z_0 z_3; z_1 z_2; z_1 z_3; z_2 z_3].$$

Let X be the image of $\iota(M)$ via a generic projection to $\mathbb{P}^4$, and $f : M \to X \subset \mathbb{P}^4$ the obtained stable map (Mather [12]). Since $c(TM) = (1 + a)^4 \in H^*(M) = \mathbb{Q}[a]/\langle a^4 \rangle$ and the pullback of the hyperplane class is $2a$, thus $d = 8$, $\xi_1 = 16$, $\xi_2 = 32$, $\xi_3 = 64$, $\xi_{01} = 12$, $\xi_{11} = 24$, $\xi_{001} = 4$. By Table 3 and Example 4.3 above, we have

$$\mu_0 = 20, \ t = 20, \ \gamma = 20, \ q = 5, \ s_t = 40, \ \chi_C = -20, \ m_3 = 4.$$

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