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A geometric representation of the generalized mean curvature

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Abstract

A varifold is a generalization of a differential manifold using Radon measures. The theory of varifolds is a central topic in geometric measure theory. Any varifold possesses a notion similar to "the area", and the generalized mean curvature is defined through the first variation of "the area". If a varifold has C^2 regularity, then the generalized mean curvature coincides with the classical mean curvature. Furthermore, if the generalized mean curvature vector has some integrablity, then we obtain some regularity of the varifold. In this sense the generalized mean curvature contains information concerning its shape. However, it is not known that generalized mean curvature vector is represented without the first variation. In this paper, under the $C^{1,\alpha}$ regularity condition, for $\alpha > 1/3$, we give a geometric representation of the generalized mean curvature using a limit of integral averages suggested by the Menger curvature.

1. Introduction

The theory of varifolds is a branch of geometric measure theory. It is a generalized notion of differentiable submanifolds using Radon measures, and, for example, we know the monotonicity formula, the compactness theorem and the isoperimetric inequality for varifolds. Today, varifold theory is used for the study of minimal surfaces as well as mean curvature flows, and some regularity theorems are known.

A varifold V is a Radon measure on the product topological space of $\mathbb{R}^n$ and the Grassmannian $\mathrm{G}(n,k)$. We consider a differentiable submanifold M on $\mathbb{R}^n$ as a varifold $\mathrm{v}(M)$ defined by

$$\mathrm{v}(M)(f) = \int f(x, T_x M)\, d\mathscr{H}^k x, \tag{1.1}$$

for a continuous function f on $\mathbb{R}^n \times \mathrm{G}(n,k)$ with compact support. Here $\mathscr{H}^k$ is the k-dimensional Hausdorff measure. For a varifold V, we can define the first variation $\delta V(g)$ by

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$$\delta V(g) = \int Dg(x)\cdot S\, dV(x,S),$$

where g is a C^1 vector field. In particular, if the total variation measure of δV is absolutely continuous with respect to the area of V, then there exists a V measurable vector field $\mathrm{h}(V,\cdot)$ such that

$$\delta V(g) = -\int \mathrm{h}(V,x)\cdot g(x)\, dV(x,S) \tag{1.2}$$

for any C^1 vector field g. The vector field $\mathrm{h}(V,\cdot)$ is called the generalized mean curvature vector of V, and coincides with the classical mean curvature vector when V has C^2 regularity. In [1], Allard showed the regularity theorem which says that if the generalized mean curvature vector has the L^p integrability, then the varifold can be locally written by the graph of a $C^{1,1-k/p}$ function, where p is greater than dimension of the varifold. The monotonicity formula and the isoperimetric inequality mentioned above are represented by the first variation and the generalized mean curvature. A varifold is called *integral* if it is represented by a countable summation of (1.1) type varifolds, and we know that the generalized mean curvature vector of such an integral varifold is contained in the orthogonal space almost everywhere ([2]). Hence the generalized mean curvature reflects the geometric shape of varifolds in this sense. Other than (1.2), we are not aware of different representations of the generalized mean curvature. In this paper, we give a representation of the generalized mean curvature vector using a limit of integral averages of a discretization of the classical mean curvature vector. Roughly speaking, the assertion of our main theorem is as follows. If a varifold V is locally $C^{1,\alpha}$ with $\alpha > 1/3$, then the generalized mean curvature vector satisfies

$$\begin{aligned}&\frac{1}{k}\operatorname{Tan}^k(\|V\|,a)^\perp(\mathrm{h}(V,a))\\ &= \lim_{R\downarrow 0}\frac{2}{\|V\|\mathrm{B}^n(a,R)}\int_{\mathrm{B}^n(a,R)}\frac{\operatorname{Tan}^k(\|V\|,a)^\perp(x-a)}{|x-a|_n^2}\, d\|V\|x,\end{aligned}$$

where $|\cdot|_n$ is the Euclidean norm of $\mathbb{R}^n$, $\mathrm{B}^n(a,r)$ is the n-dimensional closed ball with radius r and center a. In paticular, if $V = \mathrm{v}(M)$, then we have

$$\frac{1}{k}\mathrm{h}(\mathrm{v}(M),a) = \lim_{R\downarrow 0}\frac{2}{\omega_k R^k}\int_{\mathrm{B}^n(a,R)\cap M}\frac{T_aM^\perp(x-a)}{|x-a|_n^2}\, d\mathscr{H}^k x, \tag{1.3}$$

where $\omega_k = \mathscr{H}^k(\mathrm{B}^k(0,1))$. The norm of the integrand of (1.3) is the inverse of the radius of the k-dimensional sphere which is tangent to T_aM at a and passes through x. That is, it is just the Menger curvature. For details, see section 4. Hence, we expect to obtain the quantity of mean curvature by some limiting procedure. The main theorem realizes it by use of the limit of integral averages. The author hopes that our theorem contributes to understanding and development of [1], [6] and related works in this field.

In section 2, we prepare definitions and theorems for the proof of the main theorem. In section 3, we state the main theorem (Theorem 3.1) and prove it. In section 4, we explain a vector-valued version of the inverse of a tangent-point radius and provide a geometric meaning of the main theorem, as well as some examples.

2. Preliminaries

We refer the reader to [1], [2], [3], [4] and [5] for facts given in this section. Throughout this section, k and n are always integers satisfying $2 \le k \le n$. For r, $s \ge 0$, we use the notation $r \lesssim s$ when there exists $C > 0$ independent of r and s such that $r \le Cs$. Let $\mathrm{G}(n,k)$ be the space of k-dimensional subspaces of $\mathbb{R}^n$. If $S \in \mathrm{G}(n,k)$, we also use "S" to denote the orthogonal projection from $\mathbb{R}^n$ onto S. Let $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$ be the space of linear mappings from $\mathbb{R}^n$ to itself, and the inner product on $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$ is defined by

$$A \cdot B = \mathrm{Tr}\,(A^* \circ B)$$

for A, $B \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$. Whenever U is topological space, let $\mathscr{K}(U)$ be the space of continuous functions on U with compact support. Let $\mathscr{X}(\mathbb{R}^n)$ be the vector space of smooth mappings $g : \mathbb{R}^n \to \mathbb{R}^n$ with compact support. For $m \in \mathbb{N}$, let $|\cdot|_m$ be the Euclidean norm of $\mathbb{R}^m$. Whenever $a \in \mathbb{R}^m$, $r > 0$, let

$$\mathrm{B}^m(a,r) = \{\, x \in \mathbb{R}^m \,:\, |x-a|_m \le r \,\}, \ \mathrm{U}^m(a,r) = \{\, x \in \mathbb{R}^m \,:\, |x-a|_m < r \,\}.$$

Whenever $S \subset \mathbb{R}^n$, $a \in \bar{S}$, let

$$\mathrm{Tan}\,(S,a) = \bigcap_{\varepsilon>0} \bigcup_{x \in S} \bigcup_{r>0} \{\, v \in \mathbb{R}^n \,:\, |x-a|_n < \varepsilon,\ |r(x-a) - v|_n < \varepsilon \,\}.$$

Suppose V and W are finite-dimensional linear spaces, with $\dim V = m$, $\dim W < \infty$. Let $\Lambda_k V$ be the k-th exterior power of V. The inner product on $\Lambda_k V$ is induced from V when V has an inner product as follows. For $u_1 \wedge \cdots \wedge u_k$, $v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$, we define the inner product between them by

$$(u_1 \wedge \cdots \wedge u_k) \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{\sigma \in \mathrm{S}_k} \mathrm{sgn}\,\sigma \prod_{i=1}^{k} u_i \cdot v_{\sigma(i)},$$

where S_k is the symmetric group of degree k. For $f \in \mathrm{Hom}\,(V,W)$, we define $\Lambda_k f \in \mathrm{Hom}\,(\Lambda_k V, \Lambda_k W)$ by

$$\Lambda_k f(u_1 \wedge \cdots \wedge u_k) = f(u_1) \wedge \cdots \wedge f(u_k),$$

when $k \geq 2$, and $\Lambda_1 f = f$. Whenever μ is a Radon measure on U and A, $B \subset U$, let

$$(\mu \llcorner A)(B) = \mu(A \cap B).$$

Let $\mathscr{H}^k$ be the k-dimensional Hausdorff measure on $\mathbb{R}^n$ and $\omega_k = \mathscr{H}^k(\mathrm{B}^k(0,1))$. If $U \subset \mathbb{R}^n$ and $a \in U$, let

$$\Theta^k(\mu, a) = \lim_{r \downarrow 0} \frac{\mu(\mathrm{B}^n(a,r))}{\omega_k r^k}$$

and

$$\mathrm{Tan}^k(\mu, a) = \bigcap \left\{ \mathrm{Tan}(S, a) \, : \, S \subset U, \ \Theta^k(\mu \llcorner U \setminus S, a) = 0 \right\}.$$

Next, we give some definitions and properties of varifolds; we refer the reader to [1] for further details.

Definition 2.1 (see [1, 3.1 and 3.5]) *We say that V is a k-dimensional varifold in $\mathbb{R}^n$ if V is a Radon measure on $\mathbb{R}^n \times \mathrm{G}(n,k)$. Let $\mathrm{V}_k(\mathbb{R}^n)$ be the weakly topologized space of k-dimensional varifolds in $\mathbb{R}^n$. Whenever $V \in \mathrm{V}_k(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$, let $\|V\|(A) = V(A \times \mathrm{G}(n,k))$. Whenever M is a k-dimensional C^1 submanifold of $\mathbb{R}^n$, let $\mathrm{v}(M) \in \mathrm{V}_k(\mathbb{R}^n)$ be defined by*

$$\mathrm{v}(M)(f) = \int f(x, T_x M) \, d(\mathscr{H}^k \llcorner M)x$$

for $f \in \mathscr{K}(\mathbb{R}^n \times \mathrm{G}(n,k))$. We say that V is a k-dimensional rectifiable varifold in $\mathbb{R}^n$ if there exist a positive real number sequence $(a_l)_{l=1}^{\infty}$ and k-dimensional C^1 submanifolds $(M_l)_{l=1}^{\infty}$ in $\mathbb{R}^n$ such that

$$V = \sum_{l=1}^{\infty} a_l \mathrm{v}(M_l).$$

If the a_l may be taken to be positive integers, we say that V is an integral varifold in $\mathbb{R}^n$. Let $\mathrm{RV}_k(\mathbb{R}^n)$ and $\mathrm{IV}_k(\mathbb{R}^n)$ be the spaces of k-dimensional rectifiable varifolds in $\mathbb{R}^n$ and k-dimensional integral varifolds in $\mathbb{R}^n$, respectively.

Definition 2.2 (see [1, 3.2]) *Suppose that k, l and m are integers with $0 < k \leq l$, m. Let $V \in \mathrm{V}_k(\mathbb{R}^l)$ and let $F : \mathbb{R}^l \to \mathbb{R}^m$ be continuously differentiable. Then $F_\sharp V \in \mathrm{V}_k(\mathbb{R}^m)$ is defined by*

$$F_\sharp V(f) = \int f(F(x), DF(x)(S)) |\Lambda_k DF(x) \circ S| \, dV(x,S)$$

for $f \in \mathscr{K}(\mathbb{R}^m \times \mathrm{G}(m,k))$.

Definition 2.3 (see [1, 4.1]) *Suppose $V \in \mathrm{V}_k(\mathbb{R}^n)$. We define a linear functional $\delta V : \mathscr{X}(\mathbb{R}^n) \to \mathbb{R}$, called the* first variation *of V, by*

$$\delta V(g) = \int Dg(x) \cdot S \, dV(x, S)$$

for $g \in \mathscr{X}(\mathbb{R}^n)$. The total variation of V is defined by

$$\|\delta V\|(G) = \sup \{ \, \delta V(g) \, : \, g \in \mathscr{X}(\mathbb{R}^n), \ \operatorname{spt} g \subset G, \ |g|_n \leq 1 \, \}$$

whenever $G \subset \mathbb{R}^n$ is an open subset. For $A \subset \mathbb{R}^n$, not necessarily an open set, we define $\|\delta V\|(A)$ by

$$\|\delta V\|(A) = \inf \{ \, \|\delta V\|(G) \, : \, A \subset G \textit{ is open } \} .$$

Theorem 2.4 (see [1, 4.2]) *Suppose $V \in \mathrm{V}_k(\mathbb{R}^n)$ and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Then there exists a locally $\|V\|$ summable vector field $\mathrm{h}(V, \cdot)$ such that*

$$\delta V(g) = -\int \mathrm{h}(V, x) \cdot g(x) \, d\|V\| x \tag{2.1}$$

for $g \in \mathscr{X}(\mathbb{R}^n)$.

Definition 2.5 *We call $\mathrm{h}(V, \cdot)$ in Theorem 2.4 the* generalized mean curvature vector.

Proposition 2.6 is the first variation formula of a varifold restricted to a ball.

Proposition 2.6 (see [1, 4.10(1)]) *Suppose $V \in \mathrm{V}_k(\mathbb{R}^n)$, $g \in \mathscr{X}(\mathbb{R}^n)$, $R > 0$. Then it holds that*

$$\begin{aligned} \delta V(\chi_{\mathrm{B}^n(0,R)} g) &= \delta(V \llcorner \mathrm{B}^n(0, R) \times \mathrm{G}(n, k))(g) \\ &\quad - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R) \times \mathrm{G}(n,k)} \frac{S(g(x)) \cdot x}{|x|_n} \, dV(x, S). \end{aligned} \tag{2.2}$$

Proposition 2.7 is concerned with the direction of the generalized mean curvature vector.

Proposition 2.7 (see [2, 5.8]) *Suppose that $V \in \mathrm{IV}_k(\mathbb{R}^n)$ and that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Then*

$$S^{\perp}(\mathrm{h}(V, x)) = \mathrm{h}(V, x) \tag{2.3}$$

holds for V a.e. $(x, S) \in \mathbb{R}^n \times \mathrm{G}(n, k)$.

Proposition 2.8 is an elementary but important fact; use this result in the proof of Lemma 3.3 below.

Proposition 2.8 (see [3, 3.4.4]) *Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is a $\mathscr{H}^k$-summable function. Then we have*

$$\int f(x)\, d\mathscr{H}^k x = \int_{\partial \mathrm{B}^k(0,1)} \int_0^\infty f(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega. \tag{2.4}$$

3. The main theorem and its proof

Throughout this section, k and n are integers satisfying $2 \le k < n$ and α is a real number satisfying

$$\alpha > 1/3. \tag{3.1}$$

In this section, we prove the following main theorem in this paper.

Theorem 3.1 *Suppose that $V \in \mathrm{RV}_k(\mathbb{R}^n)$, $T \in \mathrm{G}(n,k)$, $a \in T$. Let $f : T \to T^\perp$ be a continuous differentiable map, and let $F : T \to \mathbb{R}^n$. Assume that there exists $C_1 > 0$ and $\delta > 0$ such that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ on $\mathrm{U}^n(a,\delta)$, and*

$$\mathrm{Im} F \cap \mathrm{U}^n(a,\delta) = \mathrm{spt}\, \|V\| \cap \mathrm{U}^n(a,\delta), \tag{3.2}$$

$$T(F(x)) = x,\ \ T^\perp(F(x)) = f(x),\ \ \|\nabla f(x) - \nabla f(a)\| \le C_1 |x-a|_n^\alpha \tag{3.3}$$

for $x \in T \cap \mathrm{U}^n(a,\delta)$. Let $a \in T$ be a Lebesgue point of $\mathrm{h}(V,\cdot)$; that is,

$$\mathrm{h}(V,a) = \lim_{r \downarrow 0} \frac{1}{\|V\|\mathrm{B}^n(a,r)} \int_{\mathrm{B}^n(a,r)} \mathrm{h}(V,x)\, d\|V\|x, \tag{3.4}$$

and assume that $0 < \Theta^k(\|V\|,a) < \infty$. For some $R_0 > 0$, we assume that

$$\left|\Theta^k(\|V\|,x) - \Theta^k(\|V\|,a)\right| \le C_1 R^{1-\alpha} \tag{3.5}$$

for any $0 < R < R_0$ and $\|V\|$ almost every $x \in \mathrm{U}^n(a,R)$. Then the normal component of $\mathrm{h}(V,a)$ is given by

$$\begin{aligned}&\frac{1}{k}\mathrm{h}(V,a)\cdot v \\ &= \lim_{R\downarrow 0} \frac{2}{\|V\|\mathrm{B}^n(a,R)} \int_{\mathrm{B}^n(a,R)} \frac{\mathrm{Tan}^k(\|V\|,a)^\perp(x-a)\cdot v}{|x-a|_n^2}\, d\|V\|x\end{aligned} \tag{3.6}$$

for $v \in \mathrm{Tan}^k(\|V\|,a)^\perp$.

When f is in the class of C^2 and $T = \operatorname{Tan}^k(\|V\|, a)$, it is known that

$$\frac{1}{k}\mathrm{h}(V,a)\cdot v = \lim_{R\downarrow 0} \frac{2}{\mathscr{H}^k(T\cap \mathrm{B}^n(b,R))}\int_{T\cap\mathrm{B}^n(b,R)} \frac{(F(y)-F(b))\cdot v}{|y-b|_k^2}\, d\mathscr{H}^k y, \quad (3.7)$$

where $b = T(a)$. (3.6) implies (3.7) and vice versa if $f \in C^2$. In this sense (3.6) is a generalization of (3.7) under less reguralty condition.

We prepare a few lemmas which we need for the proof of main theorem. By use of $f(\cdot) - f(a)$, if necessary, we may assume $f(a) = 0$. Moreover, by translation, we may assume

$$a = 0. \quad (3.8)$$

The map f is defined on T, but not necessary on $\mathrm{Im}DF(0)$. However, we regard the map f as a map defined on $\mathrm{Im}DF(0)$ provided δ sufficiently small. Next lemma is a rigorous statement of this fact.

Lemma 3.2 *Let δ, f and F satisfy (3.2) and (3.3) with $a = 0$. Then there exist $\tilde\delta > 0$, $C_2 > 0$, continuous differentiable maps $\tilde f : \mathrm{Im}DF(0) \to \mathrm{Im}DF(0)^\perp$, and $\tilde F : \mathrm{Im}DF(0) \to \mathbb{R}^n$ such that*

$$\mathrm{Im}\tilde F \cap \mathrm{U}^n(0,\tilde\delta) = \operatorname{spt}\|V\| \cap \mathrm{U}^n(0,\tilde\delta) \quad (3.9)$$

and

$$\begin{gathered}\mathrm{Im}DF(0)(\tilde F(x)) = x,\ \mathrm{Im}DF(0)^\perp(\tilde F(x)) = \tilde f(x),\ |\tilde f(0)| = \|\nabla \tilde f(0)\| = 0,\\ |\tilde f(x)|_n \le C_2|x|_n^{1+\alpha},\ \|\nabla\tilde f(x) - \nabla\tilde f(y)\| \le C_2|x-y|_n^\alpha.\end{gathered} \quad (3.10)$$

Proof. Let $S = \mathrm{Im}DF(0)$. Take $0 < \tilde\delta < \delta$. Let $\bar x$, $\bar y \in \operatorname{spt}\|V\| \cap \mathrm{U}^n(0,\tilde\delta)$ with $\bar x \ne \bar y$. By (3.9), there exist x, $y \in T \cap \mathrm{U}^n(0,\tilde\delta)$ uniquely such that $x \ne y$, and

$$F(x) = \bar x,\ F(y) = \bar y. \quad (3.11)$$

Let $X = (x, x\cdot\nabla f(0))$ and $Y = (y, y\cdot\nabla f(0))$. Then

$$|S(\bar x) - S(\bar y)|_n \ge |X-Y|_n - |(X-Y) - (S(\bar x) - S(\bar y))|_n. \quad (3.12)$$

Since $X - Y \in S$, and since $|\bar x - \bar y - (S(\bar x) - S(\bar y))|_n = \operatorname{dist}(\bar x - \bar y, S)$, we have

$$\begin{aligned}&|(X-Y) - (S(\bar x) - S(\bar y))|_n\\ =\ &|(X-Y) - (\bar x - \bar y) + (\bar x - \bar y) - (S(\bar x) - S(\bar y))|_n\\ \le\ &2|(X-Y) - (\bar x - \bar y)|_n\\ =\ &2|(x-y, (x-y)\cdot\nabla f(0)) - (x-y, f(x)-f(y))|_n\\ =\ &2|(x-y)\cdot\nabla f(0) - (f(x)-f(y))|_{n-k}\end{aligned}$$

$$
\begin{aligned}
&= 2\left|\int_0^1 (\nabla f(y+t(x-y)) - \nabla f(0))\cdot (x-y)\,dt\right|_{n-k} \\
&\leq 2C_1\tilde{\delta}^\alpha |x-y|_k.
\end{aligned} \tag{3.13}
$$

Substituting (3.13) into (3.12), we have

$$
|S(\bar{x}) - S(\bar{y})|_n \geq (1 - 2C_1\tilde{\delta}^\alpha)|x-y|_k. \tag{3.14}
$$

Hence, we obtain $|S(\bar{x}) - S(\bar{y})|_n > 0$ for sufficiently small $\tilde{\delta}$. Then, whenever $\bar{x} \in \operatorname{spt}\|V\| \cap \mathrm{U}^n(0,\tilde{\delta})$ there exists uniquely $\tilde{x} \in S \cap \mathrm{U}^n(0,\tilde{\delta})$ such that $F(\tilde{x}) = \bar{x}$. Using this correspondence, we define $\tilde{F}$ by $\tilde{F}(\tilde{x}) = \bar{x}$ and observe that, by definition, $\tilde{F}$ satisfies (3.9). Furthermore $\tilde{F}$ is continuously differentiable by the continuous differentiablity of F. Using [1, 8.9(5)], (3.9) and (3.14), we have

$$
\begin{aligned}
&\left(1 - \|\mathrm{Im}D\tilde{F}(\tilde{x}) - \mathrm{Im}D\tilde{F}(\tilde{y})\|^2\right) \|D\tilde{f}(\tilde{x}) - D\tilde{f}(\tilde{y})\|^2 \\
&\leq \|\mathrm{Im}D\tilde{F}(\tilde{x}) - \mathrm{Im}D\tilde{F}(\tilde{y})\|^2 \\
&= \|\mathrm{Im}DF(x) - \mathrm{Im}DF(y)\|^2 \\
&= \|Df(x) - Df(y)\|^2 \\
&\leq C_1^2 |x-y|_k^{2\alpha} \\
&\leq C_1^2\left(1 - 2C_1\tilde{\delta}^\alpha\right)^{-2\alpha} |\tilde{x} - \tilde{y}|_n^{2\alpha}
\end{aligned}
$$

for $\tilde{x}$ and $\tilde{y} \in S \cap \mathrm{U}^n(0,\tilde{\delta})$. Consequently, if $\tilde{\delta}$ is sufficiently small, then we have (3.10). □

By Lemma 3.2, we may assume (3.9) and (3.10) with

$$
\tilde{f} = F,\ \tilde{F} = F,\ \mathrm{Im}DF(0) = \mathbb{R}^k \times \{0\}. \tag{3.15}
$$

In what follows, we always assume (3.8) and (3.15). Furthermore, we identify $\mathbb{R}^k \times \{0\}$ with $\mathbb{R}^k$.

Lemma 3.3 *Let f and F be functions as in Theorem 3.1. Then there exist $R_0 > 0$ and $C_3 > 0$ satisfying the following properties. Whenever $R \in (0, R_0)$ and $\omega \in \partial\mathrm{B}^k(0,1)$, there exists $r(R,\omega) \in (0, R]$ such that*

$$
\begin{aligned}
r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2 = R^2,\ R \leq C_3 r(R,\omega), \\
2(r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega) \geq r(R,\omega).
\end{aligned} \tag{3.16}
$$

The function $r(R,\omega)$ is continuously differentiable with respect to R, and satisfies

$$
\frac{\partial}{\partial R} r(R,\omega) = \frac{R}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}. \tag{3.17}
$$

Furthemore for any continuous function $G:\mathbb{R}^k\to\mathbb{R}$, we have

$$\int_{F^{-1}(\mathrm{B}^n(0,R))} G(x)\,d\mathscr{H}^k x=\int_{\partial\mathrm{B}^k(0,1)}\int_0^{r(R,\omega)} G(r\omega)r^{k-1}\,dr\,d\mathscr{H}^{k-1}\omega, \tag{3.18}$$

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon))\setminus F^{-1}(\mathrm{B}^n(0,R))} G(x)\,d\mathscr{H}^k x\\ =&\int_{\partial\mathrm{B}^k(0,1)} G(r(R,\omega)\omega)\frac{r(R,\omega)^{k-1}R}{r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\omega}\,d\mathscr{H}^{k-1}\omega.\end{aligned} \tag{3.19}$$

Proof. By (3.10),

$$\frac{\partial}{\partial r}\left(r^2+|f(r\omega)|^2_{n-k}\right)=2\left(r+f(r\omega)\cdot\nabla f(r\omega)\cdot\omega\right)\geq 2r\left(1-C_2^2r^{2\alpha}\right).$$

By (3.1), there exists $R_0>0$ such that

$$\frac{\partial}{\partial r}\left(r^2+|f(r\omega)|^2_{n-k}\right)>0 \tag{3.20}$$

for $r\in(0,R_0)$. Note that R_0 is independent of ω. Hence, for $R\in(0,R_0)$ and $\omega\in\partial\mathrm{B}(0,1)$, there exists $r(R,\omega)\in(0,R]$ uniquely such that

$$r(R,\omega)^2+|f(r(R,\omega)\omega)|^2_{n-k}=R^2. \tag{3.21}$$

By the implicit function theorem, we find that $r(R,\omega)$ is continuously differentiable with respect to R. Hence, there exists $C_3>0$ such that

$$\begin{aligned}R^2=r(R,\omega)^2+|f(r(R,\omega)\omega)|^2_{n-k}&\leq r(R,\omega)^2\left(1+C_2^2r(R,\omega)^{2\alpha}\right)\\&\leq r(R,\omega)^2\left(1+C_2^2R_0^{2\alpha}\right)\\&\leq C_3^2r(R,\omega)^2.\end{aligned}$$

Since $r(R,\omega)\leq R$, for sufficiently small $R_0>0$, we have $1-C_2^2r(R,\omega)^{2\alpha}>r(R,\omega)/2$ whenever $0<R<R_0$, and

$$\begin{aligned}r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\omega&\geq r(R,\omega)\left(1-C_2^2r(R,\omega)^{2\alpha}\right)\\&\geq r(R,\omega)/2.\end{aligned}$$

Consequently, (3.16) holds. (3.17) is obtained by differentiating (3.21) with respect to R. By Proposition 2.8, we have

$$\begin{aligned}&\int_{F^{-1}(\mathrm{B}^n(0,R))} G(x)\,d\mathscr{H}^k x\\ =&\int_{\partial\mathrm{B}^k(0,1)}\int_0^\infty \chi_{\left\{x\,:\,|x|_k^2+|f(x)|^2_{n-k}\leq R^2\right\}}(r\omega)G(r\omega)r^{k-1}\,dr\,d\mathscr{H}^{k-1}\omega.\end{aligned} \tag{3.22}$$

Hence, we have (3.18) by (3.21) and (3.22). (3.19) is proved similarly to (3.18). □

Next we prove Theorem 3.1.

Proof. We assume that $f(a) = 0$, $\nabla f(a) = 0$, $a = 0$ and $T = \mathbb{R}^k \times \{0\} \in \mathrm{G}(n,k)$. Let $(e_i)_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$, with $(e_i)_{i=1}^k$ being an orthonormal basis of $\mathbb{R}^k \times \{0\}$. Since

$$\frac{1}{\|V\|\mathrm{B}^n(0,R)} = \frac{R^k}{\|V\|\mathrm{B}^n(0,R)} \cdot \frac{1}{R^k}$$

by (3.4), it is enough to show

$$\lim_{R\downarrow 0} \frac{1}{R^k} \left\{ 2\int_{\mathrm{B}^n(0,R)} \frac{T^\perp(x)\cdot e_l}{|x|_n^2}\, d\|V\|x - \frac{1}{k}\int_{\mathrm{B}^n(0,R)} \mathrm{h}(V,x)\cdot e_l\, d\|V\|x \right\}$$
$$= 0 \tag{3.23}$$

for any natural number l with $k+1 \le l \le n$. By (3.10), there exists $J : T \to \mathbb{R}$ such that

$$|\Lambda_k DF(y)| = \sqrt{1+J(y)},\ \ J(y) \lesssim \|\nabla f(y)\|^2. \tag{3.24}$$

For simplicity, we write $\Theta(x) = \Theta^k(\|V\|, x)$. By (3.24), (2.1), Proposition 2.6 and (3.9), we have

$$\begin{aligned}
&\int_{\mathrm{B}^n(0,R)} \mathrm{h}(V,x)\cdot e_l\, d\|V\|x \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R)\times \mathrm{G}(n,k)} \frac{S(e_l)\cdot x}{|x|_n}\, dV(x,S) \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{g^{ij}(y)(e_l\cdot \partial_i F(y))(\partial_j F(y)\cdot F(y))}{|F(y)|_n} \\
&\qquad\qquad \times \sqrt{1+J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{g^{ij}\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \\
&\qquad\qquad \sqrt{1+J(y)}\Theta(F(y))\, d\mathscr{H}^k y,
\end{aligned} \tag{3.25}$$

where $f_l = f\cdot e_l$, g^{ij} is the (i,j) element of the inverse matrix of $(\partial_i F\cdot \partial_j F)_{i,j=1}^k$, and we sum i, j over repeated indices from 1 to k. Since $\partial_i F\cdot \partial_j F = \delta_{ij} + \partial_i f\cdot \partial_j f$, and since $\|(\partial_i f\cdot \partial_j f)\| \lesssim \|\nabla f\|^2$, we have

$$\|(g^{ij}) - (\delta^{ij})\| \leq \frac{\|\nabla f\|^2}{1 - \|\nabla f\|^2} \lesssim \|\nabla f\|^2 \tag{3.26}$$

provided $\|\nabla f\| < 1$. Using (3.26), (3.24) and (3.10), we have

$$\begin{aligned}
&\left| \frac{(g^{ij} - \delta^{ij})\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \sqrt{1 + J(y)} \right| \\
&\lesssim \|\nabla f(y)\|^3 \frac{|y|_k + \|\nabla f(y)\| |f(y)|}{|y|_k} \\
&\lesssim |y|_k^{3\alpha}(1 + |y|_k^{2\alpha}). \qquad (3.27)
\end{aligned}$$

By (3.19),

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R))} |y|_k^{3\alpha}(1 + |y|_k^{2\alpha})\, d\mathscr{H}^k y \\
= &\int_{\partial \mathrm{B}^k(0,1)} \frac{|r(R,\omega)\omega|_k^{3\alpha}(1 + |r(R,\omega)\omega|_k^{2\alpha}) r(R,\omega)^{k-1} R}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega, \qquad (3.28)
\end{aligned}$$

and using (3.16) for (3.28), we have

$$(3.28) \lesssim R^{3\alpha - 1 + k}. \tag{3.29}$$

Note that $3\alpha - 1 > 0$ when (3.1). Consequently we have

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R))} \frac{(g^{ij} - \delta^{ij})\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \\
&\qquad \times \sqrt{1 + J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
= &\ o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.30)
\end{aligned}$$

Hence, we use (3.30), then (3.25) can be written as

$$\begin{aligned}
(3.25) = &\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R))} \frac{\delta^{ij}\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \\
&\qquad \times \sqrt{1 + J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
&+ \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R))} \frac{(g^{ij} - \delta^{ij})\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \\
&\qquad \times \sqrt{1 + J(y)}\Theta(F(y))\, d\mathscr{H}^k y
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{(\nabla f_l(y)\cdot y + \nabla f_l(y)\cdot \nabla(f^2)(y)/2)}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}} \\
&\qquad\qquad \times \sqrt{1+J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
&\quad + o(R^k) \quad \text{as } R\downarrow 0.
\end{aligned} \tag{3.31}
$$

By (3.10), we have

$$
\begin{aligned}
&\left|\frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\right| \\
&= \left|\sum_{i=1}^k \frac{(\partial_i f_l(y)-\partial_i f_l(0))(\partial_i f(y)-\partial_i f(0))\cdot (f(y)-f(0))}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\right| \\
&\lesssim \frac{\|\nabla f(y)-\nabla f(0)\|^2 |f(y)-f(0)|_{n-k}}{|y|_k} \\
&\lesssim |y|_k^{3\alpha}.
\end{aligned} \tag{3.32}
$$

We use (3.19) for (3.32), we have

$$
\begin{aligned}
&\left|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\, d\mathscr{H}^k y\right| \\
&\lesssim \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} |y|_k^{3\alpha} d\mathscr{H}^k y \\
&= \int_{\partial \mathrm{B}^k(0,1)} \frac{|r(R,\omega)\omega|_k^{3\alpha} r(R,\omega)^{k-1}R}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega.
\end{aligned} \tag{3.33}
$$

Using (3.16) for (3.33), we have

$$
(3.33) \lesssim R^{3\alpha-1+k}, \tag{3.34}
$$

and noting that $3\alpha-1>0$ when (3.1), we have

$$
\begin{aligned}
(3.31) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{\nabla f_l(y)\cdot y\sqrt{1+J(y)}}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\Theta(F(y))\, d\mathscr{H}^k y \\
&\quad + \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\, d\mathscr{H}^k y \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{\nabla f_l(y)\cdot y\sqrt{1+J(y)}}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\Theta(F(y))\, d\mathscr{H}^k y
\end{aligned}
$$

$$+ o(R^k) \quad \text{as } R \downarrow 0. \tag{3.35}$$

Using (3.19), we have

$$\begin{aligned}(3.35) = \int_{\partial \mathrm{B}^k(0,1)} & \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R\Theta(F(r(R,\omega)\omega))}{(r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \\ & \times \frac{r(R,\omega)^k\sqrt{1+J(r(R,\omega)\omega)}}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}}\, d\mathscr{H}^{k-1}\omega \\ & + o(R^k) \quad \text{as } R \downarrow 0. \end{aligned} \tag{3.36}$$

Since

$$\begin{aligned} & \frac{r(R,\omega)^k\sqrt{1+J(r(R,\omega)\omega)}}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}} \\ = & \frac{r(R,\omega)^k(\sqrt{1+J(r(R,\omega)\omega)}-1)}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}} \\ & + r(R,\omega)^k\left(\frac{1}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}} - \frac{1}{r(R,\omega)}\right) \\ & + \left(r(R,\omega)^{k-1} - R^{k-1}\right) + R^{k-1}, \end{aligned} \tag{3.37}$$

using (3.10), (3.16) and the fundamental theorem of calculus, we have

$$\begin{aligned} \left|\sqrt{1+J(r(R,\omega)\omega)}-1\right| &\lesssim \|\nabla f(r(R,\omega)\omega)\|^2 \lesssim R^{2\alpha}, \\ \left|\frac{1}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}} - \frac{1}{r(R,\omega)}\right| &\lesssim R^{2\alpha-1}, \\ \left|r(R,\omega)^{k-1} - R^{k-1}\right| &\lesssim R^{k-1+2\alpha}. \end{aligned} \tag{3.38}$$

Substituting (3.37), (3.38) and (3.1) into (3.36), we have

$$\begin{aligned}(3.36) &= \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k\Theta(F(r(R,\omega)\omega))}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega + o(R^k) \\ &= \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k(\Theta(F(r(R,\omega)\omega)) - \Theta(0))}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega \\ &\quad + \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k\Theta(0)}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega \\ &\quad + o(R^k) \quad \text{as } R \downarrow 0. \end{aligned} \tag{3.39}$$

We divide both sides of (3.39) by R^k, and take the limit as $R \downarrow 0$. It follows from

(3.4), (3.5) and (3.16) that

$$\lim_{R\downarrow 0}\int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega\Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega \in \mathbb{R}. \tag{3.40}$$

By L'Hospital's rule and (3.17), we have

$$\begin{aligned}
&\lim_{R\downarrow 0}\frac{2}{R^2}\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&= \lim_{R\downarrow 0}\frac{1}{R}\int_{\partial \mathrm{B}^k(0,1)} \nabla f_l(r(R,\omega)\omega)\cdot \omega\Theta(0)\frac{\partial r(R,\omega)}{\partial R}\, d\mathscr{H}^{k-1}\omega \\
&= (3.40).
\end{aligned} \tag{3.41}$$

On the other hand, using (3.1), (3.18), (3.38) and (3.5), we have

$$\begin{aligned}
&\int_{\mathrm{B}^n(0,R)} \frac{T^{\perp}(x)\cdot e_l}{|x|_n^2}\, d\|V\|x \\
&= \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2+|f(y)|_{n-k}^2}\sqrt{1+J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
&= \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2}\Theta(0)\, d\mathscr{H}^k y \\
&\quad + \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2}(\Theta(F(y))-\Theta(0))\, d\mathscr{H}^k y \\
&\quad + \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2+|f(y)|_{n-k}^2}\left(\sqrt{1+J(y)}-1\right)\Theta(F(y))\, d\mathscr{H}^k y \\
&\quad + \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y)\left(\frac{1}{|y|_k^2+|f(y)|_{n-k}^2}-\frac{1}{|y|_k^2}\right)\Theta(F(y))\, d\mathscr{H}^k y \\
&= \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2}\Theta(0)\, d\mathscr{H}^k y + o(R^k) \\
&= \int_{\partial \mathrm{B}^k(0,1)}\int_0^{r(R,\omega)} f_l(r\omega)r^{k-3}\Theta(0)\, dr\, d\mathscr{H}^{k-1}\omega \\
&\quad + o(R^k) \quad \text{as } R\downarrow 0.
\end{aligned} \tag{3.42}$$

The derivative of the first term of (3.42) with respect to R is

$$\int_{\partial \mathrm{B}^k(0,1)} \frac{f_l(r(R,\omega)\omega)r(R,\omega)^{k-3}R\Theta(0)}{r(R,\omega)+f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega. \tag{3.43}$$

Similarly, we have

$$(3.43) = R^{k-3}\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\, d\mathscr{H}^{k-1}\omega$$

$$
\begin{aligned}
&+\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega) r(R,\omega)^{k-3} R \\
&\qquad \times \left(\frac{\Theta(0)}{r(R,\omega)+f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega} - \frac{\Theta(0)}{r(R,\omega)} \right) d\mathscr{H}^{k-1}\omega \\
&+\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega) R \left(r(R,\omega)^{k-4} - R^{k-4} \right) \Theta(0)\, d\mathscr{H}^{k-1}\omega \\
=\ & R^{k-3} \int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&+ o(R^{k-1}) \quad \text{as } R \downarrow 0.
\end{aligned} \tag{3.44}
$$

We divide both sides of (3.42) by R^k, and use L'Hospital's rule. The assertion of Theorem 3.1 follows from (3.41) and (3.44). □

If the varifold has higher regularity, then the assertion of Theorem 3.1 becomes simpler. Corollary 3.4 associates the Laplacian of a graph which represents a varifold with the generalized mean curvature of the varifold.

Corollary 3.4 *Let f and F satisfy (3.2) and (3.3). We assume that $\alpha > 1/2$. Then*

$$
\lim_{R\downarrow 0} \frac{1}{R} \int_{\partial \mathrm{B}^k(0,1)} \nabla F(R\omega)\cdot \omega \Theta(0)\, d\mathscr{H}^{k-1}\omega \in \mathbb{R}^n \tag{3.45}
$$

and this value coincides with the generalized mean curvature.

Proof. We rewrite the integrand of (3.40) as

$$
\begin{aligned}
&\frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega} \\
=\ & \nabla f_l(r(R,\omega)\omega)\cdot \omega \left(\frac{1}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega} - \frac{1}{r(R,\omega)} \right) \\
&+\nabla f_l(r(R,\omega)\omega)\cdot \omega \left(\frac{1}{r(R,\omega)} - \frac{1}{R} \right) \\
&+\frac{1}{R} \left(\nabla f_l(r(R,\omega)\omega) - \nabla f_l(R\omega) \right)\cdot \omega + \frac{\nabla f_l(R\omega)\cdot \omega}{R}.
\end{aligned} \tag{3.46}
$$

Using (3.10) and (3.16) for the third term, we have

$$
\begin{aligned}
\frac{1}{R} \left| \nabla f_l(r(R,\omega)\omega) - \nabla f_l(R\omega) \right| &\leq \frac{C_2}{R} \left| r(R,\omega)\omega - R\omega \right|^\alpha \\
&= \frac{C_2}{R} \left(\frac{|f(r(R,\omega)\omega)|^2}{r(R,\omega)+R} \right)^\alpha \\
&\lesssim R^{2\alpha^2+\alpha-1},
\end{aligned} \tag{3.47}
$$

and we note that $2\alpha^2+\alpha-1>0$ is positive when $\alpha>1/2$. Estimating the first and second terms similarly, we have

$$\begin{aligned}&\int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot\omega\Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\cdot\omega}\, d\mathscr{H}^{k-1}\omega\\ =\ &\frac{1}{R}\int_{\partial \mathrm{B}^k(0,1)} \nabla f_l(R\omega)\cdot\omega\Theta(0)\, d\mathscr{H}^{k-1}\omega \ +o(1) \quad \text{as } R\downarrow 0.\end{aligned}$$

□

4. Inverse of a tangent-point radius and some examples

In this section, we explain the vector-valued tangent-point radius, and we introduce a generalization of the mean curvature vector different from [1]. Theorem 3.1 says that this coincides with that of [1] when the varifold V is locally the graph of a $C^{1,\alpha}$ function with $\alpha>1/3$. Furthermore we give some examples.

Suppose that $S\in \mathrm{G}(n,k)$, $x\in\mathbb{R}^n$ and $a\in S$. Then we say

$$\frac{|x-a|_n^2}{2|S^\perp(x-a)|_n}$$

is a tangent-point radius. We consider a curve which is tangent to S at a. If a point x on the curve approaches a along the curve, then inverse of the tangent-point radius tends to its curvature. The quantity

$$\frac{2S^\perp(x-a)}{|x-a|_n^2} \tag{4.1}$$

in integrand of (3.6) corresponds to the vector-valued inverse of the tangent-point radius. Hence its integral average might approximate the mean curvature, which is the heart of (3.6). The above idea is based on the Menger curvature. (4.1) is a modification of the Menger curvature for calculation ([7]). By Theorem 3.1, the classical mean curvature is represented by (3.6). It is a generalization of the generalized mean curvature without using the variation, and this is different from the corresponding generalization in [1].

Finally, we present examples with the help of (3.6); it is easier to calculate than the generalization of the mean curvature in [1]. For these examples, we prepare Propositon 4.1 below. It is proved by similar argument to the proof of Theorem 3.1, so we omit the details.

Proposition 4.1 *Under the same assumptions as Theorem 3.1 with* $T=\mathrm{Im}DF(0)=\mathbb{R}^k\times\{0\}$, $F(0)=0$. *Then we have*

$$\mathrm{h}(V,0)=\lim_{R\downarrow 0}\frac{2k}{\omega_k R^k}\int_{\mathrm{B}^k(0,R)}\frac{T^\perp(F(y))}{|y|_k^2}\, d\mathscr{H}^k y. \tag{4.2}$$

We assume that $0 < \beta < 1$ in the following examples.

Example 4.2 *Suppose $k \geq 2$, $n = k+1$, $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, $|y|_k = r$ and $y_1 = r\sigma_1$. We let*

$$f(y) = r^{1+\beta}\sigma_1.$$

Then $f \notin C^2$, and the mean curvature in the classical sense of f cannot be defined. However, by Proposition 2.8, we have

$$\begin{aligned}\frac{1}{R^k}\int_{\mathrm{B}^k(0,R)} \frac{F(y)\cdot e_{k+1}}{|y|_k^2}\, d\mathscr{H}^k y &= \frac{1}{R^k}\int_{\mathrm{B}^k(0,R)} \frac{f(y)}{|y|_k^2}\, d\mathscr{H}^k y \\ &= \frac{1}{R^k}\int_0^R r^{k+\beta-2}\, dr \int_{\partial\mathrm{B}^k(0,1)} \sigma_1\, d\mathscr{H}^{k-1}\sigma.\end{aligned}$$

Since σ_1 is odd with respect to y_1, its integration over $\partial\mathrm{B}^k(0,1)$ vanishes. Consequently the generalization of mean curvature exists.

Example 4.3 *Suppose that $k = 2$, $n = 3$ and set $y_1 = r\cos\theta$. We let*

$$f(y) = r^{1+\beta}\cos 2\theta.$$

Then $f \notin C^2$, and the mean curvature in the classical sense of f cannot be defined. However, by Proposition 2.8, we have

$$\frac{1}{R^2}\int_{\mathrm{B}^2(0,R)} \frac{f(y)}{|y|_2^2}\, d\mathscr{H}^2 y = \frac{1}{R^2}\int_0^R r^\beta\, dr \int_0^{2\pi} \cos 2\theta\, d\theta = 0$$

and the generalization of mean curvature exists.

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