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On p-unipotent triangular automorphisms of polynomial rings in positive characteristic p

Ryuji Tanimoto

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Abstract

Let k be a field of positive characteristic p and let $k[x_1, \dots, x_n]$ denote the polynomial ring in n variables over k. In this article, we treat two topics. The first topic is to give a method of constructing p-unipotent triangular automorphism of $k[x_1, \dots, x_n]$. The second topic is to give a necessary and sufficient condition for a p-unipotent automorphism σ of $k[x_1, \dots, x_n]$ to be triangular in terms of the pseudo-derivation Δ of $k[x_1, \dots, x_n]$ corresponding to σ.

0. Introduction

Let k be a field of positive characteristic p and let A be a k-algebra. For a k-algebra homomorphism $\sigma : A \to A$, we say that σ is *p-unipotent* if $\sigma^p = \mathrm{id}_A$, where $\mathrm{id}_A : A \to A$ denotes the identity map. Clearly, if a k-algebra homomorphism $\sigma : A \to A$ is p-unipotent, then σ is a k-algebra automorphism of A.

Let $k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k and let σ be a k-algebra automorphism of $k[x_1, \dots, x_n]$. We say that σ is *triangular* if σ can be written as $\sigma(x_i) = u_i x_i + f_i$ for some $u_i \in k\backslash\{0\}$ and $f_i \in k[x_1, \dots, x_{i-1}]$ $(1 \leq i \leq n)$. Especially when $u_i = 1$ for all $1 \leq i \leq n$, we say that σ is a *unitriangular* automorphism. Any p-unipotent triangular automorphism of $k[x_1, \dots, x_n]$ is a unitriangular automorphism of $k[x_1, \dots, x_n]$. We say that σ is *triangulable* if σ is conjugate to a triangular automorphism, i.e., $\varphi^{-1} \circ \sigma \circ \varphi$ is a triangular automorphism of $k[x_1, \dots, x_n]$ for some polynomial automorphism φ of $k[x_1, \dots, x_n]$.

In [4], we proved that for any p-unipotent triangular automorphism σ of the polynomial ring $k[x_1, x_2, x_3]$ in three variables, the modular invariant ring $k[x_1, x_2, x_3]^{\langle \sigma \rangle}$ is a hypersurface ring, where $\langle \sigma \rangle$ is the cyclic group generated by σ. We wish to extend this result for p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$, where $n \geq 4$. Now, we hope to express the forms of p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$. However, little is known about such forms, except for linear p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$.

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In [4], we also proved that there is a one-to-one correspondence between the set of all p-unipotent k-algebra automorphisms of A and the set of all pseudo-derivations of A. So, we are interested in translating triangularity of a p-unipotent automorphism σ of $k[x_1, \dots, x_n]$ into a property of the pseudo-derivation Δ of $k[x_1, \dots, x_n]$ corresponding to σ.

We summarise the article, as follows:

In Section 1, we give a method of constructing p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$. We may perform the method by hand calculations. We can run, in principle, the method on computer with the aid of Kemper's algorithm [2] and Gröbner bases theory. Anyway, we just started to study expressing p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$, where $n \geq 4$.

In Section 2, we give a necessary and sufficient condition for a p-unipotent automorphism σ of $k[x_1, \dots, x_n]$ to be triangular in terms of the pseudo-derivation Δ of $k[x_1, \dots, x_n]$ corresponding to σ.

1. A method of constructing p-unipotent triangular automorphisms of polynomial rings

Let k be a field of positive characteristic p and let A be a k-algebra. Given a k-algebra homomorphism $\sigma : A \to A$, we can define a k-linear map $D_\sigma : A \to A$ as $D_\sigma(f) := \sigma(f) - f$ for all $f \in A$. We have

$$D_\sigma(fg) = D_\sigma(f)\,\sigma(g) + f\,D_\sigma(g) \qquad \text{for all} \quad f, g \in A.$$

For each $\ell \geq 1$, we can define the *kernel* $A^{D_\sigma^\ell}$ of D_σ^ℓ as

$$A^{D_\sigma^\ell} := \{ f \in A \mid D_\sigma^\ell(f) = 0 \}.$$

Clearly, A^{D_σ} becomes a k-subalgebra of A, and each $A^{D_\sigma^\ell}$ becomes an A^{D_σ}-module.

1.1 On p-unipotent triangular automoprhisms

Lemma 1 *Let $\sigma : A \to A$ be a k-algebra homomorphism. Then σ is p-unipotent if and only if $D_\sigma^p = 0$. In particular when $A = k[x_1, \dots, x_n]$ is the polynomial ring in n variables over k, σ is p-unipotent if and only if $D_\sigma^p(x_i) = 0$ for all $1 \leq i \leq n$.*

Proof. The proof follows from $D_\sigma^p = D_{\sigma^p}$. Q.E.D.

By the following Lemmas 2 and 3, we can inductively construct p-unipotent triangular automorphisms of $k[x_1, \dots, x_n]$, where $n \geq 1$.

Lemma 2 *Let σ be a k-algebra endomorphism of $k[x_1]$. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *σ is a p-unipotent triangular automorphism.*

(2) *$\sigma(x_1) = x_1 + f_1$ for some $f_1 \in k$.*

Proof. The proof is straightforward. Q.E.D.

For any k-algebra homomorphism $\sigma : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ and any $f \in k[x_1, \ldots, x_n]$, we can define a k-algebra homomorphism $\varepsilon_{\sigma,f} : k[x_1, \ldots, x_n, x_{n+1}] \to k[x_1, \ldots, x_n, x_{n+1}]$ as

$$\varepsilon_{\sigma,f}(x_i) = \begin{cases} \sigma(x_i) & \text{if} \quad 1 \leq i \leq n, \\ x_{n+1} + f & \text{if} \quad i = n+1. \end{cases}$$

Lemma 3 *For any integer $n \geq 1$, the following assertions (1) and (2) hold true:*

(1) *Let σ be a p-unipotent triangular automorphism of $k[x_1, \ldots, x_n]$. Take any element f of the kernel $k[x_1, \ldots, x_n]^{D_\sigma^{p-1}}$. Then the k-algebra endomorphism $\varepsilon_{\sigma,f}$ of $k[x_1, \ldots, x_n, x_{n+1}]$ is a p-unipotent triangular automorphism of $k[x_1, \ldots, x_n, x_{n+1}]$.*

(2) *Let τ be a p-unipotent triangular automorphism of $k[x_1, \ldots, x_n, x_{n+1}]$. Let $\tau|_{k[x_1,\ldots,x_n]}$ be the k-algebra endomorphism of $k[x_1, \ldots, x_n]$ defined by $\tau|_{k[x_1,\ldots,x_n]}(f) := \tau(f)$ for all $f \in k[x_1, \ldots, x_n]$. Then $\tau|_{k[x_1,\ldots,x_n]}$ is a p-unipotent triangular automorphism of $k[x_1, \ldots, x_n]$, and $\tau(x_{n+1}) - x_{n+1} \in k[x_1, \ldots, x_n]^{D^{p-1}_{\tau|_{k[x_1,\ldots,x_n]}}}$.*

Proof. (1) Note that

$$D^p_{\varepsilon_{\sigma,f}}(x_i) = \begin{cases} D^p_\sigma(x_i) & \text{if} \quad 1 \leq i \leq n, \\ D^{p-1}_\sigma(f) & \text{if} \quad i = n+1. \end{cases}$$

Thus, we have $D^p_{\varepsilon_{\sigma,f}}(x_i) = 0$ for all $1 \leq i \leq n+1$, which implies that $\varepsilon_{\sigma,f}$ is p-unipotent by Lemma 1. Clearly, $\varepsilon_{\sigma,f}$ is a triangular automorphism of $k[x_1, \ldots, x_n, x_{n+1}]$.

(2) Clearly, $\tau|_{k[x_1,\ldots,x_n]}$ is a p-unipotent triangular automorphism of $k[x_1, \ldots, x_n]$. We can express $\tau(x_{n+1})$ as $\tau(x_{n+1}) = x_{n+1} + f$ for some $f \in k[x_1, \ldots, x_n]$. So, $\tau(x_{n+1}) - x_{n+1} = f \in k[x_1, \ldots, x_n] \cap k[x_1, \ldots, x_n, x_{n+1}]^{D^{p-1}_\tau} = k[x_1, \ldots, x_n]^{D^{p-1}_{\tau|_{k[x_1,\ldots,x_n]}}}$. Q.E.D.

We denote by $\mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ the set of all p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n]$, and let $\mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ be the set defined by

$$\mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n]) := \left\{ (\sigma, f) \in \mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n]) \times k[x_1, \ldots, x_n] \;\middle|\; f \in k[x_1, \ldots, x_n]^{D_\sigma^{p-1}} \right\}.$$

By Lemma 3, we can define a map $\Phi : \mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n]) \to \mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$ as

$$\Phi(\sigma, f) := \varepsilon_{\sigma,f},$$

and also define a map $\Psi : \mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}]) \to \mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ as

$$\Psi(\tau) := (\tau|_{k[x_1,\ldots,x_n]}, \tau(x_{n+1}) - x_{n+1}).$$

We denote by $\mathrm{id}_{\mathbb{U}^{p,\triangle}(k[x_1,\ldots,x_n])}$ the identity map from $\mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ to itself, and denote by $\mathrm{id}_{\mathrm{U}^{p,\triangle}(k[x_1,\ldots,x_n,x_{n+1}])}$ the identity map form $\mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$ to itself.

The following theorem implies that there exists a one-to-one correspondence between the set $\mathbb{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ and the set $\mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$. So, we obtain a method of constructing p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n, x_{n+1}]$ from p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n]$, for any $n \geq 1$.

Theorem 4 *For any $n \geq 1$, we have*

$$\Psi \circ \Phi = \mathrm{id}_{\mathbb{U}^{p,\triangle}(k[x_1,\ldots,x_n])} \qquad \textit{and} \qquad \Phi \circ \Psi = \mathrm{id}_{\mathrm{U}^{p,\triangle}(k[x_1,\ldots,x_n,x_{n+1}])}.$$

Proof. The proof follows from Lemma 3. Q.E.D.

1.2 On forms of p-unipotent triangular automorphisms

The following lemma gives a form of any p-unipotent triangular automorphism of $k[x_1, x_2]$.

Lemma 5 *Let τ be a k-algebra endomorphism of $k[x_1, x_2]$. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *τ is a p-unipotent triangular automorphism of $k[x_1, x_2]$.*

(2) *τ has one of the following forms* (2.1) *and* (2.2)*:*

$$(2.1) \quad \begin{cases} \tau(x_1) = x_1, \\ \tau(x_2) = x_2 + f_2(x_1) \end{cases}$$

for some $f_2(x_1) \in k[x_1]$; and

$$(2.2) \quad \begin{cases} \tau(x_1) = x_1 + f_1, \\ \tau(x_2) = x_2 + \displaystyle\sum_{i=0}^{p-2} \phi_i(x_1^p - f_1^{p-1} x_1)\, x_1^i \end{cases}$$

for some $f_1 \in k \backslash \{0\}$ and $\phi_i(T) \in k[T]$ $(0 \leq i \leq p-2)$, where $k[T]$ is the *polynomial ring in one variable over k.*

Proof. The proof of the implication (2) $\Longrightarrow$ (1) is straightforward. We shall prove (1) $\Longrightarrow$ (2). By assertion (2) of Lemma 3, $\tau|_{k[x_1]}$ is a p-unipotent automorphism of $k[x_1]$ and $\tau(x_2) - x_2 \in k[x_1]^{D^{p-1}_{\tau|_{k[x_1]}}}$. By Lemma 2, $\tau|_{k[x_1]}(x_1) = x_1 + f_1$ for some $f_1 \in k$. We know from [4, Lemma 2.8] that

$$k[x_1]^{D^{p-1}_{\tau|_{k[x_1]}}} = \begin{cases} k[x_1] & \text{if} \quad f_1 = 0, \\ \displaystyle\sum_{i=0}^{p-2} k[x_1^p - f_1^{p-1} x_1]\, x_1^i & \text{if} \quad f_1 \neq 0. \end{cases}$$

So, if $f_1 = 0$, then τ has the form (2.1); and if $f_1 \neq 0$, then τ has the form (2.2). Q.E.D.

We shall give an example of non-linear p-unipotent triangular automorphisms of $k[x_1, x_2, x_3, x_4]$. Assume that the characteristic of k is three, let $A := k[x_1, x_2, x_3]$ be the polynomial ring in three variables over k, and let σ be the k-algebra automorphism σ of $k[x_1, x_2, x_3]$ defined by

$$\sigma(x_i) := \begin{cases} x_1 & \text{if} \quad i = 1, \\ x_i + x_{i-1} & \text{if} \quad i > 1. \end{cases}$$

Clearly, σ is a p-unipotent triangular automorphism of A. We know from [1] that the kernel A^{D_σ} is generated as a k-algebra by the following four polynomials f_1, f_2, f_3, f_4:

$$\begin{cases} f_1 := x_1, \\ f_2 := x_1 x_2 + 2x_2^2 + 2x_1 x_3, \\ f_3 := 2x_1^2 x_2 + x_2^3, \\ f_4 := x_1 x_2 x_3 + 2x_2^2 x_3 + x_1 x_3^2 + x_3^3. \end{cases}$$

Lemma 6 *Let σ be as above. Let τ be a p-unipotent triangular automorphism of $k[x_1, x_2, x_3, x_4]$ satisfying $\tau|_{k[x_1, x_2, x_3]} = \sigma$. Then τ has the following form:*

$$\begin{cases} \tau(x_1) = x_1, \\ \tau(x_2) = x_2 + x_1, \\ \tau(x_3) = x_3 + x_2, \\ \tau(x_4) = x_4 + \displaystyle\sum_{i=1}^{4} \beta_i(f_1, f_2, f_3, f_4)\, g_i, \end{cases}$$

for some polynomials $\beta_i(y_1, y_2, y_3, y_4) \in k[y_1, y_2, y_3, y_4]$ $(1 \leq i \leq 4)$, where $k[y_1, y_2, y_3, y_4]$ is the polynomial ring in four variables over k and the polynomials g_1, g_2, g_3, g_4 are defined by

$$\begin{cases} g_1 := 1, \\ g_2 := x_2, \\ g_3 := 2x_1x_2 + x_2^2 + 2x_2x_3, \\ g_4 := x_1x_2^2 + 2x_2^3 + 2x_1x_2x_3 + x_2^2x_3 + 2x_1x_3^2. \end{cases}$$

Proof. We know from [5, Theorem 5] that $A^{D_\sigma^2} = \sum_{i=1}^4 A^{D_\sigma} g_i$. By Lemma 3, τ has the desired form. Q.E.D.

1.3 A method of constructing a generating set of the kernel $k[x_1, \ldots, x_n]^{D_\sigma^\ell}$ $(1 \le \ell \le p-1)$

Let $A := k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, where k is a field of positive characteristic p. Let σ be a k-algebra automorphism of A of order p, i.e., $\sigma \neq \mathrm{id}_A$ and $\sigma^p = \mathrm{id}_A$. Let $B := A^{D_\sigma}$ be the kernel of D_σ. So, we can take a finitely generated k-subagebra C of B such that A is a finite C-module. In fact, we know the following C and A: For any $f \in A$, we define a polynomial $\varphi_f(T) := \prod_{i=0}^{p-1}(T - \sigma^i(f))$ of $A[T]$. Expand $\varphi(T)$ as $\sum_{i=0}^p s_i(f)\, T^i$, where $s_i(f) \in A^{D_\sigma}$ for all $0 \le i \le p-1$. Let $C := k[s_i(x_j) \mid 1 \le j \le n,\ 0 \le i \le p-1]$. Clearly, $A = \sum_{0 \le i_1, \ldots, i_n \le p-1} C\, x_1^{i_1} \cdots x_n^{i_n}$.

We shall give a method of constructing a generating set of $A^{D_\sigma^\ell}$ as a C-module for each $1 \le \ell \le p-1$, as follows:

We can write C and A as

$$\begin{cases} C = k[c_1, \ldots, c_r] & \text{for some} \quad c_1, \ldots, c_r \in C, \\ A = \displaystyle\sum_{i=1}^s C\, a_i & \text{for some} \quad a_1, \ldots, a_s \in A. \end{cases}$$

We have an increasing sequence

$$C \subset B = A^{D_\sigma} \subsetneq A^{D_\sigma^2} \subsetneq \cdots \subsetneq A^{D_\sigma^{p-1}} \subsetneq A^{D_\sigma^p} = A$$

of C-modules, and each $A^{D_\sigma^\ell}$ is a finite C-module. Let $\pi : C^{\oplus s} \to A$ be the surjective C-module homomorphism defined by

$$\pi(\gamma_1, \ldots, \gamma_s) := \sum_{i=1}^s \gamma_i\, a_i.$$

Clearly, we have

$$\begin{aligned} A^{D_\sigma^\ell} &= \pi\Big(\mathrm{Syz}_C\big(D_\sigma^\ell(a_1), \ldots, D_\sigma^\ell(a_s)\big)\Big) \\ &= \pi\Big(\mathrm{Syz}_A\big(D_\sigma^\ell(a_1), \ldots, D_\sigma^\ell(a_s)\big) \cap C^{\oplus s}\Big) \qquad \text{for all} \quad 1 \le \ell \le p-1. \end{aligned}$$

For each $1 \le \ell \le p-1$, we let

$$M_\ell := \mathrm{Syz}_A\big(D_\sigma^\ell(a_1), \ldots, D_\sigma^\ell(a_s)\big).$$

Clearly, we have $A^{D_\sigma^\ell} = \pi(M_\ell \cap C^{\oplus s})$ for all $1 \le \ell \le p-1$.

Now, we explain how to calculate a generating set of $A^{D_\sigma^\ell}$ as a C-module. Since A is a polynomial ring over k, we know an algorithm for calculating a generating set $\{m_{\ell,1}, \ldots, m_{\ell,t_\ell}\}$ of the syzygy module M_ℓ as an A-module (see, for example, [3]). And we also know an algorithm for calculating a generating set of the intersection $M_\ell \cap C^{\oplus s}$ as a C-module, by the algorithm of Kemper [2, Lemma 6]. So, let $\{\mu_{\ell,1}, \ldots, \mu_{\ell,u_\ell}\}$ be a generating set of $M_\ell \cap C^{\oplus s}$ as a C-module. Then the set $\{\pi(\mu_{\ell,1}), \ldots, \pi(\mu_{\ell,u_\ell})\}$ forms a generating set of $A^{D_\sigma^\ell}$ as a C-module.

For the convenience of the reader, we write Kemper's algorithm for calculating a generating set of the intersection $M_\ell \cap C^{\oplus s}$, as follows: Let $P = k[x_1, \ldots, x_n, y_1, \ldots, y_r]$ be the polynomial ring in $n + r$ variables over k, and let $Q := k[y_1, \ldots, y_r]$ be the polynomial subring of P. Define maps $\Phi : P^{\oplus s} \to A^{\oplus s}$ and $\Psi : Q^{\oplus s} \to C^{\oplus s}$ as

$$\begin{aligned}
\Phi(\alpha_1(x, y_1, \ldots, y_r), \ldots, \alpha_s(x, y_1, \ldots, y_r)) &:= (\alpha_1(x, c_1, \ldots, c_r), \ldots, \alpha_s(x, c_1, \ldots, c_r)),\\
\Psi(\beta_1(y_1, \ldots, y_r), \ldots, \beta_s(y_1, \ldots, y_r)) &:= (\beta_1(c_1, \ldots, c_r), \ldots, \beta_s(c_1, \ldots, c_r)),
\end{aligned}$$

where $x = (x_1, \ldots, x_n)$. Clearly, we have the following commutative diagrams, where vertical arrows are inclusion maps:

$$\begin{array}{ccc}
P^{\oplus s} & \xrightarrow{\ \Phi\ } & A^{\oplus s} \\
\uparrow & & \uparrow \\
Q^{\oplus s} & \xrightarrow[\ \Psi\ ]{} & C^{\oplus s}
\end{array}$$

Let $N_\ell := \Phi^{-1}(M_\ell)$. It follows that

$$N_\ell = \left(\sum_{i=1}^{t_\ell} P\, m_{\ell,i}\right) + \left(\sum_{j_1=1}^{r} \sum_{j_2=1}^{s} P\,(y_{j_1} - c_{j_1})\, e_{j_2}\right),$$

where e_{j_2} is the element of $P^{\oplus s}$ whose j_2-th entry is one and the other entries are zero. Since Ψ is surjective, we have

$$\Psi(N_\ell \cap Q^{\oplus s}) = M_\ell \cap C^{\oplus s}.$$

Using Gröbner bases theory for submodules of free modules over the polynomial ring Q, we can calculate a generating set of $N_\ell \cap Q^{\oplus s}$ as a Q-module, and let $\{\nu_{\ell,1}, \ldots \nu_{\ell,u_\ell}\}$ be the generating set. Let $\mu_{\ell,i} := \Psi(\nu_{\ell,i})$ for all $1 \le i \le u_\ell$. Clearly, the set $\{\mu_{\ell,1}, \ldots, \mu_{\ell,u_\ell}\}$ forms a generating set of $M_\ell \cap C^{\oplus s}$ as a C-module.

2. Triangular pseudo-derivations and triangulable pseudo-derivations

Let A be a k-algebra, where k is a field of positive characteristic p. A k-linear transformation Δ of A is said to be a *pseudo-derivation* if Δ satisfies the following conditions (1), (2) and (3):

(1) $\Delta(fg) = \Delta(f)g + f\Delta(g) + \sum_{i=1}^{p-1} \frac{(-1)^i}{i} \Delta^i(f)\Delta^{p-i}(g)$ for all $f, g \in A$.

(2) $\Delta(1_A) = 0$, where 1_A is the unity of A.

(3) $\Delta^p = 0$.

Given a pseudo-derivation Δ of A, we can define a k-linear transformation $\mathrm{Exp}(\Delta) : A \to A$ as

$$\mathrm{Exp}(\Delta)(f) := \sum_{i=0}^{p-1} \frac{\Delta^i(f)}{i!}.$$

We know that $\mathrm{Exp}(\Delta)$ is a k-algebra automorphism of A satisfying $\mathrm{Exp}(\Delta)^p = \mathrm{id}_A$ (see [4, Lemma 1.2]).

Given a k-algebra automorphism σ of A. We can define a k-linear transformation $\mathrm{Log}(\sigma) : A \to A$ as

$$\mathrm{Log}(\sigma)(f) := \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} D_\sigma^i(f).$$

For any p-unipotent automorphism σ of A, the truncated logarithm $\mathrm{Log}(\sigma)$ is a pseudo-derivation of A (see [4, Lemma 1.6]).

We denote by $U_k^p(A)$ the set of all p-unipotent automorphisms of A, and by $\mathrm{PDer}_k(A)$ the set of all pseudo-derivations of A. Let $\mathrm{Exp} : \mathrm{PDer}_k(A) \to U_k^p(A)$ be the map defined by $\Delta \mapsto \mathrm{Exp}(\Delta)$ and let $\mathrm{Log} : U_k^p(A) \to \mathrm{PDer}_k(A)$ be the map defined by $\sigma \mapsto \mathrm{Log}(\sigma)$. We denote by $\mathrm{id}_{\mathrm{PDer}_k(A)}$ the identity map from $\mathrm{PDer}_k(A)$ to itself and by $\mathrm{id}_{U_k^p(A)}$ the identity map from $U_k^p(A)$ to itself.

We know the following theorem (see [4, Theorem 1.7]), which states that there is a one-to-one correspondence between the set of all p-unipotent automorphisms of A and the set of all pseudo-derivations of A.

Theorem 7 *We have*

$$\mathrm{Log} \circ \mathrm{Exp} = \mathrm{id}_{\mathrm{PDer}_k(A)} \quad \textit{and} \quad \mathrm{Exp} \circ \mathrm{Log} = \mathrm{id}_{U_k^p(A)}.$$

And we have $\mathrm{Exp}(\Delta) = \mathrm{id}_A$ *if and only if* $\Delta = 0$.

2.1 Triangular pseudo-derivations

Let Δ be a pseudo-derivation of $k[x_1, \ldots, x_n]$ and let σ be the p-unipotent automorphism of $k[x_1, \ldots, x_n]$ corresponding to Δ, i.e., $\sigma = \mathrm{Exp}(\Delta)$. We say that Δ is *triangular* if σ is a triangular automorphism of $k[x_1, \ldots, x_n]$.

The following theorem gives a necessary and sufficient condition for a p-unipotent automorphism σ of $k[x_1, \ldots, x_n]$ to be triangular in terms of the pseudo-derivation Δ of $k[x_1, \ldots, x_n]$ corresponding to σ.

Theorem 8 *Let σ be a p-unipotent automorphism of $k[x_1, \ldots, x_n]$ and let Δ be the pseudo-derivation of $k[x_1, \ldots, x_n]$ corresponding to σ. Then the following conditions (1) and (2) are equivalent:*

(1) *σ is triangular.*

(2) *$\Delta(x_i) \in k[x_1, \ldots, x_{i-1}]$ for all $1 \leq i \leq n$.*

In order to prove Theorem 8, we prepare the following lemma. After we proved Lemma 9, we prove Theorem 8.

Lemma 9 *Let i be an integer satisfying $1 \leq i \leq n$, and let Δ be a pseudo-derivation of $k[x_1, \ldots, x_n]$ satisfying the following conditions (1) and (2):*

(1) $\Delta(x_i) \in k[x_1, \ldots, x_{i-1}]$.

(2) $\Delta(k[x_1, \ldots, x_{i-1}]) \subset k[x_1, \ldots, x_{i-1}]$.

Then, for all $r \geq 1$, we have

$$\begin{cases} \Delta(x_i^r) \in \displaystyle\sum_{\ell=0}^{r-1} k[x_1, \ldots, x_{i-1}]\, x_i^\ell, \\ \Delta\left(\displaystyle\sum_{\ell=0}^{r-1} k[x_1, \ldots, x_{i-1}]\, x_i^\ell\right) \subset \displaystyle\sum_{\ell=0}^{r-1} k[x_1, \ldots, x_{i-1}]\, x_i^\ell. \end{cases}$$

Proof. We proceed by induction on $r \geq 1$. If $r = 1$, the proof is clear. So let $r \geq 2$ and suppose that

$$\begin{cases} \Delta(x_i^{r-1}) \in \displaystyle\sum_{\ell=0}^{r-2} k[x_1, \ldots, x_{i-1}]\, x_i^\ell, \\ \Delta\left(\displaystyle\sum_{\ell=0}^{r-2} k[x_1, \ldots, x_{i-1}]\, x_i^\ell\right) \subset \displaystyle\sum_{\ell=0}^{r-2} k[x_1, \ldots, x_{i-1}]\, x_i^\ell. \end{cases}$$

We have

$$\Delta(x_i^r) = \Delta(x_i)\, x_i^{r-1} + x_i\, \Delta(x_i^{r-1}) + \sum_{\ell=1}^{p-1} \frac{(-1)^\ell}{\ell}\, \Delta^\ell(x_i)\, \Delta^{p-\ell}(x_i^{r-1})$$

$$\in \sum_{\ell=0}^{r-1} k[x_1,\ldots,x_{i-1}]\, x_i^{\ell}.$$

For any $f \in k[x_1,\ldots,x_{i-1}]$, we have

$$\begin{aligned}\Delta(f\, x_i^{r-1}) &= \Delta(f)\, x_i^{r-1} + f\,\Delta(x_i^{r-1}) + \sum_{\ell=1}^{p-1} \frac{(-1)^{\ell}}{\ell}\, \Delta^{\ell}(f)\, \Delta^{p-\ell}(x_i^{r-1}) \\ &\in \sum_{\ell=0}^{r-1} k[x_1,\ldots,x_{i-1}]\, x_i^{\ell}.\end{aligned}$$

Q.E.D.

Now, we prove Theorem 8. We first prove (1) $\Longrightarrow$ (2). For all $1 \le i \le n$, we have

$$\Delta(x_i) = \mathrm{Log}(\sigma)(x_i) = \sum_{\ell=1}^{p-1} \frac{(-1)^{\ell-1}}{\ell}\, D_\sigma^{\ell}(x_i).$$

Since σ is triangular, we know that $D_\sigma(x_i) \in k[x_1,\ldots,x_{i-1}]$ and $D_\sigma(k[x_1,\ldots,x_{i-1}]) \subset k[x_1,\ldots,x_{i-1}]$. So, we have $D_\sigma^{\ell}(x_i) \in k[x_1,\ldots,x_{i-1}]$ for all $1 \le \ell \le p-1$. It follows that $\Delta(x_i) \in k[x_1,\ldots,x_{i-1}]$.

We next prove (2) $\Longrightarrow$ (1). We begin with proving that $\Delta(k[x_1,\ldots,x_j]) \subset k[x_1,\ldots,x_j]$ for all $0 \le j \le n-1$. We proceed by induction on j. If $j=0$, the proof is clear. So, let $j \ge 1$ and suppose that $\Delta(k[x_1,\ldots,x_{j-1}]) \subset k[x_1,\ldots,x_{j-1}]$. Since $\Delta(x_j) \in k[x_1,\ldots,x_{j-1}]$, we know from Lemma 9 that $\Delta(k[x_1,\ldots,x_j]) \subset k[x_1,\ldots,x_j]$. So, for all $1 \le i \le n$, we have

$$\sigma(x_i) = \mathrm{Exp}(\Delta)(x_i) = \sum_{\ell=0}^{p-1} \frac{\Delta^{\ell}(x_i)}{\ell!} \in x_i + k[x_1,\ldots,x_{i-1}],$$

where we denote by $x_i + k[x_1,\ldots,x_{i-1}]$ $(1 \le i \le n)$ the set of all polynomials F which can be written in the form $F = x_i + f$ for some $f \in k[x_1,\ldots,x_{i-1}]$. This compietes the proof of Theorem 8.

2.2 Triangulable pseudo-derivations

Let Δ be a pseudo-derivation of $k[x_1,\ldots,x_n]$. Clearly, for any automorphism φ of $k[x_1,\ldots,x_n]$, $\varphi^{-1}\Delta\varphi$ is a pseudo-derivation of $k[x_1,\ldots,x_n]$. We say that Δ is *triangulable* if $\varphi^{-1}\Delta\varphi$ is triangular for some automorphism φ of $k[x_1,\ldots,x_n]$.

We have the following lemma:

Lemma 10 *Let σ be a p-unipotent automorphism of $k[x_1,\ldots,x_n]$ and let Δ be the pseudo-derivation of $k[x_1,\ldots,x_n]$ corresponding to σ. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) *σ is triangulable.*

(2) Δ *is triangulable.*

Proof. The proof follows from the fact that $\varphi^{-1}\sigma\varphi = \mathrm{Exp}(\varphi^{-1}\Delta\varphi)$ for any automorphism φ of $k[x_1, \ldots, x_n]$. Q.E.D.

References

[1] L. E. Dickson, On Invariants and the Theory of Numbers, The Madison Colloquium (1913, Part 1), Amer. Math. Society, reprinted by Dover, New York, 1966.

[2] G. Kemper, Calculating invariant rings of finite groups over arbitrary fields, J. Symbolic Comput. **21** (1996), no. 3, 351–366.

[3] M. Kreuzer, L. Robbiano, Computational commutative algebra 1, Corrected reprint of the 2000 original, Springer-Verlag, Berlin, 2008. x+321 pp.

[4] R. Tanimoto, Pseudo-derivations and modular invariant theory, Transform. Groups **23** (2018), no. 1, 271–297.

[5] R. Tanimoto, The image membership algorithm for twisted derivations in modular invariant theory, Saitama Math. J. **29** (2012), 55–64.

Faculty of Education,
Shizuoka University,
836 Ohya, Suruga-ku,
Shizuoka 422–8529, Japan
e-mail: tanimoto.ryuji@shizuoka.ac.jp