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Osculating spheres to a family of curves

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Abstract

We study extrinsic conformal geometry of space forms involving pencils of circles or of spheres. We consider curves orthogonal to a foliation of an open set of a 3-sphere by spheres and prove that the osculating spheres to the curves at points of a leaf form a pencil.

1. Introduction

Let $\mathcal{F}$ be a codimension 1 foliation of an open set of $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$ by spheres. The orthogonal curves form a one-dimensional foliation $\mathcal{F}^\perp$. The holonomy maps of $\mathcal{F}^\perp$ are conformal maps defined from one spherical leaf of $\mathcal{F}$ to another (see [3]). We study the local geometry of the curves of $\mathcal{F}^\perp$ at the points of a given spherical leaf of $\mathcal{F}$. We first consider a "baby case", foliations $\mathcal{F}$ of the sphere $\mathbb{S}^2$ by circles and the orthogonal foliations $\mathcal{F}^\perp$, and prove our main theorem in this dimension (Theorem 2.5) using only classical tools. Then we prove our main theorem (Theorem 3.4) for $\mathbb{S}^3$ using the de Sitter space Λ^4 which is a model of the set of oriented spheres of $\mathbb{S}^3$. The n-dimensional case can be shown by the same argument. The Euclidean and hyperbolic cases can be reduced to the case of $\mathbb{S}^n$ since $\mathbb{R}^n$ and $\mathbb{H}^n$ are conformally equivalent to $\mathbb{S}^n \setminus \infty$ and to an open ball of $\mathbb{S}^n$ respectively.

The topic of this paper is in the intersection of conformal (Möbius) geometry and the theory of foliations. The reader is referred to [6] for advanced studies of the topic, and to [2] and [4] for the basics of conformal (Möbius) geometry. Related studies can be found in [1], where transformations of plane that map circles to circes are studied, and in [7], where holomorphic maps of the complex plane are characterized in terms of the images of pencils of circles.

2. Euclidean space view of osculating circles and pencils

A foliation is *Riemannian* if and only if the orthogonal trajectories are geodesics. In the Euclidean plane a Riemannian foliation is just a foliation by "*parallel*" curves. Along a common normal, these parallel curves have the same

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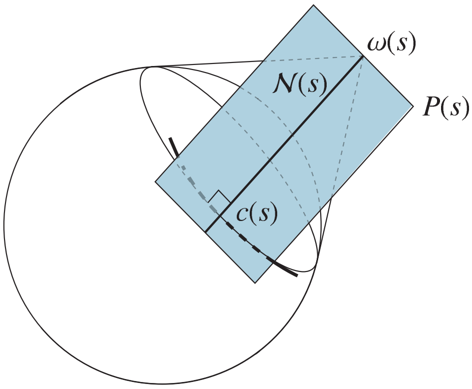

Figure 1 The singular locus of the envelope of planes tangent to $\mathbb{S}^2$ along a curve determines the osculating circles to the curve

center of curvature, namely, they have concentric osculating circles. Thus we have

Proposition 2.1. *If $\mathcal{F}$ is a foliation of a planar open set U by affine lines, all the orthogonal trajectories of the leaves have concentric osculating circles at the points of the same leaf.*

For example, if $\mathcal{E}$ is the evolute of a plane curve C, the tangent lines to $\mathcal{E}$ form a foliation by affine lines, and the orthogonal trajectories of the leaves are the involutes of $\mathcal{E}$, which are the curves parallel to C. We want to give first a generalization of that result concerning a foliation of the plane by circles:

Theorem 2.2. *If $\mathcal{F}$ is a foliation of a planar open set U by circles, the osculating circles of all the orthogonal trajectories at the points of the same leaf are in a same pencil of circles* (*see* Definition 2.3 *for the definition of a pencil*).

To prove this purely conformal theorem, it is convenient to replace $\mathbb{R}^2$ by the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. A circle in $\mathbb{S}^2$ has a *vertex* which is the vertex of the cone tangent to $\mathbb{S}^2$ along the circle, or the "point at infinity" when the circle is a geodesic circle (the tangent cone is in that case a tangent cylinder).

Notice that the curve C drawn on $\mathbb{S}^2$ has an osculating circle $\mathcal{O}_{C,t}$ at each point $c(t)$ which is a circle of $\mathbb{S}^2$. Therefore it is the intersection of $\mathbb{S}^2$ with the osculating plane of the curve. This intersection is always a circle and not a point, as the osculating plane of a smooth curve drawn on $\mathbb{S}^2$ cannot be tangent to $\mathbb{S}^2$. The vertex of the osculating circle at $c(t)$ is called the *focal point* at $c(t)$.

For a second definition of the osculating circle, we parametrize C by $s \mapsto c(s)$, and we associate to each s the affine tangent plane $P(s)$ of $\mathbb{S}^2$ at $c(s)$. This family of planes admits an envelope as follows, where the envelope is a surface that is

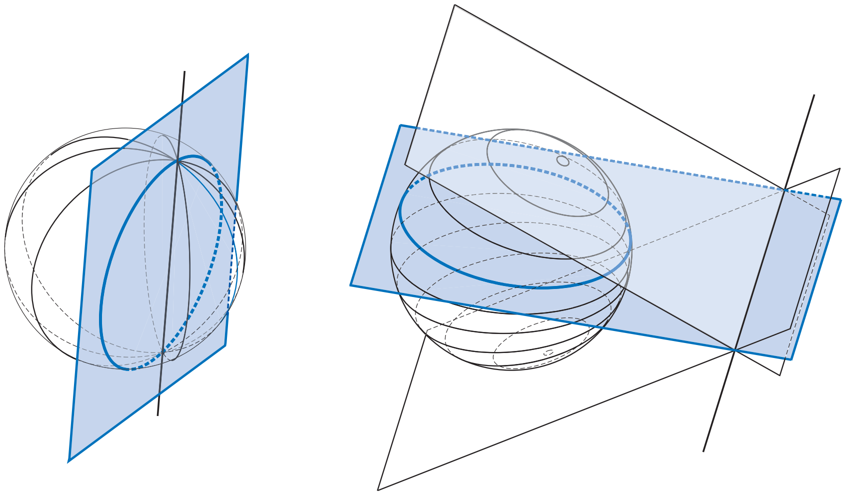

Figure 2 Two pencils of circles on $\mathbb{S}^2$

tantgent to each $P(s)$ at a line. As every point of $\mathbb{S}^2$ is umbilical, the Weingarten map of $\mathbb{S}^2$ at $c(s)$ is the identity and all the curves of $\mathbb{S}^2$ are lines of principal curvature. The characteristic direction $\mathcal{N}(s)$ of the envelope of planes $P(s)$ at s, which is the limit of $P(s) \cap P(s+\Delta s)$ as Δs goes to 0, is the line tangent to $\mathbb{S}^2$ at $c(s)$ which is normal to C. Thus, the union of the normal lines $\mathcal{N}(s)$ is the envelope of the planes $P(s)$ and is therefore a developable surface $\mathcal{W}$. If two curves $c_1(s)$ and $c_2(s)$ have a contact of order 2 at $c_1(s) = c_2(s)$, the focal point $\omega(s) \in \mathcal{N}(s)$ is the same for the two curves, i.e. the vertex of the osculating circle $\mathcal{O}_{C_i}$ of C_i at $c_i(s)$ is $\omega(s)$ for $i = 1, 2$.

Notice that, if we unroll the developable surface $\mathcal{W}$, the focal point of the plane curve (i.e. the characteristic point of the family of normals to the curve) corresponds to the focal point of the family of planes tangent to the sphere along the curve.

The distance of $c(s)$ to $\omega(s)$ has a classical geometric interpretation. Developing the surface $\mathcal{W}$ onto the affine tangent plane to $\mathbb{S}^2$ at $c(s)$, the development of C has an osculating circle at $c(s)$ centered at $\omega(s)$: this point is sometimes called the *geodesic curvature center* of C at $c(s)$. The distance of $c(s)$ to $\omega(s)$ is the *geodesic curvature radius* of C at $c(s)$, and its inverse k_g is the geodesic curvature of C at $c(s)$. We shall call $\omega(s)$ the *osculation vertex* of C at s.

Let us recall the classical definition of a pencil of circles.

Definition 2.3. A *pencil of circles* in $\mathbb{S}^2$ is a set of circles of intersection of $\mathbb{S}^2$ with the planes containing an affine line Δ called the *axis* of the pencil (see Figure 2).

We give a proposition which we use in the proof of the spherical version of Theorem 2.2.

Proposition 2.4. *Let Δ be a line. The set of circles of $\mathbb{S}^2$ that have vertices on Δ is a pencil of circles. If Δ intersects $\mathbb{S}^2$, it is a Poncelet pencil with limit points $\Delta \cap \mathbb{S}^2$. If $\Delta \cap \mathbb{S}^2 = \emptyset$, it is a pencil with base points. If $\Delta \cap \mathbb{S}^2$ is one point, it is a pencil of circles tangent at that point.*

Proof. For any point v in $\mathbb{R}^3$, the *polar plane of v with respect to the sphere* $\mathbb{S}^2$ is the set of points w such that $w \cdot v = 1$, for the usual scalar product in $\mathbb{R}^3$. If v is outside $\mathbb{S}^2$, the intersection of the polar plane of v with $\mathbb{S}^2$ is a circle Γ of vertex v. The intersection of the polar planes of all the points v on an affine line Δ is an affine line, called *the polar Δ^* of Δ with respect to* $\mathbb{S}^2$. The polar planes of the points $v \in \Delta$ form a pencil, the planes containing Δ^*. Thus their intersections with $\mathbb{S}^2$ form a pencil of circles of axis Δ^*. If Δ does not cut $\mathbb{S}^2$, then Δ^* cuts $\mathbb{S}^2$ and all the planes containing Δ^* must cut $\mathbb{S}^2$. In that case, the pencil of circles Γ has the intersections of Δ^* and $\mathbb{S}^2$ as base points.

If Δ cuts $\mathbb{S}^2$, then Δ^* does not cut $\mathbb{S}^2$, and the planes containing Δ^* and cutting $\mathbb{S}^2$ are contained in a dihedral sector between two planes tangent to $\mathbb{S}^2$. The corresponding pencil of circles has two limit points, the contacts of the two tangent planes with $\mathbb{S}^2$. The limit case, when Δ is tangent to $\mathbb{S}^2$ gives a pencil of circles tangent to the line Δ^* perpendicular to Δ at the common tangency point with $\mathbb{S}^2$. □

Theorem 2.5. (A spherical version of Theorem 2.2) *If $\mathcal{F}$ is a foliation of an open set U of $\mathbb{S}^2$ by circles, the osculating circles of all the orthogonal trajectories at the points of the same leaf are in a same pencil of circles on $\mathbb{S}^2$.*

Proof. The circles $\Gamma(t)$ of the foliation may be described as the intersections of planes $\Pi(t)$ with $\mathbb{S}^2$, where $\{\Pi(t)\}_{t\in\mathbb{R}}$ is a family of affine planes in $\mathbb{R}^3$ (see Figure 3). Let $\Delta(t)$ be the characteristic line of the envelope of this family of planes.

Consider a trajectory C orthogonal to the foliation, parameterized by arc length : $s \mapsto c(s) \in \mathbb{S}^2$. We denote the parameter t of the leaf which contains $c(s)$ by $t(s)$. At the point $c(s) \in C \cap \Gamma(t(s))$, C is orthogonal to $\Gamma(t(s))$ and the affine normal line $\mathcal{N}(s)$ to C (tangent to $\mathbb{S}^2$) is contained in the plane $\Pi(t(s))$. Recall that the osculation vertex $\omega(s)$ is the common limit of the two nearest points of $\mathcal{N}(s')$ and $\mathcal{N}(s)$, when s' tends to s. Then $\omega(s)$ is contained in an affine line $\Delta(t(s))$ which is the limit of the intersection of $\Pi(t(s'))$ and $\Pi(t(s))$ when s' tend to s (see Figure 3). Then $\omega(s)$ is on $\Delta(t(s))$. It implies that, for a fixed t, all the osculation vertices of the curves orthogonal to $\mathcal{F}$ at the points of the leaf $\Gamma(t)$ are on the characteristic line $\Delta(t)$.

In our case, all the osculating vertices of the orthogonal trajectories at points of $\Gamma(t)$ are in the line $\Delta(t)$, therefore the osculating circles to these orthogonal curves form a pencil of axis $\Delta(t)^*$. This pencil is orthogonal to the pencil of

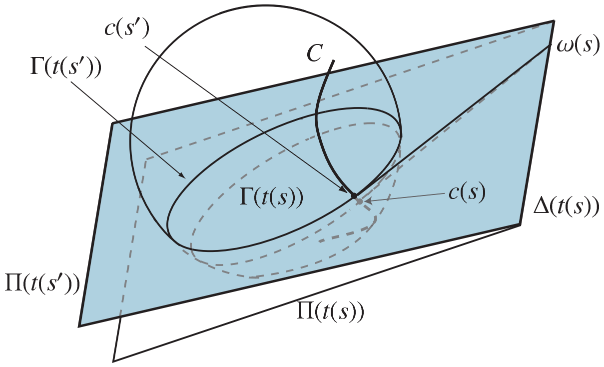

Figure 3

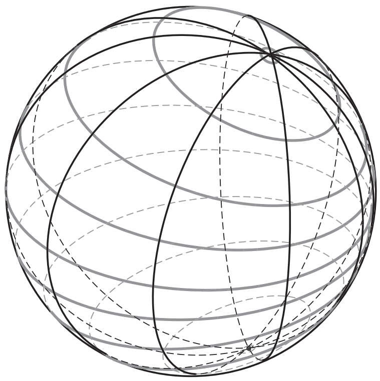

Figure 4 Two orthogonal pencils (one in black, the other in grey)

circles intersection of $\mathbb{S}^2$ with the planes containing $\Delta(t)$ (see Figure 4). $\square$

3. Conformal view of osculating spheres and pencils

3.1 Preliminaries for Möbius geometry

We will present Möbius geometry of spheres contained in $\mathbb{S}^3$. The analogous definitions, necessary to deal with circles of $\mathbb{S}^2$, or even with $(n-1)$-spheres of $\mathbb{S}^n$, will be left to the reader.

The *Lorentz quadratic form* $\mathcal{L}$ on the 5-dimensional space $\mathbb{R}^5$ and the associated *Lorentz bilinear form* $\mathcal{L}(\cdot,\cdot)$, are defined by $\mathcal{L}(\mathbf{x}) = \mathcal{L}(x_0,\cdots,x_4) = -x_0^2 + (x_1^2 + \cdots + x_4^2)$ and $\mathcal{L}(\boldsymbol{u},\boldsymbol{v}) = -u_0v_0 + (u_1v_1 + \cdots + u_4v_4)$. When $\mathbb{R}^5$ is endowed with the Lorentz quadratic form $\mathcal{L}$, we denote it by $\mathbb{R}^5_1$. The isotropy cone $\mathcal{L}ight = \{\boldsymbol{v} \in \mathbb{R}^5_1 \,|\, \mathcal{L}(\boldsymbol{v}) = 0\}$ of $\mathcal{L}$ is called the *light cone*. Its non-zero vectors are also called *light-like* vectors. The *light cone* divides the set of vec-

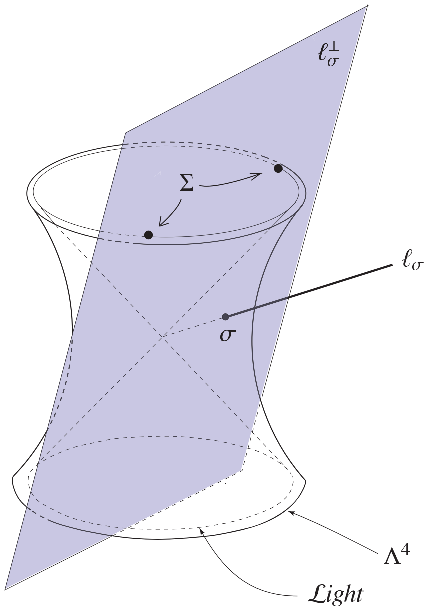

Figure 5 $\mathbb{S}^3_\infty$ and the correspondence between points of Λ^4 and 2-spheres.

tors $\boldsymbol{v} \in \mathbb{R}^5_1, \boldsymbol{v} \notin \{\mathcal{L} = 0\}$ in two classes. A vector $\boldsymbol{v}$ in $\mathbb{R}^5_1$ is called *space-like* if $\mathcal{L}(\boldsymbol{v}) > 0$ and *time-like* if $\mathcal{L}(\boldsymbol{v}) < 0$. A straight line is called space-like (or time-like) if it contains a space-like (or respectively, time-like) vector. We say that a vector $\boldsymbol{u}$ in the Minkowski space $\mathbb{R}^5_1$ is *orthogonal* to a vector $\boldsymbol{v} \in \mathbb{R}^5_1$ if $\mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) = 0$.

The points at infinity of the light cone in the upper half space $\{x_0 > 0\}$ form a 3-dimensional sphere. Let it be denoted by $\mathbb{S}^3_\infty$. Since it can be considered as the set of lines through the origin in the light cone, it is identified with the intersection $\mathbb{S}^3_1$ of the upper half light cone and the hyperplane $\{x_0 = 1\}$, which is given by $\mathbb{S}^3_1 = \{(x_1, \cdots, x_4) \,|\, x_1^2 + \cdots + x_4^2 - 1 = 0\}$.

In this Minkowski space $\mathbb{R}^5_1$, the hypersurface Λ^4 of the point σ such that $\mathcal{L}(\sigma) = 1$ gives a parameterization of the set of oriented 2-spheres in $\mathbb{S}^3$ as follows. To each point $\sigma \in \Lambda^4$ corresponds an oriented 2-sphere $\Sigma = \sigma^\perp \cap \mathbb{S}^3_\infty$ or $\Sigma = \sigma^\perp \cap \mathbb{S}^3_1$ (see Figure 5). It is convenient to have a formula giving the point $\sigma \in \Lambda^4$ in terms of the Riemannian geometry of the corresponding sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ and a point m on it. For that, we need the unit vector $\boldsymbol{n}$ tangent to $\mathbb{S}^3$ and normal to Σ at m and the geodesic curvature of the sphere Σ.

Proposition 3.1. *The point $\sigma \in \Lambda^4$ corresponding to the 2-sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ is given by*

$$\sigma = k_g m + \boldsymbol{n}. \tag{1}$$

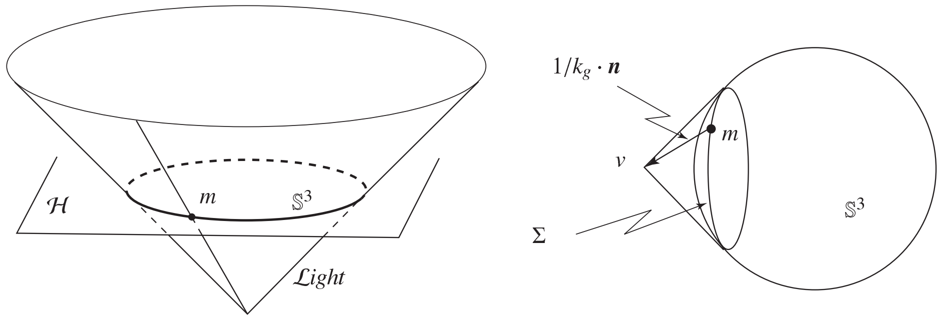

Figure 6 Spherical model in the Lorentz space $\mathbb{R}^5_1$ (left). The geodesic curvature k_g, picture in the affine 4-plane $\mathcal{H}$ (right).

Proof. The proof of Proposition 3.1 can be found in ([5], prop. 3.1). Let us give a sketch of the idea of the proof (see Figure 6).

Let $\mathcal{H}$ be the affine hyperplane such that $\mathbb{S}^3 = \mathcal{L}ight \cap \mathcal{H}$; the restriction of the Lorentz metric to $\mathcal{H}$ is an Euclidean metric. Let P be the hyperplane such that $\Sigma = \mathbb{S}^3 \cap P$. The vertex v of the cone, contained in $\mathcal{H}$, tangent to $\mathbb{S}^3$ along Σ is a point of the line $P^\perp$ which contains the point $\sigma \in \Lambda^4$.

If m is a point on Σ, its (Euclidean) distance $|\overrightarrow{mv}|$ to the vertex v is (up to sign) the inverse R_g of the geodesic curvature k_g of Σ (at m, but not dependent on m). One has $\mathcal{L}(\overrightarrow{mv}) = R_g^2$. Let $\boldsymbol{n}$ be a unit vector tangent to $\mathbb{S}^3$, normal to Σ in the direction compatible with its chosen orientation. The orientation of $\boldsymbol{n}$ and the sign of k_g depend on the orientation of the sphere. As m is light-like and orthogonal to $\overrightarrow{mv} = \dfrac{1}{k_g}\boldsymbol{n}$, one has $\mathcal{L}(v) = R_g^2$. Finally

$$\sigma \;=\; \frac{v}{R_g} \;=\; \frac{1}{R_g}(m + \overrightarrow{mv}) \;=\; k_g m + \boldsymbol{n}.$$

□

This number k_g may be named *geodesic curvature*, as $|k_g|$ is the inverse of the radius of the sphere centered at v passing through m.

Remark 3.2. A similar proposition can be stated for 2-spheres and planes in the Euclidean plane $\mathbb{E}^3$ seen as a paraboloid, a section of the light cone by an affine 4-plane parallel to a 4-plane tangent to the light cone. The argument goes parallel for any dimension.

Definition 3.3. A *pencil of spheres* is a one-parameter family of spheres that correspond to the intersection of Λ^4 with a 2-dimensional plane through the origin.

3.2 Main theorem

Using the space of oriented spheres Λ^4, we shall prove the following theorem which gives the corresponding result for a foliation of (an open set on) the sphere $\mathbb{S}^3$ by 2-dimensional spheres. In this section, curves in Λ^4 and the corresponding families of spheres will be smooth (of class $\mathcal{C}^\infty$). Therefore the leaves of the orthogonal 1-dimensional foliation $\mathcal{F}^\perp$ are also smooth in open domains where the family of spheres define a foliation.

Theorem 3.4. *Consider a one parameter family $\mathcal{S} = \{\Sigma(t)\}_{t\in\mathbb{R}}$ of (2-dimensional) spheres in $\mathbb{S}^3$, and let $\sigma(t)$ be the point of Λ^4 corresponding to $\Sigma(t)$. We assume that the vectors $\sigma(t)$, $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ are linearly independent in $\mathbb{R}^5_1$. Let us denote by $C : t \mapsto c(t)$ the parameterization of a leaf of the foliation $\mathcal{F}^\perp$, a curve such that $c(t) \in \Sigma(t)$ and $\dot{c}(t)$ is orthogonal to the tangent plane of $\Sigma(t)$ at $c(t)$ in $\mathbb{S}^3$ for any t.*

Then all the osculating spheres of the orthogonal trajectories at the points of $\Sigma(t)$, where they can be, are in a same pencil of spheres $\mathcal{P}$. The image of $\mathcal{P}$ in Λ^4 is the intersection of Λ^4 with the orthogonal complement of the 3-dimensional subspace of $\mathbb{R}^5_1$ generated by $\sigma(t)$, $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$.

In particular, although we have 2-dimensional set of orthogonal trajectories, we have only 1-dimensional set of osculating spheres along a leaf (Corollary 3.5).

In what follows, we shall consider $\mathbb{S}^3$ as the intersection of the hyperplane $\mathcal{H}$ defined in $\mathbb{R}^5_1$ by the equation $x_0 = 1$ with the light cone. Any "point in $\mathbb{S}^3$" will be supposed to be in $\mathcal{H} \cap \mathcal{L}ight$. In particular, if a curve $t \mapsto c(t)$ is drawn in $\mathbb{S}^3 \subset \mathcal{H}$, its speed $\dot{c}(t)$ is in the hyperplane defined by $x_0 = 0$ which gives the projective space "at the infinity" of $\mathcal{H}$.

Proof. For a point $m \in \mathbb{S}^3 \subset \mathcal{H}$ and $\sigma \in \Lambda^4$, we know that $m \in \Sigma$ if and only if $\mathcal{L}(\sigma, m) = 0$. Let π_t be the hyperplane of equation $\mathcal{L}(\sigma(t), m) = 0$. Near any point of $\Sigma \subset \mathbb{S}^3$, the *affine* 3*-planes* $\Pi_t = \pi_t \cap \mathcal{H}$ are all transverse to the sphere $\mathbb{S}^3 \subset \mathcal{H}$ and the spheres $\Sigma_t = \Pi_t \cap \mathbb{S}^3$ foliate a neighborhood of m in $\mathbb{S}^3$.

The curve $t \mapsto c(t) \in \mathbb{S}^3 \subset \mathcal{H}$ has a contact of order $\geq k$ with a 3-plane $H = \sigma^\perp \cap \mathcal{H}$ at $c(0)$ if and only if the function $t \mapsto \mathcal{L}(\sigma, c(t))$ and all of its derivatives of order $\leq k$ take the value 0 at t, as the function $t \mapsto \mathcal{L}(\sigma, c(t))$ is a regular smooth function in a neighborhood of 0. As the curve $t \mapsto c(t)$ is drawn on $\mathbb{S}^3$, this means that the curve $t \mapsto c(t)$ has a contact of order $\geq k$ with $H \cap \mathbb{S}^3 = \Sigma$.

For $k = 1$, it means that a sphere $\Sigma_{tang} \subset \mathcal{H}$ is tangent to the arc $C : t \mapsto c(t)$ at $c(t)$ if and only if

$$\mathcal{L}\big(\sigma_{tang}, c(t)\big) = \mathcal{L}\big(\sigma_{tang}, \dot{c}(t)\big) = 0,$$

that is, if and only if σ_{tang} is orthogonal to $c(t)$ and to $\dot{c}(t)$.

For $k = 2$, it means that the sphere $\Sigma_{circ.osc} \subset \mathcal{H}$ contains the osculating circle of the arc $C : t \mapsto c(t)$ at $c(t)$ if and only if

$$\mathcal{L}\big(\sigma_{circ.osc}, c(t)\big) = \mathcal{L}\big(\sigma_{circ.osc}, \dot{c}(t)\big) = \mathcal{L}\big(\sigma_{circ.osc}, \ddot{c}(t)\big) = 0.$$

For $k = 3$, it means that the sphere $\Sigma_{osc} \subset \mathcal{H}$ *is the osculating sphere* of the arc $C : t \mapsto c(t)$ at c if and only if

$$\mathcal{L}\big(\sigma_{osc}, c(t)\big) = \mathcal{L}\big(\sigma_{osc}, \dot{c}(t)\big) = \mathcal{L}\big(\sigma_{osc}, \ddot{c}(t)\big) = \mathcal{L}\big(\sigma_{osc}, \dddot{c}(t)\big) = 0.$$

Then, when the four vectors $c(t), \dot{c}(t), \ddot{c}(t), \dddot{c}(t)$ are independent, the osculating sphere Σ_{osc} to the arc corresponds to the point of Λ^4 orthogonal to the hyperplane $\big\langle c(t), \dot{c}(t), \ddot{c}(t), \dddot{c}(t)\big\rangle$ generated by these four points (the linear independence of the four points is equivalent to the existence and unicity of the osculating sphere).

If the arc $C = \{c(t)\} \subset \mathbb{S}^3 \subset \mathcal{H}$ is orthogonal to $\Sigma(t)$ at the point $c(t)$, we have seen that $\sigma(t) = k_g c(t) + \boldsymbol{n}(t)$, where $\boldsymbol{n}(t)$ is a unit vector orthogonal to $\Sigma(t)$ at $c(t)$.

As C is orthogonal to Σ at $c(t)$, $\boldsymbol{n}(t)$ is proportional to $\dot{c}(t)$, we can write $\dot{c}(t) = \alpha(t)\sigma(t) + \beta(t)c(t)$. This gives an other interpretation of the vector spaces constructed above :

$$\begin{aligned}
\big\langle c(t), \dot{c}(t)\big\rangle &= \ \langle c(t), \sigma(t)\rangle, \\
\big\langle c(t), \dot{c}(t), \ddot{c}(t)\big\rangle &= \ \big\langle c(t), \sigma(t), \dot{\sigma}(t)\big\rangle, \\
\big\langle c(t), \dot{c}(t), \ddot{c}(t), \dddot{c}(t)\big\rangle &= \ \big\langle c(t), \sigma(t), \dot{\sigma}(t), \ddot{\sigma}(t)\big\rangle.
\end{aligned}$$

In particular, the point σ_{osc} corresponding to the osculating sphere Σ_{osc}, orthogonal to this last hyperplane of $\mathbb{R}^5_1$, is orthogonal to the 3-dimensional subspace $\big\langle \sigma(t), \dot{\sigma}(t), \ddot{\sigma}(t)\big\rangle$ of $\mathbb{R}^5_1$ which depends only on the family $\{\Sigma(t)\}_{t\in\mathbb{R}}$. The set of points ζ orthogonal to a 3-dimensional vector space of $\mathbb{R}^5_1$ corresponds to a pencil of spheres in $\mathbb{S}^3$. ☐

Corollary 3.5. *All the curves orthogonal to the initial family of spheres along the circle $\Sigma \cap \Sigma_{osc}(t)$ have the same osculating sphere, $\Sigma_{osc}(t)$.*

Remark 3.6. The coefficients $\alpha(t)$ and $\beta(t)$ of $\dot{c}(t)$ in the proof can be given explicitly. Let e_0 be the unit vector $(1, 0, \ldots, 0)$. Since $\dot{c}(t) \in T_{c(t)}\mathbb{S}^3 \cap \big(T_{c(t)}\Sigma(t)\big)^\perp$, using Gram-Schmidt orthonormalization, we have

$$\dot{c}(t) \parallel \sigma(t) - \mathcal{L}\big(c(t) - e_0, \sigma(t)\big)\, c(t) + \mathcal{L}\big(c(t), \sigma(t)\big)\, e_0.$$

Since

$$\mathcal{L}(c(t), \sigma(t)) = 0 \quad \forall t, \tag{2}$$

we have

$$\mathcal{L}\big(\dot{c}(t), \sigma(t)\big) = -\mathcal{L}\big(c(t), \dot{\sigma}(t)\big). \tag{3}$$

Applying (2), (3) and $\mathcal{L}(\sigma(t), \sigma(t)) = 1$ we obtain

$$\dot{c}(t) = -\mathcal{L}\big(c(t), \dot{\sigma}(t)\big)\big[\sigma(t) + \mathcal{L}(\sigma(t), e_0)\, c(t)\big].$$

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