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# Computable Białynicki-Birula decomposition of the Hilbert scheme

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## Abstract

We observe that there exists a Białynicki-Birula decomposition of the Hilbert scheme $\mathrm{Hilb}_n^P$ such that the cells are homeomorphic to Gröbner strata of homogeneous ideals with fixed initial ideal. Using such a decomposition, we show that $\mathrm{Hilb}_n^P$ is singular at a monomial scheme if the corresponding Gröbner stratum is singular at $J$.


## 1. Introduction

Let $k$ be a field. We consider the Hilbert scheme $\mathrm{Hilb}_n^P$ parameterizing closed subschemes of $\mathbb{P}_k^n$ with Hilbert polynomial $P$. If we fix a term order $\prec$ on the polynomial ring $S = k[x] = k[x_0, \dots, x_n]$, each homogeneous ideal $I$ in $S$ has a unique initial ideal $\mathrm{in}_\prec(I)$ and a unique reduced Gröbner basis $G$ with respect to $\prec$. Hence if we think to the set of $k$-rational points of $\mathrm{Hilb}_n^P$ as represented by a set of homogeneous ideals in $S$, $k$-rational points can be decomposed into loci of homogeneous ideals with fixed initial ideal, called *Gröbner strata* or *Gröbner basis schemes* [NS00, Rob09, RT10]. The main purpose of this article is to discuss this decomposition and its relation with a torus group action on $\mathrm{Hilb}_n^P$ corresponding to $\prec$.

For short, we call Gröbner basis scheme by *Gröbner scheme*. We denote by $\mathrm{Gröb}_\prec^J$ the Gröbner scheme or the Gröbner stratum parameterizing homogeneous ideals in $S$ whose initial ideal is $J$ with respect to $\prec$. In this paper, we call those schemes the Gröbner strata when we think to those schemes as loci of a Hilbert scheme. Otherwise we call $\mathrm{Gröb}_\prec^J$ the Gröbner scheme when we think to $\mathrm{Gröb}_\prec^J$ as the scheme representing the *Gröbner functor* [Led11, LR16]

$$\mathcal{G}\mathrm{röb}_\prec^J : (k\text{-Alg}) \to (\mathrm{Set})$$

$$A \mapsto \left\{ G \subset A[x] \;\middle|\; \begin{array}{l} G \text{ is a homogeneous reduced Gröbner basis} \\ \text{whose } \mathrm{in}_\prec(\langle G \rangle) = J \end{array} \right\}.$$

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The Gröbner scheme $\mathrm{Gröb}^J_\prec$ is computable, in the sense that we can give defining equations of $\mathrm{Gröb}^J_\prec$ and explicitly describe the family of ideals [Rob09, RT10, Led11]. Each Gröbner scheme is a weighted cone in its tangent space on the point corresponding to the monomial ideal [Rob09, RT10]. It implies that $\mathrm{Gröb}^J_\prec$ is smooth at $J$ if and only if $\mathrm{Gröb}^J_\prec$ is isomorphic to an affine space. This property plays a relevant role in this content. For instance, it can be used to decide whether $\mathrm{Gröb}^J_\prec$ is smooth or not. We denote the computation methods in Section 3 that we used for this paper.

To the best knowledge of the author, the study of Gröbner strata began from Notari and Spreafico's results [NS00] which became basic concepts and properties nowadays. Roggero and Terracini dealt with a construction of Gröbner strata from Buchberger's criterion [RT10], Robbiano and Lederer also gave a construction of Gröbner strata from a theory of border basis [Rob09, Led11]. Moreover, [Rob09, RT10] show properties on Gröbner strata about the computability and the smoothness as above. Hence one can consider that the smoothness of Gröbner strata implies the rationality on the Hilbert scheme. Lella and Roggero show the following.

**Theorem 1.1** ([LR11]). *Let $H$ be an irreducible component of $\mathrm{Hilb}^P_n$. Let $r$ be the Gotzmann number of $P$.*

- *If $H$ is smooth, then it is rational.*
- *If $H$ contains a smooth point which corresponds to a $(r, \prec)$-segment ideal (where $\prec$ is any term order), then $H$ is rational.*
- *The Reeves and Stillman component $H_{\mathrm{RS}}$ of $\mathrm{Hilb}^P_n$ is rational.*

*Moreover, if $J$ is a $(r, \prec)$-segment ideal and $\mathrm{Hilb}^P_n$ is smooth at $\mathrm{Proj}\, S/J$, then $\mathrm{Gröb}^J_\prec$ is isomorphic to an affine space.*

The *Gotzmann number* of a Hilbert polynomial $P$ is a number determined by a combinatorial way and is an upper-bound of Castelnuovo-Munfold regularity on the Hilbert scheme $\mathrm{Hilb}^P_n$. By Gotzmann's Persistence Theorem, we have a representation of the $k$-rational points of the Hilbert scheme as the following [LR11]:

$$\mathrm{Hilb}^P_n(k) \cong \left\{ I \subset S \;\middle|\; \begin{array}{l} \bullet\ I \text{ is homogeneous generated by } I_r \\ \bullet\ \dim_k I_r = \dbinom{n+r}{r} - P(r) \\ \bullet\ \dim_k I_{r+1} = \dbinom{n+r+1}{r+1} - P(r+1) \end{array} \right\}. \tag{1.1}$$

Here we mean $I_r = \{f \in I \mid f \text{ is homogeneous of degree } r\}$.

For shot, we denote by $\mathbf{I}_{\mathrm{P,n}}$ the above three conditions. There is an easy proposition about initial ideals of ideals in $\mathrm{Hilb}^P_n(k)$.

**Proposition 1.2.** *Let $I$ be a homogeneous ideal in $S$. Let $\prec$ be a term order on $S$. Then $I$ satisfies $\mathbf{I}_{\mathrm{P,n}}$ if and only if the initial ideal $\mathrm{in}_{\prec} I$ satisfies $\mathbf{I}_{\mathrm{P,n}}$.*

*Proof.* Note that $I$ satisfies $\mathbf{I}_{\mathrm{P,n}}$ if and only if $I$ is the $r$-truncation of a saturated ideal defining an element of $\mathrm{Hilb}_n^P(k)$ [RT10, LR11]. Here we call $I_{\geq r}$ the $r$-truncation of $I$ and we say $I$ is saturated if $I_{\mathrm{sat}} := \{f \in S \mid \exists\, d \geq 0,\ \forall x^\gamma \in S_d,\ x^\gamma f \in I\} = I$.

Put $J = \mathrm{in}_{\prec} I$. Assume that $I$ satisfies $\mathbf{I}_{\mathrm{P,n}}$. For any $s \geq r$, we have $\dim_k(S/J)_s = \dim_k(S/I)_s = P(s)$. The Hilbert polynomial of $\mathrm{Proj}\, S/J$ in $\mathbb{P}_k^n$ is also $P$, then $\dim_k(S/J_{\mathrm{sat}})_s = P(s)$. Thus $(J_{\mathrm{sat}})_{\geq r} = J_{\geq r} = \mathrm{in}_{\prec}(I_{\geq r}) = \mathrm{in}_{\prec}(I) = J$. Conversely, assume that $J$ satisfies $\mathbf{I}_{\mathrm{P,n}}$. Then there exists a saturated monomial ideal $J'$ such that $J = J'_{\geq r}$ and $\mathrm{Proj}\, S/J' \in \mathrm{Hilb}_n^P(k)$. Put $I' = I_{\mathrm{sat}}$. Then for any $s \geq r$, we have $\dim_k(S/I)_s = \dim_k(S/J)_s = \dim_k(S/J')_s = P(s) = \dim_k(S/I')_s$. Therefore we obtain $I = I'_{\geq r}$. □

We define the set of monomial ideals in $S$ satisfying $\mathbf{I}_{\mathrm{P,n}}$ and denote it by $\mathcal{M}_{\mathrm{P,n}}$, i.e.

$$\mathcal{M}_{\mathrm{P,n}} = \left\{ J \subset S \;\middle|\; \begin{array}{l} \bullet\ J \text{ is a monomial ideal generated by } J_r \\ \bullet\ \dim_k J_r = \binom{n+r}{r} - P(r) \\ \bullet\ \dim_k J_{r+1} = \binom{n+r+1}{r+1} - P(r+1) \end{array} \right\} \tag{1.2}$$
$$\cong \{x \in \mathrm{Hilb}_n^P(k) \mid x \text{ corresponds to a monomial ideal}\}.$$

Then we obtain the following decomposition of the $k$-rational points of $\mathrm{Hilb}_n^P$ by Proposition 1.2:

$$\mathrm{Hilb}_n^P(k) = \coprod_{J \in \mathcal{M}_{\mathrm{P,n}}} \mathrm{Gr\ddot{o}b}_{\prec}^J(k). \tag{1.3}$$

We call this decomposition the *Gröbner decomposition* of $\mathrm{Hilb}_n^P$ with respect to $\prec$. Since the set $\mathcal{M}_{\mathrm{P,n}}$ is computable (see Section 3), the Gröbner schemes in the Gröbner decomposition (1.3) is also computable.

**Example 1.3.** We introduce an example of the Gröbner decomposition computed by the methods in Section 3. We consider the Hilbert scheme of $d$ points in $\mathbb{P}_k^2$ and let $\prec$ be the lexicographic order on $k[x_0, x_1, x_2]$. The Hilbert scheme $\mathrm{Hilb}_2^d$ is smooth and its dimension is $2d$ [Har10]. By computing (1.3), in fact, $\mathrm{Gr\ddot{o}b}_{\prec}^J$ is isomorphic to an affine space $\mathbb{A}_k^m$ for any $J \in \mathcal{M}_{d,2}$. We make Table 1 of the numbers of $J \in \mathcal{M}_{d,2}$ such that $\mathrm{Gr\ddot{o}b}_{\prec}^J \cong \mathbb{A}_k^m$. In fact, these numbers are the Betti numbers of $\mathrm{Hilb}_2^d$ [ES87].

In [ES88], Ellingsrud and Strømme give a cell decomposition of $\mathrm{Hilb}_2^d$ using a

Table 1 The numbers of $J \in \mathcal{M}_{d,2}$ such that $\mathrm{Gröb}^J_{\prec} \cong \mathbb{A}^m_k$

| $d \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 3 | 2 | 1 | | | | | | |
| 3 | 1 | 2 | 5 | 6 | 5 | 2 | 1 | | | | |
| 4 | 1 | 2 | 6 | 10 | 13 | 10 | 6 | 2 | 1 | | |
| 5 | 1 | 2 | 6 | 12 | 21 | 24 | 21 | 12 | 6 | 2 | 1 |

result of Białynicki-Birula [BB73, BB76], called *Białynicki-Birula decomposition*. The original Białynicki-Birula decomposition is a cell decomposition of a smooth projective variety with a one dimensional torus action. Since Gröbner strata are naturally equipped with a one dimensional torus action from Gröbner degeneration [Bay82, Rob09, RT10], one may wonder when the Gröbner decomposition (1.3) coincides with a Białynicki-Birula decomposition.

The interest in these object is not new. Evain and Lederer study Gröbner strata and Białynicki-Birula decomposition in Hilbert schemes parameterizing points on an affine space [EL12]. They give a functorial definition of cells of Białynicki-Birula decomposition and they show that such functors are representable by $k$-schemes called *Białynicki-Birula scheme*. For more general setting, Drinfeld [Dri13], Jelisiejew and Sienkiewicz [JS18] study about the Białynicki-Birula decomposition on any (possibly non-smooth) algebraic scheme with an action of $\mathbb{G}_{\mathrm{m}}$ or a reductive group.

Thanks to these studies, the contribution of this paper is to compare a Białynicki-Birula decomposition of $\mathrm{Hilb}^P_n$ with the Gröbner decomposition focusing on their topologies and singularities. Moreover, we will obtain another proof of a result in [LR11] as Theorem 1.5.

**Theorem 1.4.** *For any term order $\prec$, there exists a $\mathbb{G}_{\mathrm{m}}$-action on $\mathrm{Hilb}^P_n$ such that:*

- *the scheme of fixed points is 0-dimensional and the $k$-rational fixed points are the monomial schemes $\{\mathrm{Proj}\, S/J \mid J \in \mathcal{M}_{\mathrm{P,n}}\}$ in $\mathrm{Hilb}^P_n$.*
- *the Białynicki-Birula scheme for a fixed point $\mathrm{Proj}\, S/J$ is homeomorphic to $\mathrm{Gröb}^J_{\prec}$ with the same $k$-rational points in $\mathrm{Hilb}^P_n$.*

*In particular, the Gröbner decomposition*

$$\mathrm{Hilb}^P_n(k) = \coprod_{J \in \mathcal{M}_{\mathrm{P,n}}} \mathrm{Gröb}^J_{\prec}(k).$$

*coincides with the Bialyniciki-Birula decomposition of $\mathrm{Hilb}^P_n$ on the $k$-rational points.*

**Theorem 1.5.** *For any $J \in \mathcal{M}_{\mathrm{P,n}}$, if the Hilbert scheme $\mathrm{Hilb}^P_n$ is smooth at $\mathrm{Proj}\, S/J$, then the Gröbner scheme $\mathrm{Gröb}^J_{\prec}$ is isomorphic to an affine space.*

On Theorem 1.5, we provide an example in where $P = 2t + 2$, $n = 3$ (Example 7.7). Such Hilbert scheme is dealt in a Sernesi's book [Ser06] and Sernesi shows that $\mathrm{Hilb}_3^{2t+2}$ has a singular point defined by a monomial ideal using a obstruction theory. In our example, we find 18 new singular points in $\mathrm{Hilb}_3^{2t+2}$ defined by monomial ideals in $\mathcal{M}_{2t+2,3}$ using Theorem 1.5.

This paper is organized as follows. Section 2 and Section 3 describe preliminaries, notations and computation methods used in this paper. We attach a $\mathbb{G}_{\mathrm{m}}$-action on $\mathrm{Hilb}_n^P$ induced from given term order $\prec$ in Section 4. Then we define the Białynicki-Birula decomposition of $\mathrm{Hilb}_n^P$ in Section 5. In Section 6 and Section 7, we compare topologies and smoothness on $\mathrm{Hilb}_n^P$ and $\mathrm{Gröb}_{\prec}^J$. Then we will obtain Theorem 1.4 and Theorem 1.5.

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## 2. Preliminaries and Notation

- Let $k$ be a field and $S = k[x] = k[x_0, \dots, x_n]$ the polynomial ring over $k$ in $(n+1)$ variables. We always fix a term order $\prec$ on $S$ that is a total order on all monomials in $S$ with $x^\alpha \prec x^\beta \Rightarrow x^\alpha x^\gamma \prec x^\beta x^\gamma$ and $x^\alpha \succ 0$ ($x^\alpha \neq 1$).
- We equip $S$ with the ordinal total degrees of polynomials and denote it by $\deg f$ for $f \in S$. For a subset $A \subset S$, we denote by $A_r$ the homogeneous elements of $A$ with degree $r$ and denote by $\langle A \rangle$ the ideal generated by $A$ in $S$.
- For $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, let $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Using this notation, we regard $\mathbb{N}^{n+1}$ as the set of monomials in $(n+1)$ variables. The degree of $\alpha$ is $|\alpha| = \alpha_0 + \cdots + \alpha_n$. For a subset $A \subset \mathbb{N}^{n+1}$, let $A_r = \{\alpha \in A \mid |\alpha| = r\}$. We also define $A_{\geq r}$ and $A_{\leq r}$ in a similar way.
- For $k$-schemes $X$ and $Y$, let $X(Y) = \mathrm{Hom}_k(Y, X)$. If $Y = \operatorname{Spec} A$, we denote it by $X(A)$ instead.
- We denote by $\mathbb{G}_{\mathrm{m}}$ the one-dimensional algebraic torus $\operatorname{Spec} k[t, t^{-1}]$.

The *Hilbert scheme* $\mathrm{Hilb}_n^P$ is the scheme representing the following *Hilbert functor*:

$$\mathcal{H}\mathrm{ilb}_n^P : (k\text{-Sch}) \to (\mathrm{Set})$$

$$Z \mapsto \left\{ Y \subset \mathbb{P}^n_Z \;\middle|\; \begin{array}{l} Y \text{ is a closed subscheme in } \mathbb{P}^n_Z \text{ flat over } Z, \\ \text{the Hilbert polynomials of all fibers on closed} \\ \text{points of } Z \text{ are } P \end{array} \right\}.$$

The Hilbert scheme $\mathrm{Hilb}^P_n$ is a projective scheme over $k$ [Har10].

The *Gröbner scheme* $\mathrm{Gröb}^J_\prec$ is the scheme representing the following *Gröbner functor*:

$$\mathcal{G}\mathrm{röb}^J_\prec : (k\text{-Alg}) \to (\mathrm{Set})$$

$$A \mapsto \left\{ G \subset A[x] \;\middle|\; \begin{array}{l} G \text{ is a homogeneous reduced Gröbner basis} \\ \text{whose } \mathrm{in}_\prec(\langle G \rangle) = J \end{array} \right\}.$$

See [Wib07] for the definition of Gröbner basis with ring coefficient. If a monomial ideal $J$ defines a point of $\mathrm{Hilb}^P_n$, then there exists a canonical morphism $\mathrm{Gröb}^J_\prec \to \mathrm{Hilb}^P_n$ induced by the natural transformation

$$\mathcal{G}\mathrm{röb}^J_\prec \to \mathcal{H}\mathrm{ilb}^P_n$$

$$G \mapsto \mathrm{Proj}\, A[x]/\langle G \rangle.$$

If we denote a morphism $\mathrm{Gröb}^J_\prec \to \mathrm{Hilb}^P_n$, we always mean this morphism.

Important facts on $\mathrm{Gröb}^J_\prec$ for this content are the following:

**Theorem 2.1** ([LR16]). *If $J \in \mathcal{M}_{\mathrm{P,n}}$, then $\mathrm{Gröb}^J_\prec \to \mathrm{Hilb}^P_n$ is a locally closed immersion. Namely, it can be factored as $j \circ i$ where $i$ is a closed immersion and $j$ is an open immersion.*

**Theorem 2.2** ([Rob09, RT10]). *For any monomial ideal $J$ and any term order $\prec$, the $\mathrm{Gröb}^J_\prec$ is smooth at $J$ if and only if $\mathrm{Gröb}^J_\prec$ is isomorphic to an affine space over $k$.*

## 3. Computation methods

In this section, we introduce computation methods for Gröbner schemes and Gröbner decompositions. We mainly refer to [RT10] for this section.

Let $I$ be a homogeneous ideal in $S$ with $\mathrm{in}_\prec I = J$. Let $B_J$ be the minimal generators of $J$. Then the reduced Gröbner basis of $I$ is in the following form:

$$G = \left\{ g_\alpha = x^\alpha + \sum_{\substack{x^\beta \notin B_J \\ x^\beta \prec x^\alpha}} C_{\alpha,\beta} x^\beta \;\middle|\; \alpha \in B_J, g_\alpha \text{ is homogeneous} \right\}.$$

Conversely, for given $\prec$ and $J$, let us consider the form $G$ as the above with arbitrary coefficients $C_{\alpha,\beta} \in k$. Using the Buchberger's criterion, we will obtain the polynomials $F_1, \ldots, F_s$ of the coefficients $C_{\alpha,\beta}$ such that $G$ is a Gröbner basis of $\langle G \rangle$ if and only if the polynomials $F_1, \ldots, F_s$ are vanish. In fact, by regarding the coefficients $C_{\alpha,\beta}$ as the variables, the polynomials $F_1, \ldots, F_s$ only depend on the choice of $\prec$ and $J$ [RT10]. Therefore $F_1, \ldots, F_s$ define the Gröbner scheme $\mathrm{Gr\ddot{o}b}^J_\prec$.

**Theorem 3.1** ([RT10])**.** *The Gröbner scheme* $\mathrm{Gr\ddot{o}b}^J_\prec$ *is isomorphic to the affine scheme*

$$\operatorname{Spec} k[C_{\alpha,\beta} \mid x^\alpha \in B_J, x^\beta \notin B_J, x^\alpha \succ x^\beta, \deg x^\alpha = \deg x^\beta]/\langle F_1, \ldots, F_s \rangle.$$

Therefore we can compute $\mathrm{Gr\ddot{o}b}^J_\prec$ by computing syzygies of the form $G$. Note that also we can compute $\mathrm{Gr\ddot{o}b}^J_\prec$ by a theory of *border basis* [Rob09, Led11].

For computing the tangent space of $\mathrm{Gr\ddot{o}b}^J_\prec$ on $J$, denoted by $T^J_\prec$, we recall a positive grading on $k[C]$, where

$$C = \{C_{\alpha,\beta} \mid x^\alpha \in B_J, x^\beta \notin B_J, x^\alpha \succ x^\beta, \deg x^\alpha = \deg x^\beta\},$$

and a theory of homogeneous varieties given in [FR09, RT10].

**Proposition 3.2** ([Bay82, Proposition 1.8])**.** *Let $\prec$ be a term order on $S$, and let $A$ be a finite subset of $\mathbb{N}^{n+1}$. Then there exists a vector $\omega \in \mathbb{N}^{n+1}$ such that for any $\alpha, \beta \in A$, $\alpha \prec \beta$ if and only if $\omega \cdot \alpha < \omega \cdot \beta$. Here $\omega \cdot \alpha$ is the ordinary inner product $\omega_0\alpha_0 + \cdots \omega_n\alpha_n$.*

Let $r$ be the maximum degree in $B_J$. We fix a vector $\omega \in \mathbb{N}^{n+1}$ given by Proposition 3.2 for fixed term order $\prec$ and the finite subset $A = (\mathbb{N}^{n+1})_{\leq r}$. Then this vector $\omega$ implies a positive grading on $R$ as follows.

**Proposition 3.3** ([RT10, Lemma 3.2])**.** *Let $J \in \mathcal{M}_{\mathrm{P,n}}$. We define a grading $\Lambda$ on $K[C]$ such that $\Lambda(C_{\alpha,\beta}) = \omega \cdot \alpha - \omega \cdot \beta$. Then the defining ideal of* $\mathrm{Gr\ddot{o}b}^J_\prec$ *in* $\operatorname{Spec} k[C]$ *is $\Lambda$-homogeneous.*

**Definition 3.4** ([RT10, Definition 3.3])**.** Let $C$ be a set of variables and $k[C]$ be a polynomial ring with a positive grading $\Lambda$. Let $\mathcal{A}$ be a $\Lambda$-homogeneous ideal in $k[C]$.

- For a polynomial $F \in k[C]$, we denote by $L(F)$ the linear component of $F$, i.e. the sum of terms of usual degree 1 that appear in $F$. We also denote by $L(A)$ the $k$-vector space $\{L(F) \mid F \in \mathcal{A}\}$.
- A subset $C'$ of $C$ is *eliminable variables* if for any $c' \in C'$, $L(A)$ contains elements of the type $c' + l$ with $l \in k[C \setminus C']$. $C'$ is a *maximal* set of eliminable variables if $\#(C') = \dim_k(L(\mathcal{A}))$.

**Theorem 3.5** ([RT10, Proposition 3.4, Corollary 3.6]). *Let* $C = \{C_{\alpha,\beta} \mid x^\alpha \in \mathcal{B}_J, x^\beta \notin \mathcal{B}_J, x^\alpha \succ x^\beta, \deg x^\alpha = \deg x^\beta\}$*. Let* $\mathcal{A} \subset k[C]$ *be the defining ideal of the Gröbner scheme* $\mathrm{Gröb}^J_\prec$ *in* $\mathrm{Spec}\, k[C]$*. If* $C' = \{c'_1, \ldots, c'_r\}$ *is a maximal set of eliminable variables for* $\mathcal{A}$*, then the reduced Gröbner basis* $\mathcal{G}$ *of* $\mathcal{A}$ *with respect to any elimination order of the variables* $C'$ *in the following form:*

$$\mathcal{G} = \{c'_1 + g_1, \ldots, c'_r + g_r, f_1, \ldots, f_d\}$$

*with* $g_i \in k[C \setminus C']$ *and* $f_j \in \langle C \setminus C' \rangle^2 \subset k[C \setminus C']$*. In particular,* $\mathrm{Gröb}^J_\prec \cong \mathrm{Spec}\, k[C \setminus C']/\langle f_1, \ldots, f_d\rangle$ *Moreover, the affine scheme* $\mathrm{Spec}\, k[C \setminus C']$ *is isomorphic to the tangent space of* $T^J_\prec$ *as schemes. Then* $\mathrm{Gröb}^J_\prec$ *is non-singular at* $J$ *if and only if* $f_1 = \cdots = f_d = 0$.

We can find a maximal set of eliminable variables by the following method.

- A basis of $L(\mathcal{A})$: Since $J \in \mathrm{Gröb}^J_\prec$ as the origin of $\mathrm{Spec}\, k[C]$, the defining ideal $\mathcal{A}$ is contained in $\langle C \rangle$. Thus generators of $\mathcal{A}$ do not contain constant terms. Then the linear components of generators of $\mathcal{A}$ is a basis of $L(\mathcal{A})$.
- Gaussian reduction: Let us consider $L(\mathcal{A})$ as vector subspace in the vector space $V = \{\sum_{c \in C} a_c c \mid a_c \in k\}$ with canonical basis $C$. Let $\{b_1, \ldots, b_s\}$ be a basis of $L(\mathcal{A})$. Assume that $b_i = \sum_{c \in C} b_{i,c} c$ and consider the matrix $B = (b_{i,c})$. Doing Gaussian reduction on the rows of $B$, we will obtain new basis $\{e_1, \ldots, e_s\}$ of $L(\mathcal{A})$ and $s$ variables $c'_1, \ldots, c'_s$ such that $e_i = c'_i + l_i$ with $l_i \in k[C \setminus C']$. Therefore $C'$ is a maximal eliminable variables.

Then we can compute the embedding $\mathrm{Gröb}^J_\prec \hookrightarrow T^J_\prec$ with its defining ideal and can determine if $\mathrm{Gröb}^J_\prec$ is non-singular or not.

Next we introduce a computation methods for the Gröbner decomposition of the Hilbert scheme

$$\mathrm{Hilb}^P_n(k) = \coprod_{J \in \mathcal{M}_{\mathrm{P,n}}} \mathrm{Gröb}^J_\prec(k). \tag{3.1}$$

**Proposition 3.6** ([Vas98, Corollary B.5.1]). *Let* $P$ *be the Hilbert polynomial of a projective scheme in a projective space. Then there exist integers* $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ *such that*

$$P(t) = \sum_{i=1}^{r} \binom{t + a_i - i + 1}{a_i}.$$

We call the number $r$ in the above the *Gotzmann number* of $P$. The procedure for the Gotzmann number is the following.

(1) Put $a_1 = \deg P$ and $P_1 = P - \binom{t+a_1-1+1}{a_1}$.
(2) If we have obtained $P_i$ and $P_i$ is not constant, then put $a_{i+1} = \deg P_i$ and $P_{i+1} = P_i - \binom{t+a_{i+1}-(i+1)+1}{a_{i+1}}$.

(3) Repeat (2) until $P_s = c$ is constant. Then the Gotzmann number of $P$ is $r = s + c$.

Finally we introduce the procedure for the set of monomials $\mathcal{M}_{\mathrm{P,n}}$:

$$\mathcal{M}_{\mathrm{P,n}} = \left\{ J \subset S \;\middle|\; \begin{array}{l} \bullet\ J \text{ is a monomial ideal generated by } J_r \\ \bullet\ \dim_k J_r = \binom{n+r}{r} - P(r) \\ \bullet\ \dim_k J_{r+1} = \binom{n+r+1}{r+1} - P(r+1) \end{array} \right\}$$
$$\cong \{x \in \mathrm{Hilb}_n^P(k) \mid x \text{ corresponds to a monomial ideal}\}.$$

The procedure for $\mathcal{M}_{\mathrm{P,n}}$ is the following.

(1) Determine the Gotzmann number $r$ of $P$.
(2) Make the list $L$ of sets of $\left(\binom{n+r}{r} - P(r)\right)$ monomials of degree $r$ in $(n+1)$ variables.
(3) Set $\mathcal{M}_{\mathrm{P,n}} = \{\}$. For each $B \in L$, compute $\{x_0, \dots, x_n\} \cdot B = \{x_i x^\gamma \mid i = 0, \dots, n, x^\gamma \in B\}$. If $\#(\{x_0, \dots, x_n\} \cdot B) = \binom{n+r+1}{r+1} - P(r+1)$, then add $J = \langle B \rangle$ to $\mathcal{M}_{\mathrm{P,n}}$.

Therefore we can compute the Gröbner schemes in the Gröbner decomposition of $\mathrm{Hilb}_n^P$ (3.1).

## 4. $\mathbb{G}_{\mathrm{m}}$-action on the Hilbert scheme corresponding to a monomial order

Let $J \in \mathcal{M}_{\mathrm{P,n}}$. We fix a vector $\omega \in \mathbb{N}^{n+1}$ given by Proposition 3.2 for fixed term order $\prec$ and the finite subset $A = (\mathbb{N}^{n+1})_r$, where $r$ is the Gotzmann number of $P$. By Proposition 3.3, there is a grading $\Lambda$ on the polynomial ring

$$R = k[C_{\alpha,\beta} \mid x^\alpha \in B_J, x^\beta \notin B_J, x^\alpha \succ x^\beta, \deg x^\alpha = \deg x^\beta = r]$$

such that $\Lambda(C_{\alpha,\beta}) = \omega \cdot \alpha - \omega \cdot \beta$. We attach a $\mathbb{G}_{\mathrm{m}}$-action on $\operatorname{Spec} R$ such that $t \cdot C_{\alpha,\beta} = t^{\Lambda(C_{\alpha,\beta})} C_{\alpha,\beta} = t^{\omega\cdot\alpha - \omega\cdot\beta} C_{\alpha,\beta}$. Then Proposition 3.3 says that $\mathrm{Gröb}_\prec^J$ is $\mathbb{G}_{\mathrm{m}}$-invariant in $\operatorname{Spec} R$. The vector $\omega$ also defines a $\mathbb{G}_{\mathrm{m}}$-action on $S$ by $t \cdot x^\alpha = t^{-\omega\cdot\alpha} x^\alpha$. Therefore there exists a $\mathbb{G}_m$-action on the Hilbert scheme $\mathrm{Hilb}_n^P$ as the set of saturated homogenous ideals in $S$. Namely, for a element $Y = \operatorname{Proj} A[x]/I \in \mathcal{H}\mathrm{ilb}_n^P(A)$, we define $t \cdot Y = \operatorname{Proj} A[x]/(t \cdot I_{\mathrm{sat}})$.

These $\mathbb{G}_{\mathrm{m}}$-actions on $\mathrm{Gröb}_\prec^J$ and $\mathrm{Hilb}_n^P$ are compatible with each other under the canonical morphism.

**Proposition 4.1.** *If $J \in \mathcal{M}_{\mathrm{P,n}}$, then the canonical morphism* $\mathrm{Gröb}_\prec^J \to \mathrm{Hilb}_n^P$ *is a $\mathbb{G}_m$-equivariant morphism.*

*Proof.* For each reduced Gröbner basis

$$G = \left\{ g_\alpha = x^\alpha - \sum_{\beta \in N_J} a_{\alpha,\beta} x^\beta \;\middle|\; x^\alpha \in B_J \right\} \in \mathcal{G}\text{röb}^J_{\prec}(A),$$

we have

$$t \cdot g_\alpha = t^{-\omega\cdot\alpha} x^\alpha - \sum_{\beta \in N_J} t^{-\omega\cdot\beta} a_{\alpha,\beta} x^\beta \quad (t \in A^\times)$$

under the $\mathbb{G}_m$-action on $A[x]$. Let $I$ be the ideal generated by $G$, and let $Y = \operatorname{Proj} A[x]/I$. Then $t \cdot I = \{t \cdot f \mid f \in I\}$ is generated by the set $\{x^\alpha - \sum_{\beta \in N_J} t^{\omega\cdot\alpha - \omega\cdot\beta} a_{\alpha,\beta} x^\beta \mid x^\alpha \in B_J\}$. This set is $t \cdot G$ under the $\mathbb{G}_{\mathrm{m}}$-action on $\text{Gröb}^J_{\prec}$. Taking an integer $s \geq 0$ such that $I_{\geq s} = (I_{\mathrm{sat}})_{\geq s}$, then we obtain

$$t \cdot Y = \operatorname{Proj} A[x]/(t \cdot I_{\mathrm{sat}}) = \operatorname{Proj} A[x]/(t \cdot I_{\mathrm{sat}})_{\geq s} = \operatorname{Proj} A[x]/(t \cdot I).$$

Note that we use $(t \cdot I)_{\geq s} = t \cdot I_{\geq s}$ in the last. Therefore the morphism $\text{Gröb}^J_{\prec} \to \text{Hilb}^P_n$ is a $\mathbb{G}_m$-equivariant morphism. □

From now on, we always attach $\mathbb{G}_{\mathrm{m}}$-actions on $\text{Gröb}^J_{\prec}$ and $\text{Hilb}^P_n$ introduced in the above for given term order $\prec$.

## 5. Białynicki-Birula schemes in the Hilbert scheme

Let $X$ be a scheme locally of finite type over $k$ equipped with a $\mathbb{G}_{\mathrm{m}}$-action. For any $k$-scheme $Y$, we attach a $\mathbb{G}_{\mathrm{m}}$-action on $Y$ as the projection $\mathbb{G}_{\mathrm{m}} \times_k Y \to Y$. We also attach the trivial $\mathbb{G}_m$-action on $\mathbb{A}^1_k \times_k Y$ induced by the canonical $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{A}^1_k$.

The *scheme of fixed points* is defined as the subscheme $X^{\mathbb{G}_{\mathrm{m}}}$ such that for any $k$-scheme $Y$,

$$X^{\mathbb{G}_{\mathrm{m}}}(Y) = \{\varphi \in X(Y) \mid \varphi \text{ is } \mathbb{G}_{\mathrm{m}}\text{-equivariant}\}.$$

The scheme of fixed points exists and it is a closed subscheme of $X$ [Dri13, Proposition 1.2.2].

We define the *scheme of attractors* in $X$ as the scheme $X^+$ such that for any $k$-scheme $Y$,

$$X^+(Y) \cong \{\varphi : \mathbb{A}^1_k \times_k Y \to X \mid \varphi \text{ is } \mathbb{G}_{\mathrm{m}}\text{-equivariant}\}.$$

The scheme of attractors exists and it is locally of finite type over $k$ [Dri13, Corollary 1.4.3], [JS18, Theorem 6.17].

**Proposition 5.1.** *The scheme of fixed points of the Hilbert scheme* $\text{Hilb}^P_n$ *satisfies* $(\text{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}(k) = \{\operatorname{Proj} S/J \mid J \in \mathcal{M}_{\mathrm{P,n}}\}$.

*Proof.* Consider the Gröbner decomposition, we have

$$\coprod_{J\in\mathcal{M}_{\mathrm{P,n}}}\{J\}=\coprod_{J\in\mathcal{M}_{\mathrm{P,n}}}(\mathrm{Gröb}^J_{\prec})^{\mathbb{G}_\mathrm{m}}(k)=(\mathrm{Hilb}^P_n)^{\mathbb{G}_\mathrm{m}}(k).$$

The right side equality comes from $(\mathrm{Gröb}^J_{\prec})^{\mathbb{G}_\mathrm{m}}=(\mathrm{Hilb}^P_n)^{\mathbb{G}_\mathrm{m}}\cap\mathrm{Gröb}^J_{\prec}$ since $\mathrm{Gröb}^J_{\prec}\to\mathrm{Hilb}^P_n$ is $\mathbb{G}_\mathrm{m}$-equivalent (Proposition 4.1). □

We obtain a canonical morphism $i_X : X^+\to X$ by taking restrictions to 1:

$$i_X(Y):X^+(Y)\to X(Y)$$

$$\varphi\mapsto\varphi_{|\{1\}\times_k Y}.$$

If $X$ is separated, then this map $i_X(Y)$ is an injection for any $Y$ [Dri13, Proposition 1.4.11], i.e. $i_X$ is a *monomorphism* in ($k$-Sch).

We also obtain a canonical morphism $\pi_X : X^+\to X^{\mathbb{G}_\mathrm{m}}$ by taking restrictions to 0:

$$\pi_X(Y):X^+(Y)\to X^{\mathbb{G}_\mathrm{m}}(Y)$$

$$\varphi\mapsto\varphi_{|\{0\}\times_k Y}.$$

This morphism $\pi_X$ is $\mathbb{G}_\mathrm{m}$-equivariant and affine of finite type over $k$ [JS18, Theorem 6.17].

We describe the connected components of $X^{\mathbb{G}_\mathrm{m}}$ by $F_1,\dots,F_r$. The *Białynicki-Birula schemes* are defined as the preimages of components under $\pi_X$. More precisely, the Białynicki-Birula scheme $X_i^+$ is the subscheme of $X^+$ such that

$$X_i^+(Y)=\{\varphi\in X^+(Y)\mid\pi_X(\varphi)\in F_i(Y)\}.$$

For short, we call a Białynicki-Birula scheme by BB scheme.

If $X$ is the Hilbert scheme $\mathrm{Hilb}^P_n$, then each connected component of $X^{\mathbb{G}_\mathrm{m}}$ is a point corresponding to a monomial ideal in $\mathcal{M}_{\mathrm{P,n}}$ (see Section 6). We denote the BB scheme for $J\in\mathcal{M}_{\mathrm{P,n}}$ by $\mathrm{BB}^J_\omega$, where $\omega$ is fixed vector induced by fixed term order $\prec$ in Proposition 3.2.

**Theorem 5.2.** *Let $J\in\mathcal{M}_{\mathrm{P,n}}$. Then*

$$\mathrm{BB}^J_\omega(k)=\mathrm{Gröb}^J_{\prec}(k)$$

*in* $\mathrm{Hilb}^P_n(k)$.

*Proof.* Taking Gröbner degenerations [Bay82, Proposition 2.12], there exists a monomorphism $\mathrm{Gröb}^J_{\prec}\to\mathrm{BB}^J_\omega$. Then we obtain $\mathrm{Gröb}^J_{\prec}(k)\subset\mathrm{BB}^J_\omega(k)$. Conversely, for any $\varphi\in\mathrm{BB}^J_\omega(k)$, put $Y=\varphi_{|\{1\}}\in\mathrm{Hilb}^P_n(k)$ and assume that

$Y \in \mathrm{Gröb}^{J'}_{\prec}(k)$ for some $J' \in \mathcal{M}_{\mathrm{P,n}}$. Then taking the Gröbner degeneration of $Y$, there exists a $\mathbb{G}_{\mathrm{m}}$-equivariant morphism $\psi : \mathbb{A}^1_k \to \mathrm{Hilb}^P_n$ such that $\psi_{|\{1\}} = Y$ and $\psi_{|\{0\}} = \mathrm{Proj}\, S/J'$. Since $i_{\mathrm{Hilb}^P_n} : (\mathrm{Hilb}^P_n)^+ \to \mathrm{Hilb}^P_n; \rho \mapsto \rho_{|\{1\}}$ is a monomorphism, we obtain $\varphi = \psi$. Therefore $J = J'$. □

## 6. Topologies on base spaces

The schemes $\mathrm{Hilb}^P_n$, $\mathrm{Gröb}^J_{\prec}$ and $\mathrm{BB}^J_{\omega}$ are universal for base changes. Namely, we have

$$\mathrm{Hilb}^P_{n,k}(K) \cong (\mathrm{Hilb}^P_n \times_k \mathrm{Spec}\, K)(K) \cong \mathrm{Hilb}^P_{n,K}(K)$$

and so on for any field extension $k \subset K$. Then the representation (1.1), the Gröbner decomposition, Proposition 5.1 and Theorem 5.2 are still true on $K$-rational points for any field extension $k \subset K$. In particular, those propositions are true on *geometric points*. Our purpose in this section is to discuss about topologies on base spaces of $\mathrm{Hilb}^P_n$, $\mathrm{Gröb}^J_{\prec}$ and $\mathrm{BB}^J_{\omega}$ using the following lemma.

**Lemma 6.1.** ([Sta20, Tag 0485]) *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is locally of finite type. Then following are equivalent:*

(1) *$f$ is surjective, and*
(2) *for every algebraically closed field $k$ over $S$ the induced map $X(k) \to Y(k)$ is surjective.*

**Proposition 6.2.** *The scheme of fixed points $(\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}$ is 0-dimensional. In particular, $(\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}$ is finite and discrete.*

*Proof.* Put $X = \coprod_{J \in \mathcal{M}_{\mathrm{P,n}}} (\mathrm{Gröb}^J_{\prec})^{\mathbb{G}_{\mathrm{m}}}$. Clearly we have $\dim X = 0$. In particular, $X$ is finite over $k$. Thus the morphism $f : X \to (\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}$ is of finite type over $k$. By Lemma 6.1 and Proposition 5.1, we obtain that $f$ is surjective. Then for any irreducible closed subset $V$ in $(\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}$ consists of only one point since $V = f(f^{-1}(V))$ with a finite set $f^{-1}(V)$. Therefore $\dim(\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}} = 0$. □

**Corollary 6.3.** *The BB scheme $\mathrm{BB}^J_{\omega}$ is an affine scheme of finite type over $k$.*

*Proof.* It is immediate by that $\pi_{\mathrm{Hilb}^P_n} : (\mathrm{Hilb}^P_n)^+ \to (\mathrm{Hilb}^P_n)^{\mathbb{G}_{\mathrm{m}}}$ is affine of finite type over $k$ and Proposition 6.2 □

We show that $\mathrm{Gröb}^J_{\prec}$ and $\mathrm{BB}^J_{\omega}$ are homeomorphic.

**Theorem 6.4.** *The morphism taking Gröbner degenerations*

$$\mathrm{Gröb}^J_{\prec} \to \mathrm{BB}^J_{\omega}$$

*is a homeomorphism on base spaces.*

*Proof.* Since

$$\begin{array}{ccc} \mathrm{Gr\ddot{o}b}^J_{\prec} & \longrightarrow & \mathrm{BB}^J_{\omega} \\ & \searrow & \downarrow{\scriptstyle i} \\ & & \mathrm{Hilb}^P_n \end{array}$$

is commutative, if $i = i_{\mathrm{Hilb}^P_n} : \mathrm{BB}^J_{\omega} \to \mathrm{Hilb}^P_n$ is separated, then $\mathrm{Gr\ddot{o}b}^J_{\prec} \to \mathrm{BB}^J_{\omega}$ is also a locally closed immersion [Sta20, Tag 03HB], and therefore $\mathrm{Gr\ddot{o}b}^J_{\prec} \to \mathrm{BB}^J_{\omega}$ is a homeomorphism (Theorem 5.2, Lemma 6.1). Hence it is enough to show that $i : \mathrm{BB}^J_{\omega} \to \mathrm{Hilb}^P_n$ is separated. Indeed, $\mathrm{BB}^J_{\omega}$ and $\mathrm{Hilb}^P_n$ are separated over $\operatorname{Spec} k$ (Corollary 6.3) and $i : \mathrm{BB}^J_{\omega} \to \mathrm{Hilb}^P_n$ is a $k$-scheme morphism. Therefore $i : \mathrm{BB}^J_{\omega} \to \mathrm{Hilb}^P_n$ is also separated by basic properties on separated morphisms [Har77, II. Corollary 4.6]. ☐

## 7. Smoothness at monomial schemes

We recall the following Białynicki-Birula's result.

**Theorem 7.1.** ([BB73, BB76], see also [Dri13, JS18]) *Let $X$ be a smooth projective scheme over an algebraically closed field $k$ equipped with a $\mathbb{G}_{\mathrm{m}}$-action. We assume that* $\dim X^{\mathbb{G}_{\mathrm{m}}} = 0$. *Then there exist closed subschemes $Z_0 \supset \cdots \supset Z_q$ such that*

- *$Z_0 = X$ and $Z_q = \emptyset$,*
- *each $Z_i \setminus Z_{i+1}$ is a BB scheme in $X$,*
- *any BB scheme is isomorphic to an affine space over $k$.*

*Therefore $X$ has a cell decomposition (see [Ful98] for the definition).*

A part of the above Białynicki-Birula's result is generalized as follows.

**Theorem 7.2.** ([JS18, Corollary 7.3]) *Suppose that $X$ is smooth over $k$. Then $\pi_X : X^+ \to X^{\mathbb{G}_{\mathrm{m}}}$ is an affine fiber bundle. Moreover, both $X^{\mathbb{G}_{\mathrm{m}}}$ and $X^+$ are smooth.*

Using Theorem 7.2, the next purpose is to show that the smoothness of $\mathrm{Hilb}^P_n$ at $J \in \mathcal{M}_{\mathrm{P,n}}$ implies the smoothness of $\mathrm{Gr\ddot{o}b}^J_{\prec}$.

**Theorem 7.3.** *For any $J \in \mathcal{M}_{\mathrm{P,n}}$, if* $\mathrm{BB}^J_{\omega}$ *is smooth at* $\operatorname{Proj} S/J$, *then* $\mathrm{Gr\ddot{o}b}^J_{\prec}$ *is isomorphic to an affine space.*

*Proof.* Let $T_G$ be the Zariski tangent space on $\mathrm{Gr\ddot{o}b}^J_{\prec}$ at $J$ and $T_B$ the Zariski tangent space on $\mathrm{BB}^J_\omega$ at $\mathrm{Proj}\, S/J$. We claim that the $k$-linear map $T_G \to T_B$ induced by $\mathrm{Gröb}^J_\prec \to \mathrm{BB}^J_\omega$ is injective. Indeed, we can naturally regard $T_G$ and $T_B$ as the subsets of $\mathrm{Hom}_k(\mathrm{Spec}\, k[\varepsilon]/\langle\varepsilon^2\rangle, \mathrm{Gröb}^J_\prec)$ and $\mathrm{Hom}_k(\mathrm{Spec}\, k[\varepsilon]/\langle\varepsilon^2\rangle, \mathrm{BB}^J_\omega)$ respectively [Har77], and the morphism $\mathrm{Gröb}^J_\prec \to \mathrm{BB}^J_\omega$ is a monomorphism since $J \in \mathcal{M}_{\mathrm{P,n}}$. In fact, there exists a closed immersion $\mathrm{Gröb}^J_\prec \to T_G$ as schemes and it is an isomorphism if $\dim \mathrm{Gröb}^J_\prec = \dim_k T_G$ [FR09, RT10]. Therefore we obtain

$$\dim \mathrm{Gröb}^J_\prec \le \dim_k T_G \le \dim_k T_B \le \dim \mathrm{BB}^J_\omega = \dim \mathrm{Gröb}^J_\prec.$$

Note that the last equality comes from Theorem 6.4. Then $\mathrm{Gröb}^J_\prec$ is an affine space. □

**Proposition 7.4.** ([JS18, Proposition 5.2]) *Let $f : X \to Y$ be a $\mathbb{G}_\mathrm{m}$-equivariant morphism. If $f$ is an open immersion, then the induced morphism $f^+ : X^+ \to Y^+$ is also an open immersion.*

**Proposition 7.5.** *Let $X$ be a locally of finite type scheme over $k$ equipped with a $\mathbb{G}_\mathrm{m}$-action and $x \in X^{\mathbb{G}_\mathrm{m}}$. Assume that $\dim X^{\mathbb{G}_\mathrm{m}} = 0$ and $X$ is smooth at $x$. Then the BB scheme $X^+_x$ for $x$ is smooth at $x$.*

*Proof.* Let $U$ be the smooth locus of $X$. Then $U$ is $\mathbb{G}_\mathrm{m}$-invariant, smooth and open in $X$. By Proposition 7.4, $U^+$ is open in $X^+$. Then the BB scheme $U^+_x = (\pi_U)^{-1}(x) = U^+ \cap (\pi_X)^{-1}(x)$ is also open in $X^+_x$. Since $U^+_x$ is smooth by Theorem 7.2, $X^+_x$ is smooth at $x$. □

Therefore we obtain the following corollary by Theorem 7.3 and Proposition 7.5.

**Corollary 7.6.** *For any $J \in \mathcal{M}_{\mathrm{P,n}}$, if the Hilbert scheme $\mathrm{Hilb}^P_n$ is smooth at $\mathrm{Proj}\, S/J$, then the Gröbner scheme $\mathrm{Gröb}^J_\prec$ is isomorphic to an affine space.*

The converse is not true by the following example.

**Example 7.7.** In [Ser06], Sernesi shows that the Hilbert scheme $\mathrm{Hilb}^{2t+2}_3$ is singular at a point defined by a monomial ideal. To find other singular points, let us compute the Gröbner decomposition of $\mathrm{Hilb}^{2t+2}_3$ with respect to the reverse lexicographic order $\prec = \prec_{rvlex}$ on $k[x, y, z, w]$ such that $x \succ y \succ z \succ w$. Then we obtain:

- $\#(\mathcal{M}_{2t+2,3}) = 159$.
- The 144 monomial ideals in $\mathcal{M}_{2t+2,3}$ define smooth Gröbner schemes. The dimensions are in Table 2.
- The following 15 monomial ideals in $\mathcal{M}_{2t+2,3}$ define singular Gröbner schemes:

$$
\begin{aligned}
J_1 &= \langle w^3, zw^2, yw^2, yzw, y^2w, y^2z, y^3, xw^2, xyw, xyz, xy^2, x^2y\rangle,\\
J_2 &= \langle w^3, zw^2, yw^2, xw^2, xzw, xz^2, xyw, xyz, x^2w, x^2z, x^2y, x^3\rangle,\\
J_3 &= \langle w^3, zw^2, yw^2, xw^2, xzw, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle,\\
J_4 &= \langle zw^2, z^2w, yzw, xw^2, xzw, xz^2, xyw, xyz, x^2w, x^2z, x^2y, x^3\rangle,\\
J_5 &= \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyz, xy^2, x^2z\rangle,\\
J_6 &= \langle z^2w, z^3, yzw, yz^2, y^2w, y^2z, y^3, xz^2, xyw, xyz, xy^2, x^2y\rangle,\\
J_7 &= \langle z^2w, z^3, yzw, yz^2, y^2z, xzw, xz^2, xyw, xyz, xy^2, x^2z, x^2y\rangle,\\
J_8 &= \langle z^2w, z^3, yzw, yz^2, y^2z, xzw, xz^2, xyz, x^2w, x^2z, x^2y, x^3\rangle,\\
J_9 &= \langle z^2w, z^3, yz^2, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle,\\
J_{10} &= \langle yw^2, yzw, y^2w, y^2z, y^3, xw^2, xzw, xyw, xyz, xy^2, x^2w, x^2y\rangle,\\
J_{11} &= \langle yw^2, yzw, y^2w, xw^2, xzw, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle,\\
J_{12} &= \langle yzw, yz^2, y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2z, x^2y\rangle,\\
J_{13} &= \langle yzw, yz^2, y^2w, y^2z, y^3, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle,\\
J_{14} &= \langle yzw, yz^2, y^2z, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle,\\
J_{15} &= \langle y^2w, y^2z, y^3, xzw, xz^2, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle.
\end{aligned}
$$

Therefore $\mathrm{Hilb}_3^{2t+2}$ includes 15 singular points defined by the above 15 monomial ideals.

Let us change the monomial order to the lexicographic order $\prec=\prec_{lex}$. Then:

- The 143 monomial ideals in $\mathcal{M}_{2t+2,3}$ define smooth Gröbner schemes. The dimensions are in Table 2.
- The following 16 monomial ideals in $\mathcal{M}_{2t+2,3}$ define singular Gröbner schemes:

$$
\begin{aligned}
&J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_9, J_{10}, J_{11}, J_{12}, J_{14}, J_{15} \text{ and}\\
&J_{16} = \langle w^3, zw^2, yw^2, yzw, y^2w, y^2z, y^3, xw^2, xzw, xyw, xy^2, x^2w\rangle,\\
&J_{17} = \langle z^2w, z^3, yz^2, xw^2, xzw, xz^2, xyw, xyz, x^2w, x^2z, x^2y, x^3\rangle,\\
&J_{18} = \langle y^2w, y^2z, y^3, xw^2, xzw, xyw, xyz, xy^2, x^2w, x^2z, x^2y, x^3\rangle.
\end{aligned}
$$

The consequence is that $\mathrm{Hilb}_3^{2t+2}$ includes 18 singular points defined by the above 18 monomial ideals. The Sernesi's example is defined by a saturated monomial ideal $J = \langle yz, yw, zw, w^2\rangle$. In fact, the Gröbner scheme determined by $J_{\geq 3} \in \mathcal{M}_{2t+2,3}$ with respect to the lexicographic order or the reverse lexicographic order is smooth. Therefore the monomial ideals $J_1, \ldots, J_{18}$ does not define the same point with $J$ in $\mathrm{Hilb}_3^{2t+2}$.

One may care about the locus of these singular points. For example, are these singular points in intersections of irreducible components? However, we do

Table 2 The numbers of $J \in \mathcal{M}_{2t+2,3}$ such that $\text{Gröb}_{\prec}^{J} \cong \mathbb{A}_k^m$

| $\prec \backslash m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\prec_{rvlex}$ | 1 | 3 | 8 | 18 | 23 | 24 | 25 | 20 | 14 | 6 | 2 | 0 |
| $\prec_{lex}$ | 1 | 3 | 9 | 17 | 22 | 24 | 23 | 19 | 15 | 6 | 3 | 1 |

not have investigated it yet.

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