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Normal log canonical del Pezzo surfaces of rank one and type (IIb)

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Abstract

Let X be a normal del Pezzo surface of rank one with only rational log canonical singular points defined over $\mathbb{C}$, $\pi : V \to X$ the minimal resolution of X and D the reduced exceptional divisor of π. We prove that, if there exists a (-1)-curve C on V such that $CD = 1$ and X has a non-KLT singular point, then $V \setminus \operatorname{Supp}(C + D)(= X \setminus (\operatorname{Sing} X \cup \pi(C)))$ is affine ruled. Furthermore, we determine the surface X of type (IIb) with a non-KLT singular point.

1. Introduction

This paper is a continuation of the authors' papers [11], [12], [5], [6], [9] and [10] on normal del Pezzo surfaces of rank one with only rational log canonical singular points. We work over the complex number field $\mathbb{C}$ and use the intersection theory for normal surfaces due to Mumford [18] and Sakai [20]. In this paper, a normal del Pezzo surface means a normal projective surface whose anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor. A normal del Pezzo surface is said to have rank one if its Picard number equals one. A normal del Pezzo surface with only KLT (Kawamata log terminal) singular points is usually called a log del Pezzo surface. We call a normal del Pezzo surface with only rational log canonical singular points an *l.c. del Pezzo surface.*

Let X be an l.c. del Pezzo surface of rank one and let $\pi : V \to X$ be the minimal resolution of X, here we assume that $\operatorname{Sing} X \neq \emptyset$. Let $D = \sum_i D_i$ be the reduced exceptional divisor of π, where the D_i are irreducible components of D. Here we note that X is a rational surface by [11, Lemma 3.1] and that D is an SNC-divisor (a simple normal crossing divisor) by [2]. We have a unique effective $\mathbb{Q}$-divisor $D^\# = \sum_i \alpha_i D_i$ such that $K_V + D^\# \equiv \pi^* K_X$. By [11, Lemma 3.2], we know that $-(K_V + D^\#)$ is nef and big and that, for an irreducible curve E on V, $E(K_V + D^\#) = 0$ if and only if $E \subset \operatorname{Supp} D$. So, for a curve C on V not contained in $\operatorname{Supp} D$, $-C(K_V + D^\#) \in \{n/p \mid b \in \mathbb{Z}_{>0}\}$, where p is the smallest positive integer such that $pD^\#$ is an integral divisor. We can find irreducible

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curves C such that $-C(K_V + D^{\#})$ attains the smallest positive value. We denote the set of all such curves by $\mathrm{MV}(V, D)$.

Definition 1.1. (cf. [12, Definition 2.4])

(1) X (or (V, D)) is said to be of the first kind if there exists a curve $C \in \mathrm{MV}(V, D)$ such that $|K_V + C + D| \neq \emptyset$. It is said to be of the second kind if it is not of the first kind, i.e., $|K_V + C + D| = \emptyset$ for every curve $C \in \mathrm{MV}(V, D)$.

(2) Assume that X (or (V, D)) is of the second kind. It is said to be of type (IIa) if there exists a curve $C \in \mathrm{MV}(V, D)$ meeting at least two (-2)-curves in $\mathrm{Supp}\, D$. It is said to be of type (IIb) if there exists a curve $C \in \mathrm{MV}(V, D)$ meeting only one component of D but it is not of type (IIa). It is said to be of type (IIc) if there exists a curve $C \in \mathrm{MV}(V, D)$ such that $CD \geq 3$ but it is neither of type (IIa) nor of type (IIb). It is said to be of type (IId) if it is neither of type (IIa), of type (IIb) nor of type (IIc).

Let X and (V, D) be the same as above. If X is of the first kind, then it has only KLT singular points by [11, Corollary 3.5]. In this case, Zhang [21, Section 3] studied its structure and proved that $X_0 := X \setminus \mathrm{Sing}\, X$ is affine uniruled, namely, there exists a dominant morphism $\phi : \mathbb{A}^1_{\mathbb{C}} \times U \to X_0$, where U is a smooth curve. Assume that X is of the second kind. In [12], the authors determined the surfaces of type (IIa). Later on, the first author [10] determined the surfaces of type (IIc) containing at least one non-KLT singular points. In fact, every l.c. del Pezzo surface of rank one can have at most one non-KLT singular point by [9, Theorem 1]. For more details on l.c. del Pezzo surfaces of rank one and related results, see [9], [10] and their references.

In this paper, we study l.c. del Pezzo surfaces of rank one and type (IIb). In Section 2, we recall some elementary results on l.c. del Pezzo surfaces of rank one and some results on open algebraic surfaces. In Section 3, we prove the following result.

Theorem 1.1. *Let X be an l.c. del Pezzo surface of rank one, $\pi : V \to X$ the minimal resolution of X and D the reduced exceptional divisor of π. Assume that there exists a (-1)-curve C on V such that $CD = 1$. Then the following assertions hold true.*

(1) $V \setminus \mathrm{Supp}(C + D)(= X \setminus (\mathrm{Sing}\, X \cup \pi(C)))$ *is affine uniruled.*

(2) *If X has a non-KLT singular point P, then $V \setminus \mathrm{Supp}(C+D)$ is affine ruled (namely, it contains a surface isomorphic to $\mathbb{A}^1_{\mathbb{C}} \times U_0$, where U_0 is a smooth curve, as a Zariski open subset) and every singular point of X other than P is a cyclic quotient singular point.*

Note that X as in Theorem 1.1 may not be of type (IIb) since the (-1)-curve

C may not be an element of $\mathrm{MV}(V, D)$. In [22], Zhang proved the following result: for a normal algebraic surface S with only log canonical singularities and with nef and big anticanonial divisor, its smooth part is affine ruled or has finite fundamental group. We do not use this result in Section 3.

In Section 4, we determine the l.c. del Pezzo surfaces of rank one and type (IIb) with at least two singular points and non-KLT singular points. We prove the following result.

Theorem 1.2. *Let X be an l.c. del Pezzo surface of rank one and type* (IIb)*, $\pi : V \to X$ the minimal resolution of X and D the reduced exceptional divisor of π. Assume further that $\# \operatorname{Sing} X \geq 2$ and X has a non-KLT singular point. Let $C \in \mathrm{MV}(V, D)$ be a curve such that $CD = 1$. Then the following assertions hold.*

(1) *The divisor $C + D$ is an SNC-divisor and the dual graph of D is given as in (n) for $n = 1, \ldots, 8$ in Figure* 5.1.
(2) *There exists a $\mathbb{P}^1$-fibration $\Phi : V \to \mathbb{P}^1$ in such a way that the configuration of $C + D$ as well as all singular fibers of Φ can be explicitly described. The configuration is given in the configuration (n) for $n = 1, \ldots, 8$ in Figure* 5.2.

In [4] and [6], the first author determined the l.c. del Pezzo surfaces of rank one with unique singular points. That is why we assume in Theorem 1.2 that $\# \operatorname{Sing} X \geq 2$.

2. Preliminaries

We recall some elementary results on l.c. del Pezzo surfaces of rank one and some results on open algebraic surfaces. All the results of this section are well-known.

We employ the following notations.

Σ_m: the Hirzebruch surface of degree m.

K_V: the canonical divisor on V.

$\rho(V)$: the Picard number of V.

$\overline{\kappa}(S)$: the logarithmic Kodaira dimension of S. (See [15] for its definition.)

F_{red}: the reduced part of an effective divisor F.

$\#D(= \#D_{\mathrm{red}})$: the number of irreducible components of D_{red} of an effective divisor D.

$\lfloor L \rfloor$: the integral part of an effective $\mathbb{Q}$-divisor L.

2.1 Some results on l.c. del Pezzo surfaces of rank one

Let X be an l.c. del Pezzo surface of rank one and $\pi : V \to X$ the minimal resolution of X, here we assume that $\operatorname{Sing} X \neq \emptyset$. Let $D = \sum_i D_i$ be the reduced exceptional divisor of π, where the D_i are irreducible components of D.

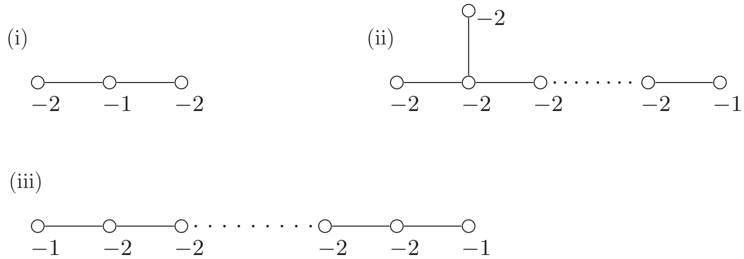

Figure 2.1

Since X has only rational singular points, D is an SNC-divisor (a simple normal crossing divisor) and consists only of smooth rational curves (cf. [2]). We often denote (V, D) and X interchangeably. There is a unique effective $\mathbb{Q}$-divisor $D^{\#} = \sum_i \alpha_i D_i$ such that $K_V + D^{\#} \equiv \pi^* K_X$.

We recall some elementary results given in [11] and [12]. They are originally given in [21] for log del Pezzo surfaces of rank one.

Lemma 2.1. *With the same notations and assumptions as above, the following assertions hold true.*

(1) *X is a rational surface.*
(2) *For any irreducible curve F, $-F(K_V + D^{\#}) = 0$ if and only if F is a component of D.*
(3) *Any $(-n)$-curve with $n \geq 2$ is a component of D.*

Proof. See [11, Lemmas 3.1 and 3.2]. □

Lemma 2.2. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration (i.e., Φ is a fibration from V onto $\mathbb{P}^1$ whose general fiber is isomorphic to $\mathbb{P}^1$). Then the following assertions hold true.*

(1) *The number of irreducible components of D not in any fiber of Φ equals $1+\sum_F(\#\{(-1)-\text{curves in } F\}-1)$, where F moves over all singular fibers of Φ.*
(2) *If a singular fiber F of Φ consists only of (-1)-curves and (-2)-curves, then the dual graph of F is given as one of the graphs* (i)–(iii) *in Figure* 2.1.

Proof. See [21, Lemma 1.5]. □

Lemma 2.3. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration. Assume that there exists a singular fiber F such that its weighted dual graph is given as one of* (i) *and* (ii) *in Figure* 2.1 *and that $C \in \mathrm{MV}(V, D)$, where C is the unique (-1)-curve in $\mathrm{Supp}\, F$. Then every singular fiber G consists only of (-1)-curves and (-2)-curves and so the dual graph of G is given as one of* (i)–(iii) *in Figure* 2.1. *Moreover, every (-1)-curve in $\mathrm{Supp}\, G$ is an element of $\mathrm{MV}(V, D)$.*

Proof. See [11, Lemma 3.7]. □

Lemma 2.4. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration and let C be a (-1)-curve in* $\mathrm{MV}(V, D)$. *Assume that Φ has a singular fiber F such that $F = 3C + \Delta$, where Δ is an effective divisor with* $\operatorname{Supp}\Delta \subset \operatorname{Supp} D$. *Then every singular fiber of Φ consists of* (-1)*-curves,* (-2)*-curves and at most one* (-3)*-curve.*

Proof. See [11, Lemma 3.8]. The assertion can be proved by using the same argument as in the proof of [21, Lemma 1.6]. □

2.2 Some results on open algebraic surfaces

A reduced effective divisor on a smooth algebraic variety is called an *SNC-divisor* if it has only simple normal crossings. Let $A = A_1 + \cdots + A_r$ be a linear chain of smooth projective rational curves on a smooth projective surface such that $A_1A_2 = A_2A_3 = \cdots = A_{r-1}A_r = 1$ and set $a_i = A_i^2$ $(i = 1, \ldots, r)$. Then we denote the weighted dual graph of A by $[a_1, a_2, \ldots, a_r]$. For an integer a and a positive integer s, we use the abbreviation $[a_s] = [a, a, \ldots, a]$ that is the weighted dual graph of a linear chain consisting of s smooth rational curves with self-intersection number a.

We recall some notions and results on open algebraic surfaces. For more details, see [15, Chapter 2] and [16, Chapter 1].

Let V be a smooth projective surface and D an SNC-divisor on V. We call such a pair (V, D) an *SNC-pair.* A connected curve consisting only of irreducible components of D is called a connected curve in D for shortness. A connected curve T in D is said to be *admissible* (resp. *rational*) if there are no (-1)-curves in $\operatorname{Supp} T$ and the intersection matrix of T is negative definite (resp. if it consists only of rational curves). A connected curve T in D is called a *twig* if its dual graph is a linear chain and T meets $D - T$ in a single point at one of the end components of T. An admissible rational twig in D is said to be *maximal* if it is not extended to an admissible rational twig with more irreducible components of D. A connected curve in D is called a *rod* (resp. a *fork*) if it is a connected component of D and its dual graph is a linear chain (resp. its dual graph is that of the exceptional curves of the minimal resolution of a KLT singular point and is not a linear chain).

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of D as a sum of effective $\mathbb{Q}$-divisors $D = D^\# + \mathrm{Bk}(D)$ such that the following two conditions (i) and (ii) are satisfied:

(i) $\operatorname{Supp}(\mathrm{Bk}(D)) = (\cup_\lambda T_\lambda) \cup (\cup_\mu R_\mu) \cup (\cup_\nu F_\nu)$.
(ii) $(K_V + B^\#)Z = 0$ for every irreducible component Z of $\operatorname{Supp}(\mathrm{Bk}(D))$.

Remark 2.1. Let X be a normal projective surface with only rational log canonical singular points, $\pi : V \to X$ the minimal resolution of X and D the reduced exceptional divisor of π. Then D is an SNC-divisor. Since X has only log canonical singular points, the $\mathbb{Q}$-divisor $D^{\#}$ defined as in the last paragraph is the same as that defined in Introduction and Section 2.1. Namely, $\pi^*(K_X) \equiv K_V + D^{\#}$.

Definition 2.1. An SNC-pair (V, D) is said to be *almost minimal* if, for every irreducible curve C on V, either $C(K_V + D^{\#}) \geq 0$ or $C(K_V + D^{\#}) < 0$ and the intersection matrix of $C + \mathrm{Bk}(D)$ is not negative definite.

For an SNC-pair (V, D), there exists a birational morphism $\mu : V \to W$ onto a smooth projective surface W such that the following conditions are satisfied:

(1) $\Delta := \mu_*(D)$ is an SNC-divisor.
(2) For any (-1)-curve $E \subset \Delta$, $E(\Delta - E) \geq 2$ and the equality holds if and only if E meets a unique irreducible component of $\Delta - E$. (Δ is then said to be SNC-minimal.)
(3) $\overline{\kappa}(V \setminus \operatorname{Supp} D) = \overline{\kappa}(X \setminus \operatorname{Supp} B)$.
(4) (V, D) is almost minimal.

See [15, Theorem 2.3.11.1 (p. 107)], which is the same as [16, Theorem 1.11], for its proof. We call the pair (W, Δ) an *almost minimal model* of (V, D).

We recall the following result on the almost minimal SNC-pairs of $\overline{\kappa} = -\infty$.

Lemma 2.5. *Let (V, D) be an almost minimal SNC-pair of $\overline{\kappa}(V \setminus \operatorname{Supp} D) = -\infty$ and assume further that D is SNC-minimal. Let $\pi : V \to \overline{V}$ be the contraction of* $\operatorname{Supp}(\mathrm{Bk}(D))$ *to normal points and set $\overline{D} := \pi_*(D)$.* (*Here we note that $\overline{V}$ has only KLT singular points.*) *Then one of the following cases takes place.*

(A) *There exists a $\mathbb{P}^1$-fibration $h : \overline{V} \to C$ onto a smooth projective curve C such that every fiber of h is irreducible and $\overline{D}F \leq 1$ for a fiber F of h.*
(B) *$\rho(\overline{V}) = 1$ and $-(K_{\overline{V}} + \overline{D})$ is an ample $\mathbb{Q}$-Cartier divisor.*

Proof. See [15, Lemmas 2.3.14.3 and 2.3.14.4 (pp. 113–114)], which is the same as [16, Lemmas 2.7 and 2.8]. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let X, $\pi : V \to X$, D and C be the same as in Theorem 1.1. Let $P_1, \ldots, P_\ell$ ($\ell = \# \operatorname{Sing} X$) be all singular points of X and $D = \sum_{k=1}^{\ell} D^{(k)}$ the decomposition of D into connected components such that $D^{(k)} = \pi^{-1}(P_k)$ ($k = 1, \ldots, \ell$) as a reduced divisor. We may assume that $CD = CD^{(1)} = 1$. Set $S := X \setminus \pi(C)$.

Lemma 3.1. *With the same notations and assumptions as above, the following assertions hold true.*

(1) $\overline{\kappa}(V \setminus \mathrm{Supp}(C+D))(=\overline{\kappa}(S \setminus \mathrm{Sing}\, S)) = -\infty$.
(2) *S is a $\mathbb{Q}$-homology plane, i.e., it is a normal affine surface with Betti numbers of the affine plane $\mathbb{A}^2_{\mathbb{C}}$.*

Proof. By [9, Lemma 2], $\overline{\kappa}(V \setminus \mathrm{Supp}\, D) = -\infty$. Since $CD = 1$, we have

$$\overline{\kappa}(V \setminus \mathrm{Supp}(C+D)) = \overline{\kappa}(V \setminus \mathrm{Supp}\, D) = -\infty,$$

which proves the assertion (1). The assertion (2) follows from [12, Lemma 3.7]. □

We consider the case where P_1 is a non-KLT singular point. Then the weighted dual graph of $D^{(1)}$ is given as one of the dual graphs (6)–(8) in [1, p. 58]. Then we have the following lemma.

Lemma 3.2. *Assume that P_1 is a non-KLT singular point. Then there exists a birational morphism $\mu : \tilde{V} \to V$ from a smooth projective surface $\tilde{V}$ such that the following conditions* (1) *and* (2) *are satisfied*:

(1) *μ is a composite of blowing-ups at a point on* $\mathrm{Supp}(C + D^{(1)})$ *and its infinitely near points.*
(2) *There exists a $\mathbb{P}^1$-fibration $\Phi : \tilde{V} \to \mathbb{P}^1$ such that $F\mu^*(C+D)_{\mathrm{red}} = 1$ for a fiber F of Φ.*

In particular, $V \setminus \mathrm{Supp}(C + D)$ is affine ruled and $P_2, \ldots, P_\ell$ are cyclic quotient singular points.

Proof. The existence of $\mu : \tilde{V} \to V$ satisfying the conditions (1) and (2) follows from [9, Lemma 4]. Here, the curve E in [9, Lemma 4] is a (-1)-curve on V and $ED = ED^{(1)} = 1$ with the notations in [9]. In fact, the curve E satisfies the conditions which are $E(K_V + D^{\#}) < 0$ and the intersection matrix of $E + \mathrm{Bk}(D)$ is negative definite. However, the proof of [9, Lemma 4] does not use the latter condition.

As seen from the conditions (1) and (2), we easily see that $V \setminus \mathrm{Supp}(C + D)$ is affine ruled. The last assertion then follows from [13, Theorem 1]. □

From now on, we consider the case where P_1 is a KLT singular point. We note that the intersection matrix of $C + D^{(1)}$ is neither negative definite nor negative semi-definite because $C + D^{(1)}$ supports a big divisor. We prove the following lemma.

Lemma 3.3. *Suppose that P_1 is a KLT singular point. Then X is a log del Pezzo surface of rank one, i.e., every singular point on X is a KLT singular point.*

Proof. Let $f : V \to W$ be the contraction of C and all subsequently (smoothly) contractible curves in $\mathrm{Supp}(D^{(1)})$ such that $f_*(C+D^{(1)})(= f_*(D^{(1)}))$ is an SNC-divisor and $E'(f_*(C+D^{(1)})-E') \geq 3$ for any (-1)-curve $E' \subset \mathrm{Supp}(f_*(C+D^{(1)}))$

(i.e., $f_*(C + D^{(1)})$ is SNC-minimal). Since the weighted dual graph of $C + D^{(1)}$ is a tree (i.e., $C + D^{(1)}$ is a connected tree of $\mathbb{P}^1$'s), such a birational morphism f exists. Set $\Delta := f_*(D)$ and $\Delta^{(i)} := f_*(D^{(i)})$ for $i = 2, \ldots, \ell$. Then $W \setminus \operatorname{Supp}\Delta = V \setminus \operatorname{Supp}(C + D)$. So $\#\Delta = \rho(W)$ and $\overline{\kappa}(V \setminus \operatorname{Supp}\Delta) = \overline{\kappa}(V \setminus \operatorname{Supp}(C + D)) = -\infty$.

Suppose to the contrary that X has a non-KLT singular point. We may assume that $\ell \geq 2$ and P_2 is a non-KLT singular point.

Claim 1. The SNC-pair (W, Δ) is almost minimal.

Proof. Suppose to the contrary that (W, Δ) is not almost minimal. Then there exists an irreducible curve, say $\tilde{E}$, on W such that $\tilde{E}(K_W + \Delta^{\#}) < 0$ and the intersection matrix of $\tilde{E} + \operatorname{Bk}(\Delta)$ is negative definite. Since $S = X \setminus \pi(C)$, which can be constructed by contracting $\Delta^{(2)}, \ldots, \Delta^{(\ell)}$ from $W \setminus \operatorname{Supp}(\Delta^{(1)})$, is a normal affine surface, we have $\tilde{E}\Delta^{(1)} > 0$. By [15, Lemmas 2.3.6.3 and 2.3.8.4 (p. 96, p. 102)] (that is the same as [16, Lemmas 1.6.2 and 1.8.3]), we know that:

(a) $\tilde{E}\Delta \leq 2$.
(b) $\tilde{E}\Delta^{(1)} = 1$. In particular, $\tilde{E}$ meets an admissible rational maximal twig, say T, in $\Delta^{(1)}$.
(c) If $\tilde{E}\Delta = 2$, then the connected component $\Delta^{(j)}$ of $\Delta - \Delta^{(1)}$ meeting $\tilde{E}$ is an admissible rational rod, i.e., P_j is a cyclic quotient singular point, and $\tilde{E} + T + \Delta^{(j)}$ can be contracted to either an admissible rational rod or a smooth point.
(d) $\overline{\kappa}(W \setminus \operatorname{Supp}(\tilde{E} + \Delta)) = \overline{\kappa}(W \setminus \operatorname{Supp}\Delta) = -\infty$.

By (d) and $\#(\tilde{E} + \Delta) = 1 + \rho(W)$, we infer from [8, Lemma 2.8] that the surface $W \setminus \operatorname{Supp}(\tilde{E} + \Delta)$ is affine ruled. Hence $S \setminus \operatorname{Sing} S$ is affine ruled, too. By [13, Theorem 1], every singular point of S is a cyclic quotient singular point. However, this is a contradiction because $\operatorname{Sing} S = \{P_2, \ldots, P_\ell\}$ and P_2 is not a KLT singular point. $\square$

We set $\Delta^{\#} = \sum_{k=1}^{\ell} \Delta^{(k)\#}$. Since $\Delta^{(2)}$ can be contracted to the non-KLT singular point P_2, $\lfloor \Delta^{(2)\#} \rfloor \neq 0$. Further, since the intersection matrix of $\Delta^{(1)}$ is not negative definite, we have $\lfloor \Delta^{(1)\#} \rfloor \neq 0$. So $\lfloor \Delta^{(1)\#} \rfloor$ and $\lfloor \Delta^{(2)\#} \rfloor$ are contained in $\operatorname{Supp}(\lfloor \Delta^{\#} \rfloor)$.

Let $\pi' : W \to \overline{W}$ be the contraction of $\operatorname{Supp}(\operatorname{Bk}(\Delta)) = \operatorname{Supp}(\Delta - \lfloor \Delta^{\#} \rfloor)$. Then $\overline{W}$ is a normal projective surface with only KLT singular points. By Lemma 2.5, one of the following cases (A) and (B) takes place.

(A) There exists a $\mathbb{P}^1$-fibration $\Phi : \overline{W} \to \mathbb{P}^1$ onto $\mathbb{P}^1$ such that every fiber of Φ is irreducible and $\pi'_*(\lfloor \Delta^{\#} \rfloor)F \leq 1$ for a fiber F of Φ. In fact, $\pi'_*(\lfloor \Delta^{\#} \rfloor)F = 1$ since the intersection matrix of Δ is neither negative definite nor negative semi-definite.
(B) $\rho(\overline{W}) = 1$ and $-(K_{\overline{W}} + \pi'_*(\lfloor \Delta^{\#} \rfloor))$ is an ample $\mathbb{Q}$-Cartier divisor.

If the case (A) takes place, then $W \setminus \operatorname{Supp} \Delta$ is affine ruled. However, by using the argument as in the last paragraph of the proof of Claim 1, we derive a contradiction.

Suppose that the case (B) takes place. We may assume further that $U := W \setminus \operatorname{Supp} \Delta$ is not affine ruled.

Claim 2. The surface U has a structure of platonic $\mathbb{A}^1_*$-fiber space over $\mathbb{P}^1$, where $\mathbb{A}^1_* = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. More precisely, there exists a surjective morphism $g : U \to \mathbb{P}^1$ from U onto $\mathbb{P}^1$ such that the following conditions are satisfied:

(i) g has no singular fibers except for three multiple fibers $F_i = \mu_i G_i$, $i = 1, 2, 3$, such that $G_i \cong \mathbb{A}^1_*$ and that $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, m\}$ $(m \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

(ii) There exist an SNC-pair $(\overline{U}, B)$ and a $\mathbb{P}^1$-fibration $\overline{g} : \overline{U} \to \mathbb{P}^1$ such that:
  (a) $\overline{U} \setminus \operatorname{Supp} B = U$.
  (b) B contains two irreducible components B_1 and B_2 that are sections of $\overline{g}$ with $B_1 \cap B_2 = \emptyset$, and the other irreducible components of B are contained in fibers of $\overline{g}$.
  (c) Every fiber of $\overline{g}$ has a linear chain as its weighted dual graph and contains a unique (-1)-curve if the fiber is reducible.

Proof. Since $\lfloor \Delta^{\#} \rfloor \neq 0$ and (W, Δ) is not affine ruled, we infer from Claim 1 and [15, Theorem 2.5.1.2 (p. 143)] (that is the same as [17, Main Theorem]) that U has a structure of platonic $\mathbb{A}^1_*$-fiber space over $\mathbb{P}^1$. The other assertion follows from the definition of a platonic $\mathbb{A}^1_*$-fiber space over $\mathbb{P}^1$. □

We can determine the weighted dual graph of B. For more details, see [14, Section 2] (see also [7, pp. 37–38]). We may assume that $B_2^2 = -b \leq -2$ by interchanging the role of B_1 and B_2. Then B consists of two connected components, say $B^{(1)}$ and $B^{(2)}$, containing B_1 and B_2, respectively. Furthermore, the weighted dual graph of $B^{(1)}$ looks like that in Figure 3.1 and that $B^{(2)}$ is an admissible rational fork. In Figure 3.1, $b_i^{(j)} \geq 2$ for $i = 1, \ldots, s_j$ and $j = 1, 2, 3$. (In fact, we can determine the weighted dual graph of B more precisely. However, we do not need the precise result.)

We easily see that $\operatorname{Supp} B$ contains no irreducible components B' with $B'^2 \geq 0$ and $B'(B - B') \leq 2$. Since the divisor Δ satisfies the same conditions as B, we know that the pair $(\overline{U}, B)$ is isomorphic to (W, Δ). Namely, there exists an isomorphism $\Psi : \overline{U} \to W$ whose restriction on B gives rise to an isomorphism between B and Δ. Since $\Delta^{(1)}$ supports a big divisor, the weighted dual graph of $\Delta^{(1)}$ is the same as that of $B^{(1)}$. So $\ell = 2$ and $\Delta^{(2)}$ is an admissible rational fork. This is a contradiction because $\Delta^{(2)}$ can be contracted to the non-KLT singular point P_2.

Therefore, every singular point of X is a KLT-singular point. □

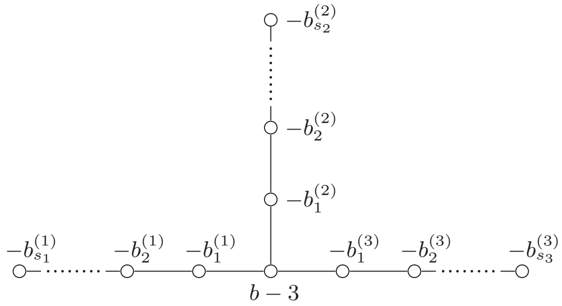

Figure 3.1

The assertion (2) of Theorem 1.1 is thus verified. The assertion (1) of Theorem 1.1 follows from the assertion (2) and [21, Theorem 6.1]. Here we note that, in [21, Section 6], the (-1)-curve C is an element of $\mathrm{MV}(V, D)$ and $CD = 1$, but the condition $C \in \mathrm{MV}(V, D)$ is not used in the proof of [21, Theorem 6.1].

The proof of Theorem 1.1 is thus completed.

Remark 3.1. We can prove Lemma 3.3 by using Palka's result on the classification of $\mathbb{Q}$-homology planes with non-KLT singular points in [19, Theorem 4.5] instead by using [15, Theorem 2.5.1.2 (p. 143)] (that is the same as [17, Main Theorem]).

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let X, π, V, D and $C \in \mathrm{MV}(V, D)$ be the same as in Theorem 1.2. Let $P_1, \ldots, P_\ell$ ($\ell = \#\,\mathrm{Sing}\,X$) be all singular points of X and $D = \sum_{k=1}^{\ell} D^{(k)}$ the decomposition of D into connected components such that $D^{(k)} = \pi^{-1}(P_k)$ ($k = 1, \ldots, \ell$) as a reduced divisor. We may assume that $CD = CD^{(1)} = 1$. Since X contains a non-KLT singular point, we infer from Lemma 3.3 that P_1 is not a KLT singular point. Further, by [9, Theorem 1], $P_2, \ldots, P_\ell$ are KLT singular points. The weighted dual graph of $D^{(1)}$ is given as one of the graphs (6)–(8) in [1, p. 58].

4.1

In this section, we consider the case where the weighted dual graph of $D^{(1)}$ is given as one of (6) and (7) in [1, p. 58]. Let $D^{(1)} = \sum_{i=1}^{r} D_i$ be the decomposition of $D^{(1)}$ into irreducible components and set $a_i := -D_i^2$ for $i = 1, \ldots, r$. In this case, $r \geq 5$ and the weighted dual graph of $D^{(1)}$ is given as in Figure 4.1.

Then $D^{(1)\#} = \frac{1}{2}(D_1 + D_2 + D_{r-1} + D_r) + \sum_{i=3}^{r-2} D_i$ and $\max\{a_3, \ldots, a_{r-2}\} \geq 3$ since the intersection matrix of $D^{(1)}$ is negative definite. Since $CD^{(1)} = 1$ and $CD^{\#} < -CK_V = 1$, we may assume that $CD^{(1)} = CD_1 = 1$. Then $CD^{\#} = \frac{1}{2}$. Since the intersection matrix of $C + D^{(1)}$ (resp. $D^{(1)}$) is not negative definite (resp. negative definite), we know that $a_3 = a_4 = 2$ and $r \geq 7$. Then the divisor

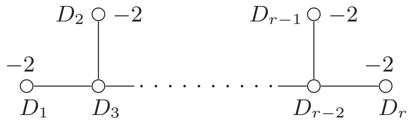

Figure 4.1

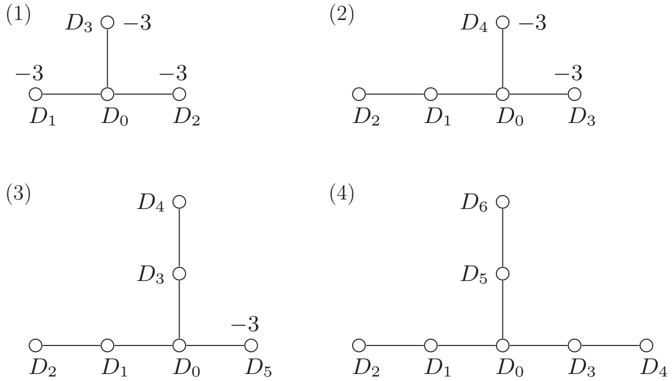

Figure 4.2

$F_0 := 2(C + D_1 + D_3) + D_2 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_5 becomes a section of Φ and $D - D_5$ is contained in fibers of Φ. By Lemma 2.3, every fiber of Φ consists only of (-1)-curves and (-2)-curves. By Lemma 2.2, we know that the weighted dual graph of every singular fiber of Φ is given as one of (i) and (ii) in Figure 2.1. If Φ has a singular fiber F_1 whose weighted dual graph is given as (i) in Figure 2.1, then the (-1)-curve E in $\operatorname{Supp} F_1$ is an element of $\mathrm{MV}(V, D)$ and E meets at least two (-2)-curves in $\operatorname{Supp} D$. So the pair (V, D) is of type (IIa), which is a contradiction. Hence, the weighted dual graph of every fiber of Φ is given as (ii) in Figure 2.1.

We know that $D = D^{(1)}$ is connected, which contradicts the hypothesis $\#\operatorname{Sing} X \geq 2$. Therefore, this case does not take place.

4.2

In Sections 4.2–4.4, we consider the case where the weighted dual graph of $D^{(1)}$ is given as (6) in [1, p. 58]. Then $(\Delta_1, \Delta_2, \Delta_3) = (3, 3, 3), (2, 4, 4), (2, 3, 6)$ with the notations in [1, p. 58].

In this section, we consider the case $(\Delta_1, \Delta_2, \Delta_3) = (3, 3, 3)$. Then the weighted dual graph of $D^{(1)}$ is given as one of (1)–(4) in Figure 4.2, where we omit the self-intersection number corresponding to a (-2)-curve and set $a_0 = -D_0^2$.

We consider the following four cases 1–4 separately.

Case 1: The weighted dual graph of $D^{(1)}$ is given as (1) in Figure 4.2.

In this case, $D^{(1)\#} = D_0 + \frac{2}{3}(D_1 + D_2 + D_3)$ and so $CD^{(1)} = CD_i = 1$

for some $i \in \{1,2,3\}$. This is a contradiction because the intersection matrix of $C + D^{(1)}$ is then negative definite. Therefore, this case does not take place.

Case 2: The weighted dual graph of $D^{(1)}$ is given as (2) in Figure 4.2.

In this case, $D^{(1)\#} = D_0 + \frac{2}{3}(D_1 + D_3 + D_4) + \frac{1}{3}D_2$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we know that $CD^{(1)} = CD_1 = 1$ and $a_0 = 2$. Then the divisor $F_0 = 2(C + D_1) + D_0 + D_2$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_3 and D_4 become sections of Φ and $D - (D_3 + D_4)$ is contained in fibers of Φ. Let $F_0, F_1, \ldots, F_r$ exhaust all singular fibers of Φ, here we note that $r \geq 1$ since $\#\operatorname{Sing} X \geq 2$. Then each $\operatorname{Supp}(F_i)$ $(i = 1, \ldots, r)$ contains at least two (-1)-curves because the irreducible component of $\operatorname{Supp}(F_i)$ meeting D_3, which is a section of Φ, is a (-1)-curve. We infer from Lemmas 2.2 and 2.3 that the weighted dual graph of F_i $(i = 1, \ldots, r)$ is given as (iii) in Figure 2.1 and that $r = 1$. Write $F_1 = E_1 + G_1 + \cdots + G_k + E_2$, where $E_1G_1 = G_1G_2 = \cdots = G_kE_2 = 1$, E_1 and E_2 are (-1)-curves and $G_1, \ldots, G_k$ are (-2)-curves. By Lemma 2.3, $E_1, E_2 \in \operatorname{MV}(V, D)$ and so $E_1D^{(1)\#} = E_2D^{(1)\#} = CD^{\#} = \frac{2}{3}$. Hence we may assume that $E_1D_3 = E_2D_4 = 1$. Then $E_1D^{(1)} = E_1D_3 = 1$, $E_2D^{(1)} = E_2D_4 = 1$.

Let $u : V \to \Sigma_3$ be the contraction of all (-1)-curves and consecutively (smoothly) contractible curves in fibers of Φ except for those meeting D_3. Then we have

$$(3 =)u(D_4)^2 = -3 + 1 + k, \quad u(D_4)u(D_3) = 0.$$

So, $k = 5$. Hence the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (1) in Figure 5.1 (resp. Figure 5.2).

Case 3: The weighted dual graph of $D^{(1)}$ is given as (3) in Figure 4.2.

In this case, $D^{(1)\#} = D_0 + \frac{2}{3}(D_1 + D_3 + D_5) + \frac{1}{3}(D_2 + D_4)$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we may assume that $CD^{(1)} = CD_1 = 1$ and $a_0 = 2$. Then the divisor $F_0 = 2(C + D_1) + D_0 + D_2$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_3 and D_5 become sections of Φ and $D - (D_3 + D_5)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_4. By Lemmas 2.2 and 2.3, we know that:

- F_0 and F_1 exhaust all singular fibers of Φ.
- The weighted dual graph of F_1 is one of (i) and (iii) in Figure 2.1.

Since D_4 is isolated in $\operatorname{Supp}(D - (D_0 + D_1 + D_2 + D_3 + D_5))$, we see that $\#F_1 = 3$. Let E be a (-1)-curve in $\operatorname{Supp}(F_1)$. Then $ED_4 = 1$ and $E \in \operatorname{MV}(V, D)$ by Lemma 2.3. Since $ED^{(1)\#} < 1$ and the coefficient of D_5 in $D^{(1)\#}$ equals $\frac{2}{3}$, $ED_5 = 0$. Hence we conclude that $ED^{\#} = \frac{1}{3}ED_4 = \frac{1}{3}$. This is a contradiction because $E \in \operatorname{MV}(V, D)$ and $CD^{\#} = CD^{(1)\#} = \frac{2}{3}CD_1 = \frac{2}{3}$. Therefore, this case does not take place.

Case 4: The weighted dual graph of $D^{(1)}$ is given as (4) in Figure 4.2.

In this case, $D^{(1)\#} = D_0 + \frac{2}{3}(D_1 + D_3 + D_5) + \frac{1}{3}(D_2 + D_4 + D_6)$. Since the intersection matrix of $D^{(1)}$ is negative definite, $a_0 = -D_0^2 \geq 3$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we may assume that $CD^{(1)} = CD_1 = 1$ and $a_0 = 3$. Then the divisor $F_0 := 4(C+D_1)+2(D_0+D_2)+D_3+D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_4 and D_6 become sections of Φ and $D - (D_4 + D_6)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D^{(2)}$, which exists by $\#\operatorname{Sing} X \geq 2$. Then the curve E_1 of $\operatorname{Supp}(F_1)$ meeting D_4 (that is a section of Φ) is a (-1)-curve. So there exists a (-1)-curve E_2 ($\neq E_1$) in $\operatorname{Supp}(F_1)$. By Lemma 2.2 (1), we know that E_1 and E_2 exhaust all (-1)-curves in $\operatorname{Supp}(F_1)$ and that F_0 and F_1 exhaust all singular fibers of Φ.

Suppose that E_1 meets D_6. Then

$$E_1 D^{\#} \geq E_1 D^{(1)\#} \geq \frac{1}{3} E_1(D_4 + D_6) = \frac{2}{3} = CD^{\#}.$$

So $E_1 \in \mathrm{MV}(V, D)$ and E_1 meets two (-2)-curves D_4 and D_6. Hence (V, D) is of type (IIa), a contradiction. Therefore, $E_1 D_6 = 0$ and $E_2 D_6 = 1$.

Set $\# F_1 = 2 + m$. Then $m = \#(D - D^{(1)})$. Let $u : V \to \Sigma_2$ be the contraction of all (-1)-curves and consecutively (smoothly) contractible curves in fibers of Φ except for those meeting D_4. Then $u_*(F_0) = u(D_3)$ and $u_*(F_1) = u(E_1)$. Further, $u(D_4)u(D_6) = 0$. So $2 = u(D_6)^2 = -2 + 1 + m + 1$ and hence $m = 2$.

Since $\# F_1 = 4$, $\operatorname{Supp}(F_1)$ contains two (-1)-curves E_1 and E_2 and the coefficients of E_1 and E_2 in F_1 are equal to one, we know that $\operatorname{Supp}(F_1)$ is a linear chain of four $\mathbb{P}^1$'s, E_1 and E_2 are end components of $\operatorname{Supp}(F_1)$ and the other two irreducible components of $\operatorname{Supp}(F_1)$ are (-2)-curves. Namely, the weighted dual graph of F_1 is $[-1, -2, -2, -1]$ (see Section 2.2 for this notion). Hence the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (2) in Figure 5.1 (resp. Figure 5.2).

4.3

In this section, we consider the case $(\Delta_1, \Delta_2, \Delta_3) = (2, 4, 4)$. Then the weighted dual graph of $D^{(1)}$ is given as one of (1)–(3) in Figure 4.3, where we omit the self-intersection number corresponding to a (-2)-curve and set $a_0 = -D_0^2$.

We consider the following three cases 1–3 separately.

Case 1: The weighted dual graph of $D^{(1)}$ is given as (1) in Figure 4.3.

By using the same argument as in Case 1 in Section 4.2, we know that this case does not take place.

Case 2: The weighted dual graph of $D^{(1)}$ is given as (2) in Figure 4.3.

In this case, $D^{(1)\#} = D_0 + \frac{3}{4}(D_2 + D_5) + \frac{1}{2}(D_1 + D_3) + \frac{1}{4} D_4$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is neither negative definite nor negative semi-definite, we know that $CD^{(1)} = CD_i = 1$ for some $i \in \{2, 3\}$.

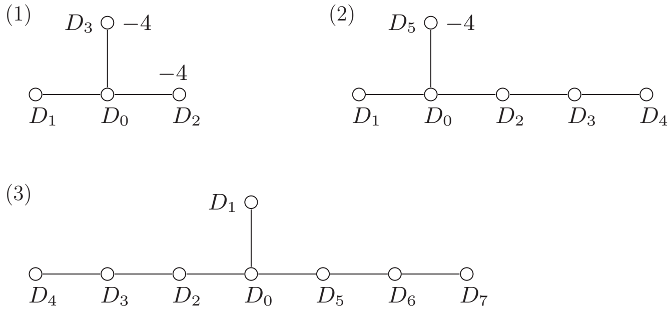

Figure 4.3

Suppose that $i = 3$, i.e., $CD_3 = 1$. Then the divisor $F = 2(C+D_3)+D_2+D_4$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$. Then D_5, which is a (-4)-curve, becomes a fiber component of $\Phi_{|F|}$. This is a contradiction by Lemma 2.3. Hence, $i = 2$, i.e., $CD_2 = 1$. Then $a_0 \in \{2, 3\}$ since the intersection matrix of $C + D^{(1)}$ is neither negative definite nor negative semi-definite.

Suppose that $a_0 = 2$. Then the divisor $F' := 2(C + D_2) + D_0 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi' := \Phi_{|F'|} : V \to \mathbb{P}^1$, D_1, D_4 and D_5 become sections of Φ' and $D - (D_1 + D_4 + D_5)$ is contained in fibers of Φ'. Since $\#\operatorname{Sing} X \geq 2$, Φ' has a singular fiber $F'_1 \neq F'$. By Lemma 2.3, the weighted dual graph of F'_1 is one of (i)–(iii) in Figure 2.1 and every (-1)-curve in $\operatorname{Supp}(F'_1)$ is an element of $\operatorname{MV}(V, D)$. Since the irreducible component of $\operatorname{Supp}(F'_1)$ meeting D_1, which is a section of Φ', is a (-1)-curve, $\operatorname{Supp}(F'_1)$ has at least two (-1)-curves. So the weighted dual graph of F'_1 is given as (iii) in Figure 2.1. Let E'_1 and E'_2 be the two (-1)-curves in $\operatorname{Supp}(F'_1)$. Then either E'_1 or E'_2 meets at least two of D_1, D_4 and D_5. We may assume that $E'_1(D_1 + D_4 + D_5) \geq 2$. Then

$$E'_1 D^{\#} \geq E'_1 D^{(1)\#} \geq \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = CD^{\#}.$$

Then, $E'_1 \in \operatorname{MV}(V, D)$ and E'_1 meets D_1 and D_4, which are (-2)-curves. Hence (V, D) is of type (IIa), which is a contradiction. Therefore, $a_0 = 3$.

Then the divisor $F_0 := 3(C + D_2) + 2D_3 + D_0 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_5 become sections of Φ and $D - (D_1 + D_5)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D^{(2)}$, which exists. Then F_1 contains at least two (-1)-curves (see the preceding paragraph). By Lemma 2.2 (1), we know that F_0 and F_1 exhaust all singular fibers of Φ and $\operatorname{Supp}(F_1)$ contains just two (-1)-curves, say E_1 and E_2.

We may assume that $E_1 D_5 = 1$. Then $E_1 D^{\#} \geq \frac{3}{4} E_1 D_5 = \frac{3}{4} = CD^{\#}$. So $E_1 \in \operatorname{MV}(V, D)$. Then $E_1 D_1 = 0$ and $E_2 D_1 = 1$. Since the coefficients of E_1 and E_2 in F_1 are equal to one, we know that $\operatorname{Supp}((F_1)_{\mathrm{red}} - (E_1 + E_2))$ is connected, namely, $D^{(2)} = (F_1)_{\mathrm{red}} - (E_1 + E_2)$.

If $D^{(2)}$ contains a curve of self-intersection number ≤ -3, then the coefficient of every component of $D^{(2)}$ in $D^{(2)\#} > 0$ and so

$$E_1 D^{\#} = E_1(D^{(1)\#} + D^{(2)\#}) > E_1 D^{(1)\#} = \frac{3}{4} = CD^{\#}.$$

This is a contradiction. Hence $D^{(2)}$ consists only of (-2)-curves. By Lemma 2.2 (2), the weighted dual graph of F_1 is given as (iii) in Figure 2.1. In particular, $F_1 = E_1 + D^{(2)} + E_2$ is a linear chain and E_1 and E_2 are end components of $\mathrm{Supp}(F_1)$. By the same argument as in Case 2 in Section 4.2 (the last paragraph in Case 2 in Section 4.2), we know that the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (3) in Figure 5.1 (resp. Figure 5.2).

Case 3: The weighted dual graph of $D^{(1)}$ is given as (3) in Figure 4.3.

In this case, $D^{(1)\#} = D_0 + \frac{3}{4}(D_2 + D_5) + \frac{1}{2}(D_1 + D_3 + D_6) + \frac{1}{4}(D_4 + D_7)$. Since the intersection matrix of $D^{(1)}$ is negative definite, $a_0 = -D_0^2 \geq 3$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we may assume that $CD^{(1)} = CD_i = 1$ for some $i \in \{2, 3\}$. We consider the following two subcases 3-1 and 3-2 separately.

Subcase 3-1: $i = 3$, i.e., $CD_3 = 1$. (The argument of this subcase is slightly different from that in the second paragraph in Case 2.) In this subcase, the divisor $F_0 := 2(C + D_3) + D_2 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_0 becomes a section of Φ and $D - D_0$ is contained in fibers of Φ. Let F_1 (resp. F_2) be the fiber of Φ containing D_1 (resp. $D_5 + D_6 + D_7$), here we note that $F_1 \neq F_2$. By Lemmas 2.2 and 2.3, we know that:

- F_0, F_1 and F_2 exhaust all singular fibers of Φ.
- For $j = 1, 2$, the weighted dual grapf of F_j is given as one of (i) and (ii) in Figure 2.1 and the (-1)-curve in $\mathrm{Supp}(F_j)$ is an element of $\mathrm{MV}(V, D)$.

Since F_1 contains D_1, we know that the weighted dual graph of F_1 is given as (i) in Figure 2.1. So the (-1)-curve in $\mathrm{Supp}(F_1)$ is an element of $\mathrm{MV}(V, D)$ and meets at least two (-2)-curves, which imply that (V, D) is of type (IIa). This is a contradiction. Therefore, this subcase does not take place.

Subcase 3-2: $i = 2$, i.e., $CD_2 = 1$. Since the intersection matrix of $C + D^{(1)}$ is not negative definite, $a_0 \in \{3, 4\}$.

Suppose that $a_0 = 3$. Then the divisor $F := 3(C + D_2) + 2D_3 + D_0 + D_4$ defines a $\mathbb{P}^1$-fibration $\Psi := \Phi_{|F|} : V \to \mathbb{P}^1$, D_1 and D_5 become sections of Ψ and $D - (D_1 + D_5)$ is contained in fibers of Ψ. Let G be the fiber of Ψ containing $D_6 + D_7$. Then the irreducible component of $\mathrm{Supp}\, G$ meeting D_1 is a (-1)-curve. Since D_1 is a section of Ψ, $\mathrm{Supp}\, G$ contains at least two (-1)-curves. By Lemma 2.2 (1), we know that F and G exhaust all singular fibers of Ψ and that $\mathrm{Supp}\, G$ contains just two (-1)-curves, say E and E'. Since D is not connected,

$\mathrm{Supp}(D - D^{(1)})$ is contained in $\mathrm{Supp}\, G$. We may assume that E meets $D_6 + D_7$. Then $E(D_6 + D_7) = 1$. Since $\mathrm{Supp}(E + E' + D_6 + D_7) \neq \mathrm{Supp}(G)$, we know that $E'(D_6 + D_7) = 0$. If E meets D_1, then

$$ED^{\#} \geq ED^{(1)\#} \geq E\left(\frac{1}{2}(D_1 + D_6) + \frac{1}{4}D_7\right) \geq \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = CD^{\#}.$$

So $E \in \mathrm{MV}(V, D)$ and E meets at least two (-2)-curves, which imply that (V, D) is of type (IIa), a contradiction. Hence, $ED_1 = 0$. We know that E meets a component, say D', in $\mathrm{Supp}(D - D^{(1)})$. Since the intersection matrix of $E + D_6 + D_7 + D'$ is negative definite, $D'^2 \leq -4$. However, by Lemma 2.4, this is a contradiction. Therefore, $a_0 = 4$.

Then the divisor $F_0 := 6(C + D_2) + 4D_3 + 2(D_0 + D_4) + D_1 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_6 becomes a section of Φ and $D - D_6$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_7. By Lemma 2.2 (1), we know that:

- F_0 and F_1 exhaust all singular fibers of Φ.
- $\mathrm{Supp}(F_1)$ contains a unique (-1)-curve, say E_1.

It is clear that $E_1 D_7 = 1$. Since the coefficient of D_7 in F_1 is equal to one, we know that $F_1 = 2E_1 + D_7 + D_8$, where D_8 is a (-2)-curve and is not a component of $D^{(1)}$ and $D_8 E_1 = D_7 E_1 = 1$. So $D = D^{(1)} + D^{(2)}$ and $D^{(2)} = D_8$. Therefore, the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (4) in Figure 5.1 (resp. Figure 5.2).

4.4

We finally consider the case $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 6)$. Then the weighted dual graph of $D^{(1)}$ is given as one of (1)–(4) in Figure 4.4, where we omit the self-intersection number corresponding to a (-2)-curve and set $a_0 = -D_0^2$.

We consider the following four cases 1–4 separately.

Case 1: The weighted dual graph of $D^{(1)}$ is given as (1) in Figure 4.4.

By using the same argument as in Case 1 in Section 4.2, we know that this case does not take place.

Case 2: The weighted dual graph of $D^{(1)}$ is given as (2) in Figure 4.4.

In this case, $D^{(1)\#} = D_0 + \frac{1}{2}D_1 + \frac{2}{3}D_2 + \frac{1}{3}D_3 + \frac{5}{6}D_4$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we know that $CD^{(1)} = CD_2 = 1$ and $a_0 = 2$. Then the divisor $F_0 := 2(C + D_2) + D_0 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_4 become sections of Φ and $D - (D_1 + D_4)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D^{(2)}$, which exists by $\#\,\mathrm{Sing}\, X \geq 2$. The irreducible component E_1 of $\mathrm{Supp}(F_1)$ meeting D_4, a section of Φ, is a (-1)-curve. We have

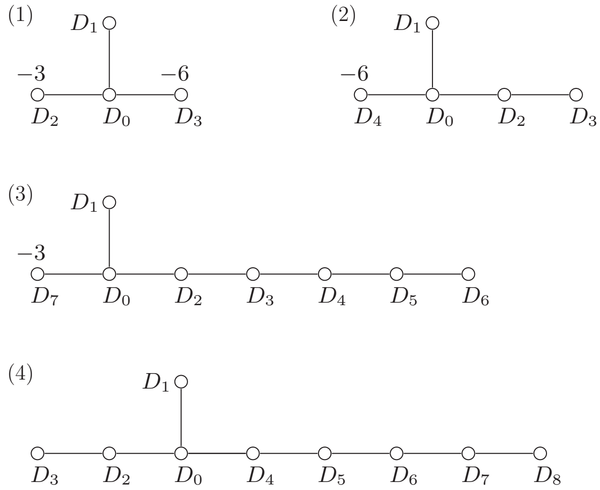

Figure 4.4

$$E_1D^{\#} \geq E_1D^{(1)\#} \geq \frac{5}{6}E_1D_4 = \frac{5}{6} > \frac{2}{3} = CD^{\#},$$

which is a contradiction. Therefore, this case does not take place.

Case 3: The weighted dual graph of $D^{(1)}$ is given as (3) in Figure 4.4.

In this case, $D^{(1)\#} = D_0 + \frac{5}{6}D_2 + \frac{2}{3}(D_3 + D_7) + \frac{1}{2}(D_1 + D_4) + \frac{1}{3}D_5 + \frac{1}{6}D_6$. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we know that $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4, 5\}$. We consider the following three subcases 3-1–3-3 separately.

Subcase 3-1: $i \in \{3, 4, 5\}$. (See the second paragraph in Case 2 in Section 4.3.) In this subcase, the divisor $F_0 := 2(C + D_i) + D_{i-1} + D_{i+1}$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F_0|} : V \to \mathbb{P}^1$. Then, D_7, which is a (-3)-curve, is a fiber component of $\Phi_{|F_0|}$. This is a contradiction by Lemma 2.3. Hence, this subcase does not take place.

Subcase 3-2: $i = 1$. Then $a_0 = 2$ since the intersection matrix of $C + D^{(1)}$ is not negative definite. So the divisor $F_0 := 3(C + D_1 + D_0) + 2D_2 + D_3 + D_7$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_4 becomes a section of Φ and $D - D_4$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D_5 + D_6$. By Lemma 2.2 (1), we know that:

- F_0 and F_1 exhaust all singular fibers of Φ.
- $\mathrm{Supp}(F_1)$ contains a unique (-1)-curve, say E_1.

Since $\ell = \#\,\mathrm{Sing}\,X \geq 2$, $\mathrm{Supp}(F_1)$ consists of E_1, D_5, D_6 and the components of $D^{(2)} + \cdots + D^{(\ell)}$. Since $E_1(D_5 + D_6) = E_1(D^{(2)} + \cdots + D^{(\ell)}) = 1$ and E_1 is the unique (-1)-curve in $\mathrm{Supp}(F_1)$, we have $\ell = 2$. It follows from Lemma 2.4 that $D^{(2)}$ consists of (-2)-curves and at most one (-3)-curve. Since $E_1+D_5+D_6+D^{(2)}$

has negative semi-definite intersection matrix, we know that the irreducible component of $D^{(2)}$ meeting E_1 is a (-3)-curve. Further, $D^{(2)}$ is an irreducible (-3)-curve and $D^{(2)\#} = \frac{1}{3}D^{(2)}$. Since $E_1 D^{\#} = E_1 D^{(1)\#} + \frac{1}{3} \le CD^{\#} = \frac{1}{2}$, we conclude that $E_1 D_6 = 1$. Therefore, the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (5) in Figure 5.1 (resp. Figure 5.2).

Subcase 3-3: $i = 2$. Then $a_0 \in \{2, 3, 4, 5\}$ since the intersection matrix of $C + D^{(1)}$ is not negative definite. Let $f : V \to W'$ be the contraction of $C, D_2, D_3, D_4, D_5, D_6$. Then $f_*(C + D^{(1)}) = f(D_1) + f(D_0) + f(D_7)$ is a linear chain of three $\mathbb{P}^1$'s and has the weighted dual graph $[-2, 5 - a_0, -3]$, where $f(D_1)^2 = -2$, $f(D_0)^2 = 5 - a_0$ and $f(D_7)^2 = -3$.

Then we obtain a birational morphism $g : \tilde{W} \to W'$ from a smooth projective surface $\tilde{W}$ such that the following conditions are satisfied:

- g is a composite of blowing-ups at $f(D_0) \cap f(D_7)$ and its infinitely near points.
- $g^{-1}(f_*(D^{(1)}))$ is a linear chain and its weighted dual graph is

$$\begin{cases} [-2, -1, -2, -1, -3, (-2)_{4-a_0}, -4] & (2 \le a_0 \le 4), \\ [-2, -1, -2, -1, -5] & (a_0 = 5), \end{cases}$$

  where $g'(f(D_1))$ is a (-2)-curve and is one of the end components of $g^{-1}(f_*(D^{(1)}))$ and $g'(f(D_0))$ is a (-1)-curve next to $g'(f(D_1))$. The subgraph $[(-2)_{4-a_0}]$ means the weighted dual graph of the linear chain consisting of $(4 - a_0)$ vertices of weight (-2).

Let $\tilde{E}$ be the (-2)-curve in $\mathrm{Supp}(g^{-1}(f_*(D^{(1)}))$ that is next to $g'(f(D_0))$ but not $g'(f(D_1))$. Let $h := \tilde{W} \to W$ be the contraction of $g'(f(D_0))$ and $\tilde{E}$. Then $\Gamma^{(1)} := h_*(g^{-1}(f_*(D^{(1)})))$ is a linear chain whose weighted dual graph is

$$\begin{cases} [0, 0, -3, (-2)_{4-a_0}, -4] & (2 \le a_0 \le 4), \\ [0, 0, -5] & (a_0 = 5). \end{cases}$$

Let $\Gamma^{(1)} = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_{7-a_0}$ be a decomposition of $\Gamma^{(1)}$ into irreducible components such that $\Gamma_0 = h(g'(f(D_1)))$, $\Gamma_0\Gamma_1 = \Gamma_1\Gamma_2 = \cdots = \Gamma_{6-a_0}\Gamma_{7-a_0} = 1$.

Set $\phi := h \circ g^{-1} \circ f : V \cdots \to W$ and let Γ be the total transform of $C + D$ via ϕ. We note that all the components of $D - D^{(1)}$ are not affected by the birational map ϕ and $\rho(W) = \#\Gamma = 8 - a_0 + \#(D - D^{(1)})$. The divisor Γ_0 defines a $\mathbb{P}^1$-fibration $\Psi := \Phi_{|\Gamma_0|} : W \to \mathbb{P}^1$, Γ_1 becomes a section of Ψ and $\Gamma - \Gamma_1$ is contained in fibers of Ψ. Let G_1 be the fiber of Ψ containing $\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{7-a_0}$. We prove the following claim.

Claim. (1) G_1 is the unique singular fiber of Ψ.

(2) $\mathrm{Supp}(G_1)$ is a linear chain of $\mathbb{P}^1$'s and contains a unique (-1)-curve, say E_1.

Proof. Let $G_1, \ldots, G_k$ ($k \geq 1$) exhaust all singular fibers of Ψ. Each $\mathrm{Supp}(G_j)$ ($j = 1, \ldots, k$) contains at least one (-1)-curve, which is not contained in the image of $D - D^{(1)}$ via ϕ. Since all components of $\Gamma - \Gamma_0$ are fiber components of Ψ, we have

$$\rho(W) - 2 = \sum_{j=1}^{k}(\#G_j - 1) \geq \#(D - D^{(1)}) + (\#\Gamma^{(1)} - 2) = \rho(W) - 2.$$

Hence, for each $j = 1, \ldots, k$, we know that:

- $\mathrm{Supp}(G_j)$ contains a unique (-1)-curve, say E_j.
- The proper transform of E_j on V is not a component of D.
- The proper transform of every component of $(G_j)_{\mathrm{red}} - E_i$ on V is a component of D.

Suppose that $k \geq 2$. Then $\mathrm{Supp}(G_2)$ consists only of E_2 and some components of the image of $\mathrm{Supp}(D - D^{(1)})$ by ϕ. Then the component of $\mathrm{Supp}(G_2)$ meeting Γ_1, which is a section of Ψ, must be E_2. So $\mathrm{Supp}(G_2)$ contains at least two (-1)-curves. This is a contradiction. Therefore, $k = 1$, which proves the assertion (1).

We prove the assertion (2). It is clear that $E_1(\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{7-a_0}) = 1$. Let $\mu : W \to Z$ be a successive contraction of E_1 and consecutively (smoothly) contractible curves in $\mathrm{Supp}(G_1)$ such that $\mu(\Gamma_2)$ becomes a (-1)-curve. Since the coefficient of Γ_2 in G_1 is equal to one and $\mathrm{Supp}(G_1)$ contains a unique (-1)-curve, it follows that $\mu_*(G_1)$ consists only of two (-1)-curves. By considering the possibility of μ, we obtain the assertion (2). □

By virtue of Claim, we know that $\#\,\mathrm{Sing}\,X = 2$, $D^{(2)}$ is an admissible rational rod and its weighted dual graph is $[-2, -2, a_0 - 7, -2]$, which is the adjoint of the dual graph of $\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{7-a_0}$ (see [3, (3.9)] for its definition). Let $D^{(2)} = D_8 + D_9 + D_{10} + D_{11}$ be the irreducible decomposition of $D^{(2)}$ such that $D_8^2 = D_9^2 = D_{11}^2 = -2$, $D_{10}^2 = a_0 - 7$, $D_8 D_9 = D_9 D_{10} = D_{10} D_{11} = 1$. Let E be the proper transform of E_1 on V. Then E is a (-1)-curve, $ED = 2$ and $ED_7 = ED_8 = 1$. By simple computation, we know that the coefficient α_8 of D_8 in $D^{\#}$ equals $\frac{10-2a_0}{35-6a_0}$. Since $C \in \mathrm{MV}(V, D)$,

$$CD^{\#} = \frac{5}{6} \geq ED^{\#} \geq \frac{2}{3}ED_7 + \alpha_8 ED_8 = \frac{2}{3} + \alpha_8.$$

Hence, $a_0 = 5$.

The divisor $F_0 := 5(C + D_2) + 4D_3 + 3D_4 + 2D_5 + D_0 + D_6$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_7 become sections of Φ and $D - (D_1 + D_7)$ is contained in fibers of Φ. By using the argument similar to that in Case 2 in Section 4.2 (see also Case 4 in Section 4.2 and Case 2 in Section 4.3), we know that the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (6) in Figure 5.1 (resp. Figure 5.2).

Case 4: The weighted dual graph of $D^{(1)}$ is given as (4) in Figure 4.4.

In this case, $D^{(1)\#} = D_0 + \frac{5}{6}D_4 + \frac{2}{3}(D_2+D_5) + \frac{1}{2}(D_1+D_6) + \frac{1}{3}(D_3+D_7) + \frac{1}{6}D_8$ and $a_0 \geq 3$ since the intersection matrix of $D^{(1)}$ is negative definite. Since $CD^{(1)\#} < 1$ and the intersection matrix of $C + D^{(1)}$ is not negative definite, we know that $CD^{(1)} = CD_i = 1$ for some $i \in \{2,4,5,6,7\}$. We consider the following three subcases 4-1–4-3 separately.

Subcase 4-1: $i \in \{5,6,7\}$. In this subcase, the divisor $F_0 := 2(C+D_i) + D_{i-1} + D_{i+1}$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$. Since $a_0 \geq 3$, D_0 is not a fiber component of Φ by Lemma 2.3 (see Subcase 3-1 in Case 3). Hence $i = 5$, D_0 and D_7 become sections of Φ and $D - (D_0 + D_7)$ is contained in fibers of Φ. Let F_1 (resp. F_2) be the fiber of Φ containing D_1 (resp. $D_2 + D_3$), here $F_1 \neq F_2$. By Lemmas 2.2 and 2.3, we know that:

- F_0, F_1 and F_2 exhaust all singular fibers of Φ.
- For $j = 1, 2$, the weighted dual grapf of F_j is given as one of (i)–(iii) in Figure 2.1 and every (-1)-curve in $\mathrm{Supp}(F_j)$ is an element of $\mathrm{MV}(V, D)$.

Since F_1 contains D_1, we know that the weighted dual graph of F_1 is given as one of (i) and (iii) in Figure 2.1. If it is given as (iii) in Figure 2.1, then $F_1 = E_1 + D_1 + E_1'$, where E_1 and E_1' are (-1)-curves and $E_1D_1 = E_1'D_1 = 1$. We may assume that $E_1D_7 = 1$ since D_7 is a section of Φ. Then

$$E_1D^{\#} \geq E_1\left(\frac{1}{2}D_1 + \frac{1}{3}D_7\right) = \frac{5}{6} > \frac{2}{3} = CD^{\#},$$

which is a contradiction. So the weighted dual graph of F_1 is given as (i) in Figure 2.1. In particular, $\mathrm{Supp}(F_1)$ has a unique (-1)-curve, say E_1. Since E_1 has coefficient two in F_1, $E_1D_7 = 0$. So $\mathrm{Supp}(F_1)$ contains D_8, in particular, $F_1 = 2E_1 + D_1 + D_8$ and $E_1D_1 = E_1D_8 = 1$. Then

$$E_1D^{\#} \geq E_1\left(\frac{1}{2}D_1 + \frac{1}{6}D_8\right) = \frac{2}{3} = CD^{\#}.$$

So $E_1 \in \mathrm{MV}(V, D)$ and E_1 meets two (-2)-curves D_1 and D_8, which imply that (V, D) is of type (IIa). This is a contradiction. Therefore, this subcase does not take place.

Subcase 4-2: $i = 2$. Then $a_0 = 3$ since the intersection matrix of $C + D^{(1)}$ is not negative definite. So the divisor $F_0 := 4(C + D_2) + 2(D_0 + D_3) + D_1 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_5 becomes a section of Φ and $D - D_5$ is contained in fibers of Φ. By the same argument as in Subcase 3-2 in Case 3, we know that:

- F_0 and F_1 exhaust all singular fibers of Φ.

- $\mathrm{Supp}(F_1)$ contains a unique (-1)-curve, say E_1.

Since $\ell = \#\,\mathrm{Sing}\,X \geq 2$, $\mathrm{Supp}(F_1)$ consists of E_1, D_6, D_7, D_8 and the components of $D^{(2)} + \cdots + D^{(\ell)}$. It is clear that $\ell = 2$ and $E_1(D_6 + D_7 + D_8) = ED^{(2)} = 1$. Let D_j be the irreducible component of $D^{(2)}$ meeting E_1. Since the intersection matrix of $E_1 + D_6 + D_7 + D_8$ is negative definite, we know that $E_1 D_7 = 0$ and $D_j^2 \leq -4$. Then the coefficient α_j of D_j in $D^{\#} \geq \frac{1}{2}$ and the equality holds if and only if $D_j = D^{(2)}$ and $D_j^2 = -4$. Since $CD^{\#} = \frac{2}{3}$ and $CD^{\#} \geq E_1 D^{\#}$, we know that $E_1 D_8 = 1$ and $D_j^2 = -4$. So, we may set $j = 9$ and have $F_1 = 4E_1 + 3D_8 + 2D_7 + D_6 + D_9$. Therefore, we know that the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (7) in Figure 5.1 (resp. Figure 5.2).

Subcase 4-3: $i = 4$. Then $a_0 \in \{3, 4, 5, 6\}$ since the intersection matrix of $C + D^{(1)}$ is not negative definite. Suppose that $a_0 = 6$. Then the divisor $F := 10(C + D_4) + 8D_5 + 6D_6 + 4D_7 + 2(D_0 + D_8) + D_1 + D_2$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$, D_3 becomes a section of Φ and $D - D_3$ is contained in fibers of $\Phi_{|F|}$. Let F' be the singular fiber of $\Phi_{|F|}$ containing $D^{(2)}$, which exists by $\#\,\mathrm{Sing}\,X \geq 2$. Then the irreducible component of $\mathrm{Supp}(F')$ meeting D_3, a section of $\Phi_{|F|}$, must be a (-1)-curve. So $\mathrm{Supp}(F')$ contains at least two (-1)-curves. This is a contradiction by Lemma 2.2 (1). Therefore, $3 \leq a_0 \leq 5$. Let $f : V \to W'$ be the contraction of $C, D_4, D_5, D_6, D_7, D_8$. Then $f_*(C + D^{(1)}) = f(D_1) + f(D_0) + f(D_2) + f(D_3)$ is a linear chain of four $\mathbb{P}^1$'s and has the weighted dual graph $[-2, 5 - a_0, -2, -2]$, where $f(D_1)^2 = f(D_2)^2 = f(D_3)^2 = -2$ and $f(D_0)^2 = 5 - a_0$.

Then we obtain a birational morphism $g : \tilde{W} \to W'$ from a smooth projective surface $\tilde{W}$ such that the following conditions are satisfied:

- g is a composite of blowing-ups at $f(D_0) \cap f(D_2)$ and its infinitely near points.
- $g^{-1}(f_*(D^{(1)}))$ is a linear chain and its weighted dual graph is

$$\begin{cases} [-2, -1, -2, -1, -3, -2, -3, -2] & (a_0 = 3), \\ [-2, -1, -2, -1, -3, -3, -2] & (a_0 = 4), \\ [-2, -1, -2, -1, -4, -2] & (a_0 = 5), \end{cases}$$

  where $g'(f(D_1))$ is a (-2)-curve and is one of the end components of $g^{-1}(f_*(D^{(1)}))$ and $g'(f(D_0))$ is a (-1)-curve next to $g'(f(D_1))$.

Let $\tilde{E}$ be the (-2)-curve in $\mathrm{Supp}(g^{-1}(f_*(D^{(1)})))$ that is next to $g'(f(D_0))$ but not $g'(f(D_1))$. Let $h := \tilde{W} \to W$ be the contraction of $g'(f(D_0))$ and $\tilde{E}$. Then $\Gamma^{(1)} := h_*(g^{-1}(f_*(D^{(1)})))$ is a linear chain whose weighted dual graph is

$$\begin{cases} [0, 0, -3, -2, -3, -2] & (a_0 = 3), \\ [0, 0, -3, -3, -2] & (a_0 = 4), \\ [0, 0, -4, -2] & (a_0 = 5). \end{cases}$$

In particular, $\#\Gamma^{(1)} = 9 - a_0$. Let $\Gamma^{(1)} = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_{8-a_0}$ be a decomposition of $\Gamma^{(1)}$ into irreducible components such that $\Gamma_0 = h(g'(f(D_1)))$, $\Gamma_0\Gamma_1 = \Gamma_1\Gamma_2 = \cdots = \Gamma_{7-a_0}\Gamma_{8-a_0} = 1$.

Set $\phi := h \circ g^{-1} \circ f : V \cdots \to W$ and let Γ be the total transform of $C + D$ via ϕ. We note that all the components of $D - D^{(1)}$ are not affected by the birational map ϕ and $\rho(W) = \#\Gamma = 9 - a_0 + \#(D - D^{(1)})$. The divisor Γ_0 defines a $\mathbb{P}^1$-fibration $\Psi := \Phi_{|\Gamma_0|} : W \to \mathbb{P}^1$, Γ_1 becomes a section of Ψ and $\Gamma - \Gamma_1$ is contained in fibers of Ψ. Let G_1 be the fiber of Ψ containing $\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{8-a_0}$. By using the same argument as in the proof of Claim in Subcase 3-3 in Case 3, we obtain the following claim.

Claim. (1) G_1 is the unique singular fiber of Ψ.
(2) $\mathrm{Supp}(G_1)$ is a linear chain of $\mathbb{P}^1$'s and contains a unique (-1)-curve, say E_1.

By virtue of Claim, we know that $\#\,\mathrm{Sing}\,X = 2$ and $D^{(2)}$ is an admissible rational rod and its weighted dual graph is $[-3, a_0 - 7, -2]$, which is the adjoint of the dual graph of $\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{8-a_0}$. Let $D^{(2)} = D_9 + D_{10} + D_{11}$ be the irreducible decomposition of $D^{(2)}$ such that $D_9^2 = -3$, $D_{10}^2 = a_0 - 7$, $D_{11}^2 = -2$, $D_9D_{10} = D_{10}D_{11} = 1$. Let E be the proper transform of E_1 on V. Then E is a (-1)-curve, $ED = 2$ and $ED_3 = ED_9 = 1$. By simple computation, we know that the coefficient α_9 of D_9 in $D^\#$ equals $\frac{23-4a_0}{37-6a_0}$. Since $C \in \mathrm{MV}(V, D)$,

$$CD^\# = \frac{5}{6} \geq ED^\# \geq \frac{1}{3}ED_3 + \alpha_9 ED_9 = \frac{1}{3} + \alpha_9.$$

Hence, $a_0 = 5$. Furthermore, $ED_1 = 0$.

The divisor $F_0 := 5(C + D_4) + 4D_5 + 3D_6 + 2D_7 + D_0 + D_8$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_2 become sections of Φ and $D-(D_1+D_2)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_3. Then the component of $\mathrm{Supp}(F_1)$ meeting D_1 is a (-1)-curve. So, $\mathrm{Supp}(F_1)$ contains at least two (-1)-curves. By Lemma 2.2, we know that:

- F_0 and F_1 exhaust all singular fibers of Φ.
- $\mathrm{Supp}(F_1)$ contains just two (-1)-curves.

Hence, $\mathrm{Supp}(F_1)$ contains E and $D^{(2)} = D_9 + D_{10} + D_{11}$. By using the similar argument to that in Case 2 in Section 4.2 (see also Case 4 in Section 4.2 and Case 2 in Section 4.3), we know that $F_1 = 2E + D_9 + D_{10} + D_{11} + D_1 + E'$, $E'D_{11} = 1$ and $\mathrm{Supp}(F_1)$ is a linear chain. Therefore, the weighted dual graph of D (resp. the configuration of $C + D$ and all the singular fibers of Φ) is given as (8) in Figure 5.1 (resp. Figure 5.2).

The proof of Theorem 1.2 is thus completed.

5. The dual graphs and the configurations in Theorem 1.2

In Figure 5.1, the numbers in brackets coincide with the classifying numbers

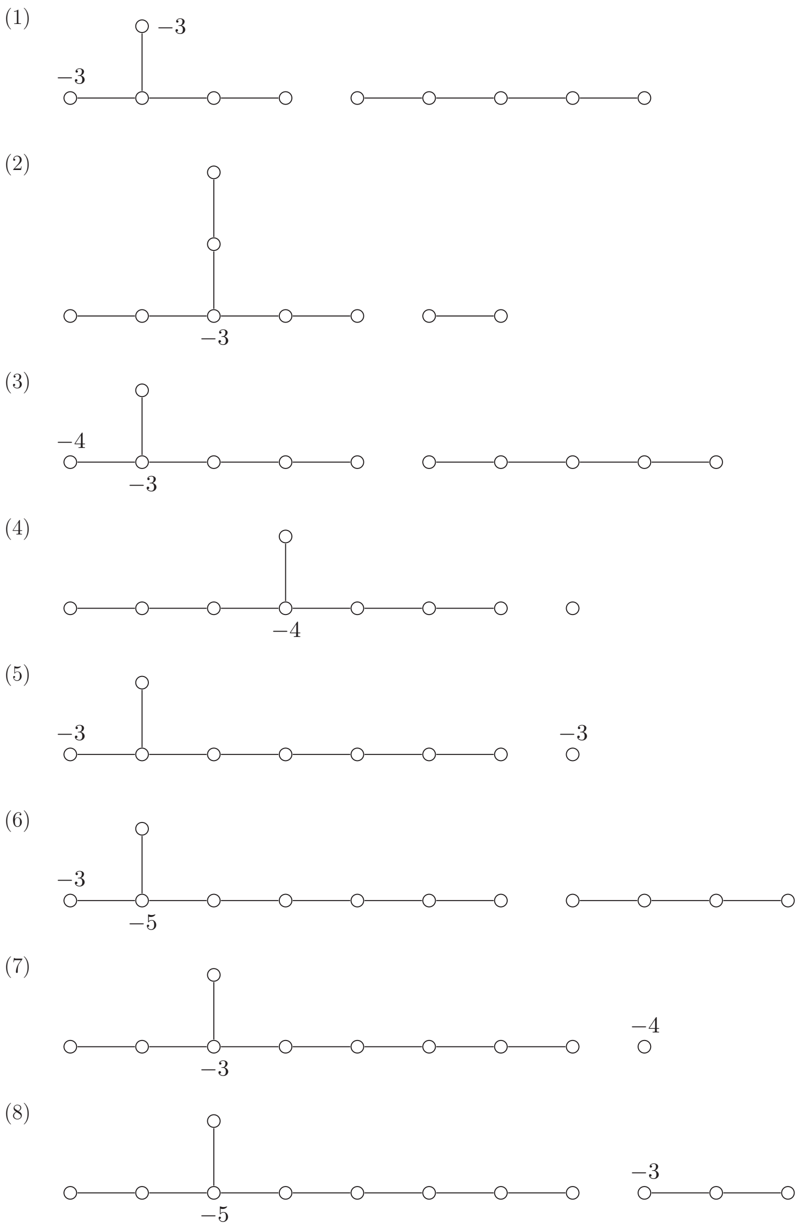

Figure 5.1

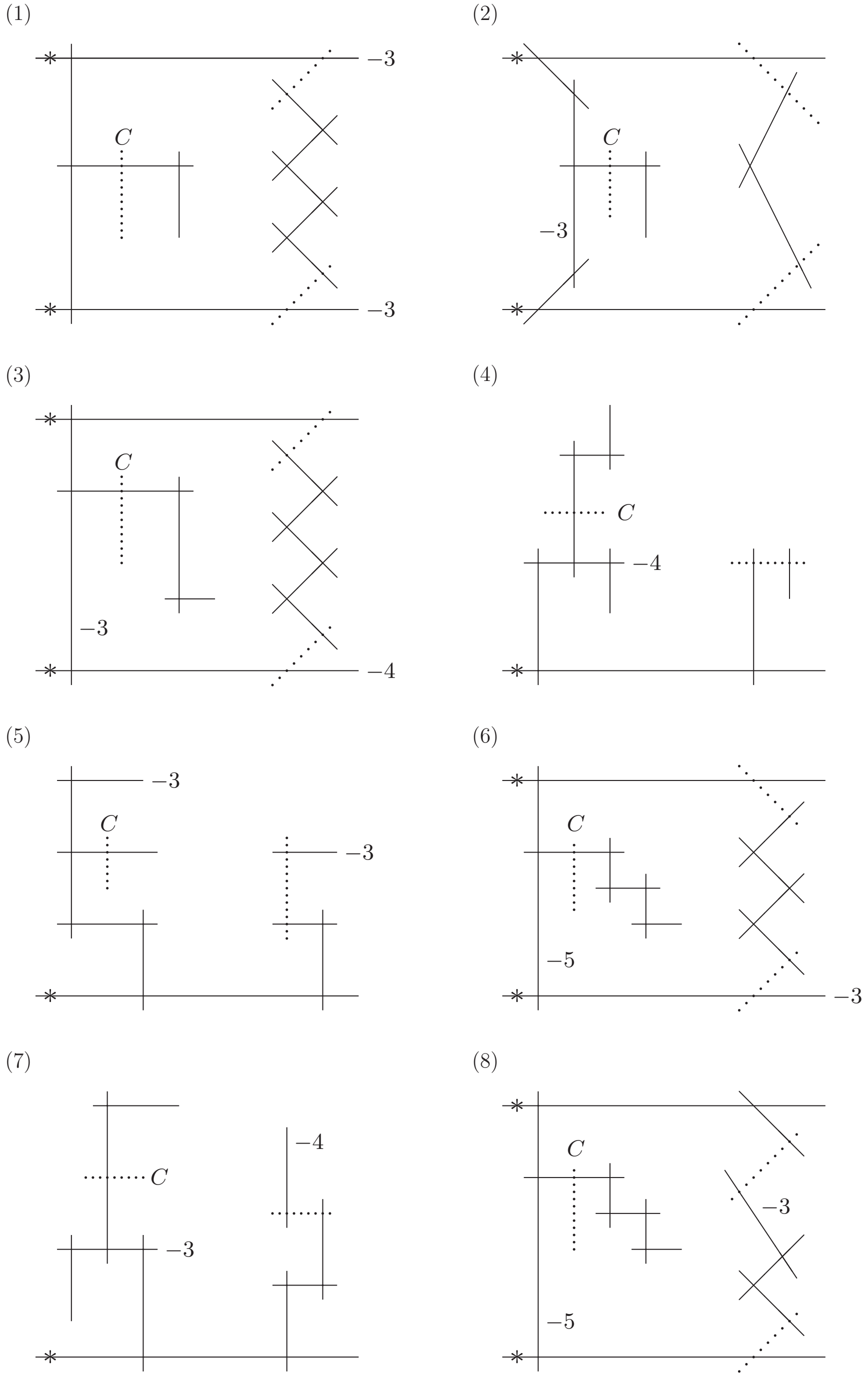

Figure 5.2

in Theorem 1.2; we omit the weight corresponding to a (-2)-curve.

In Figure 5.2, the numbers in brackets coincide with the classifying numbers in Theorem 1.2; a dotted line stands for a (-1)-curve; a solid line stands for a component of D; the self-intersection number of a (-2)-curve is omitted; a line with $*$ on it is a section of the vertical $\mathbb{P}^1$-fibration on V.

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