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Triangular $\mathbb{Z}/3\mathbb{Z}$-actions on the affine four-space in characteristic three

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Abstract

Let k be a field of characteristic three, let $\mathbb{Z}/3\mathbb{Z}$ denote the cyclic group of order three and let $\mathbb{A}_k^4$ denote the affine four-space over k. In this article, we describe triangular $\mathbb{Z}/3\mathbb{Z}$-actions on $\mathbb{A}_k^4$, up to conjugation of automorphisms of $\mathbb{A}_k^4$.

Introduction

Let k be a field of positive characteristic p, let $\mathbb{A}_k^n$ denote the affine n-space over k and let $k[x_1, \ldots, x_n]$ denote the coordinate ring of $\mathbb{A}_k^n$. Given an algebraic action of a p-cyclic group $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{A}_k^n$, we can regard any element σ of $\mathbb{Z}/p\mathbb{Z}$ as a k-algebra automorphism of $k[x_1, \ldots, x_n]$ satisfying $\sigma^p = \mathrm{id}_{k[x_1,\ldots,x_n]}$, where $\mathrm{id}_{k[x_1,\ldots,x_n]}$ denotes the identity map from $k[x_1, \ldots, x_n]$ to itself. Conversely, given a k-algebra automorphism σ of $k[x_1, \ldots, x_n]$ satisfying $\sigma^p = \mathrm{id}_{k[x_1,\ldots,x_n]}$, we can naturally define an algebraic $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{A}_k^n$. We say that an algebraic $\mathbb{Z}/p\mathbb{Z}$-action Θ on $\mathbb{A}_k^n$ is *triangular* if there exists an element θ of $\mathbb{Z}/p\mathbb{Z}$ such that the following conditions (1) and (2) are satisfied:

(1) θ is a triangular automorphism of $k[x_1, \ldots, x_n]$, i.e., θ can be expressed as $\theta(x_i) = a_i x_i + f_i$ for some $a_i \in k\backslash\{0\}$ and $f_i \in k[x_1, \ldots, x_{i-1}]$ $(1 \le i \le n)$.

(2) $\Theta(\ell, x) = \theta^\ell(x)$ for all $\ell \ge 0$ and $x \in \mathbb{A}_k^n$.

Clearly, $\theta^p = \mathrm{id}_{k[x_1,\ldots,x_n]}$ and $a_i = 1$ for all $1 \le i \le n$.

We are interested in describing triangular $\mathbb{Z}/p\mathbb{Z}$-actions Θ on $\mathbb{A}_k^n$ up to conjugation of automorphisms of $\mathbb{A}_k^n$ as well as describing triangular automorphisms θ of $k[x_1, \ldots, x_n]$ satisfying $\theta^p = \mathrm{id}_{k[x_1,\ldots,x_n]}$ up to conjugation of automorphisms of $k[x_1, \ldots, x_n]$. At present, we can describe triangular automorphisms θ of $k[x_1, \ldots, x_n]$ satisfying $\theta^p = \mathrm{id}_{k[x_1,\ldots,x_n]}$ only for $1 \le n \le 3$ (cf. [1] for $n = 2$, and [2] for $n = 3$).

In this article, assuming $p = 3$, we describe triangular automorphisms τ of $k[x, y, z, w]$ satisfying $\tau^3 = \mathrm{id}_{k[x,y,z,w]}$, up to conjugation of automorphisms of

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$k[x,y,z,w]$, where $k[x,y,z,w]$ is a polynomial ring in four variables over k. For this description, we introduce special triangular automorphisms of $k[x,y,z,w]$ with the following forms ①, ②, ③, ④:

① Let $\gamma(x,y,z) \in k[x,y,z]$. We can define a triangular automorphism τ of $k[x,y,z,w]$ as

$$\begin{cases} \tau(x) := x, \\ \tau(y) := y, \\ \tau(z) := z, \\ \tau(w) := w + \gamma(x,y,z). \end{cases}$$

② Given a non-zero polynomial $\beta(x,y) \in k[x,y]$, we let $N(z) := z^3 - \beta(x,y)^2\, z$. Choose arbitrary $\gamma_i \in k[x,y,N(z)]$ $(i = 0,1)$. We can define a triangular automorphism τ of $k[x,y,z,w]$ as

$$\begin{cases} \tau(x) := x, \\ \tau(y) := y, \\ \tau(z) := z + \beta(x,y), \\ \tau(w) := w + \gamma_0 + \gamma_1\, z. \end{cases}$$

③ Given two non-zero polynomials $\alpha(x) \in k[x]$ and $\beta_0(s,t) \in k[s,t]$, we let $N(y) := y^3 - \alpha^2 y$ and let $d(x)$ be the greatest common divisor of $\alpha(x)$ and $\beta_0(x, N(y))$ in $k[x,y]$. We can express $\alpha(x)$ and $\beta_0(x,N(y))$ as $\alpha(x) = d(x)\, a(x)$ and $\beta_0(x,N(y)) = d(x)\, b_0(x,N(y))$ for some $a(x) \in k[x]$ and $b_0(s,t) \in k[s,t]$. Write $\beta_0 := \beta_0(x,N(y))$ and $b_0 := b_0(x,N(y))$. Let

$$\begin{cases} f_1 := x, \\ f_2 := y^3 - \alpha^2 y, \\ f_3 := z^3 - \beta_0(x,N(y))^2\, z, \\ f_4 := az - b_0(x,N(y))\, y, \end{cases}$$

and let

$$\begin{cases} g_1 := 1, \\ g_2 := y, \\ g_3 := z, \\ g_4 := ayz^2 + b_0(x,N(y))\, y^2 z. \end{cases}$$

Choose arbitrary $\gamma_i \in k[f_1, f_2, f_3, f_4]$ $(1 \le i \le 4)$. Then we can define a triangular automorphism τ of $k[x, y, z, w]$ as

$$\begin{cases} \tau(x) := x, \\ \tau(y) := y + \alpha(x), \\ \tau(z) := z + \beta_0(x, N(y)), \\ \tau(w) := w + \displaystyle\sum_{i=1}^{4} \gamma_i \cdot g_i. \end{cases}$$

④ Given three polynomials $\alpha(x) \in k[x]$, $\beta_0(s,t), \beta_1(s,t) \in k[s,t]$ such that $\alpha(x) \neq 0$ and $\beta_1(s,t) \neq 0$, we let $N(y) := y^3 - \alpha^2 y$ and let $d(x)$ be the greatest common divisor of $\alpha(x)$ and $\beta_0(x, N(y)) + \beta_1(x, N(y))\, y$ in $k[x, y]$. We can express $\alpha(x)$ and $\beta_i(x, N(y))$ $(i = 0, 1)$ as $\alpha(x) = d(x)\, a(x)$ and $\beta_i(x, N(y)) = d(x)\, b_i(x, N(y))$ for some $a(x) \in k[x]$ and $b_i(s,t) \in k[s,t]$ $(i = 0, 1)$. Write $\beta_i := \beta_i(x, N(y))$ $(i = 0, 1)$ and $b_i := b_i(x, N(y))$ $(i = 0, 1)$. Let

$$\begin{cases} f_1 := x, \\ f_2 := y^3 - \alpha^2 y, \\ f_3 := z^3 + \alpha\beta_1 z^2 - \beta_1^2 y^2 z + (\alpha\beta_1^2 + \beta_0\beta_1) yz + (-\beta_0^2 + \alpha\beta_0\beta_1) z, \\ f_4 := az + b_1 y^2 - a\beta_1 y - b_0 y, \end{cases}$$

and let

$$\begin{cases} g_1 := 1, \\ g_2 := y, \\ g_3 := b_0\beta_1 y^2 - 2b_0 yz + 2az^2 + b_1 y^2 z, \\ g_4 := b_1 f_2 z + (a\beta_1 + b_0) y^2 z + ayz^2. \end{cases}$$

For any $(\lambda_1, \lambda_2, \lambda_3) \in k[x, y]^{\oplus 3}$, we define a polynomial $g(\lambda_1, \lambda_2, \lambda_3)$ of $k[x, y, z]$ as

$$g(\lambda_1, \lambda_2, \lambda_3) := \lambda_1 z + \lambda_2 y^2 + \lambda_3 (yz - \beta_1 y^2).$$

Consider polynomials

$$g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}) \quad (1 \le j \le r)$$

of $k[x, y, z]$, where $\{ (\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}) \in k[f_1, f_2]^{\oplus 3} \mid 1 \le j \le r \}$ is a generating set of the syzygy $k[f_1, f_2]$-module $\mathrm{Syz}_{k[f_1, f_2]}(b_1, -a, -b_0)$. Choose arbitrary

$\gamma_{1,i} \in k[f_1, f_2, f_3, f_4]$ $(1 \le i \le 4)$ and $\gamma_{2,j} \in k[f_1, f_2, f_3, f_4]$ $(1 \le j \le r)$. Then we can define a triangular automorphism τ of $k[x, y, z, w]$ as

$$\begin{cases} \tau(x) := x, \\ \tau(y) := y + \alpha(x), \\ \tau(z) := z + \beta_0(x, N(y)) + \beta_1(x, N(y))\, y, \\ \tau(w) := w + \displaystyle\sum_{i=1}^{4} \gamma_{1,i}\, g_i + \sum_{j=1}^{r} \gamma_{2,j}\, g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}). \end{cases}$$

The aim of this article is to prove the following theorem:

Theorem 1 *Assume that the characteristic p of k is three. Then the following assertions* (1) *and* (2) *hold true:*

(1) *Any triangular automorphism τ of $k[x, y, z, w]$ with one of the above forms* ①, ②, ③, ④ *satisfies $\tau^3 = \mathrm{id}_{k[x,y,z,w]}$.*

(2) *For any triangular automorphism τ of $k[x, y, z, w]$ satisfying $\tau^3 = \mathrm{id}_{k[x,y,z,w]}$, there exists an automorphism Ψ of $k[x, y, z, w]$ such that $\Psi^{-1} \circ \tau \circ \Psi$ is a triangular automorphism of $k[x, y, z, w]$ with one of the above forms* ①, ②, ③, ④.

Definitions and notations.

Let k be a field of positive characteristic p and let A be a k-algebra. For a k-algebra homomorphism $\sigma : A \to A$, we say that σ is *p-unipotent* if $\sigma^p = \mathrm{id}_A$, where $\mathrm{id}_A : A \to A$ denotes the identity map. Clearly, if a k-algebra homomorphism $\sigma : A \to A$ is p-unipotent, then σ is a k-algebra automorphism of A.

Let A be a k-algebra. Let $D_\sigma : A \to A$ be the map defined by $D_\sigma(a) := \sigma(a) - a$ for all $a \in A$. Clearly, D_σ is a σ-twisted k-derivation of A, i.e., D_σ is a k-linear transformation of A satisfying $D_\sigma(a_1 \cdot a_2) = D_\sigma(a_1) \cdot \sigma(a_2) + a_1 \cdot D_\sigma(a_2)$ for all $a_1, a_2 \in A$. For any $\ell \ge 1$, we denote by $\mathrm{Ker}(D_\sigma^\ell)$ the kernel of D_σ^ℓ, i.e.,

$$\mathrm{Ker}(D_\sigma^\ell) := \{\, a \in A \mid D_\sigma^\ell(a) = 0 \,\}.$$

Clearly, $\mathrm{Ker}(D_\sigma)$ is a k-subalgebra of A, and $\mathrm{Ker}(D_\sigma^\ell)$ is a $\mathrm{Ker}(D_\sigma)$-module for any $\ell \ge 1$. If A is finitely generated as a k-algebra and $D_\sigma^p = 0$, then $\mathrm{Ker}(D_\sigma)$ is finitely generated as a k-algebra, and each $\mathrm{Ker}(D_\sigma^\ell)$ is finitely generated as a $\mathrm{Ker}(D_\sigma)$-module.

1. *p*-unipotent triangular automorphisms

1.1 An inductive construction of *p*-uniptotent triangular automorphisms

We can construct p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n, x_{n+1}]$ from p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n]$. We shall recall this inductive construction (see the following Lemma 2).

We denote by $\mathrm{U}^{p,\triangle}(k[x_1, \ldots, x_n])$ the set of all p-unipotent triangular automorphisms of $k[x_1, \ldots, x_n]$.

For any k-algebra homomorphism $\sigma : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ and any polynomial δ of $k[x_1, \ldots, x_n]$, we can define a k-algebra homomorphism $E_{\sigma,\,\delta} : k[x_1, \ldots, x_n, x_{n+1}] \to k[x_1, \ldots, x_n, x_{n+1}]$ as

$$E_{\sigma,\,\delta}(x_i) := \begin{cases} \sigma(x_i) & \text{if} \quad 1 \le i \le n, \\ x_{n+1} + \delta & \text{if} \quad i = n+1. \end{cases}$$

Lemma 2 *Let $n \ge 1$. Then the following assertions* (1) *and* (2) *hold true:*

(1) *Let $\sigma \in U^{p,\triangle}(k[x_1, \ldots, x_n])$. Take any polynomial δ of $\mathrm{Ker}(D_\sigma^{p-1})$. Then $E_{\sigma,\,\delta} \in U^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$.*

(2) *Let $\tau \in U^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$. We can define a k-algebra homomorphism $\sigma : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ as $\sigma(f) := \tau(f)$ for all $f \in k[x_1, \ldots, x_n]$. Let $\delta := \tau(x_{n+1}) - x_{n+1} \in k[x_1, \ldots, x_n]$. Then we have $\sigma \in U^{p,\triangle}(k[x_1, \ldots, x_n])$, $\delta \in \mathrm{Ker}(D_\sigma^{p-1})$ and $\tau = E_{\sigma,\,\delta}$.*

Proof. See [1, Lemma 3]. Q.E.D.

The following lemma implies that if two p-unipotent triangular automorphisms $\sigma_1, \sigma_2 \in U^{p,\triangle}(k[x_1, \ldots, x_n])$ are conjugate to each other, i.e., $\psi^{-1} \circ \sigma_1 \circ \psi = \sigma_2$ for some k-algebra automorphism ψ of $k[x_1, \ldots, x_n]$, then for any $\delta \in \mathrm{Ker}(D_{\sigma_1}^{p-1})$, the two p-unipotent triangular automorphisms $E_{\sigma_1,\,\delta}$ and $E_{\sigma_2,\,\psi^{-1}(\delta)}$ of $U^{p,\triangle}(k[x_1, \ldots, x_n, x_{n+1}])$ are conjugate to each other.

Lemma 3 *Let $\sigma_i : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ $(i = 1, 2)$ be two k-algebra homomorphisms such that $\psi^{-1} \circ \sigma_1 \circ \psi = \sigma_2$ for some k-algebra automorphism ψ of $k[x_1, \ldots, x_n]$. We can extend ψ to the k-algebra automorphism Ψ of $k[x_1, \ldots, x_n, x_{n+1}]$ defined by*

$$\Psi(x_i) := \begin{cases} \psi(x_i) & \textit{if} \quad 1 \le i \le n, \\ x_{n+1} & \textit{if} \quad i = n+1. \end{cases}$$

Let $\delta \in k[x_1, \ldots, x_n]$. Then the following assertions (1) *and* (2) *hold true:*

(1) $\Psi^{-1} \circ E_{\sigma_1, \delta} \circ \Psi = E_{\sigma_2, \psi^{-1}(\delta)}$.

(2) *For any* $\ell \geq 1$, $\delta \in \mathrm{Ker}(D_{\sigma_1}^{\ell})$ *if and only if* $\psi^{-1}(\delta) \in \mathrm{Ker}(D_{\sigma_2}^{\ell})$.

Proof. The proofs of assertions (1) and (2) are straightforward. Q.E.D.

1.2 $U^{p,\triangle}(k[x, y, z])$

Any p-unipotent triangular automorphism σ of $k[x, y, z]$ has one of the following forms (I) and (II), up to conjugataion of automorphisms of $k[x, y, z]$ (see [2, Theorem 2]):

$$\text{(I)} \quad \begin{cases} \sigma(x) = x, \\ \sigma(y) = y, \\ \sigma(z) = z + \beta(x, y) \qquad (\,\beta(x, y) \in k[x, y]\,). \end{cases}$$

$$\text{(II)} \quad \begin{cases} \sigma(x) = x, \\ \sigma(y) = y + \alpha(x) \qquad (\,\alpha(x) \in k[x] \backslash \{0\}\,), \\ \sigma(z) = z + \displaystyle\sum_{i=0}^{p-2} \beta_i(x, N_y) \cdot y^i \\ \qquad\qquad\qquad \left(\begin{array}{l} N_y := y^p - \alpha(x)^{p-1} y \in k[x, y], \\ \beta_i(s, t) \in k[s, t] \text{ for all } 0 \leq i \leq p-2, \\ \beta_i(s, t) \neq 0 \text{ for some } 0 \leq i \leq p-2 \end{array} \right). \end{cases}$$

Clearly, any k-algebra homomorphism σ of $k[x, y, z]$ with one of the above forms (I) and (II) is a p-unipotent triangular automorphism of $k[x, y, z]$.

For any polynomial $f \in k[x, y, z]$ and any k-algebra automorphism σ of $k[x, y, z]$, we can define a polynomial $N(f)$ of $k[x, y, z]$ as

$$N(f) := \prod_{i=0}^{p-1} \sigma^i(f).$$

If σ has the form (II), we have

$$N_y = N(y).$$

2. $\mathrm{Ker}(D_\sigma)$

From now on until the last section of this article, we assume $p = 3$ and σ is a 3-unipotent triangular automorphism of $k[x, y, z]$ with one of the forms (I) and (II).

In this section, we recall a generating set of the kernel $\mathrm{Ker}(D_\sigma)$ of D_σ, where the map $D_\sigma : k[x,y,z] \to k[x,y,z]$ is defined as $D_\sigma(f) = \sigma(f) - f$ (see the Introduction).

If σ has the form (II), we let $\beta(x,y) := \beta_0(x,N(y)) + \beta_1(x,N(y)) \cdot y$ and let $d(x)$ be the greatest common divisor of the two polynomials $\alpha(x)$ and $\beta(x,y)$ of $k[x,y]$. Then we can express $\alpha(x)$, $\beta(x,y)$ and $\beta_i(x,y)$ $(i = 1,2)$ as

$$\begin{cases} \alpha(x) = d(x)\,a(x) & (\,a(x) \in k[x]\backslash\{0\}\,), \\ \beta(x,y) = d(x)\,b(x,y) & (\,b(x,y) \in k[x,y]\,), \\ \beta_i(x,y) = d(x)\,b_i(x,N(y)) & (\,b_i[s,t] \in k[s,t] \quad \text{for} \quad i = 0,1\,). \end{cases}$$

Clearly, $b(x,y) = b_0(x,N(y)) + b_1(x,N(y))\,y$. For simplicity, we write β_0, β_1, b_0, b_1 in place of $\beta_0(x,N(y))$, $\beta_1(x,N(y))$, $b_0(x,N(y))$, $b_1(x,N(y))$, respectively. Now, we can define the four polynomials f_i $(1 \le i \le 4)$ of $k[x,y,z]$ as

$$\begin{cases} f_1 := x, \\ f_2 := N(y) = y^3 - \alpha^2 y, \\ f_3 := N(z) = z^3 + \alpha\beta_1 z^2 - \beta_1^2 y^2 z + (\alpha\beta_1^2 + \beta_0\beta_1)yz + (-\beta_0^2 + \alpha\beta_0\beta_1)z, \\ f_4 := az + b_1 y^2 - a\beta_1 y - b_0 y. \end{cases}$$

Lemma 4 *The following assertions* (1), (2), (3) *hold true:*

(1) *If σ has the form* (I) *and $\beta = 0$, then*

$$\mathrm{Ker}(D_\sigma) = k[\,x,\,y,\,z\,].$$

(2) *If σ has the form* (I) *and $\beta \neq 0$, the kernel* $\mathrm{Ker}(D_\sigma)$ *is generated by he set* $\{x,\,y,\,N(z)\}$ *as a k-algebra, i.e.,*

$$\mathrm{Ker}(D_\sigma) = k[\,x,\,y,\,N(z)\,].$$

(3) *If σ has the form* (II), *the kernel* $\mathrm{Ker}(D_\sigma)$ *is generated by the set* $\{f_1,\,f_2,\,f_3,\,f_4\}$ *as a k-algebra, i.e.,*

$$\mathrm{Ker}(D_\sigma) = k[\,f_1,\; f_2,\; f_3,\; f_4\,],$$

and $\mathrm{Ker}(D_\sigma)$ *is a hypersurface ring with the relation*

$$f_4^3 + a\alpha\beta_1 f_4^2 - (a^2\beta_0^2 - a^2\alpha\beta_0\beta_1)f_4 - a^3 f_3 + 2b_1^3 f_2^2 + (b_0^3 - a^2 b_0 \beta_1^2)f_2 = 0.$$

Proof. (1) is clear.

(2) See [2, Lemma 2.8].

(3) See [2, Lemmas 4.2 and 4.6]. Q.E.D.

3. $\mathrm{Ker}(D_\sigma^2)$

In the following theorem, by separating four cases, we give, in each case, a generating set of $\mathrm{Ker}(D_\sigma^2)$.

Theorem 5 *Let $B := \mathrm{Ker}(D_\sigma)$. Then the following assertions* (1), (2), (3), (4) *hold true:*

(1) *If σ has the form* (I) *and $\beta = 0$, we have $D_\sigma = 0$ and*

$$\mathrm{Ker}(D_\sigma^2) = \mathrm{Ker}(D_\sigma) = B.$$

(2) *If σ has the form* (I) *and $\beta \neq 0$, the kernel* $\mathrm{Ker}(D_\sigma^2)$ *is generated by the set* $\{1,\ z\}$ *as a B-module, i.e.,*

$$\mathrm{Ker}(D_\sigma^2) = B \cdot 1 + B \cdot z.$$

(3) *If σ has the form* (II) *and $\beta_1(s,t) = 0$, the kernel* $\mathrm{Ker}(D_\sigma^2)$ *is generated by the set* $\{1,\ y,\ z,\ ayz^2 + by^2z\}$ *as a B-module, i.e.,*

$$\mathrm{Ker}(D_\sigma^2) = B \cdot 1 + B \cdot y + B \cdot z + B \cdot (ayz^2 + by^2z).$$

(4) *If σ has the form* (II) *and $\beta_1(s,t) \neq 0$, we let g_i* $(1 \leq i \leq 4)$ *and* $g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)})$ $(1 \leq j \leq r)$ *be the polynomials given in* ④ *in the Introduction. Then the kernel* $\mathrm{Ker}(D_\sigma^2)$ *is generated by the polynomials g_i* $(1 \leq i \leq 4)$ *and* $g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)})$ $(1 \leq j \leq r)$ *as a B-module, i.e.,*

$$\mathrm{Ker}(D_\sigma^2) = \sum_{i=1}^{4} B \cdot g_i + \sum_{j=1}^{r} B \cdot g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}).$$

We remark that a generating set of $\mathrm{Ker}(D_\sigma^2)$ is already given for the linear triangular automorphism σ of $k[x,y,z]$ defined by $\sigma(x) := x$, $\sigma(y) := y + x$, $\sigma(z) := z + y$ (see [3, Theorem 5]).

The proofs of (1) and (2) of the above Theorem 5 are clear (see [2, Lemma 2.8] for (2)). We prepare the following Section 4 for proving assertions (3) and (4) of the above Theorem 5, give in Section 5 a proof of (3) of Theorem 5, and give in Section 6 a proof of (4) of Theorem 5.

4. $D_\sigma(y^iz^j)$ and $D_\sigma^2(y^iz^j)$ $(0 \leq i, j \leq 2)$, where σ is of the form (II)

Assume that σ has the form (II). In the following lemma, we express both $D_\sigma(y^iz^j)$ $(0 \leq i,j \leq 2)$ and $D_\sigma^2(y^iz^j)$ $(0 \leq i,j \leq 2)$.

Lemma 6 *The following assertions* (1) *and* (2) *hold true:*

(1) *We can express* $D_\sigma(y^iz^j)$ $(0 \le i, j \le 2)$ *as*

$$\begin{cases} D_\sigma(1) &= 0, \\ D_\sigma(y) &= \alpha, \\ D_\sigma(z) &= \beta, \\ D_\sigma(y^2) &= 2\alpha y + \alpha^2, \\ D_\sigma(yz) &= \alpha z + \beta y + \alpha\beta, \\ D_\sigma(z^2) &= 2\beta z + \beta^2, \\ D_\sigma(y^2z) &= 2\alpha yz + \beta y^2 + \alpha^2 z + 2\alpha\beta y + \alpha^2\beta, \\ D_\sigma(yz^2) &= \alpha z^2 + 2\beta yz + 2\alpha\beta z + \beta^2 y + \alpha\beta^2, \\ D_\sigma(y^2z^2) &= 2\alpha yz^2 + 2\beta y^2 z + \alpha^2 z^2 + \alpha\beta yz + \beta^2 y^2 + 2\alpha^2\beta z + 2\alpha\beta^2 y + \alpha^2\beta^2. \end{cases}$$

(2) *We can express* $D_\sigma^2(y^iz^j)$ $(0 \le i, j \le 2)$ *as*

$$\begin{cases} D_\sigma^2(1) &= 0, \\ D_\sigma^2(y) &= 0, \\ D_\sigma^2(z) &= \alpha\beta_1, \\ D_\sigma^2(y^2) &= 2\alpha^2, \\ D_\sigma^2(yz) &= 2\alpha\beta_0 + 2\alpha^2\beta_1, \\ D_\sigma^2(z^2) &= -d\beta_1 f_4 + \alpha^2\beta_1^2 + \alpha\beta_0\beta_1 + 2\beta_0^2, \\ D_\sigma^2(y^2z) &= -d\alpha f_4 + \alpha^3\beta_1, \\ D_\sigma^2(yz^2) &= (d\alpha\beta_1 + d\beta_0)f_4 - \beta_1^2 f_2 - \alpha^3\beta_1^2 - \alpha^2\beta_0\beta_1, \\ D_\sigma^2(y^2z^2) &= -d^2 f_4^2 - d\alpha^2\beta_1 f_4 - (\alpha\beta_1^2 + \beta_0\beta_1)f_2 + 2\alpha^2\beta_0^2 + \alpha^3\beta_0\beta_1 + \alpha^4\beta_1^2. \end{cases}$$

Proof. The proofs of (1) and (2) are straightforward. Q.E.D.

5. Proof of assertion (3) of Theorem 5

Assume that σ has the form (II) and $\beta_1(s,t) = 0$. So, $\beta = \beta_0 \ne 0$, $b = b_0 \ne 0$ and $\beta, b \in k[f_1, f_2]$.

5.1 $\mathrm{Syz}_B(D_\sigma^2(y^2), D_\sigma^2(yz), D_\sigma^2(z^2), D_\sigma^2(y^2z), D_\sigma^2(yz^2), D_\sigma^2(y^2z^2))$

Since $\beta_1(s,t) = 0$, we can express the polynomials f_i $(1 \le i \le 4)$ as

$$\begin{cases} f_1 = x, \\ f_2 = y^3 - \alpha^2 y, \\ f_3 = z^3 - \beta^2 z, \\ f_4 = az - by. \end{cases}$$

We know that B is generated by the set $\{f_1, f_2, f_3, f_4\}$ as a k-algebra, i.e., $B = k[f_1, f_2, f_3, f_4]$, and have the relation

$$f_4^3 - a^2b^2d^2 f_4 - a^3 f_3 + b^3 f_2 = 0. \tag{R}$$

We can express $D_\sigma^2(y^i z^j)$ $(0 \le i, j \le 2)$ as

$$\begin{cases} D_\sigma^2(1) &= 0, \\ D_\sigma^2(y) &= 0, \\ D_\sigma^2(z) &= 0, \\ D_\sigma^2(y^2) &= 2\alpha^2 = 2a^2d^2, \\ D_\sigma^2(yz) &= 2\alpha\beta = 2abd^2, \\ D_\sigma^2(z^2) &= 2\beta^2 = 2b^2d^2, \\ D_\sigma^2(y^2z) &= -d\alpha f_4 = -ad^2 f_4, \\ D_\sigma^2(yz^2) &= d\beta f_4 = bd^2 f_4, \\ D_\sigma^2(y^2z^2) &= 2d^2 f_4^2 + 2\alpha^2\beta^2 = d^2(2f_4^2 + 2a^2b^2d^2). \end{cases}$$

Thus, we have

$$\begin{aligned} &\mathrm{Syz}_B(D_\sigma^2(y^2),\ D_\sigma^2(yz),\ D_\sigma^2(z^2),\ D_\sigma^2(y^2z),\ D_\sigma^2(yz^2),\ D_\sigma^2(y^2z^2)) \\ &\quad = \mathrm{Syz}_B(2a^2,\ 2ab,\ 2b^2,\ -af_4,\ bf_4,\ 2f_4^2 + 2a^2b^2d^2). \end{aligned}$$

The following lemma gives a generating set of the above syzygy B-module.

Lemma 7 *Let u_i $(1 \le i \le 15)$ be the elements of $B^{\oplus 6}$ defined by*

$$\begin{cases} u_1 &:= (-af_3f_4 + b^2d^2f_4^2,\ 0,\ bf_2f_4,\ 0,\ 0,\ f_4^2), \\ u_2 &:= (-af_3 - 2b^2d^2f_4,\ 0,\ bf_2,\ 0,\ 0,\ f_4), \\ u_3 &:= (abf_3 + b^3d^2f_4,\ 0,\ -b^2f_2, 0,\ f_4^2,\ 0), \\ u_4 &:= (-a^2f_3 - ab^2d^2f_4,\ 0,\ abf_2,\ f_4^2,\ 0,\ 0), \\ u_5 &:= (f_4,\ 0,\ 0,\ -a,\ 0,\ 0), \\ u_6 &:= (0,\ f_4,\ 0,\ -b,\ 0,\ 0), \\ u_7 &:= (0,\ 0,\ f_4,\ 0,\ b,\ 0), \\ u_8 &:= (0,\ 0,\ 0,\ b,\ a,\ 0), \\ u_9 &:= (-b,\ a,\ 0,\ 0,\ 0,\ 0), \\ u_{10} &:= (0,\ -b,\ a,\ 0,\ 0,\ 0), \\ u_{11} &:= (f_4^2,\ -2a^3bd^2,\ 0,\ 0,\ 0,\ 2a^2), \\ u_{12} &:= (0,\ f_4^2 - 2a^2b^2d^2,\ 0,\ 0,\ 0,\ 2ab), \\ u_{13} &:= (0,\ -2ab^3d^2,\ f_4^2,\ 0,\ 0,\ 2b^2), \\ u_{14} &:= (0,\ a^2bd^2,\ 0,\ f_4,\ 0,\ -a), \\ u_{15} &:= (0,\ -ab^2d^2,\ 0,\ 0,\ f_4,\ b). \end{cases}$$

Then we have

$$\mathrm{Syz}_B(2a^2,\ 2ab,\ 2b^2,\ -af_4,\ bf_4,\ 2f_4^2+2a^2b^2d^2)=\sum_{i=1}^{15} B\cdot u_i.$$

Proof. Using the relation (R), we can prove $u_i \in \mathrm{Syz}_B(2a^2,\ 2ab,\ 2b^2,\ -af_4,\ bf_4,\ 2f_4^2+2a^2b^2d^2)$ for all $1 \le i \le 15$.

Now, take any element $c=(c_1,c_2,c_3,c_4,c_5,c_6)$ of $\mathrm{Syz}_B(2a^2,\ 2ab,\ 2b^2,\ -af_4,\ bf_4,\ 2f_4^2+2a^2b^2d^2)$ ($\subset B^{\oplus 6}$). Since $B=\bigoplus_{\ell=0}^{2} k[f_1,f_2,f_3]\,f_4^{\ell}$ is a free $k[f_1,f_2,f_3]$-module, we can write each c_i as

$$c_i=\sum_{\ell=0}^{2} c_{i,\ell}\,f_4^{\ell} \qquad \text{for some} \quad c_{i,\ell}\in k[f_1,f_2,f_3] \qquad (\ 1\le i\le 6,\quad 0\le \ell\le 2\ ).$$

Let

$$\gamma := c-(c_{6,2}u_1+c_{6,1}u_2+c_{5,2}u_3+c_{4,2}u_4).$$

Then γ has the following form:

$$\begin{aligned}\gamma =&(\gamma_{1,0}+\gamma_{1,1}f_4+\gamma_{1,2}f_4^2,\ \ \gamma_{2,0}+\gamma_{2,1}f_4+\gamma_{2,2}f_4^2,\ \ \gamma_{3,0}+\gamma_{3,1}f_4+\gamma_{3,2}f_4^2,\\ &\gamma_{4,0}+\gamma_{4,1}f_4,\ \ \gamma_{5,0}+\gamma_{5,1}f_4,\ \ \gamma_{6,0})\\ &\text{for some } \gamma_{i,j}\in k[f_1,f_2,f_3]\\ &\left((i,j)\in\left\{\begin{matrix}(1,0),(1,1),(1,2),\ \ (2,0),(2,1),(2,2),\ \ (3,0),(3,1),(3,2),\\ (4,0),(4,1),\ \ (5,0),(5,1),\ \ (6,0)\end{matrix}\right\}\right).\end{aligned}$$

Clearly, $\gamma \in \mathrm{Syz}_B(2a^2,\ 2ab,\ 2b^2,\ -af_4,\ bf_4,\ 2f_4^2+2a^2b^2d^2)$. So, we have

$$\begin{cases}2\gamma_{1,2}a^2+2\gamma_{2,2}ab+2\gamma_{3,2}b^2-\gamma_{4,1}a+\gamma_{5,1}b+2\gamma_{6,0}=0,\\ 2\gamma_{1,1}a^2+2\gamma_{2,1}ab+2\gamma_{3,1}b^2-\gamma_{4,0}a+\gamma_{5,0}b=0,\\ 2\gamma_{1,0}a^2+2\gamma_{2,0}ab+2\gamma_{3,0}b^2+2\gamma_{6,0}a^2b^2d^2=0.\end{cases}$$

Thus we have

$$(*)\qquad \begin{cases}\gamma_{6,0}=2\gamma_{1,2}a^2+2\gamma_{2,2}ab+2\gamma_{3,2}b^2-\gamma_{4,1}a+\gamma_{5,1}b,\\ (\gamma_{1,1},\ \gamma_{2,1},\ \gamma_{3,1},\ \gamma_{4,0},\ \gamma_{5,0})\in \mathrm{Syz}_{k[f_1,f_2,f_3]}(2a^2,\ 2ab,\ 2b^2,\ -a,\ b),\\ (\gamma_{1,0},\ \gamma_{2,0}+\gamma_{6,0}abd^2,\ \gamma_{3,0})\in \mathrm{Syz}_{k[f_1,f_2,f_3]}(a^2,\ ab,\ b^2).\end{cases}$$

We consider the second formula of $(*)$. Using $\mathrm{GCD}_{k[x,y]}(a,b)=1$, we can obtain

$$\mathrm{Syz}_{k[f_1,f_2,f_3]}(2a^2,\ 2ab,\ 2b^2,\ -a,\ b) = \bigoplus_{i=1}^{4} k[f_1,f_2,f_3]\cdot v_i,$$

where v_1, v_2, v_3, v_4 be the four elements of $k[f_1,f_2,f_3]^{\oplus 5}$ defined by

$$\begin{cases} v_1 := (1,\ 0,\ 0,\ -a,\ 0), \\ v_2 := (0,\ 1,\ 0,\ -b,\ 0), \\ v_3 := (0,\ 0,\ 1,\ 0,\ b), \\ v_4 := (0,\ 0,\ 0,\ b,\ a). \end{cases}$$

Thus, the second formula of $(*)$ implies that

$$(\gamma_{1,1},\ \gamma_{2,1},\ \gamma_{3,1},\ \gamma_{4,0},\ \gamma_{5,0}) = \sum_{i=1}^{4} \mu_i \cdot v_i \qquad \text{for some} \quad \mu_1,\mu_2,\mu_3,\mu_4 \in k[f_1,f_2,f_3]\ .$$

So,

$$\begin{cases} \gamma_{1,1} = \mu_1, \\ \gamma_{2,1} = \mu_2, \\ \gamma_{3,1} = \mu_3, \\ \gamma_{4,0} = -\mu_1 a - \mu_2 b + \mu_4 b, \\ \gamma_{5,0} = \mu_3 b + \mu_4 a. \end{cases}$$

We consider the third formula of $(*)$. We can express the syzygy module $\mathrm{Syz}_{k[f_1,f_2,f_3]}(a^2,\ ab,\ b^2)$ as

$$\mathrm{Syz}_{k[f_1,f_2,f_3]}(a^2,\ ab,\ b^2) = \bigoplus_{i=1}^{2} k[f_1,f_2,f_3]\cdot w_i,$$

where $w_1 := (-b,\ a,\ 0)$ and $w_2 := (0,\ -b,\ a)$. Thus, the third formula of $(*)$ implies that

$$(\gamma_{1,0},\ \gamma_{2,0} + abd^2\gamma_{6,0},\ \gamma_{3,0}) = \nu_1 w_1 + \nu_2 w_2 \qquad \text{for some} \quad \nu_1,\nu_2 \in k[f_1,f_2,f_3].$$

Thus we have

$$\begin{cases} \gamma_{1,0} = -\nu_1 b, \\ \gamma_{2,0} = \nu_1 a - \nu_2 b - \gamma_{6,0} abd^2 \\ \qquad = \nu_1 a - \nu_2 b - 2\gamma_{1,2} a^3bd^2 - 2\gamma_{2,2} a^2b^2d^2 - 2\gamma_{3,2} ab^3d^2 + \gamma_{4,1} a^2bd^2 - \gamma_{5,1} ab^2d^2, \\ \gamma_{3,0} = \nu_2 a. \end{cases}$$

Now each component of $\gamma = (\gamma_1,\, \gamma_2,\, \gamma_3,\, \gamma_4,\, \gamma_5,\, \gamma_6)$ can be calculated as

$$\begin{cases} \gamma_1 = \gamma_{1,0} + \gamma_{1,1} f_4 + \gamma_{1,2} f_4^2 = -\nu_1 b + \mu_1 f_4 + \gamma_{1,2} f_4^2, \\ \gamma_2 = \gamma_{2,0} + \gamma_{2,1} f_4 + \gamma_{2,2} f_4^2 \\ \quad = \nu_1 a - \nu_2 b - 2\gamma_{1,2} a^3 b d^2 - 2\gamma_{2,2} a^2 b^2 d^2 - 2\gamma_{3,2} a b^3 d^2 + \gamma_{4,1} a^2 b d^2 - \gamma_{5,1} a b^2 d^2 \\ \qquad + \mu_2 f_4 + \gamma_{2,2} f_4^2, \\ \gamma_3 = \gamma_{3,0} + \gamma_{3,1} f_4 + \gamma_{3,2} f_4^2 = \nu_2 a + \mu_3 f_4 + \gamma_{3,2} f_4^2, \\ \gamma_4 = \gamma_{4,0} + \gamma_{4,1} f_4 = -\mu_1 a - \mu_2 b + \mu_4 b + \gamma_{4,1} f_4, \\ \gamma_5 = \gamma_{5,0} + \gamma_{5,1} f_4 = \mu_3 b + \mu_4 a + \gamma_{5,1} f_4, \\ \gamma_6 = \gamma_{6,0} = 2\gamma_{1,2} a^2 + 2\gamma_{2,2} ab + 2\gamma_{3,2} b^2 - \gamma_{4,1} a + \gamma_{5,1} b. \end{cases}$$

The above equalities imply

$$\begin{aligned} \gamma = \mu_1 u_5 + \mu_2 u_6 + \mu_3 u_7 + \mu_4 u_8 + \nu_1 u_9 + \nu_2 u_{10} \\ + \gamma_{1,2} u_{11} + \gamma_{2,2} u_{12} + \gamma_{3,2} u_{13} + \gamma_{4,1} u_{14} + \gamma_{5,1} u_{15}. \end{aligned}$$

Hence we have

$$c \in \sum_{i=1}^{15} B \cdot u_i.$$

Q.E.D.

5.2 Proof of assertion (3) of Theorem 5

Let $\Phi = (\Phi_i)_{1 \le i \le 6}$ be the element of $A^{\oplus 6}$ defined by

$$\Phi := (y^2,\ yz,\ z^2,\ y^2 z,\ yz^2,\ y^2 z^2).$$

For any $\omega = (\omega_i)_{1 \le i \le 6} \in B^{\oplus 6}$, we define the element $\omega \cdot \Phi$ of A as

$$\omega \cdot \Phi := \sum_{i=1}^{6} \omega_i \, \Phi_i.$$

The kernel $\mathrm{Ker}(D_\sigma^2)$ is generated by the set $\{1,\, y,\, z\} \cup \{u_i \cdot \Phi \mid 1 \le i \le 15\}$ as a B-module. We can calculate $u_i \cdot \Phi$ $(1 \le i \le 15)$, as follows:

$$\begin{cases} u_1 \cdot \Phi &= (b^2\beta^2 f_2 - a\alpha^2 b f_3 - \alpha^2 b\beta^2 f_4) \cdot y, \\ u_2 \cdot \Phi &= -b^3 d^2 f_2 - \alpha^2\beta^3 d^2 \cdot y - \alpha\beta \cdot (ayz^2 + by^2 z), \\ u_3 \cdot \Phi &= -b^4 d^2 f_2 - (\alpha^2 b^4 d^2 - af_3 f_4) \cdot y - a\beta^2 \cdot (ayz^2 + by^2 z), \\ u_4 \cdot \Phi &= ab^3 d^2 f_2 + a\alpha^2 b^3 d^2 \cdot y - bf_2 f_4 \cdot z + \alpha^2 b \cdot (ayz^2 + by^2 z), \\ u_5 \cdot \Phi &= -bf_2 - \alpha^2 b \cdot y, \\ u_6 \cdot \Phi &= ayz^2 + by^2 z, \\ u_7 \cdot \Phi &= af_3 + a\beta^2 \cdot z, \\ u_8 \cdot \Phi &= ayz^2 + by^2 z, \\ u_9 \cdot \Phi &= f_4 \cdot y, \\ u_{10} \cdot \Phi &= f_4 \cdot z, \\ u_{11} \cdot \Phi &= (b^2 f_2 - \alpha^2 b f_4) \cdot y - 2abf_2 \cdot z, \\ u_{12} \cdot \Phi &= a^2 f_3 \cdot y + b^2 f_2 \cdot z, \\ u_{13} \cdot \Phi &= -2abf_3 \cdot y + (a\beta^2 f_4 + a^2 f_3) \cdot z, \\ u_{14} \cdot \Phi &= -bf_2 \cdot z, \\ u_{15} \cdot \Phi &= af_3 \cdot y. \end{cases}$$

Hence $\mathrm{Ker}(D_\sigma^2)$ is generated by the set $\{1, y, z, ayz^2 + by^2 z\}$ as a B-module. This completes the proof of assertion (3) of Theorem 5.

6. Proof of assertion (4) of Theorem 5

Assume that σ has the form (II) and $\beta_1(s,t) \neq 0$.

6.1 A generating set $\{\, \varphi_i \mid 1 \le i \le 9 \,\}$ of $k[x,y,z]$ as a B-module

We know that $B = \mathrm{Ker}(D_\sigma)$ is generated by f_1, f_2, f_3, f_4 (see Lemma 4). So, a generating set of $k[x,y,z]$ as a B-module is given by $\{\, y^i z^j \mid 0 \le i \le 2,\ 0 \le j \le 2 \,\}$. Thus, the following subset $\{\, \varphi_i \mid 1 \le i \le 9 \,\}$ of $k[x,y,z]$ becomes a generating set of $k[x,y,z]$ as a B-module:

$$\begin{cases} \varphi_1 := 1, \\ \varphi_2 := y, \\ \varphi_3 := z, \\ \varphi_4 := y^2, \\ \varphi_5 := yz - \beta_1 y^2, \\ \varphi_6 := z^2 - 2\beta_1 yz, \\ \varphi_7 := y^2 z - \alpha^2 z, \\ \varphi_8 := yz^2 + \beta_1 y^2 z + \alpha\beta_0 z, \\ \varphi_9 := y^2 z^2 - \alpha\beta_1 y^2 z - \beta_0^2 y^2 - \alpha^2\beta_0 z + \beta_1 f_2 z. \end{cases}$$

Using Lemma 6, we have

$$\begin{cases} D_\sigma^2(\varphi_1) = 0, \\ D_\sigma^2(\varphi_2) = 0, \\ D_\sigma^2(\varphi_3) = \alpha\beta_1, \\ D_\sigma^2(\varphi_4) = 2\alpha^2, \\ D_\sigma^2(\varphi_5) = 2\alpha\beta_0, \\ D_\sigma^2(\varphi_6) = -d\beta_1 f_4 + 2\beta_0^2, \\ D_\sigma^2(\varphi_7) = -d\alpha f_4, \\ D_\sigma^2(\varphi_8) = d\beta_0 f_4 - \beta_1^2 f_2, \\ D_\sigma^2(\varphi_9) = -d^2 f_4^2 - \beta_0\beta_1 f_2. \end{cases}$$

6.2 $\mathbf{Syz}_B\big(D_\sigma^2(\varphi_3), D_\sigma^2(\varphi_4), D_\sigma^2(\varphi_5), D_\sigma^2(\varphi_6), D_\sigma^2(\varphi_7), D_\sigma^2(\varphi_8), D_\sigma^2(\varphi_9)\big)$

Let S be the syzygy B-module of the polynomials $D_\sigma^2(\varphi_i)$ $(3 \le i \le 9)$, i.e.

$$\begin{aligned} S &:= \mathrm{Syz}_B\big(\, D_\sigma^2(\varphi_3),\ D_\sigma^2(\varphi_4),\ D_\sigma^2(\varphi_5),\ D_\sigma^2(\varphi_6),\ D_\sigma^2(\varphi_7),\ D_\sigma^2(\varphi_8),\ D_\sigma^2(\varphi_9)\,\big) \\ &= \mathrm{Syz}_B\big(\, \alpha\beta_1,\ 2\alpha^2,\ 2\alpha\beta_0,\ -d\beta_1 f_4 + 2\beta_0^2,\ -d\alpha f_4,\ d\beta_0 f_4 - \beta_1^2 f_2, \\ &\qquad\qquad -d^2 f_4^2 - \beta_0\beta_1 f_2\big). \end{aligned}$$

The following lemma gives a generating set of S as a B-module.

Lemma 8 *Let u_i $(1 \le i \le 16)$ be the elements of $B^{\oplus 7}$ defined by*

$$\begin{aligned} u_1 &:= (-d\alpha f_4^3 - \alpha^2\beta_0 f_4^2 + \alpha b_0\beta_1 f_2 f_4,\quad -\beta_0^2 f_4^2 - a f_3 f_4,\quad 0, \\ &\qquad b_0 f_2 f_4,\quad 0,\quad -b_1 f_2 f_4,\quad f_4^2), \\ u_2 &:= (-d\alpha f_4^2 - \alpha^2\beta_0 f_4 + \alpha b_0\beta_1 f_2,\quad -\beta_0^2 f_4 - a f_3,\quad 0, \\ &\qquad b_0 f_2,\quad 0,\quad -b_1 f_2,\quad f_4), \\ u_3 &:= (\alpha\beta_0 f_4^2 + a\alpha\beta_0^2 f_4 - a b_0\beta_0\beta_1 f_2,\quad -2b_0\beta_0^2 f_4 - 2ab_0 f_3,\quad 0, \\ &\qquad -b_1 f_2 f_4 - b_0^2 f_2,\quad 0,\quad f_4^2 - 2b_0 b_1 f_2,\quad 0), \\ u_4 &:= (-2b_1^2 f_2^2,\quad \alpha\beta_1 f_4^2 - a\beta_0^2 f_4 + a\alpha\beta_0\beta_1 f_4 - a^2 f_3 - a b_0\beta_1^2 f_2,\quad -2b_0^2 f_2, \\ &\qquad 0,\quad f_4^2,\quad 0,\quad 0), \\ u_5 &:= (-\alpha\beta_1 f_4^2 + a\beta_0^2 f_4 - a\alpha\beta_0\beta_1 f_4 + a^2 f_3 + a b_0\beta_1^2 f_2,\quad 0,\quad 0, \\ &\qquad f_4^2 - 2b_0 b_1 f_2,\quad 0,\quad 2b_1^2 f_2 + b_0 f_4,\quad 0), \\ u_6 &:= (f_4,\quad 0,\quad -b_0,\quad a,\quad 0,\quad 0,\quad 0), \\ u_7 &:= (f_4^2,\quad 0,\quad 2b_1^2 f_2,\quad 0,\quad 0,\quad 0,\quad ab_1), \\ u_8 &:= (0,\quad f_4,\quad 0,\quad 0,\quad 2a,\quad 0,\quad 0), \end{aligned}$$

$$
\begin{aligned}
u_9 &:= (0,\quad f_4^2,\quad ab_1f_2,\quad 0,\quad 0,\quad 0,\quad 2a^2),\\
u_{10} &:= (0,\quad 0,\quad f_4,\quad 0,\quad 2b_0,\quad 0,\quad 0),\\
u_{11} &:= (0,\quad 0,\quad f_4^2+b_0b_1f_2,\quad 0,\quad 0,\quad 0,\quad 2ab_0),\\
u_{12} &:= (0,\quad 0,\quad 0,\quad f_4,\quad 0,\quad -2b_0,\quad -b_1),\\
u_{13} &:= (0,\quad 0,\quad b_1f_2,\quad 0,\quad f_4,\quad 0,\quad -a),\\
u_{14} &:= (0,\quad 0,\quad 0,\quad -b_1f_2,\quad 0,\quad f_4,\quad b_0),\\
u_{15} &:= (b_1f_2,\quad 0,\quad 0,\quad 0,\quad b_0,\quad a,\quad 0),\\
u_{16} &:= (0,\quad 0,\quad b_0,\quad -a,\quad b_1,\quad 0,\quad 0).
\end{aligned}
$$

Let $\{\,(\lambda_1^{(j)},\ \lambda_2^{(j)},\ \lambda_3^{(j)}) \in k[f_1,f_2]^{\oplus 3} \mid 1 \le j \le r\,\}$ *be a generating set of the syzygy* $k[f_1,f_2]$*-module* $\mathrm{Syz}_{k[f_1,f_2]}(b_1,-a,-b_0)$*. Then we have*

$$
S = \sum_{i=1}^{16} B\cdot u_i + \sum_{j=1}^{r} B\cdot(\lambda_1^{(j)},\ \lambda_2^{(j)},\ \lambda_3^{(j)},\ 0,\ 0,\ 0,\ 0).
$$

Proof. Using the relation given in assertion (3) of Lemma 4, we can prove $u_i \in S$ for all $1 \le i \le 16$. Clearly, $(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}, 0, 0, 0, 0) \in S$ for all $1 \le j \le r$.

Take any element $c = (c_i)_{1\le i\le 7}$ of S. We can write each component c_i as

$$
c_i = \sum_{\ell=0}^{2} c_{i,\ell}\, f_4^{\ell} \qquad \text{for some} \quad c_{i,\ell} \in k[f_1,f_2,f_3] \qquad (\ 1\le i\le 7,\quad 0\le \ell\le 2\ ).
$$

Let

$$
\gamma := c - (c_{7,2}u_1 + c_{7,1}u_2 + c_{6,2}u_3 + c_{5,2}u_4 + c_{4,2}u_5).
$$

Then γ has the following form:

$$
\gamma = \begin{pmatrix} \gamma_{1,0}+\gamma_{1,1}f_4+\gamma_{1,2}f_4^2, & \gamma_{2,0}+\gamma_{2,1}f_4+\gamma_{2,2}f_4^2, & \gamma_{3,0}+\gamma_{3,1}f_4+\gamma_{3,2}f_4^2, \\ \gamma_{4,0}+\gamma_{4,1}f_4, & \gamma_{5,0}+\gamma_{5,1}f_4, & \gamma_{6,0}+\gamma_{6,1}f_4, & \gamma_{7,0} \end{pmatrix}
$$

for some $\gamma_{i,j} \in k[f_1,f_2,f_3]$

$$
\left((i,j) \in \left\{ \begin{matrix} (1,0),(1,1),(1,2), & (2,0),(2,1),(2,2), & (3,0),(3,1),(3,2), \\ (4,0),(4,1), & (5,0),(5,1), & (6,0),(6,1), & (7,0) \end{matrix} \right\} \right).
$$

Since γ is an element of the syzygy module S, we can obtain

$$
(*)\qquad \begin{cases} -\gamma_{7,0}\beta_0\beta_1 f_2 - \gamma_{6,0}\beta_1^2 f_2 + 2\gamma_{4,0}\beta_0^2 + 2\gamma_{3,0}\alpha\beta_0 + 2\gamma_{2,0}\alpha^2 + \gamma_{1,0}\alpha\beta_1 = 0, \\ -\gamma_{7,0}d^2 + \gamma_{6,1}d\beta_0 - \gamma_{5,1}d\alpha - \gamma_{4,1}d\beta_1 + 2\gamma_{3,2}\alpha\beta_0 + 2\gamma_{2,2}\alpha^2 + \gamma_{1,2}\alpha\beta_1 = 0, \\ \gamma_{6,0}d\beta_0 - \gamma_{6,1}\beta_1^2 f_2 - \gamma_{5,0}d\alpha - \gamma_{4,0}d\beta_1 + 2\gamma_{4,1}\beta_0^2 + 2\gamma_{3,1}\alpha\beta_0 \\ \qquad\qquad +2\gamma_{2,1}\alpha^2 + \gamma_{1,1}\alpha\beta_1 = 0. \end{cases}
$$

From the second equality of $(*)$, we have

$$\gamma_{7,0} = \gamma_{6,1}b_0 - \gamma_{5,1}a - \gamma_{4,1}b_1 + 2\gamma_{3,2}ab_0 + 2\gamma_{2,2}a^2 + \gamma_{1,2}ab_1.$$

Let

$$\begin{aligned}\delta := \gamma - (\gamma_{1,1}u_6 + \gamma_{1,2}u_7 + \gamma_{2,1}u_8 + \gamma_{2,2}u_9 + \gamma_{3,1}u_{10} + \gamma_{3,2}u_{11} \\ + \gamma_{4,1}u_{12} + \gamma_{5,1}u_{13} + \gamma_{6,1}u_{14}).\end{aligned}$$

We can express the i-th compotents of $\delta = (\delta_i)_{1\le i\le 7}$ as

$$\begin{cases}\delta_1 = \gamma_{1,0}, \\ \delta_2 = \gamma_{2,0}, \\ \delta_3 = \gamma_{3,0} + \gamma_{1,1}b_0 - 2\gamma_{1,2}b_1^2f_2 - \gamma_{2,2}ab_1f_2 - \gamma_{3,2}b_0b_1f_2 - \gamma_{5,1}b_1f_2, \\ \delta_4 = \gamma_{4,0} - \gamma_{1,1}a + \gamma_{6,1}b_1f_2, \\ \delta_5 = \gamma_{5,0} - 2\gamma_{2,1}a - 2\gamma_{3,1}b_0, \\ \delta_6 = \gamma_{6,0} + 2\gamma_{4,1}b_0, \\ \delta_7 = \gamma_{7,0} - (\gamma_{1,2}ab_1 + 2\gamma_{2,2}a^2 + 2\gamma_{3,2}ab_0 - \gamma_{4,1}b_1 - \gamma_{5,1}a + \gamma_{6,1}b_0).\end{cases}$$

Thus $\delta_i \in k[f_1, f_2, f_2]$ for all $1 \le i \le 6$ and $\delta_7 = 0$. Since $\delta \in S$, we have

$$\begin{aligned}&\delta_1 \cdot (\alpha\beta_1) + \delta_2 \cdot (2\alpha^2) + \delta_3 \cdot (2\alpha\beta_0) \\ &\quad + \delta_4 \cdot (-d\beta_1 f_4 + 2\beta_0^2) + \delta_5 \cdot (-d\alpha f_4) + \delta_6 \cdot (d\beta_0 f_4 - \beta_1^2 f_2) = 0,\end{aligned}$$

which implies that

$$\begin{cases} b_0\delta_6 - b_1\delta_4 = a\delta_5, \\ -b_1^2f_2\delta_6 + 2b_0^2\delta_4 = -2ab_0\delta_3 - 2a^2\delta_2 - ab_1\delta_1.\end{cases}$$

Using the above equalities, we have

$$\begin{aligned}(-b_0^3 - b_1^3f_2)\delta_6 &= -b_0^2(b_0\delta_6 - b_1\delta_4) + b_1(-b_1^2f_2\delta_6 + 2b_0^2\delta_4) \\ &= -ab_0^2\delta_5 - 2ab_0b_1\delta_3 - 2a^2b_1\delta_2 - ab_1^2\delta_1.\end{aligned}$$

Recall $f_2 = y^3 - \alpha^2 y$. Since

$$\mathrm{GCD}_{k[x,y]}(b_0^3 + b_1^3f_2,\, a) = \mathrm{GCD}_{k[x,y]}((b_0 + b_1y)^3,\, a) = 1,$$

we have

$$\delta_6 = a \cdot \delta_6^\flat \quad \text{for some} \quad \delta_6^\flat \in k[f_1, f_2, f_3].$$

Let

$$\varepsilon := \delta - \delta_6^\flat u_{15}$$

and write $\varepsilon = (\varepsilon_i)_{1\leq i\leq 7}$. We have $\varepsilon_i \in k[f_1, f_2, f_3]$ for all $1 \leq i \leq 5$ and $\varepsilon_6 = \varepsilon_7 = 0$. Since $\varepsilon \in S$, we have

$$(**) \qquad \begin{cases} -b_1\varepsilon_4 = a\varepsilon_5, \\ 2b_0^2\varepsilon_4 = -2ab_0\varepsilon_3 - 2a^2\varepsilon_2 - ab_1\varepsilon_1. \end{cases}$$

Thus we have

$$\begin{aligned}(b_1^3 f_2 + b_0^3)\varepsilon_4 &= (-b_1^2 f_2)(-b_1\varepsilon_4) + 2b_0(2b_0^2\varepsilon_4) \\ &= -ab_1^2 f_2\varepsilon_5 - ab_0^2\varepsilon_3 - a^2 b_0\varepsilon_2 - 2ab_0b_1\varepsilon_1,\end{aligned}$$

which implies

$$\varepsilon_4 = a \cdot \varepsilon_4^\flat \qquad \text{for some} \qquad \varepsilon_4^\flat \in k[f_1, f_2, f_3].$$

By $(**)$, we have $\varepsilon_5 = -b_1\varepsilon_4^\flat$. Let

$$\phi := \varepsilon + \varepsilon_4^\flat u_{16}$$

and write $\phi = (\phi_i)_{1\leq i\leq 7}$. Then we have $\phi_i \in k[f_1, f_2, f_3]$ for all $1 \leq i \leq 3$ and $\phi_j = 0$ for all $4 \leq j \leq 7$, and thereby have

$$(\phi_1,\ \phi_2,\ \phi_3) \in \mathrm{Syz}_{k[f_1,f_2,f_3]}(b_1,\ 2a,\ 2b_0).$$

Hence

$$\phi = (\phi_1,\ \phi_2,\ \phi_3,\ 0,\ 0,\ 0,\ 0) \in \sum_{j=1}^{r} k[f_1,\ f_2,\ f_3] \cdot (\lambda_1^{(j)},\ \lambda_2^{(j)},\ \lambda_3^{(j)},\ 0,\ 0,\ 0,\ 0),$$

which implies $c \in S$, as desired. Q.E.D.

6.3 Proof of assertion (4) of Theorem 5

Let Φ be the element of $k[x, y, z]^{\oplus 7}$ defined by

$$\Phi := (\,\varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8,\ \varphi_9\,).$$

For any $\omega = (\omega_i)_{1\leq i\leq 7} \in B^{\oplus 7}$, we define the element $\omega \cdot \Phi$ of A as

$$\omega \cdot \Phi := \sum_{i=1}^{7} \omega_i\, \varphi_{i+2}.$$

We have

$$\mathrm{Ker}(D_\sigma^2) = \sum_{i=1}^{16} B \cdot (u_i \cdot \Phi) + \sum_{j=1}^{r} B \cdot g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}).$$

Let

$$M := \sum_{i=1}^{4} B \cdot g_i + \sum_{j=1}^{r} B \cdot g(\lambda_1^{(j)}, \lambda_2^{(j)}, \lambda_3^{(j)}).$$

Clearly, $D_\sigma^2(g_i) = 0$ for all $1 \leq i \leq 4$. In order to prove $\mathrm{Ker}(D_\sigma^2) \subset M$, we have only to show $u_i \cdot \Phi \in M$ for all $1 \leq i \leq 16$.

We can express $u_{16} \cdot \Phi$, $u_{15} \cdot \Phi$, $u_{14} \cdot \Phi$ as

$$\begin{aligned}
&u_{16} \cdot \Phi = g_3 + g(-\alpha^2 b_1,\ -2b_0\beta_1 + 2a\beta_1^2,\ 2a\beta_1),\\
&u_{15} \cdot \Phi = g_4.\\
&u_{14} \cdot \Phi = af_3 \cdot y - \alpha\beta_0 \cdot g_3 + \beta_0 \cdot g(-\alpha^2 b_0,\ -\alpha b_0\beta_1 - b_0\beta_0,\ -2\alpha b_0).
\end{aligned}$$

Since

$$\begin{aligned}
(-\alpha^2 b_1,\ -2b_0\beta_1 + 2a\beta_1^2,\ 2a\beta_1) &\in \mathrm{Syz}_{k[f_1,f_2]}(b_1,\ -a,\ -b_0),\\
(-\alpha^2 b_0,\ -\alpha b_0\beta_1 - b_0\beta_0,\ -2\alpha b_0) &\in \mathrm{Syz}_{k[f_1,f_2]}(b_1,\ -a,\ -b_0),
\end{aligned}$$

we have $u_i \cdot \Phi \in M$ for all $i = 16, 15, 14$.

Let

$$h := (-f_4\alpha^2 + a\alpha^2\beta_0)z + a\beta_0^2 y^2 + b_1 f_2 yz + (f_4 + a\alpha\beta_1)y^2 z - ay^2 z^2.$$

We can deform h as

$$h = \alpha^2 \cdot g_3 + g(\eta_1,\ \eta_2,\ \eta_3),$$

where

$$\begin{cases}
\eta_1 := a\alpha^2\beta_0 - a\beta_1 f_2 - b_0 f_2,\\
\eta_2 := \alpha^2 b_0\beta_1 + a\beta_0^2 + 2b_1\beta_1 f_2,\\
\eta_3 := 2b_1 f_2 + 2\alpha^2 b_0.
\end{cases}$$

Since $(\eta_1,\ \eta_2,\ \eta_3) \in \mathrm{Syz}_{k[f_1,f_2]}(b_1,\ -a,\ -b_0)$, we have $h \in M$.

Since

$$u_{13} \cdot \Phi = h + g(-a\beta_1 f_2,\ -b_1\beta_1 f_2,\ 0),$$

we have $u_{13} \cdot \Phi \in M$.

We can deform $u_{12} \cdot \Phi$ as

$$u_{12} \cdot \Phi = f_4\varphi_6 - 2b_0\varphi_8 - b_1\varphi_9$$

$$
\begin{aligned}
&= az^3 - 2\alpha b_0\beta_0 z + \beta_0^2 b_1 y^2 + \alpha^2\beta_0 b_1 z + \alpha^2 b_1\beta_1 yz \\
&= a\cdot(f_3 - \alpha\beta_1 z^2 + \beta_1^2 y^2 z - (\alpha\beta_1^2 + \beta_0\beta_1)yz - (-\beta_0^2 + \alpha\beta_0\beta_1)z) \\
&\quad - 2\alpha b_0\beta_0 z + \beta_0^2 b_1 y^2 + \alpha^2\beta_0 b_1 z + \alpha^2 b_1\beta_1 yz \\
&= af_3 - a\alpha\beta_1 z^2 + a\beta_1^2 y^2 z - a\beta_0\beta_1 yz - a\beta_0^2 z + \beta_0^2 b_1 y^2 \\
&= af_3 - a\alpha\beta_1 z^2 + d^2 ab_1(g_3 - b_0\beta_1 y^2 + 2b_0 yz - 2az^2) \\
&\quad - a\beta_0\beta_1 yz - a\beta_0^2 z + \beta_0^2 b_1 y^2 \\
&= af_3 + d^2 ab_1 g_3 - a\beta_0^2 z - (a\beta_0\beta_1^2 - \beta_0^2 b_1)y^2 + a\beta_0\beta_1 yz \\
&= af_3 + \alpha\beta_1 g_3 + g(-a\beta_0^2,\ b_0\beta_0\beta_1,\ a\beta_0\beta_1).
\end{aligned}
$$

Thus $u_{12}\cdot\Phi \in M$.

Let

$$H := \alpha^2 b_0 z - \beta_1 f_4 y^2 + f_4 yz - b_0 y^2 z.$$

Then

$$
\begin{aligned}
H &= \alpha^2 b_0 z - (az + b_1 y^2 - a\beta_1 y - b_0 y)\beta_1 y^2 + (az + b_1 y^2 - a\beta_1 y - b_0 y)yz - b_0 y^2 z \\
&= \alpha^2 b_0 z - 2a\beta_1 y^2 z - b_1\beta_1 y^4 + a\beta_1^2 y^3 + b_0\beta_1 y^3 + ayz^2 + b_1 y^3 z - 2b_0 y^2 z \\
&= \alpha^2 b_0 z - 2a\beta_1 y^2 z - b_1\beta_1 y(f_2 + \alpha^2 y) + a\beta_1^2(f_2 + \alpha^2 y) + b_0\beta_1(f_2 + \alpha^2 y) \\
&\quad + ayz^2 + b_1 z(f_2 + \alpha^2 y) + b_0 y^2 z \\
&= g_4 - b_1 f_2 z + \alpha^2 b_0 z - b_1\beta_1 y(f_2 + \alpha^2 y) + a\beta_1^2(f_2 + \alpha^2 y) + b_0\beta_1(f_2 + \alpha^2 y) \\
&\quad + b_1 z(f_2 + \alpha^2 y) \\
&= (a\beta_1^2 f_2 + b_0\beta_1 f_2) + (-b_1\beta_1 f_2 + a\alpha^2\beta_1^2 + \alpha^2 b_0\beta_1)\cdot y + g_4 \\
&\quad + g(\alpha^2 b_0,\ 0,\ \alpha^2 b_1).
\end{aligned}
$$

Thus $H \in M$.

We can deform $u_{11}\cdot\Phi$ as

$$
\begin{aligned}
u_{11}\cdot\Phi &= (f_4^2 + b_0 b_1 f_2)\varphi_5 + 2ab_0\varphi_9 \\
&= f_4^2 yz - f_4^2\beta_1 y^2 + b_0 b_1 f_2 yz - b_0 b_1\beta_1 f_2 y^2 + 2ab_0 y^2 z^2 - 2a\alpha b_0\beta_1 y^2 z \\
&\quad - 2ab_0\beta_0^2 y^2 - 2a\alpha^2 b_0\beta_0 z + 2ab_0\beta_1 f_2 z \\
&= f_4^2 yz - f_4^2\beta_1 y^2 \\
&\quad + b_0\cdot(h - (-f_4\alpha^2 + a\alpha^2\beta_0)z - a\beta_0^2 y^2 - (f_4 + a\alpha\beta_1)y^2 z + ay^2 z^2) \\
&\quad - b_0 b_1\beta_1 f_2 y^2 + 2ab_0 y^2 z^2 - 2a\alpha b_0\beta_1 y^2 z - 2ab_0\beta_0^2 y^2 - 2a\alpha^2 b_0\beta_0 z \\
&\quad + 2ab_0\beta_1 f_2 z \\
&= b_0 h + f_4\cdot(\alpha^2 b_0 z - f_4\beta_1 y^2 + f_4 yz - b_0 y^2 z) + g(2ab_0\beta_1 f_2,\ -b_0 b_1\beta_1 f_2,\ 0) \\
&= b_0 h + f_4 H + g(2ab_0\beta_1 f_2,\ -b_0 b_1\beta_1 f_2,\ 0) \\
&= b_0 h + (a\beta_1^2 f_2 f_4 + b_0\beta_1 f_2 f_4) + (-b_1\beta_1 f_2 f_4 + a\alpha^2\beta_1^2 f_4 + \alpha^2 b_0\beta_1 f_4)\cdot y \\
&\quad + f_4\cdot g_4 + f_4\cdot g(\alpha^2 b_0,\ 0,\ \alpha^2 b_1) + g(2ab_0\beta_1 f_2,\ -b_0 b_1\beta_1 f_2,\ 0).
\end{aligned}
$$

We have

$$u_{10} \cdot \Phi = f_4\varphi_5 + 2b_0\varphi_7 = \alpha^2 b_0 z - \beta_1 f_4 y^2 + f_4 yz - b_0 y^2 z = H,$$

which implies $u_{10} \cdot \Phi \in M$.

We can deform $u_9 \cdot \Phi$ as

$$\begin{aligned} u_9 \cdot \Phi &= f_4^2\varphi_4 + ab_1 f_2\varphi_5 + 2a^2\varphi_9 \\ &= f_4^2 y^2 + ab_1 f_2 yz - ab_1\beta_1 f_2 y^2 \\ &\quad + 2a^2y^2z^2 - 2a^2\alpha\beta_1 y^2 z - 2a^2\beta_0^2 y^2 - 2a^2\alpha^2\beta_0 z + 2a^2\beta_1 f_2 z \\ &= f_4^2 y^2 + a(h - (-f_4\alpha^2 + a\alpha^2\beta_0)z - a\beta_0^2 y^2 - (f_4 + a\alpha\beta_1)y^2 z + ay^2z^2) \\ &\quad - ab_1\beta_1 f_2 y^2 + 2a^2y^2z^2 - 2a^2\alpha\beta_1 y^2 z - 2a^2\beta_0^2 y^2 - 2a^2\alpha^2\beta_0 z + 2a^2\beta_1 f_2 z \\ &= ah + f_4(f_4y^2 + a\alpha^2 z - ay^2 z) + g(2a^2\beta_1 f_2,\ -ab_1\beta_1 f_2,\ 0). \end{aligned}$$

Since

$$\begin{aligned} &f_4 y^2 + a\alpha^2 z - ay^2 z \\ &\quad = (az + b_1 y^2 - a\beta_1 y - b_0 y)y^2 + a\alpha^2 z - ay^2 z \\ &\quad = b_1 y^4 - a\beta_1 y^3 - b_0 y^3 + a\alpha^2 z \\ &\quad = b_1 y(f_2 + \alpha^2 y) - a\beta_1(f_2 + \alpha^2 y) - b_0(f_2 + \alpha^2 y) + a\alpha^2 z \\ &\quad = \alpha^2 f_4 + b_1 f_2 y - a\beta_1 f_2 - b_0 f_2, \end{aligned}$$

we have

$$\begin{aligned} u_9 \cdot \Phi = ah &+ (\alpha^2 f_4^2 - a\beta_1 f_2 f_4 - b_0 f_2 f_4) + b_1 f_2 f_4 \cdot y \\ &+ g(2a^2\beta_1 f_2,\ -ab_1\beta_1 f_2,\ 0). \end{aligned}$$

Thus $u_9 \cdot \Phi \in M$.

We can deform $u_8 \cdot \Phi$ as

$$\begin{aligned} u_8 \cdot \Phi &= f_4\varphi_4 + 2a\varphi_7 \\ &= b_1 y^4 - a\beta_1 y^3 - b_0 y^3 - 2a\alpha^2 z \\ &= b_1 y(f_2 + \alpha^2 y) - a\beta_1(f_2 + \alpha^2 y) - b_0(f_2 + \alpha^2 y) - 2a\alpha^2 z \\ &= -(a\beta_1 f_2 + b_0 f_2) + (b_1 f_2 - a\alpha^2\beta_1 - \alpha^2 b_0) \cdot y + \alpha^2 \cdot g(a,\ b_1,\ 0). \end{aligned}$$

Thus $u_8 \cdot \Phi \in M$.

We can deform $u_7 \cdot \Phi$ as

$$\begin{aligned} u_7 \cdot \Phi &= f_4^2\varphi_3 + 2b_1^2 f_2\varphi_5 + ab_1\varphi_9 \\ &= f_4^2 z + 2b_1^2 f_2 yz - 2b_1^2\beta_1 f_2 y^2 + ab_1 y^2 z^2 - a\alpha b_1\beta_1 y^2 z - a\beta_0^2 b_1 y^2 \\ &\quad - a\alpha^2\beta_0 b_1 z + ab_1\beta_1 f_2 z \end{aligned}$$

$$\begin{aligned}
&= f_4^2 - b_1(h - (-f_4\alpha^2 + a\alpha^2\beta_0)z - a\beta_0^2y^2 - (f_4 + a\alpha\beta_1)y^2z + ay^2z^2) \\
&\quad - 2b_1^2\beta_1f_2y^2 + ab_1y^2z^2 - a\alpha b_1\beta_1y^2z - a\beta_0^2b_1y^2 \\
&\quad - a\alpha^2\beta_0b_1z + ab_1\beta_1f_2z \\
&= -b_1h + f_4(f_4z - \alpha^2b_1z + b_1y^2z) + g(ab_1\beta_1f_2,\ -2b_1^2\beta_1f_2,\ 0).
\end{aligned}$$

Since

$$\begin{aligned}
&f_4z - \alpha^2b_1z + b_1y^2z \\
&\quad = (az + b_1y^2 - a\beta_1y - b_0y)z - \alpha^2b_1z + b_1y^2z \\
&\quad = az^2 + 2b_1y^2z - a\beta_1yz + b_0yz - \alpha^2b_1z \\
&\quad = az^2 + 2(g_3 - b_0\beta_1y^2 + 2b_0yz - 2az^2) - a\beta_1yz + b_0yz - \alpha^2b_1z \\
&\quad = -a\beta_1yz - \alpha^2b_1z - g_3 + b_0\beta_1y^2 \\
&\quad = -g_3 + g(-\alpha^2b_1,\ -a\beta_1^2 + b_0\beta_1,\ -a\beta_1),
\end{aligned}$$

we have

$$\begin{aligned}
u_7 \cdot \Phi = &-b_1h - f_4 \cdot g_3 + f_4 \cdot g(-\alpha^2b_1,\ -a\beta_1^2 + b_0\beta_1,\ -a\beta_1) \\
&+ g(ab_1\beta_1f_2,\ -2b_1^2\beta_1f_2,\ 0).
\end{aligned}$$

Thus $u_7 \cdot \Phi \in M$.

We have

$$u_6 \cdot \Phi = f_4\varphi_3 - b_0\varphi_5 + a\varphi_6 = g_3,$$

which implies $u_6 \cdot \Phi \in M$.

For proving $u_2 \cdot \Phi \in M$, it is enough to show $(u_2 + \alpha^2\beta_0u_6 + d\alpha u_7 + \beta_0^2u_8) \cdot \Phi \in M$ (since we already know $u_i \cdot \Phi \in M$ for all $i = 6, 7, 8$). Note that

$$\begin{aligned}
&u_2 + \alpha^2\beta_0u_6 + d\alpha u_7 + \beta_0^2u_8 \\
&\quad = (\alpha b_0\beta_1f_2, \quad -af_3, \quad -\alpha^2b_0\beta_0 + 2d\alpha b_1^2f_2, \quad b_0f_2 + a\alpha^2\beta_0, \\
&\qquad 2a\beta_0^2, \quad -b_1f_2, \quad f_4 + \alpha^2b_1).
\end{aligned}$$

So, letting

$$\begin{aligned}
A := &-af_3y^2 + (-\alpha^2b_0\beta_0 + 2d\alpha b_1^2f_2)(yz - \beta_1y^2) \\
&+ (b_0f_2 + a\alpha^2\beta_0)(z^2 - 2\beta_1yz) \\
&+ 2a\beta_0^2(y^2z - \alpha^2z) \\
&- b_1f_2(yz^2 + \beta_1y^2z), \\
B := &\ (f_4 + \alpha^2b_1)(y^2z^2 - \alpha\beta_1y^2z - \beta_0^2y^2 - \alpha^2\beta_0z + \beta_1f_2z),
\end{aligned}$$

we have

$$(u_2 + \alpha^2\beta_0 u_6 + d\alpha u_7 + \beta_0^2 u_8) \cdot \Phi = A + B.$$

Since $f_3 = z^3 + \alpha\beta_1 z^2 - \beta_1^2 y^2 z + (\alpha\beta_1^2 + \beta_0\beta_1)yz + (-\beta_0^2 + \alpha\beta_0\beta_1)z$, we can deform A as

$$\begin{aligned} A = & -ay^2z^3 - a\alpha\beta_1 y^2 z^2 - b_1 f_2 yz^2 + (\ a\alpha^2\beta_1^2 - a\alpha\beta_0\beta_1 - b_1\beta_1 f_2\ )y^2 z \\ & + (\ b_0 f_2 + a\alpha^2\beta_0\ )z^2 \\ & + \big(\ a\beta_1^2 f_2 - a\alpha^2(\alpha\beta_1^2 + \beta_0\beta_1) - \alpha^2 b_0\beta_0 + 2d\alpha b_1^2 f_2 + b_0\beta_1 f_2 + a\alpha^2\beta_0\beta_1\ \big)yz \\ & + \big(\ \alpha^2 b_0\beta_0 - 2d\alpha b_1^2 f_2\ \big)\beta_1 y^2 \\ & - \big(\ a(\alpha\beta_1^2 + \beta_0\beta_1)f_2 + 2a\alpha^2\beta_0^2\ \big)z. \end{aligned}$$

Since $f_4 + \alpha^2 b_1 = az + b_1 y^2 - a\beta_1 y - b_0 y + \alpha^2 b_1$ and $y^3 = f_2 + \alpha^2 y$, we can deform B as

$$\begin{aligned} B = & \ ay^2z^3 + \alpha^2 b_1 y^2 z^2 \\ & + (\ b_1 f_2 - a\alpha^2\beta_1 - \alpha^2 b_0\ )yz^2 \\ & + \big(\ -2\alpha^3 b_1\beta_1 - a\beta_0^2 - \alpha^2\beta_0 b_1 + b_1\beta_1 f_2\ \big)y^2 z \\ & + \big(\ -b_0 f_2 - a\alpha^2\beta_0\ \big)z^2 \\ & + \big(\ -\alpha b_1\beta_1 f_2 + a\alpha^3\beta_1^2 + 2a\alpha^2\beta_0\beta_1 + \alpha^2 b_0\beta_0 - a\beta_1^2 f_2 - b_0\beta_1 f_2\ \big)yz \\ & + \alpha^2\beta_0^2 b_1 y^2 \\ & + \big(\ a\alpha\beta_1^2 f_2 + \alpha b_0\beta_1 f_2 - \alpha^4\beta_0 b_1 + \alpha^2 b_1\beta_1 f_2\ \big)z \\ & + \big(\ -\beta_0^2 b_1 f_2 + a\alpha^2\beta_0^2\beta_1 + \alpha^2 b_0\beta_0^2\ \big)y \\ & + a\beta_0^2 f_2 + b_0 f_2\beta_0^2. \end{aligned}$$

Thus we can deform the sum $A + B$ as

$$\begin{aligned} A + B = & \big(\ -\alpha^2\beta_1 - \alpha\beta_0\ \big)ayz^2 \\ & + \big(\ -2\alpha^2\beta_0 b_1 - a\alpha^2\beta_1^2 - a\beta_0^2\ \big)y^2 z \\ & + \big(\ -a\alpha^2\beta_0\beta_1 + 2d\alpha b_1^2 f_2 - \alpha b_1\beta_1 f_2\ \big)yz \\ & + \big(\ \alpha^2 b_0\beta_0\beta_1 - 2\alpha b_1\beta_1^2 f_2 + \alpha^2\beta_0^2 b_1\ \big)y^2 \\ & + \big(\ -2a\alpha^2\beta_0^2 - \alpha^4\beta_0 b_1 + \alpha^2 b_1\beta_1 f_2\ \big)z \\ & + \big(\ -\beta_0^2 b_1 f_2 + a\alpha^2\beta_0^2\beta_1 + \alpha^2 b_0\beta_0^2\ \big)y \\ & + a\beta_0^2\beta_1 f_2 + b_0\beta_0^2 f_2 \\ = & \big(\ -\alpha^2\beta_1 - \alpha\beta_0\ \big)g_4 \\ & + \big(\ \alpha^2\beta_1 + \alpha\beta_0\ \big)\big(\ b_1 f_2 z + (a\beta_1 + b_0)y^2 z\ \big) \\ & + \big(\ -2\alpha^2\beta_0 b_1 - a\alpha^2\beta_1^2 - a\beta_0^2\ \big)y^2 z \\ & + \big(\ -a\alpha^2\beta_0\beta_1 + 2d\alpha b_1^2 f_2 - \alpha b_1\beta_1 f_2\ \big)yz \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\alpha^2 b_0 \beta_0 \beta_1 - 2\alpha b_1 \beta_1^2 f_2 + \alpha^2 \beta_0^2 b_1 \right) y^2 \\
&\quad + \left(-2a\alpha^2 \beta_0^2 - \alpha^4 \beta_0 b_1 + \alpha^2 b_1 \beta_1 f_2 \right) z \\
&\quad + \left(-\beta_0^2 b_1 f_2 + a\alpha^2 \beta_0^2 \beta_1 + \alpha^2 b_0 \beta_0^2 \right) y \\
&\quad + a\beta_0^2 \beta_1 f_2 + b_0 \beta_0^2 f_2 \\
&= \left(-\alpha^2 \beta_1 - \alpha\beta_0 \right) g_4 \\
&\quad + \left(-a\alpha^2 \beta_0 \beta_1 + \alpha b_1 \beta_1 f_2 \right) yz \\
&\quad + \left(2\alpha^2 b_0 \beta_0 \beta_1 - 2\alpha b_1 \beta_1^2 f_2 \right) y^2 \\
&\quad + \left(-2a\alpha^2 \beta_0^2 - \alpha^4 \beta_0 b_1 + 2\alpha^2 b_1 \beta_1 f_2 + \alpha\beta_0 b_1 f_2 \right) z \\
&\quad + \left(-\beta_0^2 b_1 f_2 + a\alpha^2 \beta_0^2 \beta_1 + \alpha^2 b_0 \beta_0^2 \right) y \\
&\quad + a\beta_0^2 \beta_1 f_2 + b_0 \beta_0^2 f_2 \\
&= \left(a\beta_0^2 \beta_1 f_2 + b_0 \beta_0^2 f_2 \right) \\
&\quad + \left(-b_1 \beta_0^2 f_2 + a\alpha^2 \beta_0^2 \beta_1 + \alpha^2 b_0 \beta_0^2 \right) \cdot y \\
&\quad - \left(\alpha^2 \beta_1 + \alpha\beta_0 \right) \cdot g_4 \\
&\quad + g(0, \quad \alpha^2 b_0 \beta_0 \beta_1 - b_0 \beta_1^2 f_2, \quad -a\alpha^2 \beta_0 \beta_1 + \alpha b_1 \beta_1 f_2) \\
&\quad + g(a\alpha^2 \beta_0^2 - \alpha^4 b_0 \beta_1 - \alpha^2 b_1 \beta_1 f_2 + \alpha\beta_0 b_1 f_2, \\
&\qquad\qquad - a\alpha^2 \beta_0 \beta_1^2 - \alpha b_1 \beta_1^2 f_2 + \alpha^2 b_0 \beta_0 \beta_1 + b_0 \beta_1^2 f_2, \quad 0).
\end{aligned}
$$

We obtain $(u_2 + \alpha^2 \beta_0 u_6 + d\alpha u_7 + \beta_0^2 u_8) \cdot \Phi \in M$.

Using

$$
\begin{aligned}
u_5 - b_1 u_2 - f_4 u_{12} &= f_4(a\beta_0^2, \quad \beta_0^2 b_1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0) \\
&\quad + f_3(a^2, \quad ab_1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0),
\end{aligned}
$$

we can show $u_5 \cdot \Phi \in M$.

Note that

$$
\begin{aligned}
&u_4 - au_2 - b_1 f_2 u_{10} - f_4 u_{13} - b_1 f_2 u_{15} \\
&\quad = f_4^2(\alpha^2, \quad \alpha\beta_1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0) \\
&\qquad + f_4(a\alpha^2 \beta_0, \quad a\alpha\beta_0 \beta_1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0) \\
&\qquad + f_2(-a\alpha b_0 \beta_0, \quad -ab_0 \beta_1^2, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0) \\
&\qquad + (0, \quad 0, \quad b_0^2 f_2 + b_1 f_2 f_4, \quad -ab_0 f_2, \quad 0, \quad 0, \quad 0).
\end{aligned}
$$

Since $f_4 = az + b_1 y^2 - a\beta_1 y - b_0 y$, we have

$$
\begin{aligned}
&(0, \quad 0, \quad b_0^2 f_2 + b_1 f_2 f_4, \quad -ab_0 f_2, \quad 0, \quad 0, \quad 0) \cdot \Phi \\
&\quad = (b_0^2 f_2 + b_1 f_2 f_4)\varphi_5 - ab_0 f_2 \varphi_6 \\
&\quad = -2b_0^2 f_2 yz + 2b_0^2 \beta_1 f_2 y^2 \\
&\qquad + ab_1 f_2 yz^2 + b_1^2 f_2 y^3 z - ab_1 \beta_1 f_2 y^2 z - b_0 b_1 f_2 y^2 z
\end{aligned}
$$

$$
\begin{aligned}
&- ab_1\beta_1 f_2 y^2 z - b_1^2\beta_1 f_2 y^4 + ab_1\beta_1^2 f_2 y^3 + b_0 b_1 \beta_1 f_2 y^3 \\
&- ab_0 f_2 z^2 + 2ab_0\beta_1 f_2 yz.
\end{aligned}
$$

Since $ab_1 f_2 y^2 = b_1 f_2 \cdot ayz^2 = b_1 f_2 \cdot (g_4 - b_1 f_2 z - (a\beta_1 + b_0)y^2 z)$, $y^3 = f_2 + \alpha^2 y$ and $b_0 b_1 f_2 y^2 z = b_0 f_2 \cdot (g_3 - b_0\beta_1 y^2 + 2b_0 yz - 2az^2)$, we have

$$
\begin{aligned}
&(0,\quad 0,\quad b_0^2 f_2 + b_1 f_2 f_4,\quad -ab_0 f_2,\quad 0,\quad 0,\quad 0)\cdot\Phi \\
&= -2b_0^2 f_2 yz - b_0^2\beta_1 f_2 y^2 + b_1 f_2 g_4 \\
&\quad + b_0 b_1 f_2 y^2 z \\
&\quad + \alpha^2 b_1^2 f_2 yz - b_1^2\beta_1 f_2^2 y - \alpha^2 b_1^2 \beta_1 f_2 y^2 \\
&\quad + \left(ab_1\beta_1^2 f_2 + b_0 b_1 \beta_1 f_2 \right)\left(f_2 + \alpha^2 y \right) \\
&\quad - ab_0 f_2 z^2 + 2ab_0\beta_1 f_2 yz \\
&= -2b_0^2 f_2 yz - b_0^2\beta_1 f_2 y^2 + b_1 f_2 g_4 \\
&\quad + b_0 f_2 \left(g_3 - b_0\beta_1 y^2 + 2b_0 yz - 2az^2 \right) \\
&\quad + \alpha^2 b_1^2 f_2 yz - b_1^2\beta_1 f_2^2 y - \alpha^2 b_1^2 \beta_1 f_2 y^2 \\
&\quad + \left(ab_1\beta_1^2 f_2 + b_0 b_1 \beta_1 f_2 \right)\left(f_2 + \alpha^2 y \right) \\
&\quad - ab_0 f_2 z^2 + 2ab_0\beta_1 f_2 yz \\
&= b_1 f_2 g_4 + b_0 f_2 g_3 - b_1^2 \beta_1 f_2^2 y \\
&\quad + \left(ab_1\beta_1^2 f_2 + b_0 b_1 \beta_1 f_2 \right)\left(f_2 + \alpha^2 y \right) \\
&\quad + \left(b_0^2 \beta_1 f_2 - \alpha^2 b_1^2 \beta_1 f_2 \right) y^2 \\
&\quad + \left(\alpha^2 b_1^2 f_2 + 2ab_0\beta_1 f_2 \right) yz \\
&= \left(ab_1\beta_1^2 f_2^2 + b_0 b_1 \beta_1 f_2^2 \right) \\
&\quad + \left(a\alpha^2 b_1 \beta_1^2 f_2 + \alpha^2 b_0 b_1 \beta_1 f_2 - b_1^2\beta_1 f_2^2 \right)\cdot y \\
&\quad + b_0 f_2 \cdot g_3 \\
&\quad + b_1 f_2 \cdot g_4 \\
&\quad + g(0,\ b_0^2\beta_1 f_2,\ 2ab_0\beta_1 f_2) \\
&\quad + g(0,\ -\alpha\beta_0 b_1^2 f_2,\ \alpha^2 b_1^2 f_2).
\end{aligned}
$$

Thus we have $u_4 \cdot \Phi \in M$.

Since $u_3 = -b_0 u_2 + f_4 u_{14}$ and $u_1 = f_4 u_2$, we can show $u_3 \cdot \Phi, u_1 \cdot \Phi \in M$. This completes the proof of assertion (4) of Thoerem 5.

7. Proof of Theorem 1

In this section, we prove Theorem 1 stated in the Introduction.

We first prove assertion (1) of Theorem 1. Assume that τ has the form ①. Let $\sigma : k[x,y,z] \to k[x,y,z]$ be the triangular automorphism defined by $\sigma(f) := \tau(f)$ for all $f \in k[x,y,z]$. Thus σ has the form (I) and $\beta = 0$ (see

Subsection 1.2). So, we can use both assertion (1) of Lemma 4 and assertion (1) of Theorem 5. Then $\mathrm{Ker}(D_\sigma^2) = k[x,y,z]$. Let $\gamma(x,y,z) \in k[x,y,z]$ be the polynomial appearing in ①. So, $\gamma(x,y,z) \in \mathrm{Ker}(D_\sigma^2)$. Using assertion (1) of Lemma 2, we have $E_{\sigma,\,\gamma(x,y,z)} \in U^{3,\triangle}(k[x,y,z,w])$. Since $\tau = E_{\sigma,\,\gamma(x,y,z)}$, we conclude $\tau^3 = \mathrm{id}_{k[x,y,z,w]}$. Similarly, we can prove $\tau^3 = \mathrm{id}_{k[x,y,z,w]}$ for τ having any one of the forms ②, ③, ④.

We next prove assertion (2) of Theorem 1. Since $\tau \in U^{3,\triangle}(k[x,y,z,w])$, we can define a k-algebra homomorphism $\sigma : k[x,y,z] \to k[x,y,z]$ as $\sigma(f) := \tau(f)$ for all $f \in k[x,y,z]$. Let $\delta := \tau(w) - w \in k[x,y,z]$. By assertion (2) of Lemma 2, we have $\sigma \in U^{3,\triangle}(k[x,y,z])$, $\delta \in \mathrm{Ker}(D_\sigma^2)$ and $\tau = E_{\sigma,\,\delta}$. There exists an automorphism ψ of $k[x,y,z]$ such that $\psi^{-1} \circ \sigma \circ \psi$ has one of the forms (I) and (II) (see Subsection 1.2). Then one of the following cases (1), (2), (3) and (4) can occur:

(1) $\psi^{-1} \circ \sigma \circ \psi$ has the form (I) and $\beta = 0$.

(2) $\psi^{-1} \circ \sigma \circ \psi$ has the form (I) and $\beta \neq 0$.

(3) $\psi^{-1} \circ \sigma \circ \psi$ has the form (II) and $\beta_1(s,t) = 0$.

(4) $\psi^{-1} \circ \sigma \circ \psi$ has the form (II) and $\beta_1(s,t) \neq 0$.

Corresponding to any one of the above four cases, we already know in Theorem 5 a generating set of $\mathrm{Ker}(D^2_{\psi^{-1}\circ\sigma\psi})$. Let $\Psi : k[x,y,z,w] \to k[x,y,z,w]$ be the k-algebra automorphism defined by $\Psi(x) := \psi(x)$, $\Psi(y) := \psi(y)$, $\Psi(z) := \psi(z)$ and $\Psi(w) := w$. By Lemma 3, we have $\Psi^{-1} \circ \tau \circ \Psi = E_{\sigma,\,\psi^{-1}(\delta)}$ and $\psi^{-1}(\delta) \in \mathrm{Ker}(D^2_{\psi^{-1}\circ\sigma\circ\psi})$. Hence $\Psi^{-1} \circ \tau \circ \Psi$ has one of the forms ①, ②, ③, ④, as desired.

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