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Classification of symplectically normal endomorphisms of 4-dimensional symplectic vector spaces

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Abstract

We classify symplectically normal endomorphisms of 4-dimensional complex symplectic vector spaces with respect to symplectic similarity.

1. Introduction

Let (V, ω) be a finite dimensional symplectic vector space over the field $\mathbb{C}$ of complex numbers. It is an interesting problem to classify endomorphisms of symplectic vector spaces with respect to symplectic similarity. Several classification results of endomorphisms are known for the following cases:

- the self-adjoint endomorphisms: [3], [8],
- the anti-self-adjoint endomorphisms: [1], [6],
- endomorphisms preserving the symplectic form ω: [7], [4].

Such endomorphisms are symplectically normal, that is, an endomorphism M and the symplectic adjoint $M^{*\omega}$ of M are commutative.

In this paper, we investigate the classification of symplectic similarity classes of symplectically normal endomorphisms of (V, ω). We have introduced in [5], the symplectic characteristic polynomial $\varphi_M^\omega(s,t)$ of an endomorphism M of V, whose square is the characteristic polynomial of $(M - sE)(M^{*\omega} - sE)$. We remark that $\varphi_M^\omega(s,0)$ is the characteristic polynomial $\varphi_M(s)$ of M.

Let M be a symplectically normal endomorphism of (V, ω). We showed ([5, Proposition 5.1]) that $\varphi_M^\omega(s,t)$ has the form

$$\varphi_M^\omega(s,t) = \prod_{i=1}^{k} \{(\lambda_i - s)(\mu_i - s) - t\}^{m_i}, \quad \lambda_i, \mu_i \in \mathbb{C}$$

so that $(\lambda_i - s)(\mu_i - s) \neq (\lambda_j - s)(\mu_j - s)$ for $i \neq j$. Moreover, we got ([5, Lemma 5.2]) the symplectically orthogonal direct sum decomposition:

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$$V = \bigoplus_{i=1}^{k} V_i, \quad M = \bigoplus_{i=1}^{k} M|_{V_i}, \quad V_i \subset V_j^{\perp\omega} \quad \text{if } i \neq j$$

and $(V_i, \omega|_{V_i})$ is a symplectic vector space of dimension $2m_i$. Here

$$V_i = (\tilde{V}_M(\lambda_i) \cap \tilde{V}_{M^{*\omega}}(\mu_i)) + (\tilde{V}_M(\mu_i) \cap \tilde{V}_{M^{*\omega}}(\lambda_i))$$

and $\tilde{V}_M(\lambda)$ is the generalized eigenspace of M with an eigenvalue λ. Therefore, to accomplish the classification of symplectically normal endomorphisms, it is enough to consider the problem on each direct sum component, that is, the case where

$$\varphi_M^\omega(s,t) = \{(\lambda - s)(\mu - s) - t\}^n \tag{1.1}$$

(see Section 3 for details).

We thus assume that a symplectically normal endomorphism M satisfies the condition (1.1). If $\lambda \neq \mu$, then Theorem 4.3 shows that the symplectic similarity class of M is determined by the simultaneous similarity class of two matrices which are associated with M. In the 4-dimensional case, this allows us to obtain the classification of such endomorphisms (see Proposition 6.3).

If $\lambda = \mu$, then $\varphi_M(s) = \varphi_M^\omega(s,0) = (\lambda - s)^{2n}$. This means that $M - \lambda E$ is nilpotent. Hence, the classification of this case reduces to the classification of nilpotent endomorphisms. It is easy to classify the case where $\dim V = 2$. Indeed, if $M - \lambda E$ is nilpotent, there exists a symplectic basis (e_1, f_1) such that M is represented by

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The 4-dimensional case is given as follows.

Theorem 1.1. *Let M, M' be symplectically normal endomorphisms of V of dimension 4. Suppose that $N = M - \lambda E$, $N' = M' - \lambda E$ are nilpotent.*

(1) *There exists a symplectic basis (e_1, e_2, f_1, f_2) such that M is represented by*

$$\begin{pmatrix} \lambda & 1 & 0 & b \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{pmatrix} \quad \textit{if } NN^{*\omega} \neq O,$$

$$\begin{pmatrix} \lambda & 1 & 0 & 1 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{pmatrix} \quad \textit{if } NN^{*\omega} = O,\ N^2 \neq O,$$

$$\begin{pmatrix} \lambda & 0 & b_1 & b_2 \\ 0 & \lambda & b_3 & b_4 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad \text{if } NN^{*\omega} = O,\ N^2 = O$$

where the matrix

$$B_2 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

is one of the following:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & b_4 \end{pmatrix}.$$

Moreover, the value b, which is denoted by $I_v(M)$, in the first 4×4 matrix is independent of the choice of symplectic basis such that M is represented by the first matrix. For the third 4×4 matrix, the congruence class $I_{cc}(M)$ of B_2 is independent of the choice of symplectic basis such that M is represented by the third matrix.

(2)
- *Suppose that $NN^{*\omega} \neq O$ and that $N'N'^{*\omega} \neq O$. Then M is symplectically similar to M' if and only if $I_v(M) = I_v(M')$.*
- *Suppose that $NN^{*\omega} = O$, $N^2 = O$ and that $N'N'^{*\omega} = O$, $N'^2 = O$. Then M is symplectically similar to M' if and only if $I_{cc}(M) = I_{cc}(M')$.*

We remark that the condition $NN^{*\omega} = O$ is symplectic invariant. The condition $N^2 = O$ is also symplectic invariant.

The paper is organized as follows. In Section 2, we recall the fundamental facts on symplectic linear algebra and the notion of the symplectic characteristic polynomial.

In Section 3, we show the decomposition theorem of symplectically normal endomorphisms and introduce the symplectic invariant in order to proceed the classification process.

In Section 4, we investigate the case $\lambda \neq \mu$, assuming (1.1).

In Section 5, we investigate the case $\lambda = \mu$, assuming (1.1).

In Section 6, we give the classification of symplectically normal endomorphisms in dimension 4.

2. Preliminaries

In this section, we recall the fundamental facts on symplectic linear algebra.

Definition 2.1. Let V be a finite dimensional vector space over $\mathbb{C}$. A bilinear form ω of V is said to be *symplectic* if

- ω is alternating, that is, $\omega(v,v)=0$ for all $v \in V$ and
- ω is non-degenerate, that is, the linear map

$$\omega^\flat : V \to V^* \quad v \mapsto \omega(\cdot, v)$$

is an isomorphism.

The set of automorphisms of V which preserve ω is denoted by $\mathrm{Sp}(V,\omega)$.

A *symplectic vector space* is a vector space equipped with a symplectic form. Let (V,ω) be a symplectic vector space and let $\dim_{\mathbb{C}} V = 2n$. By abuse of notation, we use the same letter V for (V,ω).

Definition 2.2. Let W be a subspace of V. We define the *symplectically orthogonal subspace* $W^{\perp\omega}$ by

$$W^{\perp\omega} = \{v \in V \mid \omega(v,w) = 0 \quad \forall w \in W\}.$$

The properties of symplectically orthogonal subspaces are showed in [5].

Definition 2.3. Let M be an endomorphism of V. The *symplectic adjoint endomorphism* $M^{*\omega}$ of M is the endomorphism of V defined by

$$M^{*\omega} = (\omega^\flat)^{-1} M^* \omega^\flat$$

where M^* is the dual map of M.

It is clear that $P \in \mathrm{Sp}(V,\omega)$ if and only if $P^{*\omega} = P^{-1}$.

Definition 2.4.

(1) An endomorphism M of V is said to be *symplectically normal* if $MM^{*\omega} = M^{*\omega}M$.
(2) An endomorphism M of V is *symplectically similar* to an endomorphism M' of V if there exists $P \in \mathrm{Sp}(V,\omega)$ such that $P^{-1}MP = M'$.

Definition 2.5. A basis $\mathcal{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ of V is said to be *symplectic* if

$$\omega(e_i, e_j) = 0,\ \omega(f_i, f_j) = 0,\ \omega(e_i, f_j) = \delta_{ij}, \quad i,j \in \{1, \ldots, n\}.$$

In what follows, we use the notation below. Let M be an endomorphism of V and $\mathcal{B} = (v_1, \ldots, v_{2n})$ be a basis of V.

- The symbol E denotes the identity map and E_n denotes the identity matrix of size n.
- The symbol A^T denotes the transpose matrix of a matrix A.
- The symbol $[M]_{\mathcal{B}}$ denotes the matrix representing the endomorphism M with respect to the basis $\mathcal{B}$.

- Let P be an automorphism of V. We use the symbol $P\mathcal{B}$ for $(Pv_1, \dots, Pv_{2n})$. We remark that $[P^{-1}MP]_{\mathcal{B}} = [M]_{P\mathcal{B}}$.

It is easy to show the following.

Proposition 2.6.

(1) *Let $\mathcal{B}$ be a symplectic basis of V and let P be an automorphism of V. Then $P\mathcal{B}$ is a symplectic basis of V if and only if $P \in \mathrm{Sp}(V, \omega)$.*
(2) *An endomorphism M of V is symplectically similar to an endomorphism M' of V if and only if there exist symplectic bases $\mathcal{B}$, $\mathcal{B}'$ such that $[M]_{\mathcal{B}} = [M']_{\mathcal{B}'}$.*

Let M be an endomorphism of V and $\mathcal{B}$ be a symplectic basis of V. If

$$[M]_{\mathcal{B}} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

where A_n, B_n, C_n, and D_n are $n \times n$ matrices, then

$$[M^{*\omega}]_{\mathcal{B}} = \Omega_{2n}^{-1}[M]_{\mathcal{B}}^T \Omega_{2n} = \begin{pmatrix} D_n^T & -B_n^T \\ -C_n^T & A_n^T \end{pmatrix}.$$

Here

$$\Omega_{2n} = \begin{pmatrix} O & E_n \\ -E_n & O \end{pmatrix}.$$

A $2n \times 2n$ matrix P_{2n} is said to be *symplectic* if $\Omega_{2n}^{-1} P_{2n}^T \Omega_{2n} = P_{2n}^{-1}$. The set of $2n \times 2n$ symplectic matrices is denoted by $\mathrm{Sp}(2n, \mathbb{C})$. A matrix M_{2n} is said to be *symplectically similar* to a matrix M'_{2n} if there exists $P_{2n} \in \mathrm{Sp}(2n, \mathbb{C})$ such that $P_{2n}^{-1} M_{2n} P_{2n} = M'_{2n}$.

Let $\mathcal{B}$ be a symplectic basis of V. It is clear that $\mathrm{Sp}(V, \omega) \ni P \mapsto [P]_{\mathcal{B}} \in \mathrm{Sp}(2n, \mathbb{C})$ is bijective. It is easy to show that an endomorphism M is symplectically similar to an endomorphism M' if and only if $[M]_{\mathcal{B}}$ is symplectically similar to $[M']_{\mathcal{B}}$.

We close this section to recall the notion of the symplectic characteristic polynomial, which is introduced in [5, Definition 3.3].

Definition 2.7. The symplectic characteristic polynomial $\varphi_M^\omega(s,t)$ of an endomorphism M of V is a polynomial in two variables s, t which satisfies the following:

- The square $\varphi_M^\omega(s,t)^2$ is equal to $\det((M - sE)(M^{*\omega} - sE) - tE)$.
- The coefficient of t^n is equal to $(-1)^n$ where $\dim_{\mathbb{C}} V = 2n$.

The symplectic characteristic polynomial has the following properties.

Proposition 2.8. *Let M, M' be endomorphisms of V.*

- *The polynomial $\varphi^\omega_M(s,0)$ is the characteristic polynomial $\varphi_M(s)$ of M.*
- *If M is symplectically similar to M', then $\varphi^\omega_M(s,t) = \varphi^\omega_{M'}(s,t)$.*

The Proposition 5.1 in [5] shows that

Proposition 2.9. *If M is a symplectically normal endomorphism of V, then the symplectic characteristic polynomial $\varphi^\omega_M(s,t)$ has the form*

$$\varphi^\omega_M(s,t) = \prod_{i=1}^{n} \{(\lambda_i - s)(\mu_i - s) - t\}, \quad \lambda_i, \mu_i \in \mathbb{C}.$$

3. Decomposition theorem

We state the decomposition theorem for symplectically normal endomorphisms. Let M be a symplectically normal endomorphism of V and let the symplectic characteristic polynomial $\varphi^\omega_M(s,t)$ be of the form

$$\varphi^\omega_M(s,t) = \prod_{i=1}^{k} \{(\lambda_i - s)(\mu_i - s) - t\}^{m_i}, \quad \lambda_i, \mu_i \in \mathbb{C}$$

so that $(\lambda_i - s)(\mu_i - s) \neq (\lambda_j - s)(\mu_j - s)$ for $i \neq j$. Then from [5, Lemma 5.2], we get the symplectically orthogonal direct sum decomposition:

$$V = \bigoplus_{i=1}^{k} V_i, \quad M = \bigoplus_{i=1}^{k} M|_{V_i}, \quad V_i \subset V_j^{\perp\omega} \quad \text{if } i \neq j$$

and $(V_i, \omega|_{V_i})$ is a symplectic space of dimension $2m_i$. Here

$$V_i = (\tilde{V}_M(\lambda_i) \cap \tilde{V}_{M^{*\omega}}(\mu_i)) + (\tilde{V}_M(\mu_i) \cap \tilde{V}_{M^{*\omega}}(\lambda_i)).$$

Hence if $\mathcal{B}_i = (e^i_1, \ldots, e^i_{m_i}, f^i_1, \ldots, f^i_{m_i})$ is a symplectic basis of V_i, then

$$\oplus_{i=1}^k \mathcal{B}_i = (e^1_1, \ldots, e^1_{m_1}, \ldots, e^k_1, \ldots, e^k_{m_k}, f^1_1, \ldots, f^1_{m_1}, \ldots, f^k_1, \ldots f^k_{m_k})$$

is a symplectic basis of V. It is clear that the symplectic similarity class $\overline{[M|_{V_i}]_{\mathcal{B}_i}}$ of $[M|_{V_i}]_{\mathcal{B}_i}$ is independent of the choice of symplectic basis of V_i. Let $I_m(M)$ denote a tuple $(\{\lambda_i, \mu_i\}, \overline{[M|_{V_i}]_{\mathcal{B}_i}})_{i=1}^k$.

Theorem 3.1. *Let M and M' be symplectically normal endomorphisms of V. Then M is symplectically similar to M' if and only if $I_m(M) = I_m(M')$.*

Proof. Let $(V_i, \mathcal{B}_i)_i$ and $(V'_i, \mathcal{B}'_i)_i$ represent $I_m(M)$ and $I_m(M')$ respectively. Suppose that there exists $P \in \mathrm{Sp}(V,\omega)$ such that $M' = P^{-1}MP$. Then $V_i = PV'_i$

and $P\mathcal{B}_i'$ is a symplectic basis of V_i. Let P_{2m_i} be the $2m_i \times 2m_i$ matrix which transforms $\mathcal{B}_i$ to $P\mathcal{B}_i'$. Then

$$[M'|_{V_i'}]_{\mathcal{B}_i'} = [M|_{V_i}]_{P\mathcal{B}_i} = P_{2m_i}^{-1}[M|_{V_i}]_{\mathcal{B}_i}P_{2m_i}^{-1}.$$

Hence, we get $I_m(M) = I_m(M')$.

Conversely, we suppose that $I_m(M) = I_m(M')$. Then there exist $2m_i \times 2m_i$ symplectic matrices P_{2m_i} such that

$$[M'|_{V_i'}]_{\mathcal{B}_i'} = P_{2m_i}^{-1}[M|_{V_i}]_{\mathcal{B}_i}P_{2m_i}^{-1}.$$

Let $P_{2m_i}\mathcal{B}_i$ be the basis of V_i transformed $\mathcal{B}_i$ by P_{2m_i}. Then $\mathcal{B} = \oplus_{i=1}^k P_{2m_i}\mathcal{B}_i$ is a symplectic basis of V. We define $P \in \mathrm{Sp}(V,\omega)$ by $\oplus_{i=1}^k \mathcal{B}_i' = P\mathcal{B}$. Then $M' = P^{-1}MP$. □

The symplectic characteristic polynomial of $M|_{V_i}$ is $\{(\lambda_i - s)(\mu_i - s) - t\}^{m_i}$. Hence, to achieve the classification of symplectic similarity classes of endomorphisms, it is enough to consider the case where $\varphi_M^\omega(s,t) = \{(\lambda - s)(\mu - s) - t\}^n$.

4. The case where $\lambda \neq \mu$

Let M be a symplectically normal endomorphism of V which satisfies (1.1). We study the symplectic similarity class of M under the condition $\lambda \neq \mu$.

Lemma 4.1. *Let M be a symplectically normal endomorphism of V. Suppose that*

$$\varphi_M^\omega(s,t) = \{(\lambda - s)(\mu - s) - t\}^n, \quad \lambda \neq \mu.$$

Then we have the following.

(1) $\tilde{V}_M(\lambda) = \tilde{V}_{M^{*\omega}}(\mu)$, $\tilde{V}_M(\mu) = \tilde{V}_{M^{*\omega}}(\lambda)$.

(2) *There exists a symplectic basis $\mathcal{B} = (e_1, \dots, e_n, f_1, \dots, f_n)$ of V such that $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_n)$) is a basis of $\tilde{V}_M(\lambda)$ (resp. $\tilde{V}_M(\mu)$). In particular,*

$$[M]_{\mathcal{B}} = \begin{pmatrix} A_n & O \\ O & D_n \end{pmatrix}, \quad A_n D_n^T = D_n^T A_n.$$

(3) *The simultaneous similarity class $\overline{(A_n, D_n^T)}$ of two commutative matrices A_n, D_n^T is independent of the choice of symplectic basis which satisfies the condition in* (2).

Proof. It follows from Lemmas 5.2, 2.4 and 2.6 in [5] that there exists a symplectic basis $\mathcal{B} = (e_1, \dots, e_n, f_1, \dots, f_n)$ such that $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_n)$) is a basis of $\tilde{V}_M(\lambda) \cap \tilde{V}_{M^{*\omega}}(\mu)$ (resp. $\tilde{V}_M(\mu) \cap \tilde{V}_{M^{*\omega}}(\lambda)$). From the fact that

$$\dim_{\mathbb{C}} \tilde{V}_M(\lambda) = \dim_{\mathbb{C}} \tilde{V}_M(\mu) = \dim_{\mathbb{C}} \tilde{V}_{M^{*\omega}}(\lambda) = \dim_{\mathbb{C}} \tilde{V}_{M^{*\omega}}(\mu) = n,$$

we get (1) and (2). Since M is symplectically normal, we get $A_n D_n^T = D_n^T A_n$.

Take another symplectic basis $\mathcal{B}' = (e_1', \dots, e_n', f_1', \dots, f_n')$ such that $(e_1', \dots, e_n')$ (resp. $(f_1', \dots, f_n')$) is a basis of $\tilde{V}_M(\lambda)$ (resp. $\tilde{V}_M(\mu)$). Let

$$[M]_{\mathcal{B}'} = \begin{pmatrix} A_n' & O \\ O & D_n' \end{pmatrix}.$$

We define $P \in \mathrm{Sp}(V, \omega)$ by $\mathcal{B}' = P\mathcal{B}$. Then $[M]_{\mathcal{B}'} = [P]_{\mathcal{B}}^{-1}[M]_{\mathcal{B}}[P]_{\mathcal{B}}$. Since $(A_n - \lambda E_n)^n = (A_n' - \lambda E_n)^n = O$, we have

$$[P]_{\mathcal{B}} \begin{pmatrix} O & O \\ O & (D_n' - \lambda E_n)^n \end{pmatrix} = \begin{pmatrix} O & O \\ O & (D_n - \lambda E_n)^n \end{pmatrix} [P]_{\mathcal{B}}.$$

This implies that

$$Q_n(D_n' - \lambda E_n)^n = (D_n - \lambda E_n)^n R_n = O$$

where

$$[P]_{\mathcal{B}} = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix}.$$

Since $(D_n - \lambda E_n)^n$, $(D_n' - \lambda E_n)^n$ are invertible matrices, we get $Q_n = R_n = O$. From the fact that $[P]_{\mathcal{B}}$ is a symplectic matrix, we obtain $S_n = (P_n^{-1})^T$. Hence, we conclude that $\overline{(A_n, D_n^T)} = \overline{(A_n', D_n'^T)}$. □

The simultaneous similarity class $\overline{(A_n, D_n^T)}$ is determined by the ordered pair (λ, μ). Let $I_{ssc}(M, (\lambda, \mu))$ denote $\overline{(A_n, D_n^T)}$.

Remark 4.2.

- If $(e_1 \dots, e_n, f_1, \dots, f_n)$ is a symplectic basis, then $(-f_1, \dots, -f_n, e_1, \dots, e_n)$ is also a symplectic basis. Therefore, if $I_{ssc}(M, (\lambda, \mu)) = \overline{(A_n, D_n^T)}$, then $I_{ssc}(M, (\mu, \lambda)) = \overline{(D_n, A_n^T)}$.
- Let (A_n, D_n) a pair of two $n \times n$ matrices which represents the simultaneous similarity class $I_{ssc}(M, (\lambda, \mu))$. Then there exists a symplectic basis $\mathcal{B}$ such that

$$[M]_{\mathcal{B}} = \begin{pmatrix} A_n & O \\ O & D_n^T \end{pmatrix}.$$

Theorem 4.3. *Let M and M' be symplectically normal endomorphisms of V. Suppose that*

$$\varphi_M^{\omega}(s, t) = \varphi_{M'}^{\omega}(s, t) = \{(\lambda - s)(\mu - s) - t\}^n, \quad \lambda \neq \mu.$$

Then M is symplectically similar to M' if and only if

$$I_{ssc}(M,(\lambda,\mu)) = I_{ssc}(M',(\lambda,\mu)).$$

Proof. Since $[P^{-1}MP]_{\mathcal{B}} = [M]_{P\mathcal{B}}$ for $P \in \mathrm{Sp}(V,\omega)$, we get $I_{ssc}(P^{-1}MP,(\lambda,\mu))) = I_{ssc}(M,(\lambda,\mu))$.

We assume that $I_{ssc}(M,(\lambda,\mu)) = I_{ssc}(M',(\lambda,\mu))$. Let $I_{ssc}(M,(\lambda,\mu)) = \overline{(A_n, D_n^T)}$ and $I_{ssc}(M',(\lambda,\mu)) = \overline{(A'_n, D'^T_n)}$. Then there exists an $n\times n$ invertible matrix P_n such that $P_n^{-1}A_nP_n = A'_n$, $P_n^{-1}D_n^TP_n = D'^T_n$. Suppose that symplectic bases $\mathcal{B}$, $\mathcal{B}'$ satisfy that

$$[M]_{\mathcal{B}} = \begin{pmatrix} A_n & O \\ O & D_n \end{pmatrix}, \quad [M']_{\mathcal{B}'} = \begin{pmatrix} A'_n & O \\ O & D'_n \end{pmatrix}.$$

Define $P \in \mathrm{Sp}(V,\omega)$ by

$$[P]_{\mathcal{B}} = \begin{pmatrix} P_n & O \\ O & (P_n^{-1})^T \end{pmatrix}.$$

Then $M' = P^{-1}MP$. □

5. The case where $\lambda = \mu$

Let M be a symplectically normal endomorphism of V satisfying (1.1) as in Section 4. We assume that $\lambda = \mu$. Then

$$\varphi_M(s) = \varphi_M^{\omega}(s,0) = (\lambda - s)^{2n},$$

which implies that $N = M - \lambda E$ is nilpotent. We study the symplectic similarity class of N. The 2-dimensional case is given as follows.

Lemma 5.1. *Let $\dim_{\mathbb{C}} V = 2$ and let N be a nilpotent endomorphism of V. If $N \neq O$, then there exists a symplectic basis (e_1, f_1) such that N is represented by*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proof. There exists a symplectic basis (e_1, f_1) such that $Ne_1 = 0$. Since N is nilpotent and $N \neq 0$, we get $Nf_1 = be_1$ and $b \neq 0$. Then $(\sqrt{b}e_1, \sqrt{b}^{-1}f_1)$ is a symplectic basis which is desired. □

Next we study the 4-dimensional case.

Lemma 5.2. *Let $\dim_{\mathbb{C}} V = 4$ and let N be a symplectically normal and nilpotent endomorphism of V. Then there exists a symplectic basis $\mathcal{B}$ such that*

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \end{pmatrix}.$$

Proof. Since N is symplectically normal, there exists a vector v such that $Nv = 0$, $N^{*\omega}v = 0$. Let $\mathcal{B} = (e_1, e_2, f_1, f_2)$ be a symplectic basis such that $e_1 = v$. Then

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & a_4 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & c_4 & d_3 & d_4 \end{pmatrix}.$$

This implies that the characteristic polynomial of N is $t^2\{(a_4 - t)(d_4 - t) - c_4 d_4\}$. Since N is nilpotent, the matrix

$$\begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix}$$

is nilpotent. Hence, it follows from Lemma 5.1 that there exists a symplectic matrix (p_{ij}) such that

$$(p_{ij})^{-1} \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix} (p_{ij}) = \begin{pmatrix} 0 & b_4' \\ 0 & 0 \end{pmatrix}.$$

We define the endomorphism P by

$$[P]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{11} & 0 & p_{12} \\ 0 & 0 & 1 & 0 \\ 0 & p_{21} & 0 & p_{22} \end{pmatrix}.$$

Then $P \in \mathrm{Sp}(V, \omega)$ and

$$[N]_{P\mathcal{B}} = [P^{-1}NP]_{\mathcal{B}} = [P]_{\mathcal{B}}^{-1} \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & a_4 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & c_4 & d_3 & d_4 \end{pmatrix} [P]_{\mathcal{B}} = \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 \end{pmatrix}.$$

Since N is symplectically normal, we get $(a_2 + d_3)b_4' = 0$. Suppose that $a_2 + d_3 \neq 0$. We define $Q \in \mathrm{Sp}(V, \omega)$ by

$$[Q]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q_1 & 0 & q_2^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & -q_2 & 0 & 0 \end{pmatrix}$$

where

$$q_1 = b_2 - b_3, \quad q_2 = a_2 + d_3.$$

Then

$$[Q^{-1}P^{-1}NPQ]_{\mathcal{B}} = [Q]_{\mathcal{B}}^{-1} \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 \end{pmatrix} [Q]_{\mathcal{B}} = \begin{pmatrix} 0 & -a_2' & b_1 & b_2' \\ 0 & 0 & b_3' & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_2' & 0 \end{pmatrix}$$

where

$$a_2' = a_2 b_3 + b_2 d_3, \quad b_2' = a_2(a_2 + d_3), \quad b_3' = -d_3(a_2 + d_3).$$

Hence, $PQ\mathcal{B}$ is a symplectic basis which is desired. □

Lemma 5.3. *Let* $\dim_{\mathbb{C}} V = 4$ *and let* N *be a symplectically normal and nilpotent endomorphism of* V*.*

(1) *There exists a symplectic basis* $\mathcal{B}$ *such that*

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

if and only if $NN^{*\omega} \neq O$.
Moreover, we get $N + N^{*\omega} + bNN^{*\omega} = O$*. In particular, the value* b *is independent of the choice of symplectic basis such that* N *is represented by the matrix above.*

(2) *There exists a symplectic basis* $\mathcal{B}$ *such that*

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

if and only if $NN^{*\omega} = O$ *and* $N^2 \neq O$.

(3) *There exists a symplectic basis* $\mathcal{B}$ *such that*

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

if and only if $NN^{*\omega} = O$*,* $N^2 = O$*.*

Proof. The only if parts of three cases are clear. We prove the case (3) in the next lemma.

From Lemma 5.2, we can assume that there exists a symplectic basis $\mathcal{B}$ such that

$$[N]_{\mathcal{B}} = \begin{pmatrix} 0 & a_2 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \end{pmatrix}.$$

An easy calculation shows that

$$[NN^{*\omega}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & -a_2b_4 \\ 0 & 0 & a_2b_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [N^2]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & a_2(b_3-b_2) & a_2b_4 \\ 0 & 0 & -a_2b_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, in both cases (1) and (2), we get $a_2 \neq 0$. We define $P \in \mathrm{Sp}(V, \omega)$ by

$$[P]_{\mathcal{B}} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_2^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$[P^{-1}NP]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & a_2^{-2}b_1 & a_2^{-1}b_2 \\ 0 & 0 & a_2^{-1}b_3 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We define $Q \in \mathrm{Sp}(V, \omega)$ by

$$[Q]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & -a_2^{-2}b_1 \\ 0 & 1 & -a_2^{-2}b_1 & -a_2^{-1}b_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$[(PQ)^{-1}NPQ]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & a_2^{-1}(b_2-b_3) \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

For $r \in \mathbb{C} - \{0\}$, we define $R(r) \in \mathrm{Sp}(V, \omega)$ by

$$[R(r)]_{\mathcal{B}} = \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{pmatrix}.$$

Then

$$[(PQR(r))^{-1}NPQR(r)]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & a_2^{-1}(b_2-b_3)r^2 \\ 0 & 0 & 0 & b_4 r^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

(1). We assume that $NN^{*\omega} \neq O$. Then $b_4 \neq 0$. Hence

$$[(PQR(r))^{-1}NPQR(r)]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.1}$$

where $r = \sqrt{b_4}^{-1}$ and $b = a_2^{-1}(b_2 - b_3)b_4^{-1}$. From this representation, we get

$$N + N^{*\omega} + bNN^{*\omega} = O.$$

The condition $NN^{*\omega} \neq O$ implies that the value b is independent of the choice of symplectic basis such that N is represented by the matrix in (5.1).

(2). If $NN^{*\omega} = O$ and $N^2 \neq O$, then $b_4 = 0$ and $a_2(b_2 - b_3) \neq 0$. Hence,

$$[(PQR(r))^{-1}NPQR(r)]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

where $r = \sqrt{a_2(b_2 - b_3)^{-1}}$. □

Lemma 5.4. *Let $\dim_{\mathbb{C}} V = 2n$ and let M be an endomorphism of V. Set $W = \operatorname{Ker} M \cap \operatorname{Ker} M^{*\omega}$. Then the following are equivalent.*

(1) $M^2 = O$, $MM^{*\omega} = O$, $M^{*\omega}M = O$.
(2) $W^{\perp\omega} \subset W$.
(3) *There exists a symplectic basis $\mathcal{B}$ such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} O & B_n \\ O & O \end{pmatrix}.$$

Here B_n is an $n \times n$ matrix such that

$$B_n = \begin{pmatrix} O & O \\ O & D_k \end{pmatrix}, \quad \operatorname{Ker} D_k \cap \operatorname{Ker} D_k^T = \{0\} \tag{5.2}$$

where D_k is a $k \times k$ matrix and $k = \dim_{\mathbb{C}} W^{\perp\omega}$.

Proof. (1) ⇒ (2). The condition (1) implies that

$$\operatorname{Im} M \subset \operatorname{Ker} M, \quad \operatorname{Im} M^{*\omega} \subset \operatorname{Ker} M^{*\omega}, \quad \operatorname{Im} M \subset \operatorname{Ker} M^{*\omega}, \quad \operatorname{Im} M^{*\omega} \subset \operatorname{Ker} M.$$

Hence, we get

$$\operatorname{Im} M^{*\omega} + \operatorname{Im} M \subset \operatorname{Ker} M \cap \operatorname{Ker} M^{*\omega}.$$

Therefore, we conclude that

$$\begin{aligned}(\operatorname{Ker} M \cap \operatorname{Ker} M^{*\omega})^{\perp\omega} &= (\operatorname{Ker} M)^{\perp\omega} + (\operatorname{Ker} M^{*\omega})^{\perp\omega} \\ &= \operatorname{Im} M^{*\omega} + \operatorname{Im} M \\ &\subset \operatorname{Ker} M \cap \operatorname{Ker} M^{*\omega}.\end{aligned}$$

(2) ⇒ (3). Since $W^{\perp\omega} \subset W$, there exists a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that $(e_1, \dots, e_n, f_1 \dots, f_{n-k})$ is a basis of W. This means that (3) holds.

It is clear that (3) implies (1). □

Theorem 5.5. *Let $\dim_{\mathbb{C}} V = 2n$ and let N, N' be endomorphisms of V. Suppose that there exist symplectic bases $\mathcal{B}$, $\mathcal{B}'$ such that*

$$[N]_{\mathcal{B}} = \begin{pmatrix} O & B_n \\ O & O \end{pmatrix}, \quad [N']_{\mathcal{B}'} = \begin{pmatrix} O & B'_n \\ O & O \end{pmatrix}$$

where B_n, B'_n are $n \times n$ matrices and D_k, D'_k satisfy the condition (5.2). Then the following are equivalent.

1. *D_k is congruent to D'_k.*
2. *B_n is congruent to B'_n.*
3. *N is symplectically similar to N'.*

Proof. It is clear that (1) implies (2).

(2) ⇒ (3). We assume that there exists an $n \times n$ invertible matrix P_n such that $P_n^T B_n P_n = B'_n$. We define $P \in \operatorname{Sp}(V, \omega)$ by

$$[P]_{\mathcal{B}} = \begin{pmatrix} (P_n^T)^{-1} & O \\ O & P_n \end{pmatrix}.$$

Then $N' = P^{-1} N P$.

(3) ⇒ (1). We assume that there exists $P \in \operatorname{Sp}(V, \omega)$ such that $N' =$

$P^{-1}NP$. Then

$$\Omega_{2n}[N']_{\mathcal{B}} = \Omega_{2n}[P]_{\mathcal{B}}[N]_{\mathcal{B}}[P]_{\mathcal{B}} = [P]_{\mathcal{B}}^T\Omega_{2n}[N]_{\mathcal{B}}[P]_{\mathcal{B}},$$

which means that $\Omega_{2n}[N]_{\mathcal{B}}$ is congruent to $\Omega_{2n}[N']_{\mathcal{B}}$. Since

$$\Omega_{2n}[N]_{\mathcal{B}} = \begin{pmatrix} O & O \\ O & -B_n \end{pmatrix}, \quad \Omega_{2n}[N']_{\mathcal{B}} = \begin{pmatrix} O & O \\ O & -B'_n \end{pmatrix},$$

there exists a $k \times k$ matrix P_k such that $D'_k = P_k^T D_k P_k$. From the fact that $\operatorname{Ker} D'_k \cap \operatorname{Ker} D'^T_k = \{0\}$, we conclude that $\det P_k \neq 0$. □

Remark 5.6. The classification of congruence classes is already known (see [2] and reference therein).

6. Classification of symplectically normal endomorphisms

By using the results mentioned in the previous sections, we classify symplectically normal endomorphisms of V when $\dim_{\mathbb{C}} V \leq 4$. We first state the 2-dimensional case.

Proposition 6.1. *Let $\dim_{\mathbb{C}} V = 2$ and let M be an endomorphism of V. Then there exists a symplectic basis such that M is represented by*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \ \textit{or} \ \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Proof. If $\lambda \neq \mu$, then it follows from Lemma 4.1. If $\lambda = \mu$, then it follows from Lemma 5.1. □

The 4-dimensional case is given as follows.

Theorem 6.2. *Let $\dim_{\mathbb{C}} V = 4$ and let M be a symplectically normal endomorphism of V. Suppose that the symplectic characteristic polynomial $\varphi_M^\omega(s,t)$ has the form*

$$\varphi_M^\omega(s,t) = \{(\lambda_1 - s)(\mu_1 - s) - t\}\{(\lambda_2 - s)(\mu_2 - s) - t\}.$$

(1) *If $(\lambda_1 - s)(\mu_1 - s) \neq (\lambda_2 - s)(\mu_2 - s)$, then there exists a symplectic basis such that M is represented by one of the following matrices:*

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix},$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 1 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_2 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$

(2) *If* $(\lambda_1 - s)(\mu_1 - s) = (\lambda_2 - s)(\mu_2 - s)$, $\lambda_1 \neq \mu_1$, *then there exists a symplectic basis* $\mathcal{B}$ *such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_1 \end{pmatrix} \quad \textit{if } \dim V_M(\lambda_1) = 2,\ \dim V_M(\mu_1) = 2,$$

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_1 \end{pmatrix} \quad \textit{if } \dim V_M(\lambda_1) = 1,\ \dim V_M(\mu_1) = 2,$$

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 1 & \mu_1 \end{pmatrix} \quad \textit{if } \dim V_M(\lambda_1) = 2,\ \dim V_M(\mu_1) = 1,$$

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & d_3 & \mu_1 \end{pmatrix} \quad \textit{if } \dim V_M(\lambda_1) = 1,\ \dim V_M(\mu_1) = 1.$$

Moreover, the value d_3 *in the fourth matrix is independent of the choice of symplectic basis such that* M *is represented by the fourth matrix.*

(3) *If* $\lambda_1 = \lambda_2 = \mu_1 = \mu_2$, *then there exists a symplectic basis* $\mathcal{B}$ *such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & b \\ 0 & \lambda_1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & \lambda_1 \end{pmatrix} \quad \textit{if } NN^{*\omega} \neq O,$$

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & 1 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & \lambda_1 \end{pmatrix} \quad \textit{if } NN^{*\omega} = O,\ N^2 \neq O,$$

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & b_1 & b_2 \\ 0 & \lambda_1 & b_3 & b_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \textit{if } NN^{*\omega} = O,\ N^2 = O$$

where $N = M - \lambda_1 E$ and the matrix

$$B_2 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

is one of the following:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & b_4 \end{pmatrix}.$$

Moreover, the value b in the first matrix is independent of the choice of symplectic basis such that M is represented by the first matrix. For the third matrix, the congruence class of B_2 is independent of the choice of symplectic basis such that M is represented by the third matrix.

Proof. (1). If $(\lambda_1 - s)(\mu_1 - s) \neq (\lambda_2 - s)(\mu_2 - s)$, then we get the symplectically orthogonal direct sum decomposition:

$$V = V_1 \oplus V_2, \quad M = M|_{V_1} \oplus M|_{V_2}.$$

Hence, from Proposition 6.1, there exist symplectic basis (e_1, f_1), (e_2, f_2) of V_1, V_2 such that

$$[M|_{V_1}]_{\mathcal{B}_1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix},$$
$$[M|_{V_2}]_{\mathcal{B}_2} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{pmatrix}.$$

Therefore, (e_1, e_2, f_1, f_2) is a symplectic basis which is desired.

(2). Let $I_{ssc}(M, (\lambda, \mu)) = \overline{(A_2, D_2^T)}$. Since M is symplectically normal, we have $A_2 D_2^T = D_2^T A_2$. Hence, the simultaneous similarity class $\overline{(A_2, D_2^T)}$ is represented by

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \right) \text{ if } \dim V_M(\lambda_1) = 2, \ \dim V_M(\mu_1) = 2,$$
$$\left(\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \right) \text{ if } \dim V_M(\lambda_1) = 1, \ \dim V_M(\mu_1) = 2,$$
$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix} \right) \text{ if } \dim V_M(\lambda_1) = 2, \ \dim V_M(\mu_1) = 1,$$
$$\left(\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \mu_1 & d_3 \\ 0 & \mu_1 \end{pmatrix} \right) \text{ if } \dim V_M(\lambda_1) = 1, \ \dim V_M(\mu_1) = 1.$$

(3). The case where $NN^{*\omega} \neq O$ and the case where $NN^{*\omega} = O, N^2 \neq O$ are follow from Lemma 5.3. By Lemma 5.4, there exists a symplectic basis $\mathcal{B}$ such that

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & b_1 & b_2 \\ 0 & \lambda_1 & b_3 & b_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}.$$

From [9, Corollary 2], the distinct congruence classes of 2×2 matrices are represented by

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & b_4 \end{pmatrix}.$$

Hence, by theorem 5.5 and the item (2) in Proposition 2.6, we get the conclusion. □

Proposition 6.3. *Let* $\dim_{\mathbb{C}} V = 4$ *and let* M, M' *be symplectically normal endomorphisms of* V.

(1) *If there exist symplectic bases* $\mathcal{B}$, $\mathcal{B}'$ *such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & d_3 & \mu_1 \end{pmatrix}, \quad [M']_{\mathcal{B}'} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & d_3' & \mu_1 \end{pmatrix},$$

and $\lambda_1 \neq \mu_1$, *then* M *is symplectically similar to* M' *if and only if* $d_3 = d_3'$.

(2) *If there exist symplectic bases* $\mathcal{B}$, $\mathcal{B}'$ *such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 1 & 0 & b \\ 0 & \lambda_1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & \lambda_1 \end{pmatrix}, \quad [M']_{\mathcal{B}'} = \begin{pmatrix} \lambda_1 & 1 & 0 & b' \\ 0 & \lambda_1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & \lambda_1 \end{pmatrix},$$

then M *is symplectically similar to* M' *if and only if* $b = b'$.

(3) *If there exist symplectic bases* $\mathcal{B}$, $\mathcal{B}'$ *such that*

$$[M]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & b_1 & b_2 \\ 0 & \lambda_1 & b_3 & b_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \quad [M']_{\mathcal{B}'} = \begin{pmatrix} \lambda_1 & 0 & b_1' & b_2' \\ 0 & \lambda_1 & b_3' & b_4' \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

then M *is symplectically similar to* M' *if and only if*

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad \begin{pmatrix} b_1' & b_2' \\ b_3' & b_4' \end{pmatrix}$$

are congruent.

Proof. The cases (1), (2) follow from the item (1) in Proposition 2.6. The case (3) follows from Theorem 5.5. □

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