The second term in the asymptotics of Kohn-Rossi heat kernel on contact Riemannian manifolds

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Abstract

We describe explicitly the second term in the asymptotic expansion of the pointwise trace of the heat kernel associated with the Kohn-Rossi Laplacian on a contact Riemannian manifold.

0 Introduction

Let M be a compact manifold of dimension $2n+1$ equipped with a contact Riemannian structure (θ, ξ, g, J) consisting of a contact 1-form θ, the associated Reeb vector field ξ (i.e., $\theta(\xi) = 1$, $L_\xi \theta = 0$), a Riemannian metric g and a $(1,1)$-tensor field J called an almost complex structure which satisfy $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y)$ and $J^2 X = -X + \theta(X)\xi$ for any vector fields X, Y. In this paper we adopt such a notation as $d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$. To express tensor fields and differential forms locally, we will use a local unitary frame $\xi_\bullet := (\xi_0, \xi_1, \ldots, \xi_n, \xi_{\bar{1}}, \ldots, \xi_{\bar{n}})$ ($\xi_0 = \xi$, $J\xi_\alpha = i\xi_\alpha$, $\xi_{\bar{\alpha}} = \overline{\xi_\alpha}$, $g(\xi_\alpha, \xi_{\bar{\beta}}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) and its dual frame $\theta^\bullet := (\theta^0, \theta^1, \ldots, \theta^n, \theta^{\bar{1}}, \ldots, \theta^{\bar{n}})$ ($\theta^0 = \theta$). The subbundle spanned by $\xi_1, \ldots, \xi_n$ will be denoted by $H_{1,0}M$. On the contact Riemannian manifold $M = (M, \theta, \xi, g, J)$, let us consider the Kohn-Rossi Laplacian

$$\Box_H = \bar{\partial}_H^* \bar{\partial}_H + \bar{\partial}_H \bar{\partial}_H^*$$

acting on (p, q)-forms. The differential forms φ which are described locally as $\sum \theta^{I\bar{K}} \cdot \varphi^{I\bar{K}}$ ($I = (0 < i_1 < i_2 < \cdots < i_p)$, etc., $\theta^{I\bar{K}} := \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \wedge \theta^{\bar{k}_1} \wedge \cdots \wedge \theta^{\bar{k}_q}$) are called (p, q)-forms, which gather together into the space $\Omega^{p,q}M$. For a (p, q)-form φ, $\bar{\partial}_H \varphi$ is

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defined to be the $(p, q+1)$-component of the exterior derivative $d\varphi$. The formal adjoint of $\bar{\partial}_H$ is denoted by $\bar{\partial}_H^*$.

Let us assume $0 < q < n$, under which $\Box_H$ is hypoelliptic ([4, Remark 2.2(1)]). Then, according to [4, Theorem 2.1], the initial value problem

$$\left(\frac{\partial}{\partial t} + \Box_H\right)\phi = 0, \quad \lim_{t\to 0}\phi(t) = \varphi \quad (\varphi \in \Omega^{p,q}M) \tag{0.1}$$

has a unique fundamental solution or heat kernel $e^{-t\Box_H}(P, P')$. This is a smooth double form on $M \times M$ parameterized smoothly by $t \in (0, \infty)$, described locally as

$$e^{-t\Box_H}(P, P') = \sum \theta^{I\bar{K}}(P) \boxtimes \theta^{\bar{I}'K'}(P') \cdot \left(e^{-t\Box_H}\right)^{(I\bar{K})(I'\bar{K}')}(P, P'), \tag{0.2}$$

and the (p, q)-form

$$\begin{aligned}&\int_M e^{-t\Box_H}(P, P') \wedge \star_g \varphi(P') \\ &\qquad = \sum \theta^{I\bar{K}}(P) \cdot \int_M dV_g(P')\, \left(e^{-t\Box_H}\right)^{(I\bar{K})(I'\bar{K}')}(P, P')\, \varphi^{I'\bar{K}'}(P')\end{aligned}$$

is a solution of (0.1), where we set $dV_g = \theta \wedge (d\theta)^n/n!$. Furthermore, [4, Theorem 2.3 and (5.12)] (see also (1.9), (1.10)) says that the pointwise trace $\operatorname{tr} e^{-t\Box_H}(P^0, P^0) := \sum_{I,K}\left(e^{-t\Box_H}\right)^{(I\bar{K})(I'\bar{K}')}(P^0, P^0)$ is asymptotically expanded as

$$\operatorname{tr} e^{-t\Box_H}(P^0, P^0) \sim \sum_{k=0}^{\infty} t^{-(n+1)+k} a_k(P^0)$$

when $t \to 0$, and, in particular,

$$a_0(P^0) = \binom{n}{q}\binom{n}{p}\int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s), \quad \Phi^{n-2q}(s) := \frac{e^{-(n-2q)s}}{(2\pi)^{n+1}}\left(\frac{s}{\sinh s}\right)^n.$$

Note that $\Phi^{n-2q}(s)$ is rapidly decreasing.

These assertions were first ascertained by Folland-Stein [2], Stanton-Tartakoff [6] in the case where J is integrable (i.e., $[\Gamma(H_{1,0}M), \Gamma(H_{1,0}M)] \subset \Gamma(H_{1,0}M)$), that is, M is a strictly pseudoconvex CR manifold. In [4], the second author of this paper showed that their results still hold even if the integrability condition is ignored. In fact, his research was mainly focused on presenting a new method of describing the coefficients $a_k(P^0)$ (and its every differential) explicitly. Indeed, by using only a basic knowledge of calculus added to the formula [4, Theorem 5.3], the coefficients can be described explicitly up to an arbitrarily high order. As an example, he offered the concrete description of $a_1(P^0)$ in the case J is integrable ([4, Corollary of Theorem 5.3]). The amount of calculation is rather considerable. With the aid of Mathematica, we wish to present here its concrete description with no restriction on J.

We consider the smooth function $S(t,s) = \frac{\tanh ts}{2s}$ on $(0,1)\times\mathbb{R}\,(\ni (t,s))$ and set

$$\Phi_1(s) = \int_0^1 dt\, \frac{s\,S(1-t,s)S(t,s)}{S(1-t,s)+S(t,s)} = \frac{s}{\sinh s}\frac{s\cosh s - \sinh s}{4s^2},$$
$$\Phi_2(s) = \int_0^1 dt\, \Big(\frac{s\,S(1-t,s)S(t,s)}{S(1-t,s)+S(t,s)}\Big)^2 = \Big(\frac{s}{\sinh s}\Big)^2 \frac{2s\cosh 2s - 3\sinh 2s + 4s}{64s^3}.$$

Note that the functions $\Phi_j(s)$ $(j=1,2)$ and $s^{-2}\Phi_2(s)$ are smooth and bounded on $\mathbb{R}$, and the functions $\Phi^{n-2q}(s)\Phi_j(s)$, $\Phi^{n-2q}(s)s^{-2}\Phi_2(s)$ are rapidly decreasing.

Theorem 0.1 *We have*

$$(0.3)\quad a_1(P^0) = \sum_{\alpha,\beta=1}^{n} R_{\bar\alpha\alpha\bar\beta\beta}(P^0)\cdot\Big\{\binom{n-1}{q-1}\binom{n-1}{p}$$
$$+\Big(\binom{n-1}{p-1}-\binom{n-1}{q-1}\Big)\Big(\frac{1}{2}+\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Phi_1(s)\Big)$$
$$+\binom{n}{q}\binom{n}{p}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big(\frac{4\Phi_2(s)}{3}-\frac{1}{12}\Big)\Big\}$$
$$+\sum_{\alpha,\beta,\gamma=1}^{n}\big|N_{\alpha\beta\gamma}(P^0)\big|^2$$
$$\times\binom{n}{q}\binom{n}{p}\int_{-\infty}^{\infty} ds\,\frac{\Phi^{n-2q}(s)}{16}\Big(\frac{\Phi_1(s)-\Phi_2(s)}{12}+\frac{3\Phi_2(s)}{16s^2}-\frac{5}{48}\Big),$$

where R_{ABCD} denotes the curvature coefficient of the hermitian Tanno connection ∇ (refer to §.1) and N_{ABC} denotes the Nijenhuis coefficient associated with J. Namely, we set $R_{ABCD} = g(F(\nabla)(\xi_C,\xi_D)\xi_B,\xi_A)$ $(F(\nabla)(X,Y) := [\nabla_X,\nabla_Y]-\nabla_{[X,Y]})$ and $N_{ABC} = g([J,J](\xi_B,\xi_C),\xi_A)$ $([J,J](X,Y) := [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY])$. We put $\binom{n-1}{p-1}=0$ when $p=0$.

The almost complex structure J is integrable if and only if $N_{\alpha\beta\gamma}(P^0)$ vanishes for all α, β, $\gamma\in\{1,\dots,n\}$ and all $P^0\in M$, and the hermitian Tanno connection ∇ coincides with the Tanaka-Webster connection ([1, §.1.2]) when J is integrable. Hence, the above result is certainly consistent with [4, Corollary of Theorem 5.3].

In §.1 we recall the method of describing $a_1(P^0)$ and, in §.2 we will draw the description (0.3). From now on, to simplify the description, we assume that the Greek indices α, β, … always vary from 1 to n, the block Latin indices A, B, … vary in $\{0,1,\dots,n,\bar 1,\dots,\bar n\}$ and the symbol $\sum$ may be omitted (in an unusual manner).

1 The formula for the asymptotic coefficients

In [4], on the basis of adiabatic expansion theory ([3]) he developed the method of describing the asymptotic coefficients explicitly, which will be reviewed briefly in this section.

The connection ${}^*\nabla$ introduced by Tanno ([7]), which is defined by

$$ {}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X\xi + (\nabla^g_X\theta)(Y)\xi, $$

will be widely used in studying the contact Riemannian structure. Its hermitian part

$$ \nabla_X Y = \begin{cases} {}^*\nabla_X(f\xi) & : Y = f\xi, \\ \dfrac{1}{2}\left({}^*\nabla_X Y - J\,{}^*\nabla_X JY\right) & : Y \in \Gamma(\ker\theta), \end{cases} $$

called the hermitian Tanno connection, was adopted in [4], however. (Note that the two connections and the Tanaka-Webster connection coincide with each other when J is integrable.) Certainly we have $\nabla J = 0$ and the Kohn-Rossi Laplacian has the Weitzenböck-type formula ([4, Proposition 1.3])

$$ \begin{aligned} (1.1)\qquad \Box_H = &-\sum\left(\nabla_{\xi_\alpha}\nabla_{\xi_{\bar\alpha}} - \nabla_{\nabla_{\xi_\alpha}\xi_{\bar\alpha}}\right) - \sqrt{-1}\,q\,\nabla_\xi \\ &-\sum F(\nabla)^C_D(\xi_{\bar\alpha},\xi_\beta)\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\theta^{\bar C}\wedge\theta^{\bar D}\vee \quad (\text{acting on } \Omega^{p,q}M), \end{aligned} $$

where we set $F(\nabla)(\xi_{\bar\alpha},\xi_\beta)\xi_D = \xi_C\cdot F(\nabla)^C_D(\xi_{\bar\alpha},\xi_\beta)$ and $\theta^{\bar\alpha}\wedge$, $\theta^{\bar\alpha}\vee(=\iota_{\xi_{\bar\alpha}} = \xi_{\bar\alpha}\lrcorner)$ denote their exterior, interior products, respectively. We may assume that the pair of indices (C,D) above runs only over the set of pairs (γ,δ), $(\bar\gamma,\bar\delta)$ $(1\le\gamma,\delta\le n)$.

We want to introduce another merit of adopting ∇ ([4, Proposition 2.4]). Let $z = (z_0, z_1, \ldots, z_n)$ or $z_\bullet = (z_0, z_1, \ldots, z_n, z_{\bar 1}, \ldots, z_{\bar n})$ $(z_{\bar\alpha} := \overline{z_\alpha})$ be the ∇-normal coordinates centered at P^0 defined by $\exp^\nabla(\xi_\bullet(P^0)\cdot z_\bullet(P)) = P$, which, we assume, are related to the real ∇-normal coordinates $x = (x_0, x_1, \cdots, x_{2n})$ as $z_0 = x_0$, $z_\alpha = (x_\alpha + ix_{n+\alpha})/\sqrt{2}$. Further, let the unitary frames $\xi_\bullet$, $\theta^\bullet$ be ∇-parallel along the ∇-geodesics sz $(s\ge 0)$ from P^0. Then, as to the connection forms given by $\nabla\xi_\beta = \xi_\alpha\cdot\omega^\alpha_\beta$, we have the formal series expansion

$$ (1.2)\qquad \omega^\alpha_\beta(\partial/\partial z_A) = -\sum_{\ell=1}^\infty \frac{\ell}{(\ell+1)!}\sum z_{A_1}\cdots z_{A_\ell}\frac{\partial^{\ell-1}F(\nabla)^\alpha_\beta(\partial/\partial z_A, \partial/\partial z_{A_1})}{\partial z_{A_2}\cdots\partial z_{A_\ell}}(0), $$

where we put $\partial/\partial z_0 = \partial/\partial x_0$ and $\partial/\partial z_\alpha = \left(\partial/\partial x_\alpha - i\partial/\partial x_{n+\alpha}\right)/\sqrt{2}$. Moreover, let us set

$$ (1.3)\qquad \xi_A = \sum V_{BA}\,\partial/\partial z_B, \quad \theta^A = \sum V^{BA}\,dz_B, \quad \text{hence } V_\bullet = {}^t(V^\bullet)^{-1}. $$

Then, we have the formal series expansion

$$ \begin{aligned} (1.4)\qquad V^{BA}(z) = \delta^{BA} &+ \sum_{\ell=1}^\infty \frac{\ell}{(\ell+1)!}\sum z_{A_1}\cdots z_{A_\ell}\frac{\partial^{\ell-1}T(\nabla)^A_{A_1}(\partial/\partial z_B)}{\partial z_{A_2}\cdots\partial z_{A_\ell}}(0) \\ &+ \sum_{\ell=2}^\infty \frac{\ell-1}{(\ell+1)!}\sum z_{A_1}\cdots z_{A_\ell}\frac{\partial^{\ell-2}F(\nabla)^A_{A_1}(\partial/\partial z_{A_2}, \partial/\partial z_B)}{\partial z_{A_3}\cdots\partial z_{A_\ell}}(0), \end{aligned} $$

where we set $T(\nabla)(\xi_C, \partial/\partial z_B) = \xi_A \cdot T(\nabla)^A_C(\partial/\partial z_B)$.

From now on, the unitary frames $\xi_\bullet$, $\theta^\bullet$ are always assumed to be ∇-parallel and the coordinates z are ∇-normal centered at P^0. So are the frames in the expression (0.2) of $e^{-t\Box_H}(P,P') = e^{-t\Box_H}(z,z')$ ($z := z(P)$, $z' := z(P')$). Next, let us introduce the notion of adiabatic expansion of $\Box_H$ at P^0 ([4, Proposition 5.2]).

We will transform a neighborhood of P^0 by the diffeomorphism $\iota_\varepsilon : z \mapsto \iota_\varepsilon(z) := (\varepsilon z_0, \varepsilon^{1/2}z_1, \ldots, \varepsilon^{1/2}z_n)$ ($\varepsilon > 0$), which induces a new contact Riemannian structure $(\theta^\bullet_{(\varepsilon)}, \xi^{(\varepsilon)}_\bullet, g^{(\varepsilon)}, J^{(\varepsilon)}) := (\iota^*_\varepsilon\theta^\bullet_\varepsilon, \iota^*_\varepsilon\xi^\varepsilon_\bullet, \iota^*_\varepsilon g^\varepsilon, \iota^*_\varepsilon J^\varepsilon)$ with $\theta^A_\varepsilon := \varepsilon^{-|A|_H/2}\theta^A$, $\xi^\varepsilon_A := \varepsilon^{|A|_H/2}\xi_A$, $g^\varepsilon := \sum \theta^A_\varepsilon \otimes \theta^{\bar{A}}_\varepsilon$, $J^\varepsilon \xi^\varepsilon_\alpha := i\xi^\varepsilon_\alpha$, where we set $|A|_H = 2$ if $A = 0$ and $|A|_H = 1$ if $A \neq 0$. Obviously (1.3) produces

$$\begin{aligned} \xi^{(\varepsilon)}_\bullet &= (\partial/\partial z_\bullet) \cdot V^{(\varepsilon)}_\bullet, \quad V^{(\varepsilon)}_{BA}(z) = \varepsilon^{(|A|_H - |B|_H)/2} V_{BA}(\iota_\varepsilon(z)), \\ \theta^\bullet_{(\varepsilon)} &= (dz_\bullet) \cdot V^\bullet_{(\varepsilon)}, \quad V^{BA}_{(\varepsilon)}(z) = \varepsilon^{(|B|_H - |A|_H)/2} V^{BA}(\iota_\varepsilon(z)). \end{aligned} \tag{1.5}$$

To the structure $(\theta^\bullet_\varepsilon, \xi^\varepsilon_\bullet, g^\varepsilon, J^\varepsilon)$ the Kohn-Rossi Laplacian $\Box^\varepsilon_H := \varepsilon\Box_H$ and the hermitian Tanno connection $\nabla^\varepsilon := \nabla$ are attached. These for the structure $(\theta^\bullet_{(\varepsilon)}, \xi^{(\varepsilon)}_\bullet, g^{(\varepsilon)}, J^{(\varepsilon)})$ are $\Box^{(\varepsilon)}_H := \iota^*_\varepsilon\Box^\varepsilon_H$, $\nabla^{(\varepsilon)} := \iota^*_\varepsilon\nabla^\varepsilon$. The coordinates z are the $\nabla^{(\varepsilon)}$-normal coordinates centered at 0 with $(\partial/\partial z_\bullet)_0 = \xi^{(\varepsilon)}_\bullet(0)$ and $\xi^{(\varepsilon)}_\bullet$, $\theta^\bullet_{(\varepsilon)}$ are $\nabla^{(\varepsilon)}$-parallel along the $\nabla^{(\varepsilon)}$-geodesics from P^0.

A neighborhood of P^0 equipped with these tools is roughly approximated by a neighborhood of the origin in the Heisenberg group $H_n = (\mathbb{R} \times \mathbb{C}^n, z)$ which is a typical contact Riemannian manifold. We will describe H_n briefly to adjust the notation. This is a Lie group with the group action $zz' = (z_0 + z'_0 + \operatorname{Im}\sum z_\alpha z'_{\bar{\alpha}}, z_1 + z'_1, \ldots)$, and has a contact 1-form $\theta_H = dz_0 + dz_\beta \cdot z_{\bar{\beta}}\frac{-i}{2} + dz_{\bar{\beta}} \cdot z_\beta \frac{i}{2}$ and the Reeb vector field $\xi^H = \partial/\partial z_0$. We set $\xi^H_\beta = \partial/\partial z_\beta + \partial/\partial z_0 \cdot z_{\bar{\beta}}\frac{i}{2}$, which satisfy $\theta_H(\xi^H_\beta) = 0$ and canonically provide an almost complex structure J^H. Note that the dual frame of $\xi^H_\bullet$ is $\theta^\bullet_H = (\theta_H, dz_1, \ldots, dz_{\bar{1}}, \ldots)$. These equipments, together with the metric g^H defined by $g^H(X,Y) = \theta_H(X)\theta_H(Y) + d\theta_H(X, J^H Y)$, provide a contact Riemannian structure to H_n, which, compared the results in Lemma 2.2, certainly approximates the structure of M near P^0 roughly. Note that J^H is integrable and the Kohn-Rossi Laplacian is simplified down to

$$\mathbf{L} = -\sum \xi^H_\alpha \xi^H_{\bar{\alpha}} - \sqrt{-1}\, q\, \xi^H \quad (\text{acting on } \Omega^{p,q}H_n).$$

Further, by [5] (also refer to [4, Lemma 2.6]), if $-n < n - 2q < n$, i.e., $0 < q < n$, then the initial value problem (0.1) on H_n ($\varphi \in \Omega^{p,q}_0 H_n$) has a unique fundamental solution

$$r_H(t, z, z') = \sum \theta^{I\bar{K}}_H(z) \boxtimes \theta^{\bar{I}K}_H(z') \cdot r^{n-2q}_t(z'^{-1}z),$$

where, putting $z = (z_0, z_\blacktriangle)$, we set

$$\begin{aligned} r_t^{n-2q}(z) &= \int_{-\infty}^{\infty} ds\, e^{-is\cdot(2z_0/t)} \Phi_t^{n-2q}(s, z_\blacktriangle) \\ &:= \frac{1}{(2\pi t)^{n+1}} \int_{-\infty}^{\infty} ds \Big(\frac{s}{\sinh s}\Big)^n \exp\Big(-is\frac{2z_0}{t} - \frac{|z_\blacktriangle|^2 s}{t\tanh s} - (n-2q)s\Big). \end{aligned}$$

Note that $\Phi_1^{n-2q}(s,0) = \Phi^{n-2q}(s)$.

We consider then the transformation from a neighborhood of the origin to a neighborhood of P^0

$$I_\varepsilon : \Omega^{p,q}H_n \cong \Omega^{p,q}(M, \theta_{(\varepsilon)}), \quad \sum \theta_H^{I\bar{K}}(z)\cdot\varphi^{I\bar{K}}(z) \mapsto \sum \theta_{(\varepsilon)}^{I\bar{K}}(z)\cdot\varphi^{I\bar{K}}(z),$$

which induces the differential operator $\square_{(\varepsilon)} = I_\varepsilon^* \square_H^{(\varepsilon)}$ $(:= I_\varepsilon^{-1}\circ\square_H^{(\varepsilon)}\circ I_\varepsilon)$ called the adiabatic Kohn-Rossi Laplacian at P^0. By setting $\nabla^{(H,\varepsilon)} = I_\varepsilon^*\nabla^{(\varepsilon)}$, $\xi_\bullet^{(\varepsilon)} = I_\varepsilon^*\xi_\bullet^{(\varepsilon)}$, etc., (1.1) provides the formula

$$\begin{aligned} \square_{(\varepsilon)} = &-\sum \Big(\nabla^{(H,\varepsilon)}_{\xi_\alpha^{(\varepsilon)}} \nabla^{(H,\varepsilon)}_{\xi_{\bar\alpha}^{(\varepsilon)}} - \nabla^{(H,\varepsilon)}_{\nabla^{(\varepsilon)}_{\xi_\alpha^{(\varepsilon)}}\xi_{\bar\alpha}^{(\varepsilon)}} \Big) - \sqrt{-1}\, q\, \nabla^{(H,\varepsilon)}_{\xi^{(\varepsilon)}} \\ &-\sum F(\nabla^{(\varepsilon)})^C_D(\xi_{\bar\alpha}^{(\varepsilon)}, \xi_\beta^{(\varepsilon)}) \cdot \theta_H^{\bar\alpha}\wedge\theta_H^{\bar\beta}\vee\theta_H^{\bar C}\wedge\theta_H^{\bar D}\vee \quad (\text{acting on } \Omega^{p,q}H_n), \end{aligned}$$

where

$$\begin{aligned} \nabla^{(H,\varepsilon)}_{\xi_A^{(\varepsilon)}} &= \xi_A^{(\varepsilon)} + \sum \varepsilon^{|A|_H/2}\omega_{\bar C}^{\bar B}(\xi_A)(\iota_\varepsilon(z))\cdot\theta_H^B\wedge\theta_H^C\vee, \\ \nabla^{(\varepsilon)}_{\xi_\alpha^{(\varepsilon)}}\xi_{\bar\alpha}^{(\varepsilon)} &= \sum \xi_{\bar\beta}^{(\varepsilon)}\cdot\varepsilon^{1/2}\omega_{\bar\alpha}^{\bar\beta}(\xi_\alpha)(\iota_\varepsilon(z)), \\ F(\nabla^{(\varepsilon)})^C_D(\xi_{\bar\alpha}^{(\varepsilon)}, \xi_\beta^{(\varepsilon)})(z) &= \varepsilon^{2/2}F(\nabla)^C_D(\xi_{\bar\alpha}, \xi_\beta)(\iota_\varepsilon(z)). \end{aligned}$$

Recalling (1.2), (1.4) and (1.5), we know that $\square_{(\varepsilon)}$ can be extended smoothly up to $\varepsilon^{1/2} = 0$ and has the formal series expansion

$$\square_{(\varepsilon)} = \sum_{m=0}^{\infty} \varepsilon^{m/2}\,\square_{m/2}, \quad \square_{0/2} = \mathbf{L}$$

called the **adiabatic expansion** of $\square_H$ at P^0, whose coefficients are described as

$$\square_{m/2} = \sum_{2+|\mathbb{C}|_H = |\mathbb{B}|_H + m}^{|\mathbb{B}|=0,1,2} \square_{m/2}(\mathbb{B},\mathbb{C})\cdot z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}}, \tag{1.6}$$

where, for $\mathbb{C} = (C_1, \dots, C_{|\mathbb{C}|})$, etc., we set $|\mathbb{C}|_H = \sum |C_j|_H$, $z^{\mathbb{C}} = z_{C_1}\cdots z_{C_{|\mathbb{C}|}}$ and $(\partial/\partial z)^{\mathbb{B}} = \partial/\partial z_{B_1}\cdots\partial/\partial z_{B_{|\mathbb{B}|}}$. Each $\square_{m/2}(\mathbb{B},\mathbb{C})$ is a finite sum of operators which are the composites of such operators as $\theta_H^\alpha\wedge\theta_H^\beta\vee$, $\theta_H^{\bar\gamma}\wedge\theta_H^{\bar\delta}\vee$ multiplied by constants. If we express its action as

$$\square_{m/2}(\mathbb{B},\mathbb{C})\,\theta_H^{I'\bar{K}'} = \sum_{|I|_H=|I'|_H}^{|K|_H=|K'|_H} \square_{m/2}^{(I\bar K)(I'\bar K')}(\mathbb{B};\mathbb{C})\cdot\theta_H^{I\bar K}, \tag{1.7}$$

then the coefficients $\square_{m/2}^{(I\bar{K})(I'\bar{K}')}(\mathbb{B};\mathbb{C})$ are all expressed as polynomials made of

$$(1.8)\qquad \begin{aligned} \mathcal{R}_{A_1A_2A_3A_4A_5\cdots A_\ell} &= \frac{\partial^{\ell-4} g(F(\nabla)((\partial/\partial z_{A_3},\partial/\partial z_{A_4})\partial/\partial z_{A_2},\partial/\partial z_{A_1})}{\partial z_{A_5}\cdots\partial z_{A_\ell}}(P^0),\\ \mathcal{T}_{A_1A_2A_3A_4\cdots A_\ell} &= \frac{\partial^{\ell-3} g(T(\nabla)(\partial/\partial z_{A_2},\partial/\partial z_{A_3}),\partial/\partial z_{A_1})}{\partial z_{A_4}\cdots\partial z_{A_\ell}}(P^0) \end{aligned}$$

and one can describe the polynomials explicitly up to an arbitrarily high order.

We regard the adiabatic series $\square_{0/2}$, $\square_{1/2}$, $\ldots$ naturally as differential operators defined on the whole space H_n. (Note that $\square_{(\varepsilon)}$ is well-defined as long as the point $\iota_\varepsilon(z)$ stays near the origin.) Now, on $[0,\varepsilon_0^{1/2}]\times(0,\infty)\times H_n\times H_n$, let us construct a formal power series

$$\mathfrak{p}_{(\varepsilon)}(t,z,z') = \sum_{m=0}^{\infty}\varepsilon^{m/2}\,\mathfrak{p}_{m/2}(t,z,z')$$

so as to satisfy $\left(\frac{\partial}{\partial t}+\square_{(\varepsilon)}\right)\mathfrak{p}_{(\varepsilon)}=0$. Namely, we define it inductively by

$$\begin{aligned} &\mathfrak{p}_{0/2}(t,z,z') = r_H(t,z,z'),\\ &\mathfrak{p}_{m/2}(t,z,z') = -\big(\mathfrak{p}_{0/2}\#\sum_{m_1+m_2=m}^{m_1>0}\square_{m_1/2}\mathfrak{p}_{m_2/2}\big)(t,z,z')\quad (m>0), \end{aligned}$$

where, in general, for double forms $h_i(t,z,z')$ $(i=1,2)$ on $H_n\times H_n$, we define the convolution $(h_1\#h_2)(t,z,z')$ by $(h_1\#h_2)(t,z,z')=\int_0^t ds\int_{H_n}h_1(t-s,z,z'')\wedge\star_{g^H}h_2(s,z'',z')$. Then, (by [4, Lemma 6.3]) the double forms $\mathfrak{p}_{m/2}(t,z,z') = \sum\theta_H^{I\bar{K}}(z)\boxtimes\theta_H^{\bar{I}'K'}(z')\cdot\mathfrak{p}_{m/2}^{(I\bar{K})(I'\bar{K}')}(t,z,z')$ are well-defined and smooth on $(0,\infty)\times H_n\times H_n$, and [4, (5.10)] asserts

$$(1.9)\qquad a_k(P^0)=\sum_{I,K}\mathfrak{p}_{2k/2}^{(I\bar{K})(I\bar{K})}(1,0,0).$$

(The value $\mathfrak{p}_{m/2}^{(I\bar{K})(I\bar{K})}(1,0,0)$ is equal to 0 if m is odd. Refer to [4, Theorem 5.3] for the generalized formula.) In addition, the adiabatic expansion of $\square_H$ readily implies

Proposition 1.1 (cf. [4, (5.12), (5.13)]) *Putting* $\mathbf{r}_H=\mathbf{r}_H(t,z,z')=r_t^{n-2q}(z'^{-1}z)$, *we have*

$$(1.10)\qquad \mathfrak{p}_{0/2}^{(I\bar{K})(I\bar{K})}(1,0,0)=\mathbf{r}_H(1,0,0)=\int_{-\infty}^{\infty}ds\,\Phi^{n-2q}(s),$$

$$\begin{aligned}(1.11)\qquad \mathfrak{p}_{2/2}^{(I\bar{K})(I\bar{K})}(1,0,0) = &-\sum\square_{2/2}^{(I\bar{K})(I\bar{K})}(\mathbb{C};\mathbb{B})\cdot\mathbf{r}(\mathbb{C};\mathbb{B})\\ &+\sum\square_{1/2}^{(I\bar{K})(I'\bar{K}')}(\mathbb{H};\mathbb{G})\,\square_{1/2}^{(I'\bar{K}')(I\bar{K})}(\mathbb{F};\mathbb{E})\cdot\mathbf{r}(\mathbb{H};\mathbb{G}:\mathbb{F};\mathbb{E}),\end{aligned}$$

where we set

$$\mathbf{r}(\mathbb{C};\mathbb{B})=\Big(\mathbf{r}_H\#z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}}\mathbf{r}_H\Big)(1,0,0),$$

$$\mathbf{r}(\mathbb{H};\mathbb{G}:\mathbb{F};\mathbb{E}) = \Big(\mathbf{r}_H \# z^{\mathbb{H}}(\partial/\partial z)^{\mathbb{G}}\mathbf{r}_H \# z^{\mathbb{F}}(\partial/\partial z)^{\mathbb{E}}\mathbf{r}_H\Big)(1,0,0).$$

(*For example, the index* $(\mathbb{C},\mathbb{B})$ *runs over the set of indices* $(\mathbb{C},\mathbb{B})$ *appearing in* (1.6) *and* (1.7) *with* $m/2 = 2/2$ *and* $(I\bar{K}) = (I'\bar{K}')$.)

2 Proof of Theorem 0.1

The asymptotic coefficient $a_k(P^0) = \sum \mathfrak{p}_{2k/2}^{(I\bar{K})(I\bar{K})}(1,0,0)$ has a universal polynomial expression made of (1.8), or the four types of components, $\mathcal{R}_{A_1A_2A_3A_4\cdots A_\ell}$, $\mathbf{T}_{A_1A_2\cdots A_\ell} := \mathcal{T}_{A_10A_2\cdots A_\ell}$,

$$\mathcal{N}_{A_1A_2A_3\cdots A_\ell} = \frac{1}{4}\frac{\partial^{\ell-3}g([J,J](\partial/\partial z_{A_2},\partial/\partial z_{A_3}),\partial/\partial z_{A_1})}{\partial z_{A_4}\cdots\partial z_{A_\ell}}(P^0),$$
$$\mathbf{Q}_{A_1A_2A_3\cdots A_\ell} = \frac{\partial^{\ell-4}g((\nabla_{\partial/\partial z_{A_4}}\mathcal{Q})(\partial/\partial z_{A_2},\partial/\partial z_{A_3}),\partial/\partial z_{A_1})}{\partial z_{A_5}\cdots\partial z_{A_\ell}}(P^0),$$

where $\mathcal{Q}$ is the Tanno tensor field, i.e., $\mathcal{Q}(X,Y) = ({}^*\nabla_Y J)(X)$ (refer to [7], [4, Lemma 1.2], which say that we could omit one of the last two types from the list). Note that $\mathcal{N}_{\alpha\beta\gamma} = \mathbf{Q}_{A_1A_2A_3\cdots A_\ell} = 0$ if J is integrable.

Proposition 2.1 (cf. [4, Proposition 7.1]) *We have*

$$\begin{aligned}
\mathfrak{p}_{2/2}^{(I\bar{K})(I\bar{K})}(1,0,0) &= \sum_{\alpha\in K}^{\beta\notin I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} \\
&+ \Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\}\Big\{\frac{1}{2} + \int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Phi_1(s)\Big\} \\
&+ \mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big\{-\frac{1}{12} + \frac{4}{3}\Phi_2(s)\Big\} \\
&+ \mathcal{N}_{\alpha\beta\gamma}\mathcal{N}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big\{\frac{\Phi_1(s)-\Phi_2(s)}{12} + \frac{3\Phi_2(s)}{16s^2} - \frac{5}{48}\Big\}.
\end{aligned}$$

If this is valid, then

$$\begin{aligned}
a_1(P^0) = \sum \mathfrak{p}_{2/2}^{(I\bar{K})(I\bar{K})}(1,0,0) &= \binom{n-1}{q-1}\binom{n-1}{p}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} \\
&+ \Big\{\binom{n-1}{p-1}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \binom{n-1}{q-1}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\}\Big\{\frac{1}{2} + \int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Phi_1(s)\Big\} \\
&+ \binom{n}{q}\binom{n}{p}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big\{-\frac{1}{12} + \frac{4}{3}\Phi_2(s)\Big\} \\
&+ \mathcal{N}_{\alpha\beta\gamma}\mathcal{N}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big\{\frac{\Phi_1(s)-\Phi_2(s)}{12} + \frac{3\Phi_2(s)}{16s^2} - \frac{5}{48}\Big\}.
\end{aligned}$$

Thus we obtain (0.3). The purpose in the following is, hence, to prove Proposition 2.1.

Lemma 2.2 (cf. [4, Lemma 7.2]) *Computing* (1.4), *etc., we have*

$$
\begin{aligned}
\theta = dz_0 \cdot & \left\{ 1 + z_\alpha z_\gamma \frac{-i\mathbf{T}_{\alpha\gamma}}{6} + z_{\bar\alpha} z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\gamma\bar\alpha}}{6} + \mathcal{O}\left(|z|^3\right) \right\} \\
+ dz_\beta \cdot & \left\{ z_{\bar\beta} \frac{-i}{2} + z_0 z_\alpha \frac{i\mathbf{T}_{\alpha\beta}}{6} + z_{\bar\beta} z_\alpha z_\gamma \frac{-\mathbf{T}_{\alpha\gamma}}{24} + z_{\bar\alpha} z_{\bar\gamma} z_{\bar\beta} \frac{\mathbf{T}_{\bar\alpha\bar\gamma}}{24} + z_0^2 z_{\bar\alpha} \frac{-i\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\gamma\beta}}{24} \right. \\
& + z_A z_{\bar\alpha} z_\gamma \frac{-i\mathcal{R}_{\bar\alpha\gamma A\beta}}{12} + z_0 z_A z_\alpha \frac{i\mathbf{T}_{\alpha\beta A}}{12} + z_0 z_\gamma z_\alpha \frac{i\mathcal{N}_{\alpha\mu\beta}\mathbf{T}_{\bar\mu\bar\gamma}}{24} + z_\alpha z_\gamma \frac{i\mathcal{N}_{\alpha\gamma\beta}}{6} \\
& + z_0 z_{\bar\alpha} z_{\bar\gamma} \frac{-i\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}\mathbf{T}_{\lambda\beta}}{24} + z_0 z_{\bar\alpha} z_\lambda \frac{-i\mathcal{N}_{\gamma\lambda\beta}\mathbf{T}_{\bar\alpha\bar\gamma}}{24} + z_{\bar\alpha} z_{\bar\gamma} z_\mu \frac{-i\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}\mathcal{N}_{\lambda\mu\beta}}{24} \\
& \left. + z_A z_\alpha z_\gamma \frac{i\mathcal{N}_{\alpha\gamma\beta A}}{12} + z_{\bar\gamma} z_\alpha z_\gamma \frac{-\mathbf{T}_{\alpha\beta}}{48} + z_{\bar\beta} z_\alpha z_\gamma \frac{\mathbf{T}_{\alpha\gamma}}{48} + \mathcal{O}\left(|z|^4\right) \right\} \\
+ dz_{\bar\beta} \cdot & \left\{ z_\beta \frac{i}{2} + z_0 z_{\bar\alpha} \frac{-i\mathbf{T}_{\bar\alpha\bar\beta}}{6} + z_\alpha z_\gamma z_\beta \frac{\mathbf{T}_{\alpha\gamma}}{24} + z_{\bar\alpha} z_{\bar\gamma} z_\beta \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}}{24} + z_0^2 z_\alpha \frac{i\mathbf{T}_{\bar\gamma\bar\beta}\mathbf{T}_{\alpha\gamma}}{24} \right. \\
& + z_A z_{\bar\alpha} z_\gamma \frac{i\mathcal{R}_{\bar\alpha\gamma\bar\beta A}}{12} + z_0 z_{\bar\alpha} z_A \frac{-i\mathbf{T}_{\bar\alpha\bar\beta A}}{12} + z_0 z_{\bar\alpha} z_{\bar\lambda} \frac{-i\mathcal{N}_{\bar\alpha\bar\mu\bar\beta}\mathbf{T}_{\mu\lambda}}{24} + z_{\bar\alpha} z_{\bar\gamma} \frac{-i\mathcal{N}_{\bar\alpha\bar\gamma\bar\beta}}{6} \\
& + z_0 z_\alpha z_\gamma \frac{i\mathcal{N}_{\alpha\gamma\lambda}\mathbf{T}_{\bar\lambda\bar\beta}}{24} + z_0 z_{\bar\lambda} z_\alpha \frac{i\mathcal{N}_{\bar\gamma\bar\lambda\bar\beta}\mathbf{T}_{\alpha\gamma}}{24} + z_{\bar\mu} z_\alpha z_\gamma \frac{i\mathcal{N}_{\bar\lambda\bar\mu\bar\beta}\mathcal{N}_{\alpha\gamma\lambda}}{24} \\
& \left. + z_{\bar\alpha} z_{\bar\gamma} z_A \frac{-i\mathcal{N}_{\bar\alpha\bar\gamma\bar\beta A}}{12} + z_{\bar\alpha} z_{\bar\gamma} z_\gamma \frac{-\mathbf{T}_{\bar\alpha\bar\beta})}{48} + z_{\bar\alpha} z_{\bar\gamma} z_\beta \frac{\mathbf{T}_{\bar\alpha\bar\gamma}}{48} + \mathcal{O}\left(|z|^4\right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
\theta^\alpha = dz_0 \cdot & \left\{ z_{\bar\gamma} \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}}{2} + z_0 z_\lambda \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\gamma\lambda}}{6} + z_{\bar\gamma} z_A \frac{-\mathbf{T}_{\bar\alpha\bar\gamma A}}{3} + z_{\bar\gamma} z_{\bar\lambda} \frac{-\mathcal{N}_{\bar\alpha\bar\beta\bar\gamma}\mathbf{T}_{\beta\lambda}}{6} \right. \\
& \left. + z_\gamma z_A \frac{\mathcal{R}_{\bar\alpha\gamma A0}}{6} + z_{\bar\gamma} z_\mu \frac{-\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}\mathbf{T}_{\lambda\mu}}{6} \right\} \\
+ dz_\beta \cdot & \left\{ \delta_{\alpha\beta} + z_{\bar\gamma} z_{\bar\beta} \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_0^2 \frac{\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\gamma\beta}}{6} + z_A z_\gamma \frac{\mathcal{R}_{\bar\alpha\gamma A\beta}}{6} \right. \\
& \left. + z_0 z_\lambda \frac{\mathcal{N}_{\gamma\lambda\beta}\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_0 z_{\bar\gamma} \frac{\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}\mathbf{T}_{\lambda\beta}}{6} + z_{\bar\gamma} z_\mu \frac{\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}\mathcal{N}_{\lambda\mu\beta}}{6} \right\} \\
+ dz_{\bar\beta} \cdot & \left\{ z_0 \frac{\mathbf{T}_{\bar\alpha\bar\beta}}{2} + z_{\bar\gamma} z_\beta \frac{-i\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_0 z_A \frac{\mathbf{T}_{\bar\alpha\bar\beta A}}{3} + z_0 z_{\bar\gamma} \frac{\mathcal{N}_{\bar\alpha\bar\lambda\bar\beta}\mathbf{T}_{\lambda\gamma}}{6} + z_A z_\gamma \frac{-\mathcal{R}_{\bar\alpha\gamma\bar\beta A}}{6} \right. \\
& \left. + z_{\bar\gamma} \frac{\mathcal{N}_{\bar\alpha\bar\gamma\bar\beta}}{2} + z_{\bar\gamma} z_A \frac{\mathcal{N}_{\bar\alpha\bar\gamma\bar\beta A}}{3} + z_{\bar\gamma} z_\gamma \frac{-i\mathbf{T}_{\bar\alpha\bar\beta}}{12} + z_{\bar\gamma} z_\beta \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{12} \right\} + \mathcal{O}\left(|z|^3\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\xi = \partial/\partial z_0 \cdot & \left\{ 1 + z_\alpha z_\gamma \frac{-i\mathbf{T}_{\alpha\gamma}}{12} + z_{\bar\alpha} z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{12} \right\} \\
+ \partial/\partial z_\beta \cdot & \left\{ z_{\bar\alpha} \frac{\mathbf{T}_{\bar\beta\bar\alpha}}{2} + z_0 z_\gamma \frac{-\mathbf{T}_{\bar\beta\bar\alpha}\mathbf{T}_{\alpha\gamma}}{12} + z_{\bar\alpha} z_A \frac{\mathbf{T}_{\bar\beta\bar\alpha A}}{3} + z_{\bar\alpha} z_{\bar\gamma} \frac{\mathcal{N}_{\bar\beta\bar\lambda\bar\alpha}\mathbf{T}_{\lambda\gamma}}{6} \right. \\
& \left. + z_\alpha z_A \frac{-\mathcal{R}_{\bar\beta\alpha A0}}{6} + z_{\bar\lambda} z_\gamma \frac{-\mathcal{N}_{\bar\beta\bar\lambda\bar\alpha}\mathbf{T}_{\alpha\gamma}}{12} \right\} \\
+ \partial/\partial z_{\bar\beta} \cdot & \left\{ z_\alpha \frac{\mathbf{T}_{\beta\alpha}}{2} + z_0 z_{\bar\gamma} \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\beta\alpha}}{12} + z_A z_\alpha \frac{\mathbf{T}_{\beta\alpha A}}{3} + z_\lambda z_\alpha \frac{\mathcal{N}_{\beta\gamma\alpha}\mathbf{T}_{\bar\gamma\bar\lambda}}{6} \right. \\
& \left. + z_{\bar\alpha} z_A \frac{\mathcal{R}_{\bar\alpha\beta A0}}{6} + z_{\bar\gamma} z_\lambda \frac{-\mathcal{N}_{\beta\lambda\alpha}\mathbf{T}_{\bar\alpha\bar\gamma}}{12} \right\} + \mathcal{O}\left(|z|^3\right),
\end{aligned}
$$

$$
\begin{aligned}
\xi_\beta = \partial/\partial z_0 \cdot & \left\{ z_{\bar\beta} \frac{i}{2} + z_0 z_\gamma \frac{i\mathbf{T}_{\gamma\beta}}{12} + z_0 z_A z_\gamma \frac{i\mathbf{T}_{\gamma\beta A}}{12} + z_0 z_\mu z_\gamma \frac{i\mathcal{N}_{\gamma\lambda\beta}\mathbf{T}_{\bar\lambda\bar\mu}}{24} \right. \\
& + z_A z_{\bar\lambda} z_\gamma \frac{-i\mathcal{R}_{\bar\lambda\gamma A\beta}}{12} + z_\gamma z_\lambda \frac{i\mathcal{N}_{\gamma\lambda\beta}}{12} + z_A z_\gamma z_\lambda \frac{i\mathcal{N}_{\gamma\lambda\beta A}}{12}
\end{aligned}
$$

$$
\begin{aligned}
&+z_{\bar\lambda}z_\gamma z_\lambda\frac{-\mathbf{T}_{\gamma\beta}}{48}+z_{\bar\beta}z_\gamma z_\lambda\frac{\mathbf{T}_{\gamma\lambda}}{48}+\mathcal{O}\left(|z|^4\right)\Big\}\\
+&\partial/\partial z_\alpha\cdot\Big\{\delta_{\beta\alpha}+z_{\bar\beta}z_{\bar\gamma}\frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{12}+z_0^2\frac{\mathbf{T}_{\gamma\beta}\mathbf{T}_{\bar\alpha\bar\gamma}}{12}+z_Az_\gamma\frac{-\mathcal{R}_{\bar\alpha\gamma A\beta}}{6}\\
&+z_0z_{\bar\lambda}\frac{\mathcal{N}_{\bar\alpha\bar\lambda\bar\gamma}\mathbf{T}_{\gamma\beta}}{12}+z_0z_\lambda\frac{\mathcal{N}_{\gamma\lambda\beta}\mathbf{T}_{\bar\alpha\bar\gamma}}{12}+z_{\bar\gamma}z_\mu\frac{\mathcal{N}_{\lambda\mu\beta}\mathcal{N}_{\bar\alpha\bar\gamma\bar\lambda}}{12}+\mathcal{O}\left(|z|^3\right)\Big\}\\
+&\partial/\partial z_{\bar\alpha}\cdot\Big\{z_0\frac{-\mathbf{T}_{\alpha\beta}}{2}+z_{\bar\beta}z_\gamma\frac{i\mathbf{T}_{\alpha\gamma}}{12}+z_0z_A\frac{-\mathbf{T}_{\alpha\beta A}}{3}+z_0z_\lambda\frac{-\mathcal{N}_{\alpha\gamma\beta}\mathbf{T}_{\bar\gamma\bar\lambda}}{6}\\
&+z_Az_{\bar\gamma}\frac{\mathcal{R}_{\bar\gamma\alpha A\beta}}{6}+z_\gamma\frac{-\mathcal{N}_{\alpha\gamma\beta}}{2}+z_Az_\gamma\frac{-\mathcal{N}_{\alpha\gamma\beta A}}{3}\\
&+z_{\bar\gamma}z_\gamma\frac{-i\mathbf{T}_{\alpha\beta}}{12}+z_{\bar\beta}z_\gamma\frac{i\mathbf{T}_{\alpha\gamma}}{12}+\mathcal{O}\left(|z|^3\right)\Big\}.
\end{aligned}
$$

In addition, we have

$$\mathcal{R}_{\bar\alpha\beta\bar\gamma\bar\lambda}=-i\mathbf{T}_{\bar\alpha\bar\gamma}\delta_{\lambda\beta}+i\mathbf{T}_{\bar\alpha\bar\lambda}\delta_{\gamma\beta}+\frac{-i\mathbf{Q}_{\bar\gamma\bar\lambda\bar\alpha\beta}}{2}.$$

Proposition 2.3 (cf. [4, Corollary 7.3]) *We have*

$$
\begin{aligned}
(2.1)\ \Box_{1/2}=&\Big\{z_{\bar\gamma}\frac{\mathcal{N}_{\bar\beta\bar\gamma\bar\alpha}}{2}\Big\}\partial/\partial z_\alpha\partial/\partial z_\beta+\Big\{z_\gamma\frac{\mathcal{N}_{\beta\gamma\alpha}}{2}\Big\}\partial/\partial z_{\bar\alpha}\partial/\partial z_{\bar\beta}\\
&+\Big\{z_{\bar\beta}z_{\bar\gamma}\frac{i\mathcal{N}_{\bar\beta\bar\gamma\bar\alpha}}{12}\Big\}\partial/\partial z_0\partial/\partial z_\alpha+\Big\{z_\beta z_\gamma\frac{-i\mathcal{N}_{\beta\gamma\alpha}}{12}\Big\}\partial/\partial z_0\partial/\partial z_{\bar\alpha},
\end{aligned}
$$

$$
\begin{aligned}
(2.2)\ \Box_{2/2}=&\Big\{z_0\frac{\mathbf{T}_{\bar\beta\bar\alpha}}{2}+z_{\bar\gamma}z_\gamma\frac{i\mathbf{T}_{\bar\beta\bar\alpha}}{6}+z_{\bar\gamma}z_\alpha\frac{-i\mathbf{T}_{\bar\beta\bar\gamma}}{12}+z_\gamma z_\lambda\frac{-\mathcal{R}_{\bar\beta\gamma\bar\alpha\lambda}}{6}\\
&+z_{\bar\gamma}z_b\frac{\mathcal{N}_{\bar\beta\bar\gamma\bar\alpha b}}{3}+z_{\bar\gamma}z_\gamma\frac{-i\mathbf{T}_{\bar\beta\bar\alpha}}{12}+z_{\bar\gamma}z_\alpha\frac{i\mathbf{T}_{\bar\beta\bar\gamma}}{12}+z_{\bar\lambda}z_\gamma\frac{i\mathbf{Q}_{\bar\alpha\bar\lambda\bar\beta\gamma}}{12}\Big\}\partial/\partial z_\alpha\partial/\partial z_\beta\\
+&\Big\{z_0\frac{\mathbf{T}_{\alpha\beta}}{2}+z_{\bar\gamma}z_\gamma\frac{-i\mathbf{T}_{\alpha\beta}}{6}+z_{\bar\beta}z_\gamma\frac{i\mathbf{T}_{\alpha\gamma}}{12}+z_{\bar\gamma}z_{\bar\lambda}\frac{-\mathcal{R}_{\bar\gamma\alpha\bar\lambda\beta}}{6}\\
&+z_bz_\gamma\frac{\mathcal{N}_{\alpha\gamma\beta b}}{3}+z_{\bar\gamma}z_\gamma\frac{i\mathbf{T}_{\alpha\beta}}{12}+z_{\bar\beta}z_\gamma\frac{-i\mathbf{T}_{\alpha\gamma}}{12}+z_{\bar\gamma}z_\lambda\frac{i\mathbf{Q}_{\lambda\beta\alpha\bar\gamma}}{12}\Big\}\partial/\partial z_{\bar\alpha}\partial/\partial z_{\bar\beta}\\
+&\Big\{z_{\bar\alpha}z_{\bar\gamma}\frac{-i\mathbf{T}_{\bar\beta\bar\gamma}}{4}+z_\beta z_\gamma\frac{i\mathbf{T}_{\alpha\gamma}}{4}+\delta_{\alpha\beta}\cdot z_\lambda z_\gamma\frac{-i\mathbf{T}_{\lambda\gamma}}{6}+\delta_{\alpha\beta}\cdot z_{\bar\lambda}z_{\bar\gamma}\frac{i\mathbf{T}_{\bar\lambda\bar\gamma}}{6}\\
&+z_{\bar\lambda}z_\gamma\frac{\mathcal{R}_{\bar\beta\gamma\bar\lambda\alpha}}{3}+z_{\bar\gamma}z_\mu\frac{-\mathcal{N}_{\bar\beta\bar\gamma\bar\lambda}\mathcal{N}_{\lambda\mu\alpha}}{12}+z_{\bar\gamma}z_\mu\frac{-\mathcal{N}_{\bar\lambda\bar\beta\bar\gamma}\mathcal{N}_{\alpha\lambda\mu}}{12}+z_{\bar\gamma}z_\mu\frac{\mathcal{N}_{\bar\beta\bar\gamma\bar\lambda}\mathcal{N}_{\alpha\lambda\mu}}{4}\\
&+z_\gamma z_\lambda\frac{-i\mathbf{Q}_{\lambda\alpha\gamma\bar\beta}}{12}+z_{\bar\gamma}z_{\bar\lambda}\frac{-i\mathbf{Q}_{\bar\beta\bar\lambda\bar\gamma\alpha}}{12}\Big\}\partial/\partial z_{\bar\alpha}\partial/\partial z_\beta\\
+&\Big\{z_0z_{\bar\beta}\frac{i\mathbf{T}_{\bar\alpha\bar\beta}}{3}+z_{\bar\beta}z_{\bar\gamma}z_\beta\frac{-\mathbf{T}_{\bar\gamma\bar\alpha}}{6}+z_{\bar\beta}z_{\bar\gamma}z_\alpha\frac{\mathbf{T}_{\bar\beta\bar\gamma}}{12}+z_{\bar\gamma}z_\beta z_\lambda\frac{-i\mathcal{R}_{\bar\gamma\beta\bar\alpha\lambda}}{4}\\
&+z_{\bar\beta}z_{\bar\gamma}z_b\frac{i\mathcal{N}_{\bar\beta\bar\gamma\bar\alpha b}}{12}+z_{\bar\beta}z_{\bar\gamma}z_\gamma\frac{\mathbf{T}_{\bar\beta\bar\alpha}}{48}+z_{\bar\beta}z_{\bar\gamma}z_\alpha\frac{-\mathbf{T}_{\bar\beta\bar\gamma}}{48}+z_{\bar\lambda}z_\beta z_\gamma\frac{i\mathcal{N}_{\bar\alpha\bar\lambda\bar\mu}\mathcal{N}_{\beta\gamma\mu}}{24}\\
&+z_\beta z_\gamma z_\lambda\frac{-\mathbf{Q}_{\gamma\lambda\beta\bar\alpha}}{24}+z_{\bar\gamma}z_{\bar\lambda}z_\beta\frac{-\mathbf{Q}_{\bar\lambda\bar\gamma\bar\alpha\beta}}{24}+z_{\bar\gamma}z_{\bar\lambda}z_\beta\frac{-\mathbf{Q}_{\bar\alpha\bar\lambda\bar\gamma\beta}}{24}\Big\}\partial/\partial z_0\partial/\partial z_\alpha\\
+&\Big\{z_0z_\beta\frac{-i\mathbf{T}_{\beta\alpha}}{3}+z_{\bar\beta}z_\beta z_\gamma\frac{-\mathbf{T}_{\gamma\alpha}}{6}+z_{\bar\alpha}z_\beta z_\gamma\frac{\mathbf{T}_{\beta\gamma}}{12}+z_{\bar\gamma}z_{\bar\lambda}z_\beta\frac{i\mathcal{R}_{\bar\gamma\beta\bar\lambda\alpha}}{4}\\
&+z_bz_\beta z_\gamma\frac{-i\mathcal{N}_{\beta\gamma\alpha b}}{12}+z_{\bar\gamma}z_\beta z_\gamma\frac{\mathbf{T}_{\beta\alpha}}{48}+z_{\bar\alpha}z_\beta z_\gamma\frac{-\mathbf{T}_{\beta\gamma}}{48}+z_{\bar\gamma}z_{\bar\lambda}z_\beta\frac{-i\mathcal{N}_{\bar\gamma\bar\lambda\bar\mu}\mathcal{N}_{\alpha\beta\mu}}{24}
\end{aligned}
$$

$$
\begin{aligned}
&+z_{\bar\beta}z_\gamma z_\lambda\frac{\mathbf{Q}_{\gamma\lambda\alpha\bar\beta}}{24}+z_{\bar\beta}z_\gamma z_\lambda\frac{\mathbf{Q}_{\lambda\alpha\gamma\bar\beta}}{24}+z_{\bar\beta}z_{\bar\gamma}z_{\bar\lambda}\frac{\mathbf{Q}_{\bar\lambda\bar\gamma\bar\beta\alpha}}{24}\Big\}\partial/\partial z_0\partial/\partial z_{\bar\alpha}\\
+\Big\{&z_0z_\alpha z_\beta\frac{-\mathbf{T}_{\alpha\beta}}{24}+z_0z_{\bar\alpha}z_{\bar\beta}\frac{-\mathbf{T}_{\bar\alpha\bar\beta}}{24}+z_{\bar\beta}z_{\bar\gamma}z_\alpha z_\lambda\frac{\mathcal{R}_{\bar\beta\alpha\bar\gamma\lambda}}{12}+z_{\bar\gamma}z_{\bar\lambda}z_\alpha z_\beta\frac{-\mathcal{N}_{\bar\gamma\bar\lambda\bar\mu}\mathcal{N}_{\alpha\beta\mu}}{144}\\
&+z_{\bar\alpha}z_\beta z_\gamma z_\lambda\frac{-i\mathbf{Q}_{\gamma\lambda\beta\bar\alpha}}{48}+z_{\bar\beta}z_{\bar\gamma}z_{\bar\lambda}z_\alpha\frac{-i\mathbf{Q}_{\bar\lambda\bar\gamma\bar\beta\alpha}}{48}\Big\}\partial/\partial z_0\partial/\partial z_0\\
+\Big\{&z_{\bar\nu}\frac{i(5-n-6q)\mathbf{T}_{\bar\alpha\bar\nu}}{12}+z_\nu\frac{-\mathcal{R}_{\bar\alpha\nu\bar\mu\mu}}{3}+z_{\bar\nu}\frac{\mathcal{N}_{\bar\alpha\bar\nu\bar\mu\mu}}{3}+z_{\bar\mu}\frac{-i\mathbf{T}_{\bar\alpha\bar\mu}}{12}+z_{\bar\nu}\frac{in\mathbf{T}_{\bar\alpha\bar\nu}}{12}\\
&+z_\mu\frac{-\mathcal{N}_{\bar\alpha\bar\nu\bar\beta}\mathcal{N}_{\nu\mu\beta}}{4}+z_{\bar\mu}\frac{i\mathbf{Q}_{\bar\nu\bar\mu\bar\alpha\nu}}{12}\\
&+\Big(z_{\bar\mu}\frac{-i\mathbf{T}_{\bar\nu\bar\alpha}}{2}+\delta_{\alpha\mu}\cdot z_{\bar\beta}\frac{i\mathbf{T}_{\bar\nu\bar\beta}}{2}+z_\beta\frac{\mathcal{R}_{\bar\nu\mu\bar\alpha\beta}}{2}+z_{\bar\beta}\frac{-i\mathbf{Q}_{\bar\alpha\bar\beta\bar\nu\mu}}{4}\Big)\theta_H^{\bar\nu}\wedge\theta_H^{\bar\mu}\vee\\
&+\Big(z_{\bar\nu}\frac{i\mathbf{T}_{\bar\mu\bar\alpha}}{2}+\delta_{\alpha\nu}\cdot z_{\bar\beta}\frac{-i\mathbf{T}_{\bar\mu\bar\beta}}{2}+z_\beta\frac{-\mathcal{R}_{\bar\mu\nu\bar\alpha\beta}}{2}+z_{\bar\beta}\frac{i\mathbf{Q}_{\bar\alpha\bar\beta\bar\mu\nu}}{4}\Big)\theta_H^{\nu}\wedge\theta_H^{\mu}\vee\Big\}\partial/\partial z_\alpha\\
+\Big\{&z_\nu\frac{i(-5+7n-6q)\mathbf{T}_{\alpha\nu}}{12}+z_{\bar\nu}\frac{-\mathcal{R}_{\bar\nu\alpha\bar\mu\mu}}{3}+z_{\bar\mu}\frac{-\mathcal{N}_{\bar\nu\bar\mu\bar\beta}\mathcal{N}_{\alpha\beta\nu}}{12}+z_\mu\frac{-i\mathbf{Q}_{\nu\mu\alpha\bar\nu}}{4}\\
&+\Big(z_\nu\frac{-i\mathbf{T}_{\mu\alpha}}{2}+\delta_{\alpha\nu}\cdot z_\beta\frac{i\mathbf{T}_{\mu\beta}}{2}+z_{\bar\beta}\frac{-\mathcal{R}_{\bar\nu\mu\bar\beta\alpha}}{2}+z_\beta\frac{-i\mathbf{Q}_{\alpha\beta\mu\bar\nu}}{4}\Big)\theta_H^{\bar\nu}\wedge\theta_H^{\bar\mu}\vee\\
&+\Big(z_\mu\frac{i\mathbf{T}_{\nu\alpha}}{2}+\delta_{\alpha\mu}\cdot z_\beta\frac{-i\mathbf{T}_{\nu\beta}}{2}+z_{\bar\beta}\frac{\mathcal{R}_{\bar\mu\nu\bar\beta\alpha}}{2}+z_\beta\frac{i\mathbf{Q}_{\alpha\beta\nu\bar\mu}}{4}\Big)\theta_H^{\nu}\wedge\theta_H^{\mu}\vee\Big\}\partial/\partial z_{\bar\alpha}\\
+\Big\{&z_\nu z_\mu\frac{(-2+2n-q)\mathbf{T}_{\nu\mu}}{12}+z_{\bar\nu}z_{\bar\mu}\frac{(-2+n+q)\mathbf{T}_{\bar\nu\bar\mu}}{12}\\
&+z_{\bar\nu}z_{\bar\mu}\frac{i\mathcal{N}_{\bar\nu\bar\mu\bar\alpha\alpha}}{12}+z_{\bar\nu}z_{\bar\alpha}\frac{\mathbf{T}_{\bar\nu\bar\alpha}}{48}+z_{\bar\nu}z_{\bar\mu}\frac{-n\mathbf{T}_{\bar\nu\bar\mu}}{48}\\
&+z_{\bar\alpha}z_\mu\frac{-i\mathcal{N}_{\bar\nu\bar\alpha\bar\beta}\mathcal{N}_{\beta\nu\mu}}{24}+z_{\bar\alpha}z_\mu\frac{i\mathcal{N}_{\bar\alpha\bar\beta\bar\nu}\mathcal{N}_{\nu\mu\beta}}{24}+z_{\bar\alpha}z_\mu\frac{-i\mathcal{N}_{\bar\nu\bar\alpha\bar\beta}\mathcal{N}_{\nu\mu\beta}}{24}\\
&+z_\mu z_\alpha\frac{-\mathbf{Q}_{\alpha\nu\mu\bar\nu}}{24}+z_\mu z_\alpha\frac{-\mathbf{Q}_{\nu\mu\alpha\bar\nu}}{8}+z_{\bar\mu}z_{\bar\alpha}\frac{-\mathbf{Q}_{\bar\nu\bar\alpha\bar\mu\nu}}{24}\\
&+\Big(z_{\bar\beta}z_\alpha\frac{i\mathcal{R}_{\bar\nu\mu\bar\beta\alpha}}{2}+z_{\bar\alpha}z_{\bar\beta}\frac{\mathbf{Q}_{\bar\beta\bar\alpha\bar\nu\mu}}{8}+z_\alpha z_\beta\frac{-\mathbf{Q}_{\alpha\beta\mu\bar\nu}}{8}\Big)\theta_H^{\bar\nu}\wedge\theta_H^{\bar\mu}\vee\\
&+\Big(z_{\bar\beta}z_\alpha\frac{-i\mathcal{R}_{\bar\mu\nu\bar\beta\alpha}}{2}+z_{\bar\alpha}z_{\bar\beta}\frac{-\mathbf{Q}_{\bar\beta\bar\alpha\bar\mu\nu}}{8}+z_\alpha z_\beta\frac{\mathbf{Q}_{\alpha\beta\nu\bar\mu}}{8}\Big)\theta_H^{\nu}\wedge\theta_H^{\mu}\vee\Big\}\partial/\partial z_0\\
+\Big\{&\frac{\mathcal{R}_{\bar\alpha\beta\bar\nu\nu}}{2}\theta_H^{\bar\alpha}\wedge\theta_H^{\bar\beta}\vee+\frac{-\mathcal{R}_{\bar\beta\alpha\bar\nu\nu}}{2}\theta_H^{\alpha}\wedge\theta_H^{\beta}\vee\\
&-\mathcal{R}_{\bar\nu\mu\bar\alpha\beta}\theta_H^{\bar\alpha}\wedge\theta_H^{\bar\beta}\vee\theta_H^{\bar\nu}\wedge\theta_H^{\bar\mu}\vee+\mathcal{R}_{\bar\mu\nu\bar\alpha\beta}\theta_H^{\bar\alpha}\wedge\theta_H^{\bar\beta}\vee\theta_H^{\nu}\wedge\theta_H^{\mu}\vee\Big\},
\end{aligned}
$$

where the small Latin index b *varies only in* $\{1,\ldots,n,\bar 1,\ldots,\bar n\}$.

All of the above results are obtained with the help of Mathematica: on an Intel Core i7 3.40GHz processor the computations took about 15 seconds in total. Since the four types of components bear various intrinsic relations to each other ([4, Lemma 1.2]), the universal expressions are not uniquely determined. Some difficulty in designing the program lies in the fact.

Lemma 2.4 (cf. [4, (7.6)]) *We have*

$$a_{2/2}^{(I\bar K)(I\bar K)}(P^0)=-\sum\Box_{2/2}^{(I\bar K)(I\bar K)}(\mathbb{C};\mathbb{B})\cdot\mathbf{r}(\mathbb{C};\mathbb{B}) \tag{2.3}$$

$$
\begin{aligned}
&= \mathcal{R}_{\bar\alpha\alpha\bar\alpha\alpha}\Big\{\frac{-\mathbf{r}(\bar1\bar111;00)}{12}+\frac{\mathbf{r}(\bar1\bar212;00)}{6}\Big\}+\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\{\frac{\mathbf{r}(1;1)}{3}+\frac{-\mathbf{r}(\bar1\bar212;00)}{6}\Big\}\\
&+\Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}-\sum_{\alpha\in K}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\}\frac{i\mathbf{r}(\bar11;0)}{2}+\frac{1}{2}\Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}-\sum_{\alpha\in K}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\}\\
&+\sum_{\alpha\in K}^{\beta\notin I}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}+\mathcal{N}_{\alpha\alpha\beta}\mathcal{N}_{\bar\alpha\bar\alpha\bar\beta}\Big\{\frac{\mathbf{r}(\bar11;\bar11)}{4}+\frac{-\mathbf{r}(\bar11;\bar22)}{4}+\frac{-\mathbf{r}(\bar12;\bar12)}{4}\\
&\qquad\qquad+\frac{\mathbf{r}(\bar1\bar111;00)}{144}+\frac{-\mathbf{r}(\bar1\bar212;00)}{72}\Big\}\\
&+\mathcal{N}_{\alpha\beta\gamma}\mathcal{N}_{\bar\alpha\bar\beta\bar\gamma}\Big\{\frac{\mathbf{r}(1;1)}{12}+\frac{i\mathbf{r}(\bar11;0)}{24}+\frac{\mathbf{r}(\bar11;\bar22)}{3}+\frac{\mathbf{r}(\bar12;\bar12)}{24}+\frac{\mathbf{r}(\bar1\bar212;00)}{96}\Big\},
\end{aligned}
$$

where, for example, $\mathbf{r}(\bar1\bar111;00)$ *denotes* $\mathbf{r}(\mathbb{C};\mathbb{B})$ *with* $\mathbb{C}=(\bar1,\bar1,1,1)$ *and* $\mathbb{B}=(0,0)$.

Proof. We have $\Box_{1/2}p_{0/2}(t,z,0)=0$ because of (2.1) and the formula $\mathcal{N}_{\alpha\beta\gamma}+\mathcal{N}_{\beta\gamma\alpha}+\mathcal{N}_{\gamma\alpha\beta}=0$. Indeed, for example, $\{z_\gamma\frac{\mathcal{N}_{\beta\gamma\alpha}}{2}\}\partial/\partial z_{\bar\alpha}\partial/\partial z_{\bar\beta}\,\mathbf{r}_H(t,z,0)$ is equal to

$$
\frac{\mathcal{N}_{\beta\gamma\alpha}}{2}\,\partial/\partial z_{\bar\alpha}\partial/\partial z_{\bar\beta}\partial/\partial z_{\bar\gamma}\int_{-\infty}^{\infty}ds\,e^{-is\cdot(2z_0/t)}\Big(-\frac{t\tanh s}{s}\Big)\Phi_t^{n-2q}(s,z_\blacktriangle)=0.
$$

Hence the second term in the right hand side of (1.11) vanishes, so that we obtain the first equality at (2.3). The second equality can be shown by using (2.2) in the same way as the proof of [4, (7.6)]. ∎

Proof of Proposition 2.1. It suffices to calculate $\mathbf{r}(\bar1\bar111;00)$, etc., appearing in (2.3). Some of them have already been calculated in the proof of [4, Proposition 7.1]. The remaining terms can be calculated similarly. ∎

References

[1] S. Dragomir and G. Tomassini, Differential geometry and analysis on CR manifolds, Progress in Math. **246**, Birkhäuser, Boston-Basel-Stuttgart, 2006.

[2] G. B. Folland and E. M. Stein, Estimates for the $\bar\partial_b$ complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. **27**(1974), 429–522.

[3] M. Nagase, Twistor spaces and the general adiabatic expansions, J. Funct. Anal. **251**(2007), 680–737.

[4] M. Nagase, The heat equation for the Kohn-Rossi Laplacian on contact Riemannian manifolds, preprint.

[5] N. K. Stanton, The heat equation for the $\bar\partial$-Neumann problem in a strictly pseudoconvex Siegel domain, J. d'Analyse Math. **38**(1980), 67–112.

[6] N. K. Stanton and D. S. Tartakoff, The heat equation for the $\bar{\partial}_b$-Laplacian, Comm. Partial Differential Equations **9**(7)(1984), 597–686.

[7] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. **314**(1)(1989), 349–379.

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