On the Chern-Moser connection in almost CR-geometry

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Abstract

We show that one can construct a Cartan connection on the Cartan principal bundle over a contact Riemannian manifold by using Cartan-Chern-Moser-Le's construction and it is normal in the sense of Tanaka if and only if the manifold is integrable. This has been known to hold true in the case the dimension of the manifold is three.

1 Introduction

Let (M, θ) be a $(2n+1)$-dimensional contact manifold with a contact form θ. There is a unique vector field ξ such that $\xi \rfloor \theta = 1$ and $\xi \rfloor d\theta = 0$. Equipping M with a Riemannian metric g and a $(1,1)$-tensor field J which satisfy $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y) := -X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])$ and $J^2X = -X + \theta(X)\xi$ for any vector fields X, Y, we have a contact Riemannian manifold $M = (M, \theta, \xi : g, J)$. We set $\ker_\pm \theta = \{X \in \mathbb{C} \otimes \ker\theta \mid JX = \pm iX\}$, $\ker^\pm \theta = \{\eta \in T^*M \otimes \mathbb{C} \mid X \rfloor \eta = 0 \ (X \in \mathbb{C}\xi \cup \ker_\mp \theta)\}$.

Let ω^E_α $(1 \leq \alpha \leq n)$ be local 1-forms on $E := M \times (\mathbb{R}^+, u)$ which are linear combinations of the pullbacks to E of θ and local cross-sections of $\ker^+ \theta$ and satisfy

$$d\omega^E = i \sum \omega^E_\alpha \wedge \overline{\omega^E_\alpha} + \omega \wedge \phi^E \quad (\omega^E := u\theta,\ \phi^E \text{ is real}).$$

Gathering all of the families $(\omega^E, \omega^E_\alpha, \overline{\omega^E_\alpha}, \phi^E)$ at each point, then we have the Chern-Moser principal bundle $\pi_1 : Y \to E$ of the positive definite case (cf. [5, (4.14)]). In the case where J is integrable (i.e., $[\Gamma(\ker_+ \theta), \Gamma(\ker_+ \theta)] \subset \Gamma(\ker_+ \theta)$), Chern-Moser ([5]) found out a system of everywhere linearly independent local 1-forms on Y completely determined by intrinsic conditions, whose total number equals the dimension of Y, and accordingly constructed a Cartan connection on $\pi_1 : Y \to E$. This is a generalization of Cartan's construction in the case $n = 1$ (cf. [4], [6], [3]).

In the case $n = 1$, from the connection Le ([8]) constructed a Cartan connection (cf. (4.10)) on the Cartan principal bundle $\pi := p \circ \pi_1 : Y \to M$ ($p : E \to M$ is the canonical projection), which we want to call the Chern-Moser connection (unlike in [8]), and showed that it is normal in the sense of Tanaka ([12]). His paper [8] is a

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good guide for understanding the Chern-Moser connection and led the author to the simple questions: Can one construct the Chern-Moser connection following Cartan-Chern-Moser-Le's construction in the case $n \geq 1$ and J may not be integrable? If possible, is it normal? This paper is a report on the questions and the answers are: In the genreal case there exists certainly such a system of local 1-forms on Y and one may construct the Chern-Moser connection (Theorems 2.2 and 4.3). It is then normal if and only if J is integrable (Theorem 4.4). Quoting Jacobowitz's remark ([6]), Le ([8, p.246]) indicates that in the case J is integrable the Chern-Moser connection is presumably normal but there has been no rigorous proof, and in [8, §5] he gives a rigorous one in the case $n = 1$. We want to emphasize that a part of the proof of Theorem 4.4 in §5 is a rigorous one in the case of general n.

While our interest is centered upon Cartan-Chern-Moser's construction, there is another profound work by Tanaka ([12]) on Cartan connection. Both of them aim at generalizing Cartan's work, that is, solving the pseudo-conformal equivalence problem for non-degenerate real analytic hypersurfaces in $\mathbb{C}^{n+1}$. In particular, Tanaka clarified a one-to-one correspondence between CR-manifolds and normal Cartan connections (on the Cartan principal bundles) and it seems that the study upon his theory is a major trend nowadays. There is no precise correspondence along Cartan-Chern-Moser's construction, but fortunately now we know that, for integrable contact Riemannian manifolds, certainly one can construct normal Cartan connections by using Cartan-Chern-Moser-Le's construction (cf. the comment by Le [8, p.246]). The outcomes theirs yields are so concrete that we hope it to be employed extensively in the study of CR-manifold. The author has an interest particularly in a field of study with key words such as Fefferman spin space for general J, Dirac operator, twistor operator or holonomy of Cartan connection (cf. [10], [2]), and in the near future he is going to apply the results in this paper to the study of such a field.

It is a pleasure to thank Hajime Sato for many valuable suggestions. We rely on his deep knowledge of Cartan connection.

2 The structure equations: the first main theorem

The structure group G_1 of the principal bundle $\pi_1 : Y \to E$ consists of matrices

$$\Lambda((u_{\beta\alpha}), (v_\alpha), s) := \begin{pmatrix} 1 & v_\alpha & \overline{v_\alpha} & s \\ 0 & u_{\beta\alpha} & 0 & i\, u_{\beta\alpha}\overline{v_\alpha} \\ 0 & 0 & \overline{u_{\beta\alpha}} & -i\, \overline{u_{\beta\alpha}}\, v_\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(2n+2, \mathbb{C})$$

$$((u_{\beta\alpha}) \in U(n),\ (v_\alpha) = (v_1, \ldots, v_n) \in \mathbb{C}^n,\ s \in \mathbb{R}).$$

(As for the matrix, let us assign the numbers $0, 1, \ldots, n, \bar{1}, \ldots, \bar{n}, 2n+2$ to the columns (and rows) consecutively. Then the $(0,0)$-entry, the $(0,1)$-entry, ... are 1, v_1, ..., and the $(\beta, 2n+2)$-entry $i\, u_{\beta\alpha}\overline{v_\alpha}$ means $i\sum_\alpha u_{\beta\alpha}\overline{v_\alpha}$ actually. The matrix corresponds to [5, (4.12)]. We follow the notation in [8, p.252], however.) We have $\dim E = 2(n+1)$, $\dim G_1 = \dim U(n) + 2n + 1$ and $\dim Y = (n+2)^2 - 1$, and, as local (complex) coordinates of Y, we may take some local coordinates of E together with $u_{\beta\alpha}$, v_α, $\overline{v_\beta}$ and s.

In this paper, a local frame $\theta^\bullet = (\theta, \theta^1, \ldots, \theta^n)$ of $\mathbb{C}\theta \oplus \ker^+ \theta$ on $U_{\theta^\bullet}$ is always assumed to be unitary, i.e., $g = \theta \otimes \theta + \sum_{1 \leq \alpha \leq n} (\theta^\alpha \otimes \overline{\theta^\alpha} + \overline{\theta^\alpha} \otimes \theta^\alpha)$. Then (even if J is not integrable) we have

$$d\theta = i\sum \theta^\alpha \wedge \overline{\theta^\alpha}, \quad d\theta^\alpha = \sum \theta^\beta \wedge \Omega(\nabla)^\alpha_\beta + \sum \overline{\theta^\beta} \wedge \Omega(\nabla)^\alpha_{\bar\beta} + \theta \wedge \tau^\alpha, \tag{2.1}$$

where ∇ is a generalized Tanaka-Webster connection on TM introduced by Tanno ([13]), defined by

$$\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X\theta)(Y)\xi$$

(∇^g is the Levi-Civita connection of g). Note that the generalized one coincides with the Tanaka-Webster connection if J is integrable. Let us collect some properties of the connection and explain the symbols used above. Refer to [13], [1], [9], [11], etc., for more detailed explanation: We have $\nabla\theta = 0$, $\nabla g = 0$, $T(\nabla)(Z,W) = 0$, $T(\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi$, $(Z, W \in \Gamma(\ker_+ \theta))$. Here $T(\nabla)$ is the torsion tensor and, if we set $\tau X = T(\nabla)(\xi, X)$, then $\tau \circ J + J \circ \tau = 0$. In general ∇ does not commute with the action of the almost complex structure J. In fact, Tanno indicated

$$(\nabla_X J)Y = \mathcal{Q}(Y,X) := (\nabla^g_X J)Y + (\nabla^g_X\theta)(JY)\,\xi + \theta(Y)J\nabla^g_X\xi$$

and showed that J is integrable if and only if the Tanno tensor $\mathcal{Q}$ vanishes. Let us denote by $(\xi, \xi_1, \dots, \xi_n, \overline{\xi_1}, \dots, \overline{\xi_n})$ the dual frame of $(\theta, \theta^1, \dots, \theta^n, \overline{\theta^1}, \dots, \overline{\theta^n})$. Then we have

$$\tau = \sum \tau^\alpha_{\bar\gamma}\, \xi_\alpha \otimes \overline{\theta^\gamma} + \sum \tau^{\bar\alpha}_\gamma\, \overline{\xi_\alpha} \otimes \theta^\gamma \quad (\tau^{\bar\alpha}_\gamma = \overline{\tau^{\alpha}_{\bar\gamma}}),$$

$$\mathcal{Q} = \sum \mathcal{Q}^\alpha_{\bar\beta\bar\gamma}\, \xi_\alpha \otimes \overline{\theta^\beta} \otimes \overline{\theta^\gamma} + \sum \mathcal{Q}^{\bar\alpha}_{\beta\gamma}\, \overline{\xi_\alpha} \otimes \theta^\beta \otimes \theta^\gamma \;\; (\mathcal{Q}^{\bar\alpha}_{\beta\gamma} = -\mathcal{Q}^{\bar\beta}_{\alpha\gamma} = -\mathcal{Q}^{\bar\beta}_{\gamma\alpha} - \mathcal{Q}^{\bar\gamma}_{\alpha\beta})$$

(hence, $\tau^\alpha = \sum \tau^\alpha_{\bar\gamma}\, \overline{\theta^\gamma}$), and $\nabla\xi_\beta = \sum \Omega(\nabla)^\alpha_\beta\, \xi_\alpha + \sum \Omega(\nabla)^{\bar\alpha}_\beta\, \overline{\xi_\alpha}$, etc. In particular, we have

$$\Omega(\nabla)^{\bar\alpha}_\beta = -\frac{i}{2}\sum \mathcal{Q}^{\bar\alpha}_{\beta\gamma}\, \theta^\gamma, \quad \Omega(\nabla)^\alpha_{\bar\beta} = \frac{i}{2}\sum \mathcal{Q}^\alpha_{\bar\beta\bar\gamma}\, \overline{\theta^\gamma}.$$

On $U_{\theta^\bullet} \times (\mathbb{R}^+, u)\,(\subset E)$, let us set

$$(\omega^E, \omega^E_\alpha, \overline{\omega^E_\alpha}, \phi^E) = (u\theta, \sqrt{u}\,\theta^\alpha, \overline{\sqrt{u}\,\theta^\alpha}, -\frac{du}{u}). \tag{2.2}$$

Then, (2.1) implies

$$\begin{aligned}
d\omega^E &= i\sum \omega^E_\alpha \wedge \overline{\omega^E_\alpha} + \omega^E \wedge \phi^E,\\
d\omega^E_\alpha &= \sum \omega^E_\beta \wedge \phi^E_{\beta\alpha} + \sum \overline{\omega^E_\beta} \wedge \phi^E_{\bar\beta\alpha} + \omega^E \wedge \phi^E_\alpha,\\
\phi^E_{\beta\alpha} &+ \overline{\phi^E_{\alpha\beta}} = \delta_{\beta\alpha}\,\phi^E, \quad \phi^E_{\bar\beta\alpha} + \phi^E_{\bar\alpha\beta} = 0,
\end{aligned}$$

where

$$\phi^E_{\beta\alpha} = \Omega(\nabla)^\alpha_\beta + \frac{1}{2}\delta_{\beta\alpha}\phi^E, \quad \phi^E_{\bar\beta\alpha} = \Omega(\nabla)^\alpha_{\bar\beta} = \sum \frac{i\mathcal{Q}^\alpha_{\bar\beta\bar\gamma}}{2\sqrt{u}}\,\overline{\omega^E_\gamma},$$

$$\phi^E_\alpha = \frac{1}{\sqrt{u}}\,\tau^\alpha = \sum \frac{\tau^\alpha_{\bar\beta}}{u}\,\overline{\omega^E_\beta}.$$

On $\pi_1^{-1}(U_{\theta^\bullet} \times \mathbb{R}^+)$, we consider the family of 1-forms

$$
\begin{aligned}
(2.3)\qquad (\omega, \omega_\alpha, \overline{\omega_\alpha}, \phi) &:= (\omega^E, \omega^E_\alpha, \overline{\omega^E_\alpha}, \phi^E)(x,u)\cdot\Lambda((u_{\beta\alpha}),(v_\alpha),s)\\
&= (\omega^E,\ v_\alpha\,\omega^E + \sum u_{\beta\alpha}\,\omega^E_\beta,\ \overline{v_\alpha}\,\omega^E + \sum \overline{u_{\beta\alpha}}\,\overline{\omega^E_\beta},\\
&\qquad s\,\omega^E + i\sum u_{\beta\gamma}\,\overline{v_\gamma}\,\omega^E_\beta - i\sum \overline{u_{\beta\gamma}}\,v_\gamma\,\overline{\omega^E_\beta} + \phi^E).
\end{aligned}
$$

Lemma 2.1 *We have*

$$
(2.4)\qquad d\omega = i\sum \omega_\alpha\wedge\overline{\omega_\alpha} + \omega\wedge\phi,
$$

$$
(2.5)\qquad d\omega_\alpha = \sum \omega_\beta\wedge\phi^Y_{\beta\alpha} + \sum\overline{\omega_\beta}\wedge\phi^Y_{\bar\beta\alpha} + \omega\wedge\phi^Y_\alpha
$$

with

$$
\begin{aligned}
(2.6)\quad \phi^Y_{\beta\alpha} &:= \sum \overline{u_{\mu\beta}}\,\{\sum u_{\nu\alpha}\,\phi^E_{\mu\nu} + i\,v_\alpha\,\overline{\omega^E_\mu} - du_{\mu\alpha}\}\\
&= \sum\{\sum \overline{u_{\mu\beta}}\,u_{\nu\alpha}\,\overline{u_{\lambda\gamma}}\,\frac{\Omega(\nabla)^\nu_\mu(\xi_\lambda)}{\sqrt{u}} - i\,\frac{1}{2}\delta_{\beta\alpha}\,\overline{v_\gamma}\}\,\omega_\gamma\\
&\quad + \sum\{\sum \overline{u_{\mu\beta}}\,u_{\nu\alpha}\,u_{\lambda\gamma}\,\frac{\Omega(\nabla)^\nu_\mu(\overline{\xi_\lambda})}{\sqrt{u}} + i\,\frac{1}{2}\delta_{\beta\alpha}\,v_\gamma + i\,\delta_{\gamma\beta}\,v_\alpha\}\,\overline{\omega_\gamma}\\
&\quad + \{\sum \overline{u_{\mu\beta}}\,u_{\nu\alpha}\,\frac{\Omega(\nabla)^\nu_\mu(\xi)}{u} - \sum v_\gamma\,\overline{u_{\mu\beta}}\,u_{\nu\alpha}\,\overline{u_{\lambda\gamma}}\,\frac{\Omega(\nabla)^\nu_\mu(\xi_\lambda)}{\sqrt{u}}\\
&\quad - \sum \overline{v_\gamma}\,\overline{u_{\mu\beta}}\,u_{\nu\alpha}\,u_{\lambda\gamma}\,\frac{\Omega(\nabla)^\nu_\mu(\overline{\xi_\lambda})}{\sqrt{u}} - \delta_{\beta\alpha}\,\frac{s}{2} - i\,v_\alpha\,\overline{v_\beta}\}\,\omega\\
&\quad + \frac{1}{2}\delta_{\beta\alpha}\phi - \sum \overline{u_{\mu\beta}}\,du_{\mu\alpha},
\end{aligned}
$$

$$
\begin{aligned}
(2.7)\quad \phi^Y_{\bar\beta\alpha} &:= \sum u_{\mu\beta}\,u_{\nu\alpha}\,\phi^E_{\bar\mu\nu}\\
&= \sum\{\sum u_{\mu\beta}\,u_{\nu\alpha}\,u_{\lambda\gamma}\,\frac{iQ^\nu_{\bar\mu\bar\lambda}}{2\sqrt{u}}\}\,\overline{\omega_\gamma} + \sum\{-\sum \overline{v_\gamma}\,u_{\mu\beta}\,u_{\nu\alpha}\,u_{\lambda\gamma}\,\frac{iQ^\nu_{\bar\mu\bar\lambda}}{2\sqrt{u}}\}\,\omega,
\end{aligned}
$$

$$
\begin{aligned}
(2.8)\quad \phi^Y_\alpha &:= v_\alpha\,\phi^E + \sum u_{\gamma\alpha}\,\phi^E_\gamma - dv_\alpha - \sum v_\beta\,\phi^Y_{\beta\alpha} - \sum\overline{v_\beta}\,\phi^Y_{\bar\beta\alpha}\\
&= \sum\{-i\,v_\alpha\,\overline{v_\gamma}\}\,\omega_\gamma + \sum\{i\,v_\alpha\,v_\gamma + \sum u_{\mu\alpha}\,u_{\beta\gamma}\,\frac{\tau^\mu_{\bar\beta}}{u}\}\,\overline{\omega_\gamma}\\
&\quad + \{-s\,v_\alpha - \sum\overline{v_\gamma}\,u_{\mu\alpha}\,u_{\beta\gamma}\,\frac{\tau^\mu_{\bar\beta}}{u}\}\,\omega - \sum v_\beta\,\phi^Y_{\beta\alpha} - \sum\overline{v_\beta}\,\phi^Y_{\bar\beta\alpha} + v_\alpha\,\phi - dv_\alpha.
\end{aligned}
$$

Proof. This is shown by straightforward calculations. Note that

$$
(2.9)\quad \omega^E_\beta = -\sum v_\mu\,\overline{u_{\beta\mu}}\,\omega + \sum\overline{u_{\beta\mu}}\,\omega_\mu,\quad \phi^E = -s\,\omega - i\sum\overline{v_\beta}\,\omega_\beta + i\sum v_\beta\,\overline{\omega_\beta} + \phi.
$$

∎

Starting with adding some suitable ω-, ω_ρ- or $\overline{\omega_\rho}$-terms to $\phi^Y_{\beta\alpha}$, $\phi^Y_{\bar\beta\alpha}$, ϕ^Y_α, we will obtain the first main theorem, which is a generalization of Chern-Moser's theorem [5, the first part of Theorem 4.6] (in the positive definite case) to the case of general J. Refer to the concise review of their result in [5, Appendix (by Webster), p.269]. We are interested only in the families (2.3) induced from unitary frames so that our result is described rather clearly. (Note that ω^α (or $\omega_{\bar\alpha}$), $\omega^{\bar\alpha}$ (or ω_α), $\phi_\beta{}^\alpha$ (or $\phi_{\beta\bar\alpha}$), $\phi_{\bar\beta}{}^{\bar\alpha}$ (or $\phi_{\bar\beta\alpha}$), $g_{\alpha\bar\beta}$, etc., in [5] correspond to ω_α, $\overline{\omega_\alpha}$, $\phi_{\beta\alpha}$, $\overline{\phi_{\beta\alpha}}$, $\delta_{\alpha\beta}$, etc., in this paper.)

Theorem 2.2 *On $\pi_1^{-1}(U_{\theta^\bullet} \times \mathbb{R}^+)$, for the family $(\omega, \omega_\alpha, \overline{\omega_\alpha}, \phi)$ given at (2.3) there exists a unique family of 1-forms*

$$\phi_{\beta\alpha},\ \phi_{\bar\beta\alpha},\ \phi_\alpha,\ \psi\,(=\overline{\psi})$$

which satisfies

$$d\omega = i\sum \omega_\alpha \wedge \overline{\omega_\alpha} + \omega \wedge \phi, \tag{2.10}$$

$$d\omega_\alpha = \sum \omega_\beta \wedge \phi_{\beta\alpha} + \sum \overline{\omega_\beta} \wedge \phi_{\bar\beta\alpha} + \omega \wedge \phi_\alpha, \tag{2.11}$$

$$\phi_{\beta\alpha} + \overline{\phi_{\alpha\beta}} = \delta_{\beta\alpha}\phi, \tag{2.12}$$

$$\phi_{\bar\beta\alpha} + \phi_{\bar\alpha\beta} = 0, \tag{2.13}$$

$$d\phi = i\sum \omega_\beta \wedge \overline{\phi_\beta} + i\sum \phi_\beta \wedge \overline{\omega_\beta} + \omega \wedge \psi, \tag{2.14}$$

$$\begin{aligned} d\phi_{\beta\alpha} = {} & \sum \phi_{\beta\gamma} \wedge \phi_{\gamma\alpha} + \sum \overline{\phi_{\bar\beta\gamma}} \wedge \phi_{\bar\gamma\alpha} + i\,\overline{\omega_\beta} \wedge \phi_\alpha - i\,\overline{\phi_\beta} \wedge \omega_\alpha \\ & - i\,\delta_{\beta\alpha} \sum \overline{\phi_\gamma} \wedge \omega_\gamma - \frac{1}{2}\,\delta_{\beta\alpha}\psi \wedge \omega + \Phi_{\beta\alpha}, \end{aligned} \tag{2.15}$$

$$d\phi_{\bar\beta\alpha} = \sum \phi_{\bar\beta\gamma} \wedge \phi_{\gamma\alpha} + \sum \overline{\phi_{\beta\gamma}} \wedge \phi_{\bar\gamma\alpha} + \Phi_{\bar\beta\alpha}, \tag{2.16}$$

$$d\phi_\alpha = \phi \wedge \phi_\alpha + \sum \phi_\beta \wedge \phi_{\beta\alpha} + \sum \overline{\phi_\beta} \wedge \phi_{\bar\beta\alpha} - \frac{1}{2}\,\psi \wedge \omega_\alpha + \Phi_\alpha, \tag{2.17}$$

$$d\psi = \phi \wedge \psi + 2i\sum \phi_\beta \wedge \overline{\phi_\beta} + \Psi \tag{2.18}$$

with

$$\begin{aligned} \Phi_{\beta\alpha} = {} & \sum S_{\beta\mu\alpha\bar\nu}\,\omega_\mu \wedge \overline{\omega_\nu} + \sum S_{\beta\mu\alpha\nu}\,\omega_\mu \wedge \omega_\nu \\ & + \sum S_{\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu} \wedge \overline{\omega_\nu} + \sum (V_{\beta\alpha\mu}\,\omega_\mu - \overline{V_{\alpha\beta\mu}}\,\overline{\omega_\mu}) \wedge \omega, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \Phi_{\bar\beta\alpha} = {} & \sum S_{\bar\beta\mu\alpha\bar\nu}\,\omega_\mu \wedge \overline{\omega_\nu} + \sum S_{\bar\beta\mu\alpha\nu}\,\omega_\mu \wedge \omega_\nu \\ & + \sum S_{\bar\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu} \wedge \overline{\omega_\nu} + \lambda_{\bar\beta\alpha} \wedge \omega, \end{aligned} \tag{2.20}$$

$$\Phi_\alpha = \sum (V_{\beta\alpha\mu}\,\omega_\mu - \overline{V_{\alpha\beta\mu}}\,\overline{\omega_\mu}) \wedge \omega_\beta + \sum \lambda_{\bar\beta\alpha} \wedge \overline{\omega_\beta} + \eta_\alpha \wedge \omega, \tag{2.21}$$

$$\Psi = -i\sum \omega_\beta \wedge \overline{\eta_\beta} - i\sum \eta_\beta \wedge \overline{\omega_\beta} + \varrho \wedge \omega, \tag{2.22}$$

$$\begin{aligned} \eta_\alpha = {} & \sum \eta_{\alpha\mu}\,\omega_\mu + \sum \eta_{\alpha\bar\mu}\,\overline{\omega_\mu} + \eta_{\alpha(0)}\,\phi + \sum \eta_{\alpha(\mu\nu)}\,\phi_{\mu\nu} \\ & + \sum \eta_{\alpha(\mu)}\,\phi_\mu + \sum \eta_{\alpha(\bar\mu)}\,\overline{\phi_\mu} + \eta_{\alpha(*)}\,\psi, \end{aligned} \tag{2.23}$$

$$\sum_\beta S_{\beta\mu\beta\bar\nu} = \sum_\alpha V_{\beta\alpha\alpha} = \sum_\mu \widetilde{\eta}_{\mu\mu} = 0 \quad (\widetilde{\eta}_{\alpha\mu} := \eta_{\alpha\mu} + \overline{\eta_{\mu\alpha}}). \tag{2.24}$$

Let the forms $\lambda_{\bar\beta\alpha}$, to which one may add ω-terms, have no ω-terms. Then we have:

(1) *The 1-forms*

$$\omega,\ \omega_\alpha,\ \overline{\omega_\beta},\ \phi,\ \phi_{\beta\alpha},\ \phi_\alpha,\ \overline{\phi_\beta},\ \psi,$$

whose total number equals the dimension of Y, are everywhere linearly independent.

(2) *The coefficients $S_{\beta\dots}$ and $S_{\bar\beta\dots}$ have the following relations:*

$$
\begin{aligned}
S_{\beta\mu\alpha\bar\nu} - \overline{S_{\alpha\nu\beta\bar\mu}} = 0, \quad S_{\beta\mu\alpha\nu} + \overline{S_{\alpha\bar\mu\beta\bar\nu}} = 0,\\
S_{\beta\mu\alpha\bar\nu} = S_{\mu\beta\alpha\bar\nu} + 2S_{\bar\nu\mu\alpha\beta} = S_{\beta\mu\nu\bar\alpha} + 2\,\overline{S_{\bar\mu\nu\beta\alpha}},\\
S_{\beta\mu\alpha\nu} = \frac{1}{2}(\overline{S_{\bar\nu\alpha\beta\bar\mu}} - \overline{S_{\bar\mu\alpha\beta\bar\nu}}), \quad S_{\beta\mu\alpha\nu} + S_{\mu\nu\alpha\beta} + S_{\nu\beta\alpha\mu} = 0.
\end{aligned}
\tag{2.25}
$$

(3) *Setting* $b_{\bar\beta\alpha\bar\gamma} = \sum u_{\mu\beta}\, u_{\nu\alpha}\, u_{\lambda\gamma}\, \frac{iQ^{\nu}_{\bar\mu\bar\lambda}}{2\sqrt{u}}$ (cf. (3.3)), *we have the following more explicit expressions:*

$$\phi_{\bar\beta\alpha} = \sum b_{\bar\beta\alpha\bar\mu}\,\overline{\omega_\mu}, \tag{2.26}$$

$$\lambda_{\bar\beta\alpha} = \sum \lambda_{\bar\beta\alpha\bar\mu}\,\overline{\omega_\mu} - \sum b_{\bar\beta\alpha\bar\mu}\,\overline{\phi_\mu} \quad (\lambda_{\bar\beta\alpha\bar\mu} + \lambda_{\bar\alpha\mu\bar\beta} + \lambda_{\bar\mu\beta\bar\alpha} = 0), \tag{2.27}$$

$$
\begin{aligned}
\eta_\alpha = {}& \sum \eta_{\alpha\mu}\,\omega_\mu + \sum \eta_{\alpha\bar\mu}\,\overline{\omega_\mu}\\
& - \frac{2i}{2n+1}\sum (b_{\bar\kappa\rho\bar\alpha}\,\overline{b_{\bar\kappa\rho\bar\mu}} - b_{\bar\kappa\alpha\bar\rho}\,\overline{b_{\bar\mu\rho\bar\kappa}})\,\phi_\mu - \frac{2i}{2n+1}\sum (db_{\bar\kappa\alpha\bar\mu})_\kappa\,\overline{\phi_\mu},
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
\varrho = {}& \sum \varrho_\mu\,\omega_\mu + \sum \overline{\varrho_\mu}\,\overline{\omega_\mu} + \frac{1}{n}\sum \{(d\eta_{\kappa(\mu)})_\kappa + \overline{(d\eta_{\kappa(\bar\mu)})_\kappa} + \sum \overline{b_{\bar\kappa\mu\bar\rho}}\,\eta_{\kappa(\bar\rho)}\}\,\phi_\mu\\
& + \frac{1}{n}\sum \{\overline{(d\eta_{\kappa(\mu)})_\kappa} + (d\eta_{\kappa(\bar\mu)})_\kappa + \sum b_{\bar\kappa\mu\bar\rho}\,\overline{\eta_{\kappa(\bar\rho)}}\}\,\overline{\phi_\mu} + \frac{1}{2n}\sum (\eta_{\kappa(\kappa)} + \overline{\eta_{\kappa(\kappa)}})\,\psi,
\end{aligned}
\tag{2.29}
$$

where we put $db_{\bar\kappa\alpha\bar\mu} = \sum (db_{\bar\kappa\alpha\bar\mu})_0\,\omega + \sum (db_{\bar\kappa\alpha\bar\mu})_\beta\,\omega_\beta + \cdots$, *etc.*

(4) *There are the additional formulas:*

$$
\begin{aligned}
i(2n+1)\sum V_{\beta\beta\alpha} = {}& \sum (dS_{\beta\alpha\beta\mu} - dS_{\beta\mu\beta\alpha})_{\bar\mu} + \sum \overline{b_{\bar\alpha\mu\bar\rho}}\,(S_{\beta\bar\rho\beta\bar\mu} - S_{\beta\bar\mu\beta\bar\rho})\\
& + \sum b_{\bar\rho\beta\bar\mu}\,(\overline{S_{\bar\beta\bar\alpha\rho\bar\mu}} - \overline{S_{\bar\beta\bar\mu\rho\bar\alpha}}) + \sum \overline{b_{\bar\rho\beta\bar\mu}}\,S_{\bar\beta\alpha\rho\bar\mu} - \sum \overline{b_{\bar\rho\beta\bar\alpha}}\,S_{\bar\beta\mu\rho\bar\mu},
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
i(n+2)\,(V_{\mu\nu\gamma} - V_{\gamma\nu\mu}) = {}& i\delta_{\gamma\nu}\sum V_{\beta\beta\mu} - i\delta_{\mu\nu}\sum V_{\beta\beta\gamma}\\
& + \sum (dS_{\beta\gamma\beta\mu} - dS_{\beta\mu\beta\gamma})_{\bar\nu} + \sum \overline{b_{\bar\gamma\mu\bar\rho}}\,(S_{\beta\bar\rho\beta\bar\nu} - S_{\beta\bar\nu\beta\bar\rho})\\
& + \sum b_{\bar\rho\beta\bar\nu}\,(\overline{S_{\bar\beta\bar\gamma\rho\bar\mu}} - \overline{S_{\bar\beta\bar\mu\rho\bar\gamma}}) + \sum \overline{b_{\bar\rho\beta\bar\mu}}\,S_{\bar\beta\gamma\rho\bar\nu} - \sum \overline{b_{\bar\rho\beta\bar\gamma}}\,S_{\bar\beta\mu\rho\bar\nu},
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
\frac{i(n+2)}{2}\,(\eta_{\nu\bar\mu} - \eta_{\mu\bar\nu}) = {}& \sum \overline{(dV_{\beta\beta\nu})_\mu} - \sum \overline{(dV_{\beta\beta\mu})_\nu}\\
& + \sum \overline{(dS_{\beta\mu\beta\nu} - dS_{\beta\nu\beta\mu})_0} + \sum b_{\bar\nu\mu\bar\rho}\,V_{\beta\beta\rho},
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
\frac{i(n+2)}{2}\,(\eta_{\nu\mu} - \overline{\eta_{\mu\nu}}) = {}& \sum (dV_{\beta\beta\mu})_{\bar\nu} + \sum \overline{(dV_{\beta\beta\nu})_{\bar\mu}}\\
& + \sum b_{\bar\gamma\beta\bar\nu}\,\overline{\lambda_{\bar\beta\gamma\bar\mu}} + \sum \overline{b_{\bar\gamma\beta\bar\mu}}\,\lambda_{\bar\beta\gamma\bar\nu}.
\end{aligned}
\tag{2.33}
$$

Remark 2.3 *If J is integrable (i.e., $\mathcal{Q} = 0$), then all the forms $\phi_{\bar\beta\alpha}$ vanish, and consequently the theorem is reduced to that of Chern-Moser and, moreover, the formulas* (2.30)–(2.33) *are reduced to the additional ones $\sum_\beta V_{\beta\beta\alpha} = 0$, $V_{\mu\nu\gamma} = V_{\gamma\nu\mu}$, $\eta_{\nu\bar\mu} = \eta_{\mu\bar\nu}$, $\eta_{\nu\mu} = \overline{\eta_{\mu\nu}}$ appearing in* [5, Appendix (p.271)]. *Indeed, if the forms $\phi_{\bar\beta\alpha}$ vanish, then obviously we find successively $\Phi_{\bar\beta\alpha} = 0$ (hence, $S_{\bar\beta\dots} = 0$, $\lambda_{\bar\beta\alpha} = 0$), $S_{\beta\mu\alpha\nu} = S_{\beta\bar\mu\alpha\bar\nu} = 0$* (cf. (2.25)), *and also the other reductions.*

3 The proof of Theorem 2.2

Referring to the argument in [5, §4], we will prove Theorem 2.2. The equality (2.10) has been shown in (2.4). For a family $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha)$ satisfying (2.11), we have

$$
\begin{aligned}
(3.1)\qquad 0 = dd\omega = {} & i\sum\{-\phi_{\alpha\beta} - \overline{\phi_{\beta\alpha}} + \delta_{\alpha\beta}\phi\}\wedge\omega_\alpha\wedge\overline{\omega_\beta}\\
& - i\sum\overline{\phi_{\bar\beta\alpha}}\wedge\omega_\alpha\wedge\omega_\beta - i\sum\phi_{\bar\alpha\beta}\wedge\overline{\omega_\alpha}\wedge\overline{\omega_\beta}\\
& + \{-d\phi + i\sum\phi_\beta\wedge\overline{\omega_\beta} + i\sum\omega_\beta\wedge\overline{\phi_\beta}\}\wedge\omega,
\end{aligned}
$$

$$
\begin{aligned}
(3.2)\qquad 0 = dd\omega_\alpha = {} & \sum\{-d\phi_{\beta\alpha} + \sum\phi_{\beta\gamma}\wedge\phi_{\gamma\alpha} + \sum\overline{\phi_{\bar\beta\gamma}}\wedge\phi_{\bar\gamma\alpha} + i\,\overline{\omega_\beta}\wedge\phi_\alpha\}\wedge\omega_\beta\\
& + \sum\{-d\phi_{\bar\beta\alpha} + \sum\phi_{\bar\beta\gamma}\wedge\phi_{\gamma\alpha} + \sum\overline{\phi_{\beta\gamma}}\wedge\phi_{\bar\gamma\alpha}\}\wedge\overline{\omega_\beta}\\
& + \{-d\phi_\alpha + \phi\wedge\phi_\alpha + \sum\phi_\beta\wedge\phi_{\beta\alpha} + \sum\overline{\phi_\beta}\wedge\phi_{\bar\beta\alpha}\}\wedge\omega
\end{aligned}
$$

(cf. [5, (4.19), (4.20)]). The following argument is divided into four steps.

The first step: First we wish to find out a family $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha)$ satisfying (2.11), (2.12) and (2.13) by adding some suitable terms to $(\phi^Y_{\beta\alpha}, \phi^Y_{\bar\beta\alpha}, \phi^Y_\alpha)$ given at (2.6), (2.7) and (2.8). By (2.5), the latter certainly satisfies the formula (3.1). Hence we know

$$
\begin{aligned}
-\phi^Y_{\beta\alpha} - \overline{\phi^Y_{\alpha\beta}} + \delta_{\beta\alpha}\phi &= \sum A_{\beta\alpha\gamma}\,\omega_\gamma + \sum B_{\beta\alpha\bar\gamma}\,\overline{\omega_\gamma} + C_{\beta\alpha}\,\omega,\\
\phi^Y_{\bar\beta\alpha} &= \sum a_{\bar\beta\alpha\gamma}\,\omega_\gamma + \sum b_{\bar\beta\alpha\bar\gamma}\,\overline{\omega_\gamma} + c_{\bar\beta\alpha}\,\omega
\end{aligned}
$$

with

$$
\begin{gathered}
A_{\beta\alpha\gamma} = A_{\gamma\alpha\beta},\quad B_{\beta\alpha\bar\gamma} = B_{\beta\gamma\bar\alpha},\quad \overline{A_{\beta\alpha\gamma}} = B_{\alpha\beta\bar\gamma},\quad \overline{C_{\beta\alpha}} = C_{\alpha\beta},\\
a_{\bar\beta\alpha\gamma} = 0,\quad b_{\bar\beta\alpha\bar\gamma} = -b_{\bar\alpha\beta\bar\gamma} = -b_{\bar\alpha\gamma\bar\beta} - b_{\bar\gamma\beta\bar\alpha},\quad c_{\bar\beta\alpha} = -c_{\bar\alpha\beta},\\
A_{\beta\alpha\gamma} = i(\delta_{\alpha\gamma}\,\overline{v_\beta} + \delta_{\beta\alpha}\,\overline{v_\gamma}),\quad B_{\beta\alpha\bar\gamma} = -i(\delta_{\beta\gamma}\,v_\alpha + \delta_{\beta\alpha}\,v_\gamma),\quad C_{\beta\alpha} = \delta_{\beta\alpha}s,\\
b_{\bar\beta\alpha\bar\gamma} = \sum u_{\mu\beta}\,u_{\nu\alpha}\,u_{\lambda\gamma}\,\frac{i\mathcal{Q}^\nu_{\bar\mu\bar\lambda}}{2\sqrt{u}},\quad c_{\bar\beta\alpha} = -\sum\overline{v_\gamma}\,b_{\bar\beta\alpha\bar\gamma}
\end{gathered}\tag{3.3}
$$

(cf. [5, (4.22)–(4.24)]), which obviously imply the following.

Proposition 3.1 (cf. [5, (4.25)]) *The family $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha)$ defined by*

$$
\begin{gathered}
\phi_{\beta\alpha} = \phi^Y_{\beta\alpha} + \sum A_{\beta\alpha\gamma}\,\omega_\gamma + \frac{C_{\beta\alpha}}{2}\,\omega,\quad \phi_{\bar\beta\alpha} = \phi^Y_{\bar\beta\alpha} - c_{\bar\beta\alpha}\,\omega = \sum b_{\bar\beta\alpha\bar\gamma}\,\overline{\omega_\gamma},\\
\phi_\alpha = \phi^Y_\alpha - \sum c_{\bar\beta\alpha}\,\overline{\omega_\beta} + \sum\frac{C_{\beta\alpha}}{2}\,\omega_\beta
\end{gathered}
$$

satisfies (2.11)*,* (2.12) *and* (2.13)*.*

For the $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha)$ given above, the first two lines in the third side of (3.1) vanish. Indeed, the first line vanishes because of (2.12) and so does the second line because

$$
\sum\overline{\phi_{\bar\beta\alpha}}\wedge\omega_\alpha\wedge\omega_\beta = \sum\overline{b_{\bar\beta\alpha\bar\gamma}}\,\omega_\gamma\wedge\omega_\alpha\wedge\omega_\beta = 0.
$$

Consequently, we have

$$
\{-d\phi + i\sum\phi_\beta\wedge\overline{\omega_\beta} + i\sum\omega_\beta\wedge\overline{\phi_\beta}\}\wedge\omega = 0,
$$

so that we may take a form $\psi\,(\in\mathbb{R})$ satisfying (2.14).

Proposition 3.2 (cf. [5, (4.35), (4.36)]) *Let $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ be the family given above. Then, another family $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ satisfies* (2.11), (2.12), (2.13) *and* (2.14) *if and only if there exist relations*

$$\begin{aligned}&\phi_{\beta\alpha} = \phi'_{\beta\alpha} + D_{\beta\alpha}\,\omega, \quad \phi_{\bar\beta\alpha} = \phi'_{\bar\beta\alpha}, \quad \phi_\alpha = \phi'_\alpha + \sum D_{\beta\alpha}\,\omega_\beta + \overline{E_\alpha}\,\omega,\\ &\psi = \psi' + G\,\omega + i\sum(E_\alpha\,\omega_\alpha - \overline{E_\alpha}\,\overline{\omega_\alpha}), \quad D_{\beta\alpha} + \overline{D_{\alpha\beta}} = 0, \quad G\in\mathbb{R}.\end{aligned} \tag{3.4}$$

The second step: Let $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ satisfy (2.11), (2.12), (2.13), (2.14). Referring to the terms $\sum\{\cdots\}\wedge\overline{\omega_\beta}$ and $\sum\{\cdots\}\wedge\omega_\beta$ of (3.2), we consider the forms

$$\Psi_{\bar\beta\alpha} := d\phi_{\bar\beta\alpha} - \sum\phi_{\bar\beta\gamma}\wedge\phi_{\gamma\alpha} - \sum\overline{\phi_{\beta\gamma}}\wedge\phi_{\bar\gamma\alpha}, \tag{3.5}$$

$$\begin{aligned}\Psi_{\beta\alpha} := d\phi_{\beta\alpha} &- \sum\phi_{\beta\gamma}\wedge\phi_{\gamma\alpha} - \sum\overline{\phi_{\bar\beta\gamma}}\wedge\phi_{\bar\gamma\alpha} - i\,\overline{\omega_\beta}\wedge\phi_\alpha\\ &+ i\,\overline{\phi_\beta}\wedge\omega_\alpha + i\,\delta_{\beta\alpha}\sum\overline{\phi_\gamma}\wedge\omega_\gamma\end{aligned} \tag{3.6}$$

(cf. [5, (4.31), (4.34)]).

Lemma 3.3 (cf. [5, Lemma 4.2]) *As for $\Psi_{\bar\beta\alpha}$, we have*

$$\begin{aligned}&\Psi_{\bar\beta\alpha} \equiv \sum S_{\bar\beta\mu\alpha\bar\nu}\,\omega_\mu\wedge\overline{\omega_\nu} + \sum S_{\bar\beta\mu\alpha\nu}\,\omega_\mu\wedge\omega_\nu + \sum S_{\bar\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu}\wedge\overline{\omega_\nu} \pmod{\omega},\\ &S_{\bar\beta\mu\alpha\bar\nu} + S_{\bar\alpha\mu\beta\bar\nu} = 0, \quad S_{\bar\beta\mu\alpha\nu} + S_{\bar\alpha\mu\beta\nu} = 0, \quad S_{\bar\beta\bar\mu\alpha\bar\nu} + S_{\bar\alpha\bar\mu\beta\bar\nu} = 0,\end{aligned} \tag{3.7}$$

and the coefficients $S_{\bar\beta\cdots}$ are uniquely determined (*independently of the choice of the family*). *As for $\Psi_{\beta\alpha}$, we have*

$$\begin{aligned}&\Psi_{\beta\alpha} \equiv \sum S_{\beta\mu\alpha\bar\nu}\,\omega_\mu\wedge\overline{\omega_\nu} + \sum S_{\beta\mu\alpha\nu}\,\omega_\mu\wedge\omega_\nu + \sum S_{\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu}\wedge\overline{\omega_\nu} \pmod{\omega},\\ &S_{\beta\mu\alpha\bar\nu} - \overline{S_{\alpha\nu\beta\bar\mu}} = 0, \quad S_{\beta\mu\alpha\nu} + \overline{S_{\alpha\bar\mu\beta\bar\nu}} = 0, \quad S_{\beta\bar\mu\alpha\bar\nu} + \overline{S_{\alpha\mu\beta\nu}} = 0,\end{aligned} \tag{3.8}$$

and there are relations

$$\begin{aligned}S_{\beta\mu\alpha\bar\nu} &= S_{\mu\beta\alpha\bar\nu} + 2S_{\bar\nu\mu\alpha\beta} = S_{\beta\mu\nu\bar\alpha} + 2\,\overline{S_{\bar\mu\nu\beta\alpha}},\\ S_{\beta\mu\alpha\nu} &= \frac{1}{2}(\overline{S_{\bar\nu\alpha\beta\bar\mu}} - \overline{S_{\bar\mu\alpha\beta\bar\nu}}), \quad S_{\beta\mu\alpha\nu} + S_{\mu\nu\alpha\beta} + S_{\nu\beta\alpha\mu} = 0.\end{aligned} \tag{3.9}$$

Remark: One may calculate the coefficients $S_{\bar\beta\cdots}$ concretely from the family given in the first step, and knows that they do not vanish in general.

Proof. As for $\Psi_{\bar\beta\alpha}$: The equations (2.12) and (2.13) imply $\Psi_{\bar\beta\alpha} + \Psi_{\bar\alpha\beta} = 0$, which yields (3.7). If $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi : \Psi_{\bar\beta\alpha})$ is changed into $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha,\ \psi' : \Psi'_{\bar\beta\alpha})$ by the transformation (3.4), then

$$\begin{aligned}\Psi'_{\bar\beta\alpha} &= d\phi_{\bar\beta\alpha} - \sum\phi_{\bar\beta\gamma}\wedge(\phi_{\gamma\alpha} - D_{\gamma\alpha}\,\omega) - \sum(\overline{\phi_{\beta\gamma}} - \overline{D_{\beta\gamma}}\,\omega)\wedge\phi'_{\bar\gamma\alpha}\\ &= \Psi_{\bar\beta\alpha} + \sum D_{\gamma\alpha}\,\phi_{\bar\beta\gamma}\wedge\omega - \sum\overline{D_{\beta\gamma}}\,\phi_{\bar\gamma\alpha}\wedge\omega.\end{aligned}$$

Hence, the coefficients $S_{\bar\beta\cdots}$ are uniquely determined. As for $\Psi_{\beta\alpha}$: Obviously we have

$$\Psi_{\beta\alpha} + \overline{\Psi_{\alpha\beta}} = \delta_{\alpha\beta}\,\omega\wedge\psi, \tag{3.10}$$

$$\begin{aligned}&\sum\Psi_{\beta\alpha}\wedge\omega_\beta + \sum\Psi_{\bar\beta\alpha}\wedge\overline{\omega_\beta}\\ &\quad = \{-d\phi_\alpha + \phi\wedge\phi_\alpha + \sum\phi_\beta\wedge\phi_{\beta\alpha} + \sum\overline{\phi_\beta}\wedge\phi_{\bar\beta\alpha}\}\wedge\omega,\end{aligned} \tag{3.11}$$

which imply (3.8), (3.9). ∎

Let us study the functions $D_{\beta\alpha}$ appearing in (3.4). We apply the transformation (3.4) to (3.5), (3.6) to get

$$\begin{aligned}
\Psi_{\bar\beta\alpha} &= \Psi'_{\bar\beta\alpha} + \sum \{\overline{D_{\beta\gamma}}\,\phi'_{\bar\gamma\alpha} - D_{\gamma\alpha}\,\phi'_{\bar\beta\gamma}\} \wedge \omega,\\
\Psi_{\beta\alpha} &= \Psi'_{\beta\alpha} + i \sum \{D_{\beta\alpha}\,\delta_{\mu\nu} + D_{\mu\alpha}\,\delta_{\nu\beta} - \overline{D_{\nu\beta}}\,\delta_{\mu\alpha} - \overline{D_{\nu\mu}}\,\delta_{\beta\alpha}\}\omega_\mu \wedge \overline{\omega_\nu}\\
&\quad + \{dD_{\beta\alpha} - \sum D_{\gamma\alpha}\,\phi'_{\beta\gamma} + \sum D_{\beta\gamma}\,\phi'_{\gamma\alpha}\\
&\qquad\qquad - i\,\overline{E_\alpha}\,\overline{\omega_\beta} - i\,E_\beta\,\omega_\alpha - i\,\delta_{\beta\alpha} \sum E_\nu\,\omega_\nu\} \wedge \omega.
\end{aligned}$$

The second formula says

$$S_{\beta\mu\alpha\bar\nu} = S'_{\beta\mu\alpha\bar\nu} + i(D_{\beta\alpha}\,\delta_{\mu\nu} + D_{\mu\alpha}\,\delta_{\nu\beta} - \overline{D_{\nu\beta}}\,\delta_{\mu\alpha} - \delta_{\beta\alpha}\,\overline{D_{\nu\mu}}),$$

which coincides with [5, (4.39)]. Accordingly, referring to the argument around [5, (4.43)], we know that $\sum_\alpha S'_{\alpha\mu\alpha\bar\nu}$ vanishes if and only if

$$(n+2)D_{\beta\alpha} = -i \sum S_{\rho\beta\rho\bar\alpha} + i\frac{\delta_{\beta\alpha}}{2(n+1)} \sum S_{\rho\gamma\rho\bar\gamma}$$

holds. Hence, we obtain the following.

Proposition 3.4 (cf. [5, Lemma 4.3]) *In Proposition* 3.2*, if the family* $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ *satisfies also*

$$\begin{aligned}
\text{(3.12)}\qquad d\phi_{\beta\alpha} &= \sum \phi_{\beta\gamma} \wedge \phi_{\gamma\alpha} + \sum \overline{\phi_{\bar\beta\gamma}} \wedge \phi_{\bar\gamma\alpha} + i\,\overline{\omega_\beta} \wedge \phi_\alpha\\
&\quad - i\,\overline{\phi_\beta} \wedge \omega_\alpha - i\,\delta_{\beta\alpha} \sum \overline{\phi_\nu} \wedge \omega_\nu + \sum S_{\beta\mu\alpha\bar\nu}\,\omega_\mu \wedge \overline{\omega_\nu}\\
&\quad + \sum S_{\beta\mu\alpha\nu}\,\omega_\mu \wedge \omega_\nu + \sum S_{\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu} \wedge \overline{\omega_\nu} + \lambda_{\beta\alpha} \wedge \omega,\\
\text{(3.13)}\qquad d\phi_{\bar\beta\alpha} &= \sum \phi_{\bar\beta\gamma} \wedge \phi_{\gamma\alpha} + \sum \overline{\phi_{\beta\gamma}} \wedge \phi_{\bar\gamma\alpha} + \sum S_{\bar\beta\mu\alpha\bar\nu}\,\omega_\mu \wedge \overline{\omega_\nu}\\
&\quad + \sum S_{\bar\beta\mu\alpha\nu}\,\omega_\mu \wedge \omega_\nu + \sum S_{\bar\beta\bar\mu\alpha\bar\nu}\,\overline{\omega_\mu} \wedge \overline{\omega_\nu} + \lambda_{\bar\beta\alpha} \wedge \omega,\\
\text{(3.14)}\qquad d\phi_\alpha &= \phi \wedge \phi_\alpha + \sum \phi_\beta \wedge \phi_{\beta\alpha} + \sum \overline{\phi_\beta} \wedge \phi_{\bar\beta\alpha}\\
&\quad + \sum \lambda_{\beta\alpha} \wedge \omega_\beta + \sum \lambda_{\bar\beta\alpha} \wedge \overline{\omega_\beta} + \epsilon_\alpha \wedge \omega,\\
\text{(3.15)}\qquad & \sum_\alpha S_{\alpha\mu\alpha\bar\nu} = 0,
\end{aligned}$$

then the $D_{\beta\alpha}$ *are uniquely determined. Further, such a family exists.*

Proof. Let us prove that such a family exists certainly. We take a family $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ which satisfies (2.11), (2.12), (2.13) and (2.14). It suffices to show that it has the formulas (3.12)–(3.14). First, Lemma 3.3 says it has (3.12), (3.13). Then, since (3.11) yields the equality

$$\begin{aligned}
&\{ - \sum \lambda_{\beta\alpha} \wedge \omega_\beta - \sum \lambda_{\bar\beta\alpha} \wedge \overline{\omega_\beta}\\
&\quad + d\phi_\alpha - \phi \wedge \phi_\alpha - \sum \phi_\beta \wedge \phi_{\beta\alpha} - \sum \overline{\phi_\beta} \wedge \phi_{\bar\beta\alpha}\} \wedge \omega = 0,
\end{aligned}$$

(3.14) holds inevitably. ∎

Next, let us study the functions $\lambda_{\beta\alpha}$ appearing in Proposition 3.4. Together with (3.8) and (3.10), the formula (3.12) implies $(\lambda_{\beta\alpha} + \overline{\lambda_{\alpha\beta}}) \wedge \omega = \Psi_{\beta\alpha} + \overline{\Psi_{\alpha\beta}} = \delta_{\beta\alpha}\, \omega \wedge \psi$. Thus, we have

$$\lambda_{\beta\alpha} + \overline{\lambda_{\alpha\beta}} + \delta_{\beta\alpha}\psi \equiv 0 \pmod{\omega}$$

(cf. [5, (4.46)]). Further, we have

$$\delta_{\beta\alpha} \sum \lambda_{\gamma\gamma} + \lambda_{\beta\alpha} - (n+1)\overline{\lambda_{\alpha\beta}} \equiv 0 \pmod{\omega,\, \omega_\rho,\, \overline{\omega_\rho}}$$

(cf. [5, (4.50)]), which is shown by investigating the terms $\sum\{\cdots\} \wedge \omega_\mu \wedge \overline{\omega_\nu}$ appearing in $\sum dd\phi_{\beta\beta}\, (=0)$. Hence, $\lambda_{\beta\alpha}$ can be expressed as

$$\lambda_{\beta\alpha} \equiv \sum V_{\beta\alpha\mu}\, \omega_\mu - \sum \overline{V_{\alpha\beta\mu}}\, \overline{\omega_\mu} - \frac{1}{2}\, \delta_{\beta\alpha}\psi \pmod{\omega}$$

(cf. [5, (4.51), (4.52)]). Consequently we obtain the corollary:

Corollary 3.5 (cf. [5, Lemma 4.3, (4.53), (4.54)]) *In Proposition* 3.2*, if the family* $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ *satisfies also* (2.15) *with* (2.19)*,* (2.16) *with* (2.20)*,* (2.17) *with* (2.21) *and* (3.15)*, then the* $D_{\beta\alpha}$ *are uniquely determined. Further, such a family exists.*

Proposition 3.2 is then reduced to the proposition:

Proposition 3.6 *Let* $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ *satisfy* (2.11)*,* (2.12)*,* (2.13)*,* (2.14)*,* (2.15) *with* (2.19)*,* (2.16) *with* (2.20)*,* (2.17) *with* (2.21)*, and* (3.15)*. Then, another family* $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ *satisfies those equations if and only if there exist relations*

$$\begin{gathered} \phi_{\beta\alpha} = \phi'_{\beta\alpha}, \quad \phi_{\bar\beta\alpha} = \phi'_{\bar\beta\alpha}, \quad \phi_\alpha = \phi'_\alpha + \overline{E_\alpha}\,\omega, \\ \psi = \psi' + G\,\omega + i \sum (E_\alpha\, \omega_\alpha - \overline{E_\alpha}\, \overline{\omega_\alpha}), \quad G \in \mathbb{R}. \end{gathered} \tag{3.16}$$

The third step: First, we want to investigate the functions E_α appearing in the transformation (3.16). Let $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ be such a family as in Proposition 3.6. Then we change (2.15) (with (2.19)) by (3.16) to get $V_{\beta\alpha\mu} = V'_{\beta\alpha\mu} - i(\delta_{\mu\alpha}E_\beta + \frac{1}{2}\,\delta_{\beta\alpha}E_\mu)$ and

$$\sum_\alpha V_{\beta\alpha\alpha} = \sum_\alpha V'_{\beta\alpha\alpha} - i\,(n + \frac{1}{2})\, E_\beta$$

(cf. [5, (4.57)]). (Since there are some extra terms in our case, the computations are much complicated more than those for [5, (4.56), (4.57)].) Hence, we have the proposition:

Proposition 3.7 (cf. [5, Lemma 4.4, (4.66)])

(1) *In Proposition* 3.6*, the family* $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ *satisfies also*

$$\sum_\alpha V_{\beta\alpha\alpha} = 0, \tag{3.17}$$

then the E_α *are uniquely determined. Further, such a family exists.*

(2) *Let* $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ *satisfy* (2.11)*,* (2.12)*,* (2.13)*,* (2.14)*,* (2.15) *with* (2.19)*,* (2.16) *with* (2.20)*,* (2.17) *with* (2.21)*,* (3.15) *and* (3.17)*. Then, another family* $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ *satisfies those equations if and only if there exist relations*

$$\phi_{\beta\alpha} = \phi'_{\beta\alpha}, \quad \phi_{\bar\beta\alpha} = \phi'_{\bar\beta\alpha}, \quad \phi_\alpha = \phi'_\alpha, \quad \psi = \psi' + G\,\omega \quad (G \in \mathbb{R}). \tag{3.18}$$

Next, we will investigate the function G appearing in the transformation (3.18). Let $(\phi_{\beta\alpha}, \phi_{\bar\beta\alpha}, \phi_\alpha, \psi)$ be such a family as in Proposition 3.7(2). We differentiate (2.14) to get

$$
\begin{aligned}
(3.19)\qquad 0 = \omega \wedge \{ &- d\psi + \phi \wedge \psi + 2i \sum \phi_\beta \wedge \overline{\phi_\beta} - i \sum \omega_\beta \wedge \overline{\eta_\beta} - i \sum \eta_\beta \wedge \overline{\omega_\beta}\} \\
&+ 2i \sum \phi_\alpha \wedge \omega_\beta \wedge \overline{\phi_{\bar\alpha\beta}} + 2i \sum \overline{\phi_\alpha} \wedge \phi_{\bar\alpha\beta} \wedge \overline{\omega_\beta} \\
&- i \sum \omega_\beta \wedge \overline{\lambda_{\bar\gamma\beta}} \wedge \omega_\gamma + i \sum \lambda_{\bar\gamma\beta} \wedge \overline{\omega_\gamma} \wedge \overline{\omega_\beta}.
\end{aligned}
$$

We have thus the formulas (2.18) and (2.22). Substituting (2.23) into (2.18) and then applying the transformation (3.18) to it, we have

$$
\begin{aligned}
d\psi' = \phi \wedge \psi' &+ 2i \sum \phi_\beta \wedge \overline{\phi_\beta} - i \sum (\delta_{\mu\nu}\, G + \widetilde{\eta}_{\nu\mu})\, \omega_\mu \wedge \overline{\omega_\nu} \\
&- i \sum \overline{\eta_{\mu\bar\nu}}\, \omega_\mu \wedge \omega_\nu - i \sum \eta_{\nu\bar\mu}\, \overline{\omega_\mu} \wedge \overline{\omega_\nu} - i \sum \overline{\eta_{\beta(0)}}\, \omega_\beta \wedge \phi \\
&- i \sum \eta_{\beta(0)}\, \phi \wedge \overline{\omega_\beta} - i \sum \overline{\eta_{\beta(\mu\nu)}}\, \omega_\beta \wedge \overline{\phi_{\mu\nu}} - i \sum \eta_{\beta(\mu\nu)}\, \phi_{\mu\nu} \wedge \overline{\omega_\beta} \\
&- i \sum \overline{\eta_{\beta(\mu)}}\, \omega_\beta \wedge \overline{\phi_\mu} - i \sum \eta_{\beta(\mu)}\, \phi_\mu \wedge \overline{\omega_\beta} - i \sum \overline{\eta_{\beta(\bar\mu)}}\, \omega_\beta \wedge \phi_\mu \\
&- i \sum \eta_{\beta(\bar\mu)}\, \overline{\phi_\mu} \wedge \overline{\omega_\beta} - i \sum \overline{\eta_{\beta(*)}}\, \omega_\beta \wedge \psi' - i \sum \eta_{\beta(*)}\, \psi' \wedge \overline{\omega_\beta} \\
&+ (\varrho - dG + 2G\,\phi - i \sum G\, \overline{\eta_{\beta(*)}}\, \omega_\beta + i \sum G\, \eta_{\beta(*)}\, \overline{\omega_\beta}) \wedge \omega.
\end{aligned}
$$

Hence, we get $\widetilde{\eta}'_{\nu\mu} = \delta_{\mu\nu}\, G + \widetilde{\eta}_{\nu\mu}$, that is,

$$\sum \widetilde{\eta}'_{\mu\mu} = nG + \sum \widetilde{\eta}_{\mu\mu}$$

(cf. [5, (4.69)] around). This means: In Proposition 3.7(2), if the family $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ satisfies also

$$\sum \widetilde{\eta}_{\mu\mu} = 0,$$

then the G is uniquely determined (cf. [5, Lemma 4.5]). Further, such a family exists certainly. Namely, the part of unique existence in Theorem 2.2 was proved.

The fourth step: Here we will prove the remaining assertions. Some of them are already obvious and the others are proved by straightforward computations.

On Theorem 2.2(1): Referring to Proposition 3.1, (2.6), (2.8) and (2.9), we have, modulo ω, ω_ρ, $\overline{\omega_\rho}$,

$$
\begin{aligned}
\phi_{\beta\alpha} &\equiv - \sum \overline{u_{\mu\beta}}\, du_{\mu\alpha} + \frac{1}{2}\, \delta_{\beta\alpha}\phi, \\
\phi_\alpha &\equiv -dv_\alpha + \sum v_\beta\, \overline{u_{\mu\beta}}\, du_{\mu\alpha} + \frac{1}{2}\, v_\alpha\, \phi, \\
\psi &\equiv -ds + i \sum \overline{v_\alpha}\, v_\beta\, \overline{u_{\mu\beta}}\, du_{\mu\alpha} - i \sum \overline{v_\beta}\, v_\alpha\, u_{\mu\beta}\, \overline{du_{\mu\alpha}} \\
&\qquad - i \sum \overline{v_\alpha}\, dv_\alpha + i \sum v_\alpha\, \overline{dv_\alpha} - s\phi,
\end{aligned}
$$

which imply Theorem 2.2(1).

On Theorem 2.2(2): It has been proved in Lemma 3.3.

On the formula (2.26) for $\phi_{\bar\beta\alpha}$: Proposition 3.2 implies that the $\phi_{\bar\beta\alpha}$ satisfying the conditions in Theorem 2.2 are identifed as the ones given in Proposition 3.1.

On the formula (2.27) for $\lambda_{\bar\beta\alpha}$: Refer to (3.19). We notice that the right hand side with the term $\omega\wedge\{\cdots\}$ removed vanishes, that is,

$$\begin{aligned}2\sum\phi_\alpha\wedge\omega_\beta\wedge\overline{\phi_{\bar\alpha\beta}}+2\sum\overline{\phi_\alpha}\wedge\phi_{\bar\alpha\beta}\wedge\overline{\omega_\beta}\\ -\sum\omega_\beta\wedge\overline{\lambda_{\bar\gamma\beta}}\wedge\omega_\gamma+\sum\lambda_{\bar\gamma\beta}\wedge\overline{\omega_\gamma}\wedge\overline{\omega_\beta}=0.\end{aligned}$$

The left hand side is equal to

$$\sum\big(\overline{\lambda_{\bar\gamma\beta}}+\sum\overline{b_{\bar\gamma\beta\bar\alpha}}\,\phi_\alpha\big)\wedge\omega_\beta\wedge\omega_\gamma-\sum\big(\lambda_{\bar\gamma\beta}+\sum b_{\bar\gamma\beta\bar\alpha}\,\overline{\phi_\alpha}\big)\wedge\overline{\omega_\beta}\wedge\overline{\omega_\gamma}.$$

Hence, we obtain the formula.

On the formula (2.28) for η_α (cf. [5, (4.61) and (4.62)]): We differentiate the left hand side of (2.17) and focus only on the sum $\sum\{\cdots\}\wedge\omega_\mu\wedge\overline{\omega_\nu}$ $(=0)$. By ignoring the difference of ω-, ω_ρ- and $\overline{\omega_\rho}$-terms in the $\{\cdots\}$, it is equal to

$$\begin{aligned}(3.20)\quad &\sum\Big\{-\frac{i}{2}\,\delta_{\mu\alpha}\eta_\nu-i\,\delta_{\mu\nu}\eta_\alpha+\Lambda_{\alpha\mu\bar\nu}+\overline{dV_{\alpha\mu\nu}}+\sum\overline{V_{\gamma\mu\nu}}\,\phi_{\gamma\alpha}-\overline{V_{\alpha\mu\nu}}\,\phi\\ &-\sum\overline{V_{\alpha\mu\gamma}}\,\overline{\phi_{\nu\gamma}}-\sum\overline{V_{\alpha\gamma\nu}}\,\phi_{\mu\gamma}-\sum S_{\gamma\mu\alpha\bar\nu}\,\phi_\gamma-\sum S_{\bar\gamma\mu\alpha\bar\nu}\,\overline{\phi_\gamma}\Big\}\wedge\omega_\mu\wedge\overline{\omega_\nu},\end{aligned}$$

where we put

$$\Lambda_{\alpha\mu\bar\nu}=\sum\big(b_{\bar\nu\rho\bar\alpha}\,\overline{b_{\bar\mu\rho\bar\beta}}-b_{\bar\nu\alpha\bar\rho}\,\overline{b_{\bar\beta\rho\bar\mu}}\big)\,\phi_\beta+(db_{\bar\nu\alpha\bar\beta})_\mu\,\overline{\phi_\beta}.$$

Setting $\mu=\nu$ in the terms $\{\cdots\}$ of (3.20) and summing them up, we have

$$i\,\frac{2n+1}{2}\,\eta_\alpha\equiv\sum\Lambda_{\alpha\mu\bar\mu}\quad(\mathrm{mod}\ \omega,\,\omega_\rho,\,\overline{\omega_\rho}),$$

which gives the formula (2.28).

On the formula (2.29) for ϱ (cf. [5, (4.64) and (4.72)]): Let us differentiate the left hand side of (2.18) and focus only on the sum $\sum\{\cdots\}\wedge\omega_\mu\wedge\overline{\omega_\nu}$ $(=0)$. Then, in a way similar to the above, modulo ω, ω_ρ, $\overline{\omega_\rho}$, we have

$$\begin{aligned}i\,\delta_{\mu\nu}\varrho\equiv\;&2i\sum\overline{V_{\beta\mu\nu}}\,\overline{\phi_\beta}+2i\sum V_{\beta\nu\mu}\,\phi_\beta-i\,d\eta_{\nu\mu}-i\,\overline{d\eta_{\mu\nu}}\\ &+\frac{i}{2}\,\eta_{\nu(\mu)}\,\psi+\frac{i}{2}\,\overline{\eta_{\mu(\nu)}}\,\psi+i\sum\overline{b_{\bar\mu\rho\bar\beta}}\,\eta_{\nu(\bar\rho)}\,\phi_\beta+i\sum b_{\bar\nu\rho\bar\beta}\,\overline{\eta_{\mu(\bar\rho)}}\,\overline{\phi_\beta}\\ &+i\sum\eta_{\nu\beta}\,\phi_{\mu\beta}+i\sum\overline{\eta_{\mu\beta}}\,\overline{\phi_{\nu\beta}}\\ &-i\sum\overline{b_{\bar\beta\rho\bar\mu}}\,\eta_{\nu(\bar\rho)}\,\phi_\beta-i\sum b_{\bar\beta\rho\bar\nu}\,\overline{\eta_{\mu(\bar\rho)}}\,\overline{\phi_\beta}\\ &+i\,\eta_{\nu\mu}\,\phi+i\,\overline{\eta_{\mu\nu}}\,\phi+i\sum\eta_{\beta\mu}\,\overline{\phi_{\nu\beta}}+i\sum\overline{\eta_{\beta\nu}}\,\phi_{\mu\beta}\\ &+i\sum(d\eta_{\nu(\beta)})_\mu\,\phi_\beta+i\sum\overline{(d\eta_{\mu(\bar\beta)})_\nu}\,\phi_\beta\\ &+i\sum(d\eta_{\nu(\bar\beta)})_\mu\,\overline{\phi_\beta}+i\sum\overline{(d\eta_{\mu(\beta)})_\nu}\,\overline{\phi_\beta}.\end{aligned}$$

The sum of the right hand side with $\mu=\nu$ is equal to

$$\begin{aligned}&\frac{i}{2}\sum\big(\eta_{\mu(\mu)}+\overline{\eta_{\mu(\mu)}}\big)\,\psi+i\sum\Big\{(d\eta_{\mu(\beta)})_\mu+\overline{(d\eta_{\mu(\bar\beta)})_\mu}+\sum\overline{b_{\bar\mu\beta\bar\rho}}\,\eta_{\mu(\bar\rho)}\Big\}\,\phi_\beta\\ &\qquad+i\sum\Big\{\overline{(d\eta_{\mu(\beta)})_\mu}+(d\eta_{\mu(\bar\beta)})_\mu+\sum b_{\bar\mu\beta\bar\rho}\,\overline{\eta_{\mu(\bar\rho)}}\Big\}\,\overline{\phi_\beta}.\end{aligned}$$

Thus, we obtain (2.29).

On Theorem 2.2(4) (cf. [5, Appendix: Bianchi identities]): In a way similar to the computations for [5, $(A.4'')$], we find

$$\begin{aligned}
0 &= d\Phi_{\beta\alpha} + \sum \Phi_{\beta\gamma}\wedge\phi_{\gamma\alpha} - \sum \phi_{\beta\gamma}\wedge\Phi_{\gamma\alpha} - i\,\overline{\omega_\beta}\wedge\Phi_\alpha - i\,\overline{\Phi_\beta}\wedge\omega_\alpha\\
&\quad - \frac{i}{2}\,\delta_{\beta\alpha}\{\sum\Phi_\gamma\wedge\overline{\omega_\gamma} + \sum\overline{\Phi_\gamma}\wedge\omega_\gamma\}\\
&\quad + \sum\{\overline{\Phi_{\bar\beta\gamma}} - i\,\overline{\omega_\beta}\wedge\overline{\phi_\gamma} + i\,\overline{\phi_\beta}\wedge\overline{\omega_\gamma}\}\wedge\phi_{\bar\gamma\alpha}\\
&\quad - \sum\overline{\phi_{\bar\beta\gamma}}\wedge\{\Phi_{\bar\gamma\alpha} - i\,\omega_\gamma\wedge\phi_\alpha + i\,\phi_\gamma\wedge\omega_\alpha\}\\
&\quad - \delta_{\beta\alpha}\{\frac{i}{2}\sum\overline{\lambda_{\bar\rho\gamma}}\wedge\omega_\rho\wedge\omega_\gamma - \frac{i}{2}\sum\lambda_{\bar\rho\gamma}\wedge\overline{\omega_\rho}\wedge\overline{\omega_\gamma}\}\\
&\quad + i\,\delta_{\beta\alpha}\sum\phi_{\bar\gamma\rho}\wedge\overline{\phi_\rho}\wedge\overline{\omega_\gamma} - i\,\delta_{\beta\alpha}\sum\overline{\phi_{\bar\gamma\rho}}\wedge\omega_\rho\wedge\phi_\gamma.
\end{aligned}$$

Let us substitute (2.19)–(2.21) to it and sort out the terms. Then, referring to the sum $\sum\{\cdots\}\,\omega_\gamma\wedge\omega_\mu\wedge\overline{\omega_\nu}$, we obtain (2.30) and (2.31). Similarly, referring to the sums $\sum\{\cdots\}\,\omega_\mu\wedge\omega_\nu\wedge\omega$ and $\sum\{\cdots\}\,\omega_\mu\wedge\overline{\omega_\nu}\wedge\omega$, we obtain (2.32) and (2.33) respectively.

4 The Chern-Moser connection

With reference to the arguments in [5, §5] and [8, §4], in this section we will construct a Cartan connection on the Cartan principal bundle $\pi : Y \to M$ for general J.

First, let us set

$$(4.1)\qquad \Omega_{\theta^\bullet} = \begin{pmatrix} -\frac{1}{n+2}(\sum\overline{\phi_{\gamma\gamma}}+\phi) & -\frac12\phi_\alpha & -\frac14\psi\\ -2i\,\overline{\omega_\beta} & -\phi_{\beta\alpha}+\frac{\delta_{\beta\alpha}}{n+2}(\sum\phi_{\gamma\gamma}+\phi) & -i\,\overline{\phi_\beta}\\ 2\omega & \omega_\alpha & \frac{1}{n+2}(\sum\phi_{\gamma\gamma}+\phi)\end{pmatrix}$$

$$\in \Gamma(\mathfrak{su}(n+1,1)\otimes T^*(\pi_1^{-1}(U_{\theta^\bullet}\times\mathbb{R}^+)))$$

(cf. [5, (5.30)], [8, (4.9)]). Then, Theorem 2.2 implies

$$\begin{aligned}
(4.2)\quad F(\Omega_{\theta^\bullet}) &:= d\Omega_{\theta^\bullet} + \Omega_{\theta^\bullet}\wedge\Omega_{\theta^\bullet}\\
&= \begin{pmatrix} -\frac{1}{n+2}\sum\overline{\Phi_{\mu\mu}} & -\frac12\,\Phi_\alpha & -\frac14\Psi\\ 0 & -\Phi_{\beta\alpha}+\frac{\delta_{\beta\alpha}}{n+2}\sum\Phi_{\mu\mu} & -i\,\overline{\Phi_\beta}\\ 0 & 0 & \frac{1}{n+2}\sum\Phi_{\mu\mu}\end{pmatrix}\\
&\quad + \begin{pmatrix} -\frac{1}{n+2}\,\phi_{\bar\mu\gamma}\wedge\overline{\phi_{\bar\gamma\mu}} & -\frac12\,\overline{\phi_\gamma}\wedge\phi_{\bar\gamma\alpha} & 0\\ -2i\,\omega_\gamma\wedge\overline{\phi_{\bar\gamma\beta}} & -\overline{\phi_{\bar\beta\gamma}}\wedge\phi_{\bar\gamma\alpha}+\frac{\delta_{\beta\alpha}}{n+2}\,\overline{\phi_{\bar\mu\gamma}}\wedge\phi_{\bar\gamma\mu} & -i\,\phi_\gamma\wedge\overline{\phi_{\bar\gamma\beta}}\\ 0 & \overline{\omega_\gamma}\wedge\phi_{\bar\gamma\alpha} & \frac{1}{n+2}\,\overline{\phi_{\bar\mu\gamma}}\wedge\phi_{\bar\gamma\mu}\end{pmatrix}
\end{aligned}$$

(cf. [5, (5.33)–(5.35)], [8, (4.10)]), where the summation symbols $\sum_\gamma$, $\sum_{\gamma,\mu}$ are omitted in the matrix of the last line.

Now, we will take another unitary frame $\theta'^\bullet$ defined by

$$\theta'^\bullet = (\theta, \sum U^M_{\gamma 1}\,\theta^\gamma, \ldots, \sum U^M_{\gamma n}\,\theta^\gamma),\quad U^M \in C^\infty(U_{\theta^\bullet}\cap U_{\theta'^\bullet}, U(n)).$$

We want to clarify the relations between $\Omega_{\theta'^\bullet}$, $F(\Omega_{\theta'^\bullet})$ and the ones given above. The two families $(\omega^E, \omega^E_\alpha, \overline{\omega^E_\alpha}, \phi^E)$ and $(\omega'^E, \omega'^E_\alpha, \overline{\omega'^E_\alpha}, \phi'^E)$ (i.e., (2.2) for $\theta'^\bullet$) have the transition function $\Lambda(U^M,0,0) \in C^\infty(U_{\theta^\bullet} \cap U_{\theta'^\bullet}, G_1)$, which cannot be the one between $\Omega_{\theta^\bullet}$ and $\Omega_{\theta'^\bullet}$ obviously but provides it as follows: We will refer mainly to the argument by Le ([8], in the case $n = 1$), which is well-prepared for our discussion. Let us consider the grading of the Lie algebra $\mathfrak{su} = \mathfrak{su}(n+1,1)$

$$\mathfrak{su} = \sum_{p\in\mathbb{Z}} \mathfrak{su}_p, \quad \mathfrak{su}_p := \{A \in \mathfrak{su} \mid [E,A] = pA\}, \quad [\mathfrak{su}_p, \mathfrak{su}_q] \subset \mathfrak{su}_{p+q},$$

where E is the matrix with $E_{00} = -E_{n+1,n+1} = 1$ and $E_{ij} = 0$ (otherwise). Notice that $\mathfrak{su}_p = \{0\}$ if $|p| > 2$. The Lie subgroup $\widetilde{H}$ corresponding to the Lie subalgebra $\widetilde{\mathcal{H}} := \sum_{p\geq 0} \mathfrak{su}_p$ consists of matrices

$$\{t,(U_{\beta\alpha}),(\tau_\alpha),r\} := \begin{pmatrix} t & t\,\tau_\beta\,U_{\beta\alpha} & t(r + i\sum|\tau_\alpha|^2) \\ 0 & U_{\beta\alpha} & 2i\,\overline{\tau_\beta} \\ 0 & 0 & 1/\bar{t} \end{pmatrix}$$

$$\left(t \in \mathbb{C}^*,\ (\tau_\alpha) \in \mathbb{C}^n,\ r \in \mathbb{R},\ (U_{\beta\alpha}) \in U(n),\ \det(U_{\beta\alpha}) = \bar{t}/t\right),$$

and its subgroup $\widetilde{H}_1 := \{\{t,(U_{\beta\alpha}),(\tau_\alpha),r\} \mid |t| = 1\}$ has a surjective homomorphism

$$\zeta : \widetilde{H}_1 \to G_1, \quad \{t,(U_{\beta\alpha}),(\tau_\alpha),r\} \mapsto \Lambda(\bar{t}(U_{\beta\alpha}), (2t\sum \tau_\beta\,U_{\beta\alpha}), 4r),$$
$$\ker\zeta = \{\{\varepsilon, \varepsilon E_n, 0, 0\} \in \widetilde{H} \mid \varepsilon^{n+2} = 1,\ \varepsilon \in \mathbb{C}\} = \{\{\varepsilon, \varepsilon E_n, 0, 0\} \in \widetilde{H}_1\}.$$

Via the identification

$$\zeta : H_1 := \widetilde{H}_1/\ker\zeta \ \cong\ G_1,$$

let us regard $\pi_1 : Y \to E$ as a principal H_1-bundle. Accordingly, the function $\Lambda(U^M,0,0) \in C^\infty(U_{\theta^\bullet} \cap U_{\theta'^\bullet}, G_1)$ induces

$$\lambda_{(\theta^\bullet\theta'^\bullet)} := \{1,\ U^M,\ 0,\ 0\} \in C^\infty(U_{\theta^\bullet} \cap U_{\theta'^\bullet}, H_1).$$

Here we specify it by a representative. Note that the following result does not change even if we adopt another representative.

Proposition 4.1 (cf. [8, (4.17)], [5, (5.40)]) *On $\pi_1^{-1}(U_{\theta^\bullet} \cap U_{\theta'^\bullet} \times \mathbb{R}^+)$, we have*

$$\Omega_{\theta'^\bullet} = \lambda^{-1}_{(\theta^\bullet\theta'^\bullet)}\Omega_{\theta^\bullet}\lambda_{(\theta^\bullet\theta'^\bullet)} + \lambda^{-1}_{(\theta^\bullet\theta'^\bullet)}d\lambda_{(\theta^\bullet\theta'^\bullet)}, \quad F(\Omega_{\theta'^\bullet}) = \lambda^{-1}_{(\theta^\bullet\theta'^\bullet)}F(\Omega_{\theta^\bullet})\lambda_{(\theta^\bullet\theta'^\bullet)}, \tag{4.3}$$

where we regard $\lambda_{(\theta^\bullet\theta'^\bullet)}$ as a function on the neighborhood naturally.

Proof. The family $(\omega', \omega'_\alpha, \overline{\omega'_\alpha}, \phi')$ (i.e., (2.3) for $\theta'^\bullet$) is certainly written as

$$\omega' = \omega, \quad \omega'_\alpha = \sum U^M_{\kappa\alpha}\,\omega_\kappa, \quad \phi' = \phi. \tag{4.4}$$

The corresponding family $(\phi'_{\beta\alpha}, \phi'_{\bar\beta\alpha}, \phi'_\alpha, \psi')$ is then given by

$$\begin{aligned} \phi'_{\beta\alpha} &= \sum \overline{U^M_{\rho\beta}}\,U^M_{\kappa\alpha}\,\phi_{\rho\kappa} - \sum \overline{U^M_{\rho\beta}}\,dU^M_{\rho\alpha}, \\ \phi'_{\bar\beta\alpha} &= \sum U^M_{\rho\beta}\,U^M_{\kappa\alpha}\,\phi_{\bar\rho\kappa}, \quad \phi'_\alpha = \sum U^M_{\kappa\alpha}\,\phi_\kappa, \quad \psi' = \psi, \\ \Phi'_{\beta\alpha} &= \sum \overline{U^M_{\rho\beta}}\,U^M_{\kappa\alpha}\,\Phi_{\rho\kappa}, \quad \Phi'_{\bar\beta\alpha} = \sum U^M_{\rho\beta}\,U^M_{\kappa\alpha}\,\Phi_{\bar\rho\kappa}, \\ \Phi'_\alpha &= \sum U^M_{\kappa\alpha}\Phi_\kappa, \quad \Psi' = \Psi, \quad \eta'_\alpha = \sum U^M_{\kappa\alpha}\,\eta_\kappa, \quad \varrho' = \varrho. \end{aligned} \tag{4.5}$$

In order to show it, because of the uniqueness it suffices to ascertain that they satisfy the conditions in Theorem 2.2. We will omit the calculations: In checking the condition $\sum_\mu \widetilde{\eta}'_{\mu\mu} = 0$ (cf. (2.24)), recall that the form η_α has no $\phi_{\mu\nu}$-term (cf. (2.28)). The relations (4.4) and (4.5) imply certainly the formulas (4.3). ∎

Notice that $H_1 = \widetilde{H}_1/\ker\zeta$ is a subgroup of $G := SU(n+1,1)/\ker\zeta$ and consider the bigger subgroup $H := \widetilde{H}/\ker\zeta$. Via the identifications (cf. [8, p.251])

$$\mathbb{R}^+ \times \widetilde{H}_1 \;\cong\; \widetilde{H}, \quad \mathbb{R}^+ \times H_1 \;\cong\; H, \quad (u, \{t, U, \tau, r\}) \mapsto \{\sqrt{u}\, t, U, \tau, r\},$$

let us regard the bundle $\pi : Y \to M$ as a principal H-bundle with the trivializations

$$\pi^{-1}(U_{\theta^\bullet}) = \pi_1^{-1}(U_{\theta^\bullet} \times \mathbb{R}^+) \;\cong\; U_{\theta^\bullet} \times \mathbb{R}^+ \times H_1 \;\cong\; U_{\theta^\bullet} \times H \tag{4.6}$$

(cf. [8, Theorem 4.1]). We have the cross-sections (via the trivializations)

$$\sigma_{\theta^\bullet} : U_{\theta^\bullet} \to U_{\theta^\bullet} \times H \;\cong\; \pi^{-1}(U_{\theta^\bullet}), \quad x \mapsto (x, e) \leftrightarrow (\omega^E, \omega^E_\alpha, \overline{\omega^E_\alpha}, \phi^E)(x, 1) \cdot e$$

(e is the unit element of H or G_1) with the transition functions

$$\lambda_{(\sigma_{\theta^\bullet}\sigma_{\theta'^\bullet})} := \lambda_{(\theta^\bullet\theta'^\bullet)} \in C^\infty(U_{\theta^\bullet} \cap U_{\theta'^\bullet}, H).$$

Let us consider the (sub-) algebras $\mathcal{G} := \mathfrak{su} = \sum \mathfrak{su}_p$, $\mathcal{H} := \widetilde{\mathcal{H}} = \sum_{p\geq 0} \mathfrak{su}_p$ corresponding to G, H, and decompose $\mathcal{G}$ into

$$\mathcal{G} = \mathcal{H} + \mathcal{M}, \quad \mathcal{M} := \sum_{p<0} \mathfrak{su}_p = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ -2i\,\overline{c_\beta} & 0 & 0 \\ s & c_\alpha & 0 \end{pmatrix} \;\middle|\; \begin{matrix} (c_\alpha) \in \mathbb{C}^n, \\ s \in \mathbb{R} \end{matrix} \right\}.$$

Proposition 4.2 (cf. [8, (2.3)]) *The family $\Omega_M := \{\Omega_{\sigma_{\theta^\bullet}} := \sigma^*_{\theta^\bullet}\Omega_{\theta^\bullet} \in \Gamma(\mathcal{G} \otimes T^*U_{\theta^\bullet})\}_{\theta^\bullet}$ satisfies*

$$\Omega_{\sigma_{\theta^\bullet}} = \mathrm{Ad}(\lambda^{-1}_{(\sigma_{\theta^\bullet}\sigma_{\theta'^\bullet})})\,\Omega_{\sigma_{\theta^\bullet}} + \lambda^*_{(\sigma_{\theta^\bullet}\sigma_{\theta'^\bullet})}\Omega_H \quad (\text{on } U_{\theta^\bullet} \cap U_{\theta'^\bullet}), \tag{4.7}$$

$$\mathrm{proj} \circ \Omega_{\sigma_{\theta^\bullet}} : T_x M \;\cong\; \mathcal{G}/\mathcal{H}, \tag{4.8}$$

where Ω_H is the Maurer-Cartan form of H.

Proof. The first equality of (4.3) implies (4.7). As for (4.8): For a vector $X \in T_x M$, $(\mathrm{proj} \circ \Omega_{\sigma_{\theta^\bullet}})(X)$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ -2i\,\overline{\omega_\beta(\sigma_{\theta^\bullet *}X)} & 0 & 0 \\ 2\omega(\sigma_{\theta^\bullet *}X) & \omega_\alpha(\sigma_{\theta^\bullet *}X) & 0 \end{pmatrix}$$

as an element of $\mathcal{M}$ (cf. (4.1)). The map $\mathrm{proj} \circ \Omega_{\sigma_{\theta^\bullet}}$ is thus certainly isomorphic. ∎

Hence, by a standard argument (e.g. [8, Proposition 2.1], [7, II.1]), we have the theorem:

Theorem 4.3 (cf. [8, Theorem 4.2]) *The principal bundle connection $\Omega_Y \in \Gamma(\mathcal{G}\otimes TY)$ on the Cartan principal bundle $\pi : Y \to M$ induced from Ω_M, i.e.,*

$$\begin{aligned}\Omega_Y(X, V_h) &= \mathrm{Ad}(h^{-1})\,\Omega_{\sigma_{\theta^\bullet}}(X) + \Omega_H(V_h) \\ &= \mathrm{Ad}(h^{-1})\,\Omega_{\theta^\bullet}(\iota_* X) + \Omega_H(V_h) \quad (\iota := \sigma_{\theta^\bullet}) \\ &\big(T_{\iota(x)\cdot h}Y = T_xM \oplus T_hH \ni (X, V_h) \quad (\textit{via } (4.6))\big),\end{aligned} \tag{4.9}$$

is a Cartan connection of type G/H. Namely, it satisfies

$$\begin{gathered}\Omega_Y(A^*) = A\ (A \in \mathcal{H}), \quad R_h^*\Omega_Y = \mathrm{Ad}(h^{-1})\,\Omega_Y\ (h \in H), \\ \Omega_Y : T_yY \cong \mathcal{G}\ (y \in Y).\end{gathered} \tag{4.10}$$

Further, we have the following.

Theorem 4.4 *The Chern-Moser connection Ω_Y is normal in the sense of Tanaka ([12]) if and only if J is integrable.*

The Maurer-Cartan connection Ω_H is flat and the curvature $F(\Omega_Y)$ is expressed as

$$\begin{aligned}F(\Omega_Y)((X, V_h), (X', V'_h)) &= \mathrm{Ad}(h^{-1})\,F(\Omega_{\sigma_{\theta^\bullet}})(X, X') \\ &= \mathrm{Ad}(h^{-1})F(\Omega_{\theta^\bullet})(\iota_*X, \iota_*X').\end{aligned} \tag{4.11}$$

If the curvature satisfies the condition (5.1), we say that the connection is normal.

5 The proof of Theorem 4.4

With regard to the proof of Theorem 4.4, we refer to [8, §5] constantly. The subspaces $\mathfrak{su}_2$; $\mathfrak{su}_{-2}$; $\mathfrak{su}_1$; $\mathfrak{su}_{-1}$; $\mathfrak{su}_0$ of $\mathfrak{su} = \sum_{-2\le p\le 2}\mathfrak{su}_p$ have bases e_2^1; e_{-2}^1; $e_1^1, \dots, e_1^{2n}$; $e_{-1}^1, \dots, e_{-1}^{2n}$; $e_0^0, e_0^{(11)}, \dots, e_0^{(nn)}, e_0^{(\beta\alpha:R)}, e_0^{(\beta\alpha:I)}$ $(\beta > \alpha)$ given by

$$e_2^1 = \frac{1}{2(n+2)}\left(\begin{array}{c|c|c} 0 & 0\cdots 0 & 1 \\ \hline 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & 0\cdots 0 & 0 \end{array}\right), \quad e_{-2}^1 = \left(\begin{array}{c|c|c} 0 & 0\cdots 0 & 0 \\ \hline 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 1 & 0\cdots 0 & 0 \end{array}\right),$$

$$e_1^1 = \frac{1}{8(n+2)}\left(\begin{array}{c|c|c} 0 & i\ 0\cdots 0 & 0 \\ \hline 0 & & 2 \\ 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & 0\ 0\cdots 0 & 0 \end{array}\right), \quad e_1^2 = \frac{-1}{8(n+2)}\left(\begin{array}{c|c|c} 0 & 1\ 0\cdots 0 & 0 \\ \hline 0 & & 2i \\ 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & 0\ 0\cdots 0 & 0 \end{array}\right), \dots,$$

$$e_{-1}^1 = \left(\begin{array}{c|c|c} 0 & 0\ 0\cdots 0 & 0 \\ \hline -2i & & 0 \\ 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & 1\ 0\cdots 0 & 0 \end{array}\right), \quad e_{-1}^2 = \left(\begin{array}{c|c|c} 0 & 0\ 0\cdots 0 & 0 \\ \hline -2 & & 0 \\ 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & i\ 0\cdots 0 & 0 \end{array}\right), \dots,$$

$$e_0^0 = \left(\begin{array}{c|c|c} 1 & 0\cdots 0 & 0 \\ \hline 0 & & 0 \\ \vdots & O & \vdots \\ 0 & & 0 \\ \hline 0 & 0\cdots 0 & -1 \end{array}\right), \quad e_0^{(\beta\beta)} = \left(\begin{array}{c|c|c} -i & 0\cdots 0 & 0 \\ \hline 0 & & 0 \\ \vdots & E_0^{(\beta\beta)} & \vdots \\ 0 & & 0 \\ \hline 0 & 0\cdots 0 & -i \end{array}\right),$$

$$\underset{(\beta\neq\alpha)}{e_0^{(\beta\alpha:R)}} = \left(\begin{array}{c|c|c} 0 & 0\cdots 0 & 0 \\ \hline 0 & & 0 \\ \vdots & E_0^{(\beta\alpha:R)} & \vdots \\ 0 & & 0 \\ \hline 0 & 0\cdots 0 & 0 \end{array}\right), \quad \underset{(\beta\neq\alpha)}{e_0^{(\beta\alpha:I)}} = \left(\begin{array}{c|c|c} 0 & 0\cdots 0 & 0 \\ \hline 0 & & 0 \\ \vdots & E_0^{(\beta\alpha:I)} & \vdots \\ 0 & & 0 \\ \hline 0 & 0\cdots 0 & 0 \end{array}\right),$$

where $E_0^{(\beta\beta)}$, etc., are the $n\times n$-matrices defined by $(E_0^{(\beta\beta)})_{\mu\nu} = 2i$ (if $(\mu,\nu) = (\beta,\beta)$), $= 0$ (otherwise); $(E_0^{(\beta\alpha:R)})_{\mu\nu} = 1$ (if $(\mu,\nu) = (\beta,\alpha)$), $= -1$ (if $(\mu,\nu) = (\alpha,\beta)$), $= 0$ (otherwise); $(E_0^{(\beta\alpha:I)})_{\mu\nu} = i$ (if $(\mu,\nu) = (\beta,\alpha)$ or (α,β)), $= 0$ (otherwise). Gathering them all, we obtain a basis $\{e_p^j\}$ of $\mathcal{G} = \mathfrak{su}$ which satisfies $B(e_{-p}^i, e_p^j) = \delta_{ij}$, where B is the Killing form.

We say the Chern-Moser connection Ω_Y is normal ([12, §4.4]) if

$$[e_2^1,\, \Omega_Y^{-1}(e_{-2}^1)\rfloor F(\Omega_Y)] + \sum [e_1^j,\, \Omega_Y^{-1}(e_{-1}^j)\rfloor F(\Omega_Y)] = 0. \tag{5.1}$$

Since the normality of the connection does not depend on the choice of such a basis of $\mathcal{G}$, it suffices to check the condition at each point of $\iota(U_{\theta^\bullet})$. Let us argue at a point of $\iota(U_{\theta^\bullet})$. Thus, we have

$$\begin{aligned} \Omega_Y(X,V) = \Omega_{\theta^\bullet}(\iota_* X) + \Omega_H(V) &= \begin{pmatrix} * & * & * \\ -2i\,\overline{\omega_\beta} & * & * \\ 2\omega & \omega_\alpha & * \end{pmatrix}(X,V) \\ &=: \begin{pmatrix} \Omega_0 & \frac{i}{8(n+2)}\Omega_1^\alpha & \frac{1}{2(n+2)}\Omega_2 \\ -2i\,\overline{\Omega_{-1}^\beta} & \Omega_0^{\beta\alpha} & \frac{1}{4(n+2)}\overline{\Omega_1^\beta} \\ \Omega_{-2} & \Omega_{-1}^\alpha & -\overline{\Omega_0} \end{pmatrix}(X,V), \end{aligned}$$

$$\begin{gathered} F(\Omega_Y)((X,V),(X',V')) = F(\Omega_{\theta^\bullet})(\iota_* X, \iota_* X') \\ \big(T_{\iota(x)}Y = T_xM \oplus T_eH \ni (X,V),(X',V')\big) \end{gathered} \tag{5.2}$$

(cf. (4.9), (4.11)). Denoting by $\{\eta_p^j\}$ the dual frame of $\{\Omega_Y^{-1}(e_p^j)\}$, we have

$$\Omega_Y = \sum e_p^j \otimes \eta_p^j$$

$$\left(\Omega_1^\alpha = \eta_1^{2\alpha-1} + i\eta_1^{2\alpha},\ \Omega_{-1}^\alpha = \eta_{-1}^{2\alpha-1} + i\eta_{-1}^{2\alpha},\ \Omega_2 = \eta_2^1,\ \Omega_{-2} = \eta_{-2}^1\right),$$

$$F(\Omega_Y) = \sum A_k\, \eta_{-1}^k \wedge \eta_{-2}^1 + \sum B_{k\ell}\, \eta_{-1}^k \wedge \eta_{-1}^\ell + \cdots \quad (A_k,\ B_{k\ell} = -B_{\ell k} \in \mathfrak{su}). \tag{5.3}$$

By definition, we have the following.

Lemma 5.1 *If Ω_Y is normal, then*

$$\sum [e_1^k, A_k] = 0, \quad [e_2^1, A_\ell] - 2\sum [e_1^k, B_{k\ell}] = 0 \tag{5.4}$$

for all ℓ. Moreover, if there is no extra part $+\cdots$ in the expansion (5.3)*, then the converse is also true.*

We set

$$A_k = \sum_p A_{k:p} = \sum_p \sum_j a^j_{k:p}\, e^j_p, \quad B_{k\ell} = \sum_p B_{k\ell:p} = \sum_p \sum_j b^j_{k\ell:p}\, e^j_p, \tag{5.5}$$

where, in the case $p = 0$, we assume that the superscript j runs all over in the set $\{0, (11), \ldots, (nn), (\beta\alpha : R), (\beta\alpha : I)\}$ and $a^{(\beta\alpha:R)}_{k:0} = -a^{(\alpha\beta:R)}_{k:0}$, $a^{(\beta\alpha:I)}_{k:0} = a^{(\alpha\beta:I)}_{k:0}$, $a^{(\beta\beta:R)}_{k:0} = a^{(\beta\beta:I)}_{k:0} = 0$, etc., (and $e^{(\beta\beta:R)}_0 = e^{(\beta\beta:I)}_0 = 0$). Then we have the following.

Proposition 5.2 *If Ω_Y is normal, then we have*

$$a^1_{k:-2} = b^1_{k\ell:-2} = 0, \tag{5.6}$$

$$\begin{aligned}&\sum a^{2\beta}_{2\beta:-1} = -\sum a^{2\beta-1}_{2\beta-1:-1}, \quad a^{2\beta}_{2\alpha:-1} = -a^{2\beta-1}_{2\alpha-1:-1} \ (\beta \neq \alpha),\\ &a^{2\beta}_{2\alpha-1:-1} = a^{2\beta-1}_{2\alpha:-1},\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\sum b^{2\beta}_{2\beta,\ell:-1} = -\sum b^{2\beta-1}_{2\beta-1,\ell:-1}, \quad b^{2\beta}_{2\alpha,\ell:-1} = -b^{2\beta-1}_{2\alpha-1,\ell:-1} \ (\beta \neq \alpha),\\ &b^{2\beta}_{2\alpha-1,\ell:-1} = b^{2\beta-1}_{2\alpha,\ell:-1},\end{aligned} \tag{5.8}$$

$$\begin{aligned}
&a^0_{2\alpha-1:0} + 2\,a^{(\alpha\alpha)}_{2\alpha:0} + \sum a^{(\beta\beta)}_{2\alpha:0} - 2\sum a^{(\beta\alpha:R)}_{2\beta-1:0} + 2\sum a^{(\beta\alpha:I)}_{2\beta:0} = 0,\\
&a^0_{2\alpha:0} - 2\,a^{(\alpha\alpha)}_{2\alpha-1:0} - \sum a^{(\beta\beta)}_{2\alpha-1:0} - 2\sum a^{(\beta\alpha:R)}_{2\beta:0} - 2\sum a^{(\beta\alpha:I)}_{2\beta-1:0} = 0,\\
&2\,a^{2\alpha}_{\ell:-1} = -b^0_{2\alpha-1,\ell:0} - 2\,b^{(\alpha\alpha)}_{2\alpha,\ell:0} - \sum b^{(\beta\beta)}_{2\alpha,\ell:0} + 2\sum b^{(\beta\alpha:R)}_{2\beta-1,\ell:0} - 2\sum b^{(\beta\alpha:I)}_{2\beta,\ell:0},\\
&2\,a^{2\alpha-1}_{\ell:-1} = b^0_{2\alpha,\ell:0} - 2\,b^{(\alpha\alpha)}_{2\alpha-1,\ell:0} - \sum b^{(\beta\beta)}_{2\alpha-1,\ell:0} - 2\sum b^{(\beta\alpha:R)}_{2\beta,\ell:0} - 2\sum b^{(\beta\alpha:I)}_{2\beta-1,\ell:0},\\
&\sum a^{2\beta}_{2\beta-1:1} = \sum a^{2\beta-1}_{2\beta:1},\\
&a^0_{\ell:0} = \frac{-1}{8(n+2)} \sum (b^{2\beta}_{2\beta-1,\ell:1} - b^{2\beta-1}_{2\beta,\ell:1}).
\end{aligned}$$

Moreover, if there is no extra part $+\cdots$ *in the expansion* (5.3)*, then the converse is also true.*

Proof. Checking the $\mathfrak{su}_{-1}$-component of (5.4), we obtain (5.6). Indeed, it comes from

$$\begin{aligned}
\sum [e^k_1, A_{k:-2}] &= \sum a^1_{2\gamma-1:-2}\,[e^{2\gamma-1}_1, e^1_{-2}] + \sum a^1_{2\gamma:-2}\,[e^{2\gamma}_1, e^1_{-2}]\\
&= \sum a^1_{2\gamma-1:-2}\,\frac{-1}{8(n+2)}\,e^{2\gamma}_{-1} + \sum a^1_{2\gamma:-2}\,\frac{1}{8(n+2)}\,e^{2\gamma-1}_{-1},\\
[e^1_2, A_{\ell:-3}] - 2\sum [e^k_1, B_{k\ell:-2}] &= -2\sum [e^k_1, B_{k\ell:-2}]\\
&= \sum b^1_{2\gamma-1,\ell:-2}\,\frac{1}{4(n+2)}\,e^{2\gamma}_{-1} + \sum b^1_{2\gamma,\ell:-2}\,\frac{-1}{4(n+2)}\,e^{2\gamma-1}_{-1}.
\end{aligned}$$

By checking the $\mathfrak{su}_0$-component, ... successively, the proposition is proved. ∎

Now, let us begin proving Theorem 4.4.

The first half of the proof of Theorem 4.4. We assume that J is non-integrable. We want to show Ω_Y is not normal. By (5.2), (4.2), we have

$$\sum A_{k:-1}\eta^k_{-1} \wedge \eta^1_{-2} + \sum B_{k\ell:-1}\,\eta^k_{-1} \wedge \eta^\ell_{-1} = \begin{pmatrix} 0 & 0 & 0\\ -2i\,\overline{c_\beta} & 0 & 0\\ 0 & c_\alpha & 0\end{pmatrix}, \tag{5.9}$$

$$c_\alpha = \sum \overline{\omega_\gamma} \wedge \phi_{\bar\gamma\alpha} = \sum \frac{i}{4}\, \mathcal{Q}^\rho_{\bar\gamma\bar\alpha}\, \overline{\Omega^\gamma_{-1}} \wedge \overline{\Omega^\rho_{-1}}, \tag{5.10}$$

and (5.9) says

$$\begin{aligned}
4c_\alpha &= 4\sum (a^{2\alpha-1}_{k:-1} + i\, a^{2\alpha}_{k:-1})\, \eta^k_{-1} \wedge \eta^1_{-2} + 4\sum (b^{2\alpha-1}_{k\ell:-1} + i\, b^{2\alpha}_{k\ell:-1})\, \eta^k_{-1} \wedge \eta^\ell_{-1} \\
&= 2\sum (a^{2\alpha-1}_{2\gamma-1:-1} + i\, a^{2\alpha}_{2\gamma-1:-1})\, (\overline{\Omega^\gamma_{-1}} + \Omega^\gamma_{-1}) \wedge \Omega_{-2} \\
&\qquad + 2\sum (a^{2\alpha-1}_{2\gamma:-1} + i\, a^{2\alpha}_{2\gamma:-1})\, (\overline{\Omega^\gamma_{-1}} - \Omega^\gamma_{-1}) \wedge \Omega_{-2} \\
&\quad + \sum \{(b^{2\alpha-1}_{2\gamma-1,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho-1:-1}) + (b^{2\alpha-1}_{2\gamma-1,2\rho:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho:-1}) \\
&\qquad + (b^{2\alpha-1}_{2\gamma,2\rho:-1} + i\, b^{2\alpha}_{2\gamma,2\rho:-1}) + (b^{2\alpha-1}_{2\gamma,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma,2\rho-1:-1})\}\, \overline{\Omega^\gamma_{-1}} \wedge \overline{\Omega^\rho_{-1}} \\
&\quad + \sum \{(b^{2\alpha-1}_{2\gamma-1,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho-1:-1}) - (b^{2\alpha-1}_{2\gamma-1,2\rho:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho:-1}) \\
&\qquad + (b^{2\alpha-1}_{2\gamma,2\rho:-1} + i\, b^{2\alpha}_{2\gamma,2\rho:-1}) - (b^{2\alpha-1}_{2\gamma,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma,2\rho-1:-1})\}\, \Omega^\gamma_{-1} \wedge \Omega^\rho_{-1} \\
&\quad + 2\sum \{(b^{2\alpha-1}_{2\gamma-1,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho-1:-1}) - (b^{2\alpha-1}_{2\gamma-1,2\rho:-1} + i\, b^{2\alpha}_{2\gamma-1,2\rho:-1}) \\
&\qquad - (b^{2\alpha-1}_{2\gamma,2\rho:-1} + i\, b^{2\alpha}_{2\gamma,2\rho:-1}) + (b^{2\alpha-1}_{2\gamma,2\rho-1:-1} + i\, b^{2\alpha}_{2\gamma,2\rho-1:-1})\}\, \overline{\Omega^\gamma_{-1}} \wedge \Omega^\rho_{-1}.
\end{aligned}$$

Comparing this with (5.10), we find ((5.7) holds and)

$$b^j_{2\gamma,2\rho:-1} = b^j_{2\gamma-1,2\rho:-1} = b^j_{2\gamma,2\rho-1:-1} = b^j_{2\gamma-1,2\rho-1:-1}, \tag{5.11}$$

$$4(b^{2\alpha-1}_{2\gamma,2\rho:-1} + i\, b^{2\alpha}_{2\gamma,2\rho:-1}) = i\, \mathcal{Q}^\rho_{\bar\gamma\bar\alpha}. \tag{5.12}$$

Let us assume that (5.8) holds. Then, together with (5.11), etc., it implies $b^{2\alpha-1}_{2\gamma,2\rho:-1} = b^{2\alpha}_{2\gamma,2\rho:-1} = 0$. Hence, by (5.12), the tensor $\mathcal{Q}$ vanishes. This contradicts the assumption that J is non-integrable. Thus we find (5.8) does not hold, that is, Ω_Y is not normal. ∎

Let J be integrable from now on. The purpose is to show that Ω_Y is normal. It follows from Theorem 2.2 that the matrix in the last line of (4.2) vanishes and the expansion (5.3) has no extra part $+\cdots$. In order to check the conditions of Proposition 5.2, let us calculate the coefficients of (5.5).

Proposition 5.3 *We have*

$$a^1_{k:-2} = b^1_{k\ell:-2} = a^j_{k:-1} = b^j_{k\ell:-1} = 0$$

and

$$\begin{aligned}
&a^0_{k:0} = b^0_{k\ell:0} = 0, \\
&a^{(\beta\beta)}_{2\mu-1:0} = -\frac{1}{2}\operatorname{Im}(V_{\beta\beta\mu} - \frac{\sum_\gamma V_{\gamma\gamma\mu}}{n+2}), \quad a^{(\beta\beta)}_{2\mu:0} = -\frac{1}{2}\operatorname{Re}(V_{\beta\beta\mu} - \frac{\sum_\gamma V_{\gamma\gamma\mu}}{n+2}), \\
&a^{(\beta\alpha:R)}_{2\mu-1:0} = -\frac{1}{4}\operatorname{Re}(V_{\beta\alpha\mu} - V_{\alpha\beta\mu}), \quad a^{(\beta\alpha:R)}_{2\mu:0} = \frac{1}{4}\operatorname{Im}(V_{\beta\alpha\mu} - V_{\alpha\beta\mu}) \quad (\beta \neq \alpha), \\
&a^{(\beta\alpha:I)}_{2\mu-1:0} = -\frac{1}{4}\operatorname{Im}(V_{\beta\alpha\mu} + V_{\alpha\beta\mu}), \quad a^{(\beta\alpha:I)}_{2\mu:0} = -\frac{1}{4}\operatorname{Re}(V_{\beta\alpha\mu} + V_{\alpha\beta\mu}) \quad (\beta \neq \alpha), \\
&b^{(\beta\beta)}_{2\mu-1,2\nu-1:0} = b^{(\beta\beta)}_{2\mu,2\nu:0} = -\frac{1}{2}\operatorname{Im}(S_{\beta\mu\beta\bar\nu}), \quad b^{(\beta\beta)}_{2\mu-1,2\nu:0} = \frac{1}{2}\operatorname{Re}(S_{\beta\mu\beta\bar\nu}), \\
&b^{(\beta\alpha:R)}_{2\mu-1,2\nu-1:0} = b^{(\beta\alpha:R)}_{2\mu,2\nu:0} = \frac{1}{4}\operatorname{Re}(S_{\beta\nu\alpha\bar\mu} - S_{\alpha\nu\beta\bar\mu}) \quad (\beta \neq \alpha),
\end{aligned}$$

$$
\begin{aligned}
&b^{(\beta\alpha:R)}_{2\mu-1,2\nu:0} = -\frac{1}{4}\,\mathrm{Im}(S_{\beta\nu\alpha\bar\mu} - S_{\alpha\nu\beta\bar\mu}) \quad (\beta \neq \alpha),\\
&b^{(\beta\alpha:I)}_{2\mu-1,2\nu-1:0} = b^{(\beta\alpha:I)}_{2\mu,2\nu:0} = \frac{1}{4}\,\mathrm{Im}(S_{\beta\nu\alpha\bar\mu} + S_{\alpha\nu\beta\bar\mu}) \quad (\beta \neq \alpha),\\
&b^{(\beta\alpha:I)}_{2\mu-1,2\nu:0} = \frac{1}{4}\,\mathrm{Re}(S_{\beta\nu\alpha\bar\mu} + S_{\alpha\nu\beta\bar\mu}) \quad (\beta \neq \alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
&a^{2\alpha-1}_{2\mu-1:1} = -2(n+2)\,\mathrm{Im}(\eta_{\alpha\mu} + \eta_{\alpha\bar\mu}), \quad a^{2\alpha-1}_{2\mu:1} = -2(n+2)\,\mathrm{Re}(\eta_{\alpha\mu} - \eta_{\alpha\bar\mu}),\\
&a^{2\alpha}_{2\mu-1:1} = 2(n+2)\,\mathrm{Re}(\eta_{\alpha\mu} + \eta_{\alpha\bar\mu}), \quad a^{2\alpha}_{2\mu:1} = -2(n+2)\,\mathrm{Im}(\eta_{\alpha\mu} - \eta_{\alpha\bar\mu}),\\
&b^{2\alpha}_{2\mu-1,2\nu-1:1} = 2(n+2)\,\mathrm{Re}(V_{\alpha\mu\nu} - V_{\alpha\nu\mu} + V_{\nu\alpha\mu} - V_{\mu\alpha\nu}),\\
&b^{2\alpha-1}_{2\mu-1,2\nu-1:1} = 2(n+2)\,\mathrm{Im}(V_{\alpha\mu\nu} - V_{\alpha\nu\mu} - V_{\nu\alpha\mu} + V_{\mu\alpha\nu}),\\
&b^{2\alpha}_{2\mu,2\nu:1} = 2(n+2)\,\mathrm{Re}(V_{\alpha\mu\nu} - V_{\alpha\nu\mu} - V_{\nu\alpha\mu} + V_{\mu\alpha\nu}),\\
&b^{2\alpha-1}_{2\mu,2\nu:1} = 2(n+2)\,\mathrm{Im}(V_{\alpha\mu\nu} - V_{\alpha\nu\mu} + V_{\nu\alpha\mu} - V_{\mu\alpha\nu}),\\
&b^{2\alpha}_{2\mu-1,2\nu:1} = -2(n+2)\,\mathrm{Im}(V_{\alpha\mu\nu} + V_{\alpha\nu\mu} + V_{\nu\alpha\mu} - V_{\mu\alpha\nu}),\\
&b^{2\alpha-1}_{2\mu-1,2\nu:1} = 2(n+2)\,\mathrm{Re}(V_{\alpha\mu\nu} + V_{\alpha\nu\mu} - V_{\nu\alpha\mu} + V_{\mu\alpha\nu}).
\end{aligned}
$$

Remark: One may delete the term $\sum_\gamma V_{\gamma\gamma\mu}$ (cf. Remark 2.3) and could find further symmetry relations on the coefficients.

Proof. We have

$$
\begin{aligned}
&\sum A_{k:p}\,\eta^k_{-1} \wedge \eta^1_{-2} + \sum B_{k\ell:p}\,\eta^k_{-1} \wedge \eta^\ell_{-1} = 0 \quad (p < 0),\\
&\sum A_{k:0}\,\eta^k_{-1} \wedge \eta^1_{-2} + \sum B_{k\ell:0}\,\eta^k_{-1} \wedge \eta^\ell_{-1}\\
&\qquad = \begin{pmatrix} -\frac{1}{n+2}\sum \overline{\Phi_{\mu\mu}} & 0 & 0\\ 0 & -\Phi_{\beta\alpha} + \frac{\delta_{\beta\alpha}}{n+2}\sum \Phi_{\mu\mu} & 0\\ 0 & 0 & \frac{1}{n+2}\sum \Phi_{\mu\mu} \end{pmatrix},\\
&\sum A_{k:1}\,\eta^k_{-1} \wedge \eta^1_{-2} + \sum B_{k\ell:1}\,\eta^k_{-1} \wedge \eta^\ell_{-1} = \begin{pmatrix} 0 & -\frac{1}{2}\,\Phi_\alpha & 0\\ 0 & 0 & -i\,\overline{\Phi_\beta}\\ 0 & 0 & 0 \end{pmatrix},\\
&\sum A_{k:2}\,\eta^k_{-1} \wedge \eta^1_{-2} + \sum B_{k\ell:2}\,\eta^k_{-1} \wedge \eta^\ell_{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{4}\Psi\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

In a way similar to the computation for (5.9) with (5.10), we carry out computations to get the proposition. ∎

We finish preparations and prove the rest.

The second half of the proof of Theorem 4.4. We want to show that, if J is integrable, then Ω_Y is normal. It suffices to ascertain that the conditions in Proposition 5.2 hold. By referring to Proposition 5.3 and the conditions in Theorem 2.2 (in the case J is integrable), it is checked by straightforward computations. ∎

References

[1] E. Barletta and S. Dragomir, Differential equations on contact Riemannian manifolds, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4) **30**(1) (2001), 63–95.

[2] H. Baum and A. Juhl, Conformal differential geometry: Q-curvature and conformal holonomy, Birkhäuser Verlag, Basel-Boston-Berlin, 2010.

[3] A. Čap and J. Slovák, Parabolic geometries I: Background and general theory, Math. Surveys and monographs **154**, Amer. Math. Soc., Providence, RI, 2009.

[4] É. Cartan, Sur la geómétrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes I, Ann. Math. Pura Appl. (4) **11** (1932), 17–90; II, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (2) **1** (1932), 333–354.

[5] S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math. **133** (1975), 219–271.

[6] H. Jacobowitz, An introduction to CR structures, Math. Surveys and monographs **32**, Amer. Math. Soc., Providence, RI, 1990.

[7] S. Kobayashi and K. Nomizu, Foundations of differential geometry I, Interscience Publishers, New York, 1963.

[8] A. Le, Cartan connections for CR manifolds, Manuscripta Math. **122** (2007), 245–264.

[9] M. Nagase, The heat equation for the Kohn-Rossi Laplacian on contact Riemannian manifolds, preprint.

[10] M. Nagase, Dirac operators on the Fefferman spin spaces in almost CR-geometry, to appear in Osaka J. Math.

[11] M. Nagase and D. Sasaki, Hermitian Tanno connection and Bochner type curvature tensors of contact Riemannian manifolds, to appear in J. Math. Sci. Univ. Tokyo.

[12] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan. J. Math. **2**(1) (1976), 131–190.

[13] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. **314**(1) (1989), 349–379.