A formula for the heat kernel coefficients of the Dirac Laplacians on spin manifolds

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Abstract

Based on Getzler's rescaling transformation, we obtain a formula for the heat kernel coefficients of the Dirac Laplacian on a spin manifold. One can compute them explicitly up to an arbitrarily high order by using only a basic knowledge of calculus added to the formula.

1 Introduction

Let (M, g) be an m-dimensional compact oriented Riemannian manifold equipped with a spin structure $\rho : \mathrm{Spin}(T^*M) \to SO(T^*M)$, where $SO(T^*M)$ is the principal $SO(m)$-bundle consisting of $SO(m)$-frames of T^*M and $\mathrm{Spin}(T^*M)$ is a principal $\mathrm{Spin}(m)$-bundle together with a 2-sheeted covering map ρ (e.g. [5], [2]). Then we have the fundamental spinor bundle $\not{S} = \mathrm{Spin}(T^*M) \times_{\mathrm{Spin}(m)} S_m$ $(S_{2n} = S_{2n+1} = \mathbb{C}^{2^n})$ and the Dirac operator D given by

$$D = \sum e^j \circ \nabla^{g}_{e_j} := \sum e^j \circ \{e_j + \frac{1}{4} \sum \omega(\nabla^g)^k_\ell(e_j)\, e^\ell \circ e^k \circ\}$$

acting on the cross-sections of $\not{S}$, where $e^\bullet = (e^1, \ldots, e^m)$ is a local $SO(m)$-frame of T^*M (around a point P^0 in the following argument) and $e^j\circ$ denotes the Clifford action on S_m of the element e^j of the Clifford bundle $(Cl(T^*M), \circ)$, and $e_\bullet = (e_1, \ldots, e_m)$ is the dual frame of $e^\bullet$. Further, ∇^g is the Levi-Civita connection of g and $\omega(\nabla^g)^k_\ell$ denote the connection forms defined by $\nabla^g_X e_\ell = \sum \omega(\nabla^g)^k_\ell(X)\, e_k$.

It is known that the initial value problem associated with the Dirac Laplacian D^2

$$\Big(\frac{\partial}{\partial t} + D^2\Big)\phi(t) = 0, \qquad \lim_{t\to 0} \phi(t) = \phi_0 \quad (\phi_0 \in \Gamma(\not{S}))$$

has a unique fundamental solution or heat kernel $e^{-tD^2}(P, P')$ and, when $t \to 0$, there is an asymptotic expansion

$$e^{-tD^2}(P^0, P^0) \sim (4\pi t)^{-m/2} \sum_{\ell=0}^{\infty} t^\ell K_\ell(P^0), \quad K_\ell(P^0) \in \mathbb{C}l(T^*_{P^0}M).$$

2010 *Mathematical Subject Classification.* 53C27, 58J35.

Keywords: heat kernel coefficient; Dirac Laplacian; spin manifold; Getzler's rescaling transformation.

Here we regard $\mathrm{End}(\not{S}_{P^0})\,(\ni K_\ell(P^0))$ as a subalgebra of $\mathbb{C}l(T^*_{P^0}M) := Cl(T^*_{P^0}M)\otimes\mathbb{C}$ naturally, and consider the subspaces $\mathbb{C}l^{(p)}(T^*_{P^0}M)$ well-defined by

$$\mathbb{C}l^{(p)}(T^*_{P^0}M) \cong \wedge^p T^*_{P^0}M\otimes\mathbb{C}$$
$$e^{i_1}(P^0)\circ\cdots\circ e^{i_p}(P^0) \leftrightarrow e^{i_1}(P^0)\wedge\cdots\wedge e^{i_p}(P^0) \quad (i_1<\cdots<i_p).$$

Accordingly let us set

$$\begin{aligned} K_\ell(P^0) &= \sum_{\alpha=(\alpha_1<\cdots<\alpha_{|\alpha|})} K_{\ell,\alpha}(P^0)\, e^{\alpha_1}(P^0)\circ\cdots\circ e^{\alpha_{|\alpha|}}(P^0) \\ &= \sum K_{\ell,[p]}(P^0) := \sum\sum_{|\alpha|=p} K_{\ell,\alpha}(P^0)\, e^{\alpha_1}(P^0)\circ\cdots\circ e^{\alpha_{|\alpha|}}(P^0). \end{aligned}$$

Then, employing a purely local rescaling transformation which we call Getzler's one, from which the $\hat{A}$-genus form emerges in a natural way, Getzler [4] (cf. Berline-Getzler-Vergne [2, Theorem 4.1], Getzler [3]) obtained the formulas

$$K_{\ell,[p]}(P^0) = 0 \quad (\ell < p/2), \tag{1.1}$$

$$K_{p/2,[p]}(P^0) = \det{}^{1/2}\Big(\frac{R(P^0)/2}{\sinh(R(P^0)/2)}\Big)_{[p]}, \tag{1.2}$$

where $R(P^0)$ is the Riemannian curvature at P^0, that is, the antisymmetric $m\times m$ matrix with (j,i)-entries given by

$$R_{ji}(P^0) = \frac{1}{2}\sum g(F(\nabla^g)(e_\ell,e_k)e_i,e_j)(P^0)\, e^\ell(P^0)\wedge e^k(P^0)$$

$(F(\nabla^g)(X,Y) := [\nabla^g_X,\nabla^g_Y] - \nabla^g_{[X,Y]})$, and the right hand side of (1.2) denotes the $\mathbb{C}l^{(p)}(T^*_{P^0}M)$-component of the $\hat{A}$-genus form $(\in \wedge^{4\bullet}T^*_{P^0}M \subset \mathbb{C}l(T^*_{P^0}M))$. Notice that, though restricted to the case m is even in [4], his argument holds true also for m odd.

Getzler's purpose in [4] was to present a short proof of the famous local index theorem for D (e.g. [1]), which is a straightforward consequence of the formulas, so that his study on the heat kernel coefficients was restricted to the case enough for the purpose. In this paper we will introduce a formula for the remaining ones $K_{\ell,[p]}(P^0)$ $(\ell > p/2)$ (Theorem 3.4). We want to emphasize that, together with the formulas for the Taylor expansions of connection coefficients and transition functions due to Atiyah-Bott-Patodi ([1, Appendix II]), it induces their explicit expressions. The first author has derived such formulas for some other Laplacians ([6], [7]) in a similar way, and this paper is part of studying heat kernel coefficients under such an idea.

In §2 we explain Getzler's rescaling transformation and review quickly the proofs of (1.1) and (1.2). By applying his method, which is effective also in investigating the remaining ones, and by using the formulas given by Atiyah-Bott-Patodi, a formula for $K_{\ell,[p]}(P^0)$ $(\ell > p/2)$ will be derived in §3. From the formula, we will induce concrete expressions of $K_0(P^0)$, $K_1(P^0)$, $K_2(P^0)$ by written calculation in §4. We have a plan to compute them up to a higher order with the aid of Mathematica.

2 Getzler's rescaling transformation and a review of the proofs of (1.1) and (1.2): cf. [4] and [2, Chap. 4]

First, let us review the transformation. We will work on a neighborhood U around the point P^0 with normal coordinates $x = (x_1, \ldots, x_m)$ satisfying $(\partial/\partial x_i)_{P^0} = e_i(P^0)$. The orthonormal frame $e_\bullet$ is assumed to be parallel along the geodesics from P^0 and, hence, so is also $e^\bullet$. Via the trivialization of the spinor bundle by the parallel transport map, the heat kernel $e^{-tD^2}(P, P^0)$ gives a localized one

$$K(t,x) \ \in C^\infty((\mathbb{R}^+, t) \times (U, x), \mathbb{C}l(T^*_{P^0}M)),$$

which has an asymptotic expansion

$$\begin{aligned} K(t,x) &\sim q_t(x) \sum_{\ell=0}^{\infty} t^\ell K_\ell(x), \qquad q_t(x) := \frac{1}{(4\pi t)^{m/2}} e^{-|x|^2/4t}, \\ K_\ell(x) &= \sum_{\alpha=(\alpha_1<\cdots<\alpha_{|\alpha|})} K_{\ell,\alpha}(x)\, dx_{\alpha_1} \circ \cdots \circ dx_{\alpha_{|\alpha|}} \\ &= \sum K_{\ell,[p]}(x) := \sum \sum_{|\alpha|=p} K_{\ell,\alpha}(x)\, dx_{\alpha_1} \circ \cdots \circ dx_{\alpha_{|\alpha|}} \end{aligned}$$

$(dx_{\alpha_j} = (dx_{\alpha_j})_{P^0} = e^{\alpha_j}(P^0))$ when $t \to 0$. We have thus an interest in $K_{\ell,[p]}(0) = K_{\ell,[p]}(P^0)$.

Now, let us take $\varepsilon > 0$ and, for a form $\eta(t,x) = \eta_\alpha(t,x)\, dx_{\alpha_1} \circ \cdots \circ dx_{\alpha_{|\alpha|}}$, set $(\mathcal{T}_\varepsilon \eta)(t,x) = \varepsilon^{-|\alpha|/2} \eta_\alpha(\varepsilon t, \varepsilon^{1/2} x)\, dx_{\alpha_1} \circ \cdots \circ dx_{\alpha_{|\alpha|}}$. The rescaling transformation $\mathcal{T}_\varepsilon$ then induces Getzler's one, that is, for an operator P acting on $C^\infty(\mathbb{R}^+ \times U, \mathbb{C}l(T^*_{P^0}M))$, we set $G_\varepsilon(P) = \mathcal{T}_\varepsilon \cdot P \cdot \mathcal{T}_\varepsilon^{-1}$: For example, let $f\times$ be the multiplication by a function f, and let $dx_j\circ$ be the Clifford action of dx_j on $\mathbb{C}l(T^*_{P^0}M) = \wedge^* T^*_{P^0}M \otimes \mathbb{C}$, i.e., $dx_j\circ = dx_j \wedge - dx_j \vee$, etc. Here $dx_j \vee := (\partial/\partial x_j)_{P^0} \rfloor$ is the interior production of $(\partial/\partial x_j)_{P^0}$. Then we have

$$\begin{gathered} G_\varepsilon(f\times) = f(\varepsilon^{1/2}x)\times, \quad G_\varepsilon(dx_j\circ) = \varepsilon^{-1/2}\, dx_j \circ_\varepsilon := \varepsilon^{-1/2}(dx_j \wedge - \varepsilon^{2/2} dx_j \vee), \\ G_\varepsilon(\frac{\partial}{\partial x_j}) = \varepsilon^{-1/2} \frac{\partial}{\partial x_j}, \quad G_\varepsilon(\frac{\partial}{\partial t}) = \varepsilon^{-2/2} \frac{\partial}{\partial t}. \end{gathered}$$

For the Dirac operator D acting on $C^\infty(\mathbb{R}^+ \times U, \mathbb{C}l(T^*_{P^0}M))$, i.e., $D = \sum dx_j \circ \nabla^{\mathcal{S}_{P^0}}_{e_j} := \sum dx_j \circ \{e_j + \frac{1}{4}\omega(\nabla^g)^k_\ell(e_j)\, dx_\ell \circ dx_k \circ\}$, we set $D^{(\varepsilon)} = \varepsilon^{1/2} G_\varepsilon(D)$, which is hence expressed as follows:

$$\begin{aligned} D^{(\varepsilon)} &= \sum \varepsilon^{-1/2} dx_j \circ_\varepsilon \nabla^{(\varepsilon)}_{e_j^{(\varepsilon)}}, \\ e_j^{(\varepsilon)} &:= \varepsilon^{1/2}\, G_\varepsilon(e_j) = \sum V_{ij}(\varepsilon^{1/2}x)\, \partial/\partial x_i \quad (e_j := \sum V_{ij}(x)\, \partial/\partial x_i), \\ \nabla^{(\varepsilon)}_{e_j^{(\varepsilon)}} &:= \varepsilon^{1/2}\, G_\varepsilon(\nabla^{\mathcal{S}_{P^0}}_{e_j}) = e_j^{(\varepsilon)} + \frac{\varepsilon^{-1/2}}{4} \sum \omega(\nabla^g)^k_\ell(e_j)(\varepsilon^{1/2}x)\, dx_\ell \circ_\varepsilon dx_k \circ_\varepsilon . \end{aligned} \tag{2.1}$$

Let us set $\mathbb{D}_{(\varepsilon)} = (D^{(\varepsilon)})^2 = \varepsilon^{2/2}(G_\varepsilon(D))^2$. Then the Lichnerowitz formula for D^2 yields the rescaled one

$$\mathbb{D}_{(\varepsilon)} = -\sum \Big(\nabla^{(\varepsilon)}_{e_j^{(\varepsilon)}} \nabla^{(\varepsilon)}_{e_j^{(\varepsilon)}} - \varepsilon^{1/2} \sum \omega(\nabla^g)^j_i(e_i)(\varepsilon^{1/2}x)\, \nabla^{(\varepsilon)}_{e_j^{(\varepsilon)}} \Big) + \frac{\varepsilon^{2/2}}{4} s(\nabla^g)(\varepsilon^{1/2}x), \tag{2.2}$$

where $s(\nabla^g)$ is the scalar curvature, i.e., $s(\nabla^g) := \sum g(F(\nabla^g)(e_j, e_i)e_i, e_j)$. Obviously we have $\mathbb{D}_{(\varepsilon)} = \mathbb{D}_{0/2} + O(\varepsilon^{1/2})$ with

$$\mathbb{D}_{0/2} = -\sum\Big(\frac{\partial}{\partial x_j} + \frac{1}{4}\sum x_i\, R_{ji}\wedge\Big)^2 = -\sum\Big(\frac{\partial}{\partial x_j}\Big)^2 + \frac{1}{16}\Big\langle x\Big|R^2\Big|x\Big\rangle,$$

where we set $R = R(P^0)$, etc., for short (cf. (4.4), [2, Proposition 4.19]).

Proposition 2.1 (cf. [2, Theorem 4.12]) *For any $a_0 \in \mathbb{C}l(T^*_{P^0}M)$, there exists a unique sequence $\{\Phi_{\ell/2}(x) \in C^\infty(U, \mathbb{C}l(T^*_{P^0}M))\}_{\ell=0}^\infty$ satisfying formally*

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big)q_t(x)\sum_{\ell=0}^{\infty} t^{\ell/2}\Phi_{\ell/2}(x) = 0, \quad \Phi_{0/2}(0) = a_0.$$

Further, it is determined by the formula

$$\begin{aligned} &q_t(x)\sum_{\ell=0}^{\infty} t^{\ell/2}\Phi_{\ell/2}(x) \\ &= \frac{1}{(4\pi t)^{m/2}}\det{}^{1/2}\Big(\frac{tR/2}{\sinh(tR/2)}\Big)\,\exp\Big(-\frac{1}{4t}\Big\langle x\Big|\frac{tR}{2}\coth(\frac{tR}{2})\Big|x\Big\rangle\Big)a_0. \end{aligned}$$

It follows from the argument following [2, (4.4)] that the rescaled kernel

$$K_{(\varepsilon)}(t,x) = \varepsilon^{m/2}(\mathcal{T}_\varepsilon K)(t,x)$$

satisfies

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{(\varepsilon)}\Big)K_{(\varepsilon)}(t,x) = 0, \qquad \lim_{t\to 0} K_{(\varepsilon)}(t,x) = \delta(x) \tag{2.3}$$

and is asymptotically expanded into

$$K_{(\varepsilon)}(t,x) \sim q_t(x)\sum_{\ell=0}^{\infty}\sum_{p} t^\ell \varepsilon^{\ell-p/2}\, K_{\ell,[p]}(\varepsilon^{1/2}x) \quad (t\to 0)$$

for every $\varepsilon^{1/2} > 0$. Further, [2, Lemma 4.18] says that there is an asymptotic expansion

$$K_{(\varepsilon)}(t,x) \sim q_t(x)\sum_{i=-m}^{\infty} \varepsilon^{i/2}\,\gamma_{i/2}(t,x) \quad (\varepsilon^{1/2}\to 0) \tag{2.4}$$

which is uniform for the variables $(t,x) \in [0,1]\times U$ in the sense: Each $\gamma_{i/2}(t,x)$ is smooth on $\mathbb{R}\times U$, and, for every large integer N, there exists a constant $C > 0$ such that $\Big|K_{(\varepsilon)}(t,x) - q_t(x)\sum_{i=-m}^N \varepsilon^{i/2}\,\gamma_{i/2}(t,x)\Big| < C\varepsilon^{(N+1)/2}$ on $(0,1]\times U$. Every derivative of $K_{(\varepsilon)}(t,x)$ is also asymptotically expanded into the termwise derivative of the left hand side in the same sense. Indeed, let us denote by $K^{(k)}_{\ell,[p]}(x)$ the sum of the terms of order k of the Taylor expansion of $K_{\ell,[p]}(x)$ (i.e., $K_{\ell,[p]}(x) = \sum_{k=0}^N K^{(k)}_{\ell,[p]}(x) + O(|x|^{N+1})$). Then the sequence $\{\gamma_{i/2}(t,x)\}_{i=0}^\infty$ defined by

$$\gamma_{i/2,[p]}(t,x) = \sum_{-p\le k\le i} t^{(p+k)/2}K^{(i-k)}_{(p+k)/2,[p]}(x) \tag{2.5}$$

certainly induces the expansion. Further, by definition it will be obvious that

$$\gamma_{0/2}(0,0) = 1, \quad \gamma_{i/2}(0,0) = 0 \quad (i \neq 0), \tag{2.6}$$

$$K_{\ell,[p]}(0) = \gamma_{\ell-p/2,[p]}(1,0). \tag{2.7}$$

Proposition 2.2 (cf. [2, p.163 and Theorem 4.20]) *We have*

$$\gamma_{i/2}(t,x) = 0 \quad (i < 0), \tag{2.8}$$

$$q_t(x)\gamma_{0/2}(t,x) = \frac{1}{(4\pi t)^{m/2}} \det{}^{1/2}\Big(\frac{tR/2}{\sinh(tR/2)}\Big) \exp\Big(- \frac{1}{4t}\Big\langle x \Big| \frac{tR}{2}\coth(\frac{tR}{2}) \Big| x \Big\rangle\Big). \tag{2.9}$$

Proof. The first term $\gamma_{-m/2}(t,x)$ in the expansion (2.4) is a polynomial with respect to $t^{1/2}$ (cf. (2.5)) and satisfies

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big) q_t(x)\gamma_{-m/2}(t,x) = 0, \quad \gamma_{-m/2}(0,0) = 0$$

because of (2.3) and (2.6). Hence the uniqueness assertion in Proposition 2.1 implies $\gamma_{-m/2}(t,x) = 0$. Next, the term $\gamma_{-m/2+1/2}(t,x)$ is also a polynomial and consequently satisfies

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big) q_t(x)\gamma_{-m/2+1/2}(t,x) = 0, \quad \gamma_{-m/2+1/2}(0,0) = 0.$$

Hence, similarly we have $\gamma_{-m/2+1/2}(t,x) = 0$. Inductively we obtain (2.8). Last, since

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big) q_t(x)\gamma_{0/2}(t,x) = 0, \quad \gamma_{0/2}(0,0) = 1$$

we obtain the formula (2.9). ∎

Proposition 2.2 and the identity (2.7) yield certainly the formulas (1.1) and (1.2).

3 On the coefficients $K_{\ell,[p]}(P^0)$ $(\ell > p/2)$

Here, our study focuses on the remaining terms $\gamma_{i/2}(t,x)$, $i > 0$. The operator $\mathbb{D}_{(\varepsilon)}$ is expanded formally into

$$\mathbb{D}_{(\varepsilon)} = \sum_{i=0}^{\infty} \varepsilon^{i/2}\, \mathbb{D}_{i/2}.$$

It will be obvious that we may develop the proof of Proposition 2.2 to get the following.

Proposition 3.1 *There exists a unique sequence of formal sums* $\Psi_{i/2}(t,x) = \sum_{\ell=0}^{\infty} t^{\ell/2}\Psi_{i/2,\ell/2}(x)$ *with* $\Psi_{i/2,\ell/2} \in C^\infty(U, \mathbb{C}l(T^*_{P^0}M))$ $(i = 0,1,\ldots)$ *satisfying formally*

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{(\varepsilon)}\Big) q_t \sum_{i=0}^{\infty} \varepsilon^{i/2}\Psi_{i/2} = 0, \quad \Psi_{0/2,0/2}(0) = 1, \quad \Psi_{i/2,0/2}(0) = 0 \quad (i > 0),$$

that is,

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big) q_t\Psi_{0/2} = 0, \quad \Psi_{0/2,0/2}(0) = 1,$$

$$\Big(\frac{\partial}{\partial t} + \mathbb{D}_{0/2}\Big) q_t\Psi_{i/2} + \sum_{i_1+i_2=i}^{i_2<i} \mathbb{D}_{i_1/2}(q_t\Psi_{i_2/2}) = 0, \quad \Psi_{i/2,0/2}(0) = 0 \quad (i > 0). \tag{3.1}$$

Further, the sequence $\{\gamma_{i/2}(t,x)\}_{i=0}^{\infty}$ *satisfies the conditions.*

Let us review the formulas for the Taylor expansions of connection coefficients and transition functions due to Atiyah-Bott-Patodi [1, Appendix II] (cf. Nagase [7, (1.3)–(1.7)]). We will find that the sequence $\{\mathbb{D}_{i/2}\}_{i=0}^{\infty}$ can be computed explicitly by using only a basic knowledge of calculus added to the formulas.

First, [1, Proposition 3.7 and Appendix II] says that the connection coefficients $\omega(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j) = g(\nabla^g_{\partial/\partial x_j} e_{i_2}, e_{i_1})$ are formally expanded into

$$\omega(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j)(x) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0),$$

where we put $F(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j, \partial/\partial x_{j_1}) = g(F(\nabla^g)(\partial/\partial x_j, \partial/\partial x_{j_1})e_{i_2}, e_{i_1})$. Second, set

$$e_i = \sum V_{ji}(x)\, \partial/\partial x_j \quad \text{(cf. (2.1))}, \qquad e^i = \sum V^{ji}(x)\, dx_j.$$

Then, [1, Proposition 2.11 and Appendix II] says that the transition functions V^{ji} are formally expanded into

$$V^{ji}(x) = \delta_{ji} - \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-2} F(\nabla^g)^{i}_{j_1}(\partial/\partial x_j, \partial/\partial x_{j_2})}{\partial x_{j_3} \cdots \partial x_{j_\ell}}(0).$$

Hence, the coefficients of the Taylor expansions of $\omega(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j)$, V^{ji}, V_{ji} are all expressed as universal polynomials made of

$$R_{j_1 j_2 j_3 j_4 j_5 \cdots j_\ell} := \frac{\partial^{\ell-4} g(F(\nabla^g)(\partial/\partial x_{j_3}, \partial/\partial x_{j_4})\partial/\partial x_{j_2}, \partial/\partial x_{j_1})}{\partial x_{j_5} \cdots \partial x_{j_\ell}}(0),$$

which can be concretely computed easily. For example, we have

$$\begin{aligned}
&V^{ji}(x) = \delta_{ji} + \sum x_{j_1} x_{j_2} \frac{-R_{i j_1 j j_2}}{6} + \sum x_{j_1} x_{j_2} x_{j_3} \frac{-R_{i j_1 j j_2 j_3}}{12} + O(|x|^4),\\
(3.2)\quad &V_{ji}(x) = \delta_{ji} + \sum x_{j_1} x_{j_2} \frac{R_{i j_1 j j_2}}{6} + \sum x_{j_1} x_{j_2} x_{j_3} \frac{R_{i j_1 j j_2 j_3}}{12} + O(|x|^4),\\
&\omega(\nabla^g)^{i_1}_{i_2}(\partial/\partial x_j)(x) = \sum x_{j_1} \frac{-R_{i_1 i_2 j j_1}}{2} + \sum x_{j_1} x_{j_2} \frac{-R_{i_1 i_2 j j_1 j_2}}{3}\\
&\qquad + \sum x_{j_1} x_{j_2} x_{j_3} \Big(\frac{-R_{i_1 i_2 j j_1 j_2 j_3}}{8} + \frac{\sum R_{k j_2 i_2 j_3} R_{k i_1 j j_1}}{24} + \frac{-\sum R_{k j_2 i_1 j_3} R_{k i_2 j j_1}}{24} \Big)\\
&\qquad + O(|x|^4).
\end{aligned}$$

By referring to (2.1) and (2.2), it will be now obvious that $\{\mathbb{D}_{i/2}\}_{i=0}^{\infty}$ can be computed concretely.

For suitable forms $k_i(t, x, y) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{C}l(T^*_{P0}M))$ $(i = 1, 2)$, let us define the convolution $k_1 \# k_2$ by $(k_1 \# k_2)(t, x, y) = \int_0^t ds \int_{\mathbb{R}^m} dV(x')\, k_1(t-s, x, x')\, k_2(s, x', y)$, where dV is the standard volume element. (Recall that $\mathbb{C}l(T^*_{P0}M)$ means $\wedge^* T^*_{P0}M \otimes \mathbb{C}$ and the product $k_1(t-s, x, x')\, k_2(s, x', y)$ is their exterior product.) Then, setting

$$(q_\bullet \gamma_{0/2})(t, x, y) = q_t(x-y)\gamma_{0/2}(t, x-y),$$

we want to show that the sequence $\{\gamma_{i/2}(t, x)\}_{i=0}^{\infty}$, which is the only one satisfying the conditions in Proposition 3.1, has the following formula. For the idea, refer to [7, (1.14)–(1.20)].

Theorem 3.2 *In the case $i>0$, there is a well-defined formula*

$$q_t(x)\gamma_{i/2}(t,x) = \sum_{\substack{i_1,\dots,i_k>0\\ \sum i_j = i}} (-1)^k (q_\bullet \gamma_{0/2} \# \mathbb{D}_{i_1/2}(q_\bullet\gamma_{0/2}) \# \cdots \# \mathbb{D}_{i_k/2}(q_\bullet\gamma_{0/2}))(t,x,0), \tag{3.3}$$

and the sequence $\{\gamma_{i/2}(t,x)\}_{i=0}^\infty$ can be computed explicitly up to an arbitrarily high order by using only a basic knowledge of calculus.

We will ascertain the theorem after proving a preparatory lemma.

Lemma 3.3 *There are finite sum expressions*

$$q_t(x)\gamma_{0/2}(t,x) = \sum_{\ell \ge |B|} t^\ell (\partial/\partial x)^B q_t(x)\cdot P_{(\ell,B)}(R), \tag{3.4}$$

$$\mathbb{D}_{i/2}(q_t(x)\gamma_{0/2}(t,x)) = \sum_{\ell\ge \max\{|B|-1,0\}} t^\ell (\partial/\partial x)^B q_t(x)\cdot P_{(\ell,B)}(R) \quad (i>0), \tag{3.5}$$

where we put $(\partial/\partial x)^B = \partial/\partial x_{B_1}\cdots\partial/\partial x_{B_{|B|}}$ $(B=(B_1,\dots,B_{|B|})$ and each $P_{(\ell,B)}(R)$, which is used in different senses in the two ones (and also in the following), is a polynomial made of $R_{j_1j_2j_3j_4\cdots}$, R_{ji}, $dx_k\wedge R_{ji}$, $dx_k \vee R_{ji}$, etc.

Proof. As for (3.4): By definition,

$$q_t(x)\gamma_{0/2}(t,x) = \sum_{\ell\ge 0} t^\ell x^C q_t(x)\cdot P_{(\ell,C)}(R),$$

where we put $x^C = x_{C_1}\cdots x_{C_{|C|}}$. Together with $x_i q_t(x) = -2t\,\partial/\partial x_i(q_t(x))$, it yields (3.4). As for (3.5): By definition, we have

$$\mathbb{D}_{i/2}(q_t(x)\gamma_{0/2}(t,x)) = \sum_{|C|\ge 2} t^{-1}x^C q_t(x)\cdot P_{(-1,C)}(R) + \sum_{\ell\ge 0} t^\ell x^C q_t(x)\cdot P_{(\ell,C)}(R). \tag{3.6}$$

Indeed, we have

$$\begin{aligned} &-\sum V_{i_1j}(\varepsilon^{1/2}x)\,V_{i_2j}(\varepsilon^{1/2}x)\,\partial/\partial x_{i_1}\partial/\partial x_{i_2}(q_t(x)) \\ &= \sum V_{i_1j}(\varepsilon^{1/2}x)\,V_{i_2j}(\varepsilon^{1/2}x)\left\{\frac{\delta_{i_1i_2}}{2t} - \frac{x_{i_1}}{2t}\frac{x_{i_2}}{2t}\right\}q_t(x), \end{aligned}$$

which is formally expanded into $\sum_{k\ge 2}\varepsilon^{k/2}\sum_{|C|=k} t^{-1}x^C q_t(x)\cdot c(k,C)$. Namely, the coefficient of $\varepsilon^{k/2}$ has no terms with t^{-2} because the function $\sum V_{ij}(\varepsilon^{1/2}x)x_i$ is expanded into $\sum V_{ij}(\varepsilon^{1/2}x)x_i = x_j$ formally (that is, all the coefficients of $\varepsilon^{k/2}$ $(k\ge 1)$ vanish). Accordingly we obtain (3.6) and, hence, also (3.5). ∎

Proof of Theorem 3.2. First, obviously we have a finite sum expression

$$\Psi_{0/2}(t,x) := \gamma_{0/2}(t,x) = \sum_{\ell\ge 0} t^\ell x^C\cdot P_{(\ell,C)}(R), \quad \Psi_{0/2}(0,x)=1. \tag{3.7}$$

The right hand side of (3.3), denoted by $q_t(x)\Psi_{i/2}(t,x)$, is expressed as

$$q_t(x)\Psi_{i/2}(t,x) = \sum_{\ell\ge\max\{|B|,1\}} t^\ell(\partial/\partial x)^B q_t(x)\cdot P_{(\ell,B)}(R) \tag{3.8}$$

because of Lemma 3.3 and

$$\begin{aligned}
&\int_0^t ds \int dV(x')\,(t-s)^{\ell}(\partial/\partial x)^B(q_{t-s}(x-x'))\,s^{\ell'}(\partial/\partial x')^{B'}(q_s(x'))\\
&= \int_0^t ds\,(t-s)^{\ell}s^{\ell'}(\partial/\partial x)^{B\cup B'}\int dV(x')\,q_{t-s}(x-x')q_s(x')\\
&= t^{\ell+\ell'+1}(\partial/\partial x)^{B\cup B'}(q_t(x))\int_0^1 d\sigma\,(1-\sigma)^{\ell}\sigma^{\ell'}.
\end{aligned}$$

Thus it is certainly well-defined, and (3.8) yields a finite sum expression

$$\Psi_{i/2}(t,x) = \sum_{\ell\geq 0}^{\ell+|C|>0} t^{\ell}x^{C}\cdot P_{(\ell,C)}(R) \quad (i>0). \tag{3.9}$$

Further, we have

$$q_t(x)\Psi_{i/2}(t,x) = -(q_{\bullet}\gamma_{0/2}\# \sum_{i_1+i_2=i}^{i_2<i} \mathbb{D}_{i_1/2}(q_{\bullet}\Psi_{i_2/2}))(t,x,0) \quad (i>0),$$

and, in general, for a suitable form $k(t,x,y)$ we have

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}+\mathbb{D}_{0/2}\Big)(q_{\bullet}\gamma_{0/2}\# k)(t,x,0)\\
&= \lim_{s\to t}\int dV(x')\,q_{t-s}(x-x')\gamma_{0/2}(t-s,x-x')\,k(s,x',0)\\
&\quad + \int_0^t ds\int dV(x')\,\Big(\frac{\partial}{\partial t}+\mathbb{D}_{0/2,x}\Big)q_{t-s}(x-x')\gamma_{0/2}(t-s,x-x')\,k(s,x',0)\\
&= \lim_{s\to t}\int dV(x')\,q_{t-s}(x-x')\gamma_{0/2}(t-s,x-x')\,k(s,x',0)\\
&= \gamma_{0/2}(0,0)\,k(t,x,0) = k(t,x,0).
\end{aligned}$$

Hence the first identity in (3.1) holds true for the sequence $\{\Psi_{i/2}(t,x)\}_{i>0}$. Together with (3.7) and (3.9), it implies that the sequence $\{\Psi_{i/2}(t,x)\}_{i=0}^{\infty}$ satisfies all the conditions in Proposition 3.1. By the uniqueness, thus we obtain the formula (3.3). ∎

Theorem 3.2 and (2.7) imply the main theorem:

Theorem 3.4 *In the case $\ell > p/2$, there is a well-defined formula*

$$\begin{aligned}
K_{\ell,[p]}(P^0) = (4\pi)^{m/2} &\sum_{\sum i_j = 2\ell-p}^{i_1,\ldots,i_k>0} (-1)^k\\
&\times (q_{\bullet}\gamma_{0/2}\#\,\mathbb{D}_{i_1/2}(q_{\bullet}\gamma_{0/2})\#\cdots\#\,\mathbb{D}_{i_k/2}(q_{\bullet}\gamma_{0/2}))_{[p]}(1,0,0),
\end{aligned} \tag{3.10}$$

which can be computed explicitly by using only a basic knowledge of calculus.

4 Some computations

It follows from (1.1), (1.2) and (3.10) that the coefficients $K_{\ell}(P^0)$ can be computed explicitly. For example, we have the following computations:

Corollary 4.1 *We have*

$$K_0(P^0) = 1, \tag{4.1}$$

$$K_1(P^0) = -\frac{\sum R_{jiji}}{12} = -\frac{s(\nabla^g)(P^0)}{12}, \tag{4.2}$$

$$\begin{aligned} K_2(P^0) &= \det{}^{1/2}\Big(\frac{R/2}{\sinh(R/2)}\Big)_{[4]} - \frac{5\sum R_{jijikk}}{24} - \frac{\sum R_{jijkik}}{3} \\ &\quad + \frac{\left(\sum R_{jiji}\right)^2}{432} + \frac{\sum R_{jkik}R_{jk'ik'}}{12} + \frac{2\sum R_{jkik'}(R_{jkik'}+R_{jk'ik})}{27}. \end{aligned} \tag{4.3}$$

Proof. (4.1) is obvious (i,e., $K_0(P^0) = K_{0,[0]}(P^0) = 1$). Let us prove (4.2). We have $K_1(P^0) = K_{1,[0]}(P^0) + K_{1,[1]}(P^0)$, and, by (3.2),

$$\begin{aligned} \nabla^{(\varepsilon)}_{e^{(\varepsilon)}_j} &= \varepsilon^{0/2}\Big\{\frac{\partial}{\partial x_j} + \sum x_{j_1}\frac{R_{jj_1\ell k}}{8}dx_\ell\wedge dx_k\wedge\Big\} \\ &\quad + \varepsilon^{1/2}\Big\{\sum x_{j_1}x_{j_2}\frac{R_{jj_1\ell kj_2}}{12}dx_\ell\wedge dx_k\wedge\Big\} + \varepsilon^{2/2}\Big\{\sum x_{j_1}x_{j_2}\frac{R_{jj_1ij_2}}{6}\frac{\partial}{\partial x_i} \\ &\quad + \sum x_{j_1}\frac{-R_{jj_1\ell k}}{4}dx_\ell\wedge dx_k\vee + \sum x_{j_1}x_{j_2}x_{j_3}\Big(\frac{R_{jj_1\ell kj_2j_3}}{32} \\ &\quad + \sum\frac{R_{jj_1ij_2}R_{ij_3\ell k}}{48} + \sum\frac{R_{jj_1\ell i}R_{ij_2kj_3}}{48}\Big)dx_\ell\wedge dx_k\wedge\Big\} + O(\varepsilon^{3/2}). \end{aligned} \tag{4.4}$$

Thus we have

$$\mathbb{D}_{1/2} = -\sum\Big[\sum x_{j_1}x_{j_2}\frac{R_{jj_1\ell kj_2}}{12}dx_\ell\wedge dx_k\wedge,\ \frac{\partial}{\partial x_j} + \sum x_{j'_1}\frac{R_{jj'_1\ell'k'}}{8}dx_{\ell'}\wedge dx_{k'}\wedge\Big]_+,$$

$$\begin{aligned} \mathbb{D}_{2/2} &= -\sum\Big[\sum x_{j_1}x_{j_2}\frac{R_{jj_1\ell kj_2}}{12}dx_\ell\wedge dx_k\wedge,\ \sum x_{j'_1}x_{j'_2}\frac{R_{jj'_1\ell'k'j'_2}}{12}dx_{\ell'}\wedge dx_{k'}\wedge\Big]_+ \\ &\quad - \sum\Big[\sum x_{j_1}x_{j_2}\frac{R_{jj_1ij_2}}{6}\frac{\partial}{\partial x_i} + \sum x_{j_1}\frac{-R_{jj_1\ell k}}{4}dx_\ell\wedge dx_k\vee \\ &\quad + \sum x_{j_1}x_{j_2}x_{j_3}\Big(\frac{R_{jj_1\ell kj_2j_3}}{32} + \sum\frac{R_{jj_1ij_2}R_{ij_3\ell k}}{48} \\ &\quad + \sum\frac{R_{jj_1\ell i}R_{ij_2kj_3}}{48}\Big)dx_\ell\wedge dx_k\wedge,\ \frac{\partial}{\partial x_j} + \sum x_{j'_1}\frac{R_{jj'_1\ell'k'}}{8}dx_{\ell'}\wedge dx_{k'}\wedge\Big]_+ \\ &\quad + \sum x_{j'_1}\frac{R_{jij'_1i}}{2}\Big(\frac{\partial}{\partial x_j} + \sum x_{j_1}\frac{R_{jj_1\ell k}}{8}dx_\ell\wedge dx_k\wedge\Big) + \sum\frac{R_{jiji}}{4}, \end{aligned}$$

where we set $[P,Q]_+ = P\cdot Q + Q\cdot P$. Consequently,

$$\begin{aligned} K_{1,[1]}(P^0) &= -(4\pi)^{m/2}(q_\bullet\gamma_{0/2}\#\,\mathbb{D}_{1/2}(q_\bullet\gamma_{0/2}))_{[1]}(1,0,0) = 0, \\ K_{1,[0]}(P^0) &= (4\pi)^{m/2}(q_\bullet\gamma_{0/2}\#\,\mathbb{D}_{1/2}(q_\bullet\gamma_{0/2})\#\,\mathbb{D}_{1/2}(q_\bullet\gamma_{0/2}))_{[0]}(1,0,0) \\ &\qquad - (4\pi)^{m/2}(q_\bullet\gamma_{0/2}\#\,\mathbb{D}_{2/2}(q_\bullet\gamma_{0/2}))_{[0]}(1,0,0) \\ &= -(4\pi)^{m/2}(q_\bullet\gamma_{0/2}\#\,\mathbb{D}_{2/2}(q_\bullet\gamma_{0/2}))_{[0]}(1,0,0) = -\frac{\sum R_{jiji}}{12}, \end{aligned}$$

which imply (4.2). Similarly but by a rather lengthy computation, (4.3) is shown as well. ∎

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