The heat equation for the Kohn-Rossi Laplacian on contact Riemannian manifolds

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Abstract

We study the heat kernel associated with the Kohn-Rossi Laplacian on a compact contact Riemannian manifold. We prove its unique existence and show that its every differential at each diagonal point can be asymptotically expanded for small time, and, by applying the general adiabatic expansion theory, we present a new formula for the asymptotic coefficients. All the coefficients are described as certain universal polynomials built from the curvature and the torsion of hermitian Tanno connection, and we emphasize that, by using only a basic knowledge of calculus added to the formula, one can describe the polynomials explicitly up to an arbitrarily high order. Explicit description of an asymptotic coefficient of the pointwise trace in the strictly pseudoconvex CR case is offered as an example.

Keywords: contact Riemannian structure; Kohn-Rossi Laplacian; hermitian Tanno connection; asymptotic expansion; adiabatic expansion

0 Introduction

Let us take a compact manifold M of dimension $2n+1$ equipped with a contact 1-form θ, i.e., $\theta \wedge (d\theta)^n \neq 0$. We have hence the Reeb vector field ξ, which satisfies $\theta(\xi) = 1$ and $\mathcal{L}_\xi \theta = 0$, where $\mathcal{L}_\xi$ is the Lie differentiation by ξ. Further let us equip

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M with a Riemannian metric g and a $(1,1)$-tensor field J called an almost complex structure satisfying $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y)$ and $J^2X = -X + \theta(X)\xi$ for any vector fields X, Y. (In the paper we will adopt such a notation as $d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$.) On the manifold $M = (M, \theta, \xi, g, J)$ called the contact Riemannian manifold, let us consider the Kohn-Rossi Laplacian

$$\Box_H = \bar{\partial}_H^* \bar{\partial}_H + \bar{\partial}_H \bar{\partial}_H^*$$

acting on (p,q)-forms. Namely, we decompose the complexified contact subbundle $\mathbb{C}HM = \ker\theta \otimes \mathbb{C}$ into $\mathbb{C}HM = H_{1,0}M \oplus H_{0,1}M$ with $H_{1,0}M = \{X \in \mathbb{C}HM \mid JX = iX\}$, etc., take the dual subbundle $\mathbb{C}H^*M = H^{1,0}M \oplus H^{0,1}M$ and set $H^{p,q}M = (\wedge^p H^{1,0}M) \wedge (\wedge^q H^{0,1}M)$, whose smooth cross-sections, called (p,q)-forms, gather together into the space $\Omega^{p,q}M$. For a (p,q)-form φ, $\bar{\partial}_H\varphi$ is defined to be the $(p, q+1)$-component of the exterior derivative $d\varphi$. We denote by $\bar{\partial}_H^*$ the formal adjoint of $\bar{\partial}_H$ with respect to the hermitian inner product given as follows: We take a local unitary frame $\xi_\bullet = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{\bar{1}}, \ldots, \xi_{\bar{n}})$ of $\mathbb{C}TM$ ($\xi_0 := \xi$, $\xi_{\bar{\alpha}} := \overline{\xi_\alpha} \in H_{0,1}M$, $g(\xi_\alpha, \xi_{\bar{\beta}}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) and its dual frame $\theta^\bullet = (\theta^0, \theta^1, \ldots, \theta^n, \theta^{\bar{1}}, \ldots, \theta^{\bar{n}})$ ($\theta^0 := \theta$). Thus we have $g = \theta \otimes \theta + \sum(\theta^\alpha \otimes \theta^{\bar{\alpha}} + \theta^{\bar{\alpha}} \otimes \theta^\alpha) = \theta \otimes \theta + 2\sum \theta^\alpha \theta^{\bar{\alpha}}$ and the hermitian inner product $(\varphi, \psi)_g$ for (p,q)-forms φ, ψ is defined by

$$(\varphi, \psi)_g = \int_M \varphi \wedge \bar{\star}_g \psi = \int_M dV_g \, \langle \varphi, \psi \rangle_g, \quad \langle \varphi, \psi \rangle_g = \sum \varphi^{I\bar{K}} \, \overline{\psi^{I\bar{K}}},$$

where we put $\varphi = \sum \theta^{I\bar{K}} \cdot \varphi^{I\bar{K}}$ locally ($I = (i_1 < i_2 < \cdots < i_p)$ ($i_1 > 0$), etc., and $\theta^{I\bar{K}} := \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \wedge \theta^{\bar{k}_1} \wedge \cdots \wedge \theta^{\bar{k}_q}$) and denote by dV_g the volume element, i.e., $dV_g = \bar{\star}_g 1 = \theta \wedge (d\theta)^n/n!$. From now on, the local frames $\xi_\bullet$, $\theta^\bullet$ are always assumed to be unitary.

Let us suppose $0 < q < n$ and consider the initial value problem for the heat equation

$$\Big(\frac{\partial}{\partial t} + \Box_H\Big)\phi = 0, \quad \lim_{t \to 0} \phi(t) = \varphi \quad (\varphi \in \Omega^{p,q}M), \tag{0.1}$$

where the convergence is in the L^2-norm. Its fundamental solution or heat kernel $e^{-t\Box_H}$, expressed locally as

$$e^{-t\Box_H}(P, P') = \sum \theta^{I\bar{K}}(P) \boxtimes \theta^{\bar{I'}K'}(P') \cdot \big(e^{-t\Box_H}\big)^{(I\bar{K})(I'\bar{K}')}(P, P'), \tag{0.2}$$

is a smooth cross-section of $H^{p,q}M \boxtimes H^{q,p}M$ over $M \times M$ parameterized smoothly by $t \in \mathbb{R}^+$ which solves (0.1). Namely, the (p,q)-form

$$\begin{aligned}(e^{-t\Box_H}\varphi)(P) &= \int_M e^{-t\Box_H}(P, P') \wedge \star\varphi(P') = \int_M dV_g(P') \, \langle e^{-t\Box_H}(P, P'), \overline{\varphi(P')} \, \rangle_g \\ &= \sum \theta^{I\bar{K}}(P) \cdot \int_M dV_g(P') \, \big(e^{-t\Box_H}\big)^{(I\bar{K})(I'\bar{K}')}(P, P') \, \varphi^{I'\bar{K}'}(P')\end{aligned}$$

is the solution of (0.1). Note that the heat kernel is inevitably unique if it exists. The operator $\Box_H$ is certainly not elliptic (see (1.15)). It is hypoelliptic when $0 < q < n$ (see Remark 2.2(1)), however.

In particular, if J is integrable (i.e., $[\Gamma(H_{1,0}M), \Gamma(H_{1,0}M)] \subset \Gamma(H_{1,0}M)$), that is, if M is a strictly pseudoconvex CR manifold, it is well-known (Folland-Stein [9], Stanton-Tartakoff [18]) that a heat kernel exists and when $t \to 0$ its pointwise trace at each point $P^0 \in M$ can be asymptotically expanded as

$$\operatorname{tr} e^{-t\Box_H}(P^0, P^0) \sim t^{-(n+1)} a_0(P^0) + t^{-(n+1)+1} a_1(P^0) + \cdots \tag{0.3}$$

with

$$a_0(P^0) = \binom{n}{q}\binom{n}{p} \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s), \quad \Phi^{n-2q}(s) := \frac{e^{-(n-2q)s}}{(2\pi)^{n+1}} \Big(\frac{s}{\sinh s}\Big)^n \tag{0.4}$$

(refer also to (5.12) in the case where J is not integrable). Note that $\Phi^{n-2q}(s)$ is rapidly decreasing.

In this paper, we will show that, with no restriction on J, the heat kernel exists and its every differential at each diagonal point can be expanded asymptotically (Theorems 2.1 and 2.3), and, furthermore, there exists a new formula for the asymptotic coefficients (Theorem 5.3). We wish to emphasize that, by using only a basic knowledge of calculus added to the formula, one can describe the coefficients explicitly up to an arbitrarily high order. The idea of describing them by applying the invariance theory of Gilkey, etc., ([10, §4.8], [11, §4.1]) will readily occur to us. Indeed one can find such researches (e.g. Beals-Greiner-Stanton [3, §8], Biquard-Herzlich-Rumin [4]). Such an attempt, however, will be tough because our knowledge about non-elliptic Laplacian is too limited. Our method exhibits its ability particularly for studying such an abnormal Laplacian. Similar formula for the metric Laplacian Δ exists as well and its asymptotic coefficients can be calculated easily up to an arbitrarily high order (Nagase [15]).

We wish to present here, as an example, the explicit description of $a_1(P^0)$ in the case where J is integrable, whose calculation will be given in §7. Let us set

$$\begin{aligned} S(t,s) &= \frac{\tanh ts}{2s} \quad ((t,s) \in (0,1)\times\mathbb{R}), \\ \Phi_1(s) &= \int_0^1 dt\, \frac{s\,S(1-t,s)S(t,s)}{S(1-t,s)+S(t,s)} = \frac{s}{\sinh s}\,\frac{s\cosh s - \sinh s}{4s^2}, \\ \Phi_2(s) &= \int_0^1 dt\, \Big(\frac{s\,S(1-t,s)S(t,s)}{S(1-t,s)+S(t,s)}\Big)^2 = \Big(\frac{s}{\sinh s}\Big)^2 \frac{2s\cosh 2s - 3\sinh 2s + 4s}{64s^3}. \end{aligned} \tag{0.5}$$

The functions $\Phi_j(s)$ are smooth and bounded on $\mathbb{R}$. Each $\Phi^{n-2q}(s)\Phi_j(s)$ is rapidly decreasing.

Corollary of Theorem 5.3 *Suppose J is integrable. Then, in* (0.3) *we have*

$$a_1(P^0) = \sum_{\alpha,\beta=1}^{n} R_{\bar{\alpha}\alpha\bar{\beta}\beta}(P^0) \cdot \Big\{ \binom{n-1}{q-1}\binom{n-1}{p} + \Big(\binom{n-1}{p-1} - \binom{n-1}{q-1}\Big)\Big(\frac{1}{2} + \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s)\Phi_1(s)\Big) + \binom{n}{q}\binom{n}{p} \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s)\Big(\frac{4}{3}\Phi_2(s) - \frac{1}{12}\Big)\Big\}, \tag{0.6}$$

where R_{ABCD} denotes the curvature coefficient of the Tanaka-Webster connection ∇ (refer to §1*), i.e., $R_{ABCD} = g(F(\nabla)(\xi_C, \xi_D)\xi_B, \xi_A)$. We put $\binom{n-1}{p-1} = 0$ when $p = 0$.*

The expression (0.6) is obtained by written calculation. With the aid of Mathematica, we get also its concrete description with no restriction on J (Imai-Nagase [13]).

Accordingly our main purpose is to offer the enlightening formula (5.14) for the asymptotic coefficients on the basis of the adiabatic expansion theory ([14]) (refer also to the comment following the proof of Proposition 5.2). Indeed, almost all the arguments in this paper will be devoted to ascertaining the formula. In §1 we propose utilizing a new connection called hermitian Tanno connection to conduct researches into the contact Riemannian structure. With the use of it, the Kohn-Rossi Laplacian is expressed in the style of classical Weitzenböck formula (Proposition 1.3). We expand its connection coefficients, etc., into Taylor series, which can be expressed explicitly as universal polynomials built from the curvature and the torsion up to an arbitrarily high order (Proposition 2.4). Consequently, in §5 the adiabatic expansion theory can be applied to draw the formula. The two ideas of hermitian Tanno connection and adiabatic expansion theory will afford the keys to an understanding of the abnormal Laplacian $\Box_H$.

1 Hermitian Tanno connection and Weitzenböck-type formula for the Kohn-Rossi Laplacian

If J is integrable, the Laplacian $\Box_H$ possesses the Weitzenböck-type formula (acting on $(0,q)$-forms) with the use of the Tanaka-Webster connection (Dragomir-Tomassini [7, Theorem 1.19]). In this section we show that it still holds with the use of hermitian Tanno connection introduced below even if J is not integrable.

Let ∇^g be the Levi-Civita connection associated with the metric g. Tanno ([19]) introduced a generalized Tanaka-Webster connection ${}^*\nabla$ defined by

$$ {}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X \theta)(Y)\xi, \tag{1.1} $$

which we will call the Tanno connection. By denoting its torsion tensor by $T({}^*\nabla)$ and setting ${}^*\tau(X) = T({}^*\nabla)(\xi, X)$, the connection is characterized axiomatically ([19, Proposition 3.1]) as a unique linear connection satisfying

$$ \begin{aligned} & {}^*\nabla\theta = 0, \quad {}^*\nabla g = 0, \\ & ({}^*\nabla_Y J)(X) = \mathcal{Q}(X, Y) := (\nabla^g_Y J)(X) + (\nabla^g_Y \theta)(JX)\,\xi + \theta(X)\,J\nabla^g_Y \xi, \\ & T({}^*\nabla)(Z, W) = 0, \quad T({}^*\nabla)(Z, \overline{W}) = ig(Z, \overline{W})\xi \quad (Z, W \in H_{1,0}M), \\ & {}^*\tau \circ J + J \circ {}^*\tau = 0. \end{aligned} \tag{1.2} $$

Since the Tanno tensor field $\mathcal{Q}$ vanishes if and only if J is integrable ([19, Proposition 2.1], (1.12)), the Tanno connection coincides with the Tanaka-Webster connection when J is integrable. One could describe $\Box_H$ in the type of Weitzenböck formula by using the Tanno connection, which will be, however, rather complicated. It is caused by the fact that ${}^*\nabla J \neq 0$ when J is not integrable. To amend the situation, we consider the connection ∇ defined by

$$ \nabla_X Y = {}^*\nabla_X Y - \frac{1}{2} J\mathcal{Q}(Y, X) = \begin{cases} {}^*\nabla_X(f\xi) & : Y = f\xi, \\ \frac{1}{2}\left({}^*\nabla_X Y - J\,{}^*\nabla_X JY\right) & : Y \in \Gamma(HM), \end{cases} \tag{1.3} $$

which we will call the **hermitian Tanno connection**. This obviously satisfies $\nabla J = 0$, and the two connections ∇, ${}^*\nabla$ and the Tanaka-Webster connection coincide when J is integrable. The author found the connection ∇ to have already been referred to by Seshadri [16, the proof of Lemma 4.2]. It seems, however, that its usefulness is not yet become fully aware of.

We want to point out here that the concise characterization of the hermitian Tanno connection stated at Lemma 1.1(1) coincides with that of the Tanaka-Webster connection (refer to the remark at the final page of [7, §1.2]).

Lemma 1.1

(1) *The connection ∇ is a unique linear connection satisfying the following:*

$$ \nabla\theta = 0, \quad \nabla g = 0, \quad \nabla J = 0, \tag{1.4} $$

$$ \pi_+ T(\nabla)(Z, W) = 0 \quad (Z \in H_{1,0}M,\ W \in \mathbb{C}TM), \tag{1.5} $$

where π_+ is the natural projection to the $H_{1,0}M$-part.

(2) *Let us set $\tau(X) = T(\nabla)(\xi, X)$ and consider the Nijenhuis tensor $[J, J]$, which is defined by $[J,J](X,Y) = -[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$. Under the assumption that* (1.4) *holds,* (1.5) *is equivalent to the gathering of conditions*

$$T(\nabla)(Z,W) = \frac{1}{4}[J,J](Z,W), \quad T(\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi \quad (Z,W \in H_{1,0}M), \tag{1.6}$$

$$\tau \circ J + J \circ \tau = 0. \tag{1.7}$$

Proof. For $Z, W \in \Gamma(H_{1,0}M)$ we have

$$\theta([Z,W]) = 0, \quad \theta([Z,\overline{W}]) = -ig(Z,\overline{W}), \tag{1.8}$$

$$[J,J](Z,W) = -2([Z,W] + iJ[Z,W]) \in \Gamma(H_{0,1}M), \tag{1.9}$$

and

$$T(\nabla)(Z,W) = \pi_+ T(\nabla)(Z,W) + \frac{1}{4}[J,J](Z,W) \tag{1.10}$$

for any ∇ satisfying $\nabla J = 0$. (2) will be obvious because of (1.9) and (1.10). (1): The uniqueness can be established in the same way as for the Tanaka-Webster one. The equalities at (1.2), (1.9) (see also (1.11)) imply that the connection (1.3) satisfies (1.4), (1.6) and (1.7). ∎

Let us summarize here some properties of the connections ∇ and ${}^*\nabla$. As usual the Greek indices α, β, ... vary from 1 to n, the block Latin indices A, B, ... vary in $\{0, 1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ and the symbol $\sum$ may be omitted (in an unusual manner). Referring to [19, §6], [6, §2.1, §4], [16, §3], we obtain the following lemma.

Lemma 1.2 *We have*

$$\begin{aligned}
&\nabla\xi = 0, \quad \nabla\xi_\beta = \xi_\alpha \cdot \omega^\alpha_\beta, \quad \nabla\xi_{\bar\beta} = \xi_{\bar\alpha} \cdot \omega^{\bar\alpha}_{\bar\beta}, \quad \omega^{\bar\alpha}_{\bar\beta} = -\omega^\beta_\alpha, \\
&{}^*\nabla_{\xi_{\bar\gamma}}\xi_{\bar\beta} = \xi_\alpha \cdot \omega({}^*\nabla)^\alpha_{\bar\beta}(\xi_{\bar\gamma}) + \nabla_{\xi_{\bar\gamma}}\xi_{\bar\beta}, \quad {}^*\nabla_{\xi_\gamma}\xi_\beta = \nabla_{\xi_\gamma}\xi_\beta + \xi_{\bar\alpha} \cdot \omega({}^*\nabla)^{\bar\alpha}_\beta(\xi_\gamma), \\
&{}^*\nabla_{\xi_C}\xi_B = \nabla_{\xi_C}\xi_B \ \ (\textit{otherwise}), \quad \omega({}^*\nabla)^{\bar A}_{\bar B} = -\omega({}^*\nabla)^B_A,
\end{aligned}$$

$$\begin{cases}
\omega({}^*\nabla)^\alpha_{\bar\beta}(\xi_{\bar\gamma}) = \dfrac{i}{2}\mathcal{Q}^\alpha_{\bar\beta\bar\gamma}, \quad \omega({}^*\nabla)^{\bar\alpha}_\beta(\xi_\gamma) = -\dfrac{i}{2}\mathcal{Q}^{\bar\alpha}_{\beta\gamma}, \\
\mathcal{Q} = \xi_\alpha \otimes \theta^{\bar\beta} \otimes \theta^{\bar\gamma} \cdot \mathcal{Q}^\alpha_{\bar\beta\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\beta \otimes \theta^\gamma \cdot \mathcal{Q}^{\bar\alpha}_{\beta\gamma}, \quad \overline{\mathcal{Q}^\alpha_{\bar\beta\bar\gamma}} = \mathcal{Q}^{\bar\alpha}_{\beta\gamma} = -\mathcal{Q}^{\bar\beta}_{\alpha\gamma}
\end{cases} \tag{1.11}$$

and

$${}^*\tau = \tau = \xi_\alpha \otimes \tau^\alpha + \xi_{\bar\alpha} \otimes \tau^{\bar\alpha} = \xi_\alpha \otimes \theta^{\bar\gamma} \cdot \tau^\alpha_{\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\gamma \cdot \tau^{\bar\alpha}_\gamma, \quad \tau^{\bar\alpha}_\gamma = \tau^{\bar\gamma}_\alpha,$$

$$
\begin{aligned}
[J,J] &= \xi_\alpha \otimes \theta^{\bar\beta} \otimes \theta^{\bar\gamma} \cdot [J,J]^\alpha_{\bar\beta\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\beta \otimes \theta^\gamma \cdot [J,J]^{\bar\alpha}_{\beta\gamma} \\
&\quad + \xi_\alpha \otimes \theta \wedge \theta^{\bar\gamma} \cdot 2\tau^\alpha_{\bar\gamma} + \xi_{\bar\alpha} \otimes \theta \wedge \theta^\gamma \cdot 2\tau^{\bar\alpha}_\gamma, \quad \overline{[J,J]^\alpha_{\bar\beta\bar\gamma}} = [J,J]^{\bar\alpha}_{\beta\gamma}, \\
T({}^*\nabla) &= \xi \otimes \theta^\beta \wedge \theta^{\bar\beta} \cdot i + \xi_\alpha \otimes \theta \wedge \theta^{\bar\gamma} \cdot \tau^\alpha_{\bar\gamma} + \xi_{\bar\alpha} \otimes \theta \wedge \theta^\gamma \cdot \tau^{\bar\alpha}_\gamma, \\
T(\nabla) &= T({}^*\nabla) + \frac{1}{4}\left\{ \xi_\alpha \otimes \theta^{\bar\beta} \otimes \theta^{\bar\gamma} \cdot [J,J]^\alpha_{\bar\beta\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\beta \otimes \theta^\gamma \cdot [J,J]^{\bar\alpha}_{\beta\gamma} \right\},
\end{aligned}
$$

$$
[J,J]^{\bar\alpha}_{\beta\gamma} = -2i\mathcal{Q}^{\bar\alpha}_{\beta\gamma} + 2i\mathcal{Q}^{\bar\alpha}_{\gamma\beta}, \quad 4i\mathcal{Q}^\alpha_{\bar\beta\bar\gamma} = [J,J]^\alpha_{\bar\beta\bar\gamma} - [J,J]^\beta_{\bar\alpha\bar\gamma} + [J,J]^\gamma_{\bar\beta\bar\alpha}. \tag{1.12}
$$

By setting $F(\nabla)(\xi_C, \xi_D)\xi_B = ([\nabla_{\xi_C}, \nabla_{\xi_D}] - \nabla_{[\xi_C,\xi_D]})\xi_B = \xi_A \cdot F(\nabla)^A_B(\xi_C, \xi_D) = \xi_A \cdot F(\nabla)^A_{BCD}$*, etc., the curvature coefficients are related to each other as*

$$
\begin{aligned}
F(\nabla)^\alpha_{\beta\gamma\bar\delta} &= F({}^*\nabla)^\alpha_{\beta\gamma\bar\delta} + \frac{\mathcal{Q}^\alpha_{\bar\rho\bar\delta}\,\mathcal{Q}^{\bar\rho}_{\beta\gamma}}{4}, \quad F(\nabla)^\alpha_{\beta\bar\gamma\delta} = F({}^*\nabla)^\alpha_{\beta\bar\gamma\delta} - \frac{\mathcal{Q}^\alpha_{\bar\rho\bar\gamma}\,\mathcal{Q}^{\bar\rho}_{\beta\delta}}{4}, \\
F(\nabla)^\alpha_{\beta\gamma\delta} &= F({}^*\nabla)^\alpha_{\beta\gamma\delta} = \frac{-i\mathcal{Q}^{\bar\gamma}_{\delta\beta,\bar\alpha}}{2} - i\tau^{\bar\beta}_\gamma\,\delta_{\alpha\delta} + i\tau^{\bar\beta}_\delta\,\delta_{\alpha\gamma}, \\
F(\nabla)^\alpha_{\beta\bar\gamma\bar\delta} &= F({}^*\nabla)^\alpha_{\beta\bar\gamma\bar\delta} = \frac{-i\mathcal{Q}^\gamma_{\bar\delta\bar\alpha,\beta}}{2} + i\tau^\alpha_{\bar\delta}\,\delta_{\beta\gamma} - i\tau^\alpha_{\bar\gamma}\,\delta_{\beta\delta},
\end{aligned} \tag{1.13}
$$

where we set $\mathcal{Q}^\gamma_{\bar\delta\bar\alpha,\beta} = \theta^\gamma(({}^*\nabla_{\xi_\beta}\mathcal{Q})(\xi_{\bar\delta}, \xi_{\bar\alpha})) = \theta^\gamma((\nabla_{\xi_\beta}\mathcal{Q})(\xi_{\bar\delta}, \xi_{\bar\alpha}))$*, etc.*

Now, with the use of the hermitian connection we obtain

Proposition 1.3 (Weitzenböck-type formula) *We have*

$$
\bar\partial_H = \sum \theta^{\bar\alpha} \wedge \nabla_{\xi_{\bar\alpha}}, \quad \bar\partial^*_H = -\sum \theta^{\bar\alpha} \vee \nabla_{\xi_\alpha}, \tag{1.14}
$$

$$
\begin{aligned}
\Box_H &= -\sum \left(\nabla_{\xi_\alpha}\nabla_{\xi_{\bar\alpha}} - \nabla_{\nabla_{\xi_\alpha}\xi_{\bar\alpha}} \right) - \sqrt{-1}\,q\,\nabla_\xi \\
&\quad - \sum F(\nabla)^C_D(\xi_{\bar\alpha}, \xi_\beta) \cdot \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \vee \theta^{\bar C} \wedge \theta^{\bar D} \vee \quad (\textit{acting on } \Omega^{p,q}M),
\end{aligned} \tag{1.15}
$$

where $\theta^{\bar\alpha}\wedge$, $\theta^{\bar\alpha}\vee (= \iota_{\xi_{\bar\alpha}} = \xi_{\bar\alpha}\lrcorner)$ *denote their exterior, interior products, respectively. We may assume that the pair of indices* (C, D) *above runs only over the set of pairs* (γ, δ), $(\bar\gamma, \bar\delta)$ $(1 \le \gamma, \delta \le n)$. *(Notice that the action of* ∇ *on forms is expressed as* $\nabla_{\xi_\alpha} = \xi_\alpha + \omega^{\bar B}_{\bar C}(\xi_\alpha) \cdot \theta^B \wedge \theta^C \vee$*.)*

Proof. Since (1.6) and (1.9) imply

$$
T(\nabla)(Z, W) \in H_{0,1}M \quad (Z, W \in H_{1,0}M), \tag{1.16}
$$

the proposition is proved in the same way as in the strictly pseudoconvex CR case ([7, §1.7.6]). (In their book the proof is based on the property $T(\nabla)(Z,W) = 0$ which the integrable case has, but the property (1.16) is obviously enough for it.) Indeed, since (1.16) and the second equality at (1.6) yield

$$
\nabla_{\xi_{\bar\beta_i}}\xi_{\bar\beta_j} - \nabla_{\xi_{\bar\beta_j}}\xi_{\bar\beta_i} = \pi_-[\xi_{\bar\beta_i}, \xi_{\bar\beta_j}], \qquad \nabla_{\xi_{\bar\beta_i}}\xi_{\alpha_j} = -\pi_+[\xi_{\alpha_j}, \xi_{\bar\beta_i}],
$$

the first equality at (1.14) is certainly correct. Green's formula and (1.16) imply the second one. Accordingly we have

$$\begin{aligned}
&\bar\partial_H \bar\partial_H^* = \sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \vee \cdot \{\nabla_{\nabla_{\xi_{\bar\alpha}}\xi_\beta} - \nabla_{\xi_{\bar\alpha}} \nabla_{\xi_\beta}\},\\
&\bar\partial_H^* \bar\partial_H = -\sum \{\nabla_{\xi_\alpha}\nabla_{\xi_{\bar\alpha}} - \nabla_{\nabla_{\xi_\alpha}\xi_{\bar\alpha}}\} + \sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \vee \cdot \{\nabla_{\xi_\beta}\nabla_{\xi_{\bar\alpha}} - \nabla_{\nabla_{\xi_\beta}\xi_{\bar\alpha}}\},\\
&\sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \vee \cdot \{\nabla_{\nabla_{\xi_{\bar\alpha}}\xi_\beta} - \nabla_{\xi_{\bar\alpha}}\nabla_{\xi_\beta} + \nabla_{\xi_\beta}\nabla_{\xi_{\bar\alpha}} - \nabla_{\nabla_{\xi_\beta}\xi_{\bar\alpha}}\}\\
&\qquad = \sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \vee \cdot \Big(\nabla_{T(\nabla)(\xi_{\bar\alpha},\xi_\beta)} - F(\nabla, \theta^\bullet)(\xi_{\bar\alpha}, \xi_\beta)\Big)\\
&\qquad = -\sqrt{-1}\, q\, \nabla_\xi - \sum \theta^{\bar\alpha}\wedge \theta^{\bar\beta} \vee \cdot \theta^{\bar C} \wedge \theta^{\bar D} \vee \cdot F(\nabla)^C_D(\xi_{\bar\alpha}, \xi_\beta),
\end{aligned}$$

which assert that the formula (1.15) holds. Note that the last line follows from (1.6) and

$$\begin{aligned}
&(F(\nabla, \theta^\bullet)(\xi_A, \xi_B)\theta^{\bar D})(\xi_{\bar C}) := \Big(\nabla_{\xi_A}\nabla_{\xi_B}\theta^{\bar D} - \nabla_{\xi_B}\nabla_{\xi_A}\theta^{\bar D} - \nabla_{[\xi_A,\xi_B]}\theta^{\bar D}\Big)(\xi_{\bar C})\\
&\qquad = -\theta^{\bar D}\,(F(\nabla)(\xi_A, \xi_B)\xi_{\bar C}) = -F(\nabla)^{\bar D}_{\bar C}(\xi_A, \xi_B) = F(\nabla)^C_D(\xi_A, \xi_B).
\end{aligned}$$

∎

One can find some investigation into $\bar\partial_H^2$, which does not vanish in general (e.g. Barletta-Dragomir [2, §5.1]). Here we offer its formula with the use of the connection ∇. We have

$$\begin{aligned}
\bar\partial_H^2 &= \sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot \{\nabla_{\xi_{\bar\alpha}}\nabla_{\xi_{\bar\beta}} - \nabla_{\nabla_{\xi_{\bar\alpha}}\xi_{\bar\beta}}\}\\
&= \frac{1}{2}\sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot \{-\nabla_{T(\nabla)(\xi_{\bar\alpha},\xi_{\bar\beta})} + F(\nabla, \theta^\bullet)(\xi_{\bar\alpha}, \xi_{\bar\beta})\}\\
&= -\frac{1}{8}\sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot \nabla_{[J,J](\xi_{\bar\alpha},\xi_{\bar\beta})} + \frac{1}{2}\sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot \theta^{\bar C} \wedge \theta^{\bar D} \vee \cdot F(\nabla)^C_D(\xi_{\bar\alpha}, \xi_{\bar\beta}),
\end{aligned}$$

which, together with (1.13), implies the following formula.

Corollary 1.4 *We have*

$$\begin{aligned}
\bar\partial_H^2 = &-\frac{1}{8}\sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot \nabla_{[J,J](\xi_{\bar\alpha},\xi_{\bar\beta})}\\
&+ \sqrt{-1}\sum \theta^{\bar\alpha}\wedge \theta^{\bar\beta}\wedge \cdot \left\{ \frac{-\mathcal{Q}^\alpha_{\bar\beta\bar\gamma,\delta}}{4}\,\theta^{\bar\gamma}\wedge\theta^{\bar\delta}\vee + \frac{\mathcal{Q}^\alpha_{\bar\beta\bar\delta,\gamma}}{4}\,\theta^\gamma\wedge\theta^\delta\vee + \tau^\alpha_{\bar\delta}\,\theta^\beta\wedge\theta^\delta\vee \right\}.
\end{aligned}$$

2 Unique existence of the heat kernel and its asymptotic expansion

In this section we will prove the assertions about the existence of heat kernel and the possibility of its asymptotic expansion.

Theorem 2.1 *The initial value problem* (0.1) *has a unique heat kernel* $e^{-t\Box_H}(P,P')$. *As to the initial condition, added to* $\lim_{t\to 0}\int e^{-t\Box_H}(P,P')\wedge\star\varphi(P')=\varphi(P)$, *we have* $\lim_{t\to 0}\int\bar\varphi(P)\wedge\star e^{-t\Box_H}(P,P')=\bar\varphi(P')$.

We will prove the theorem by constructing the kernel according to the iteration method of E. E. Levi so as to turn the result to the study of its asymptotic behavior, but, in fact, we can prove it also by functional analysis method as below.

Remark 2.2 *Most of the researches on contact Riemannian manifolds are focused on the case where J is integrable. Careful assessment of their validity when J is not integrable will be needed. We will state two valid assertions* (cf. Folland-Stein [9, Theorem 2.4]) *related to this paper. Assume* $0<q<n$ *and J may not be integrable. Then we have:* (1) $\Box_H$ *is hypoelliptic.* (2) *There exists a constant* $C>0$ *such that* $\|\varphi\|_{s+1}\le C\{\|\Box_H\varphi\|_s+\|\varphi\|_0\}$ $(\varphi\in\Omega^{p,q}M)$, *where* $\|\cdot\|_s$ *is the Sobolev norm of order* s. (*These are easily ascertained by referring to* Folland-Kohn [8, §5.4].) *Hence, one can show the unique existence of heat kernel certainly by functional analysis method.*

Next, near a given point P^0, let us take local unitary frames $\xi_\bullet$, $\theta^\bullet$ which are ∇-parallel along the ∇-geodesics from P^0. Further let $z_\bullet=(z_0,z_1,\cdots,z_n,z_{\bar 1},\cdots,z_{\bar n})$ or $z=(z_0,z_1,\cdots,z_n)$ be the ∇-normal coordinates centered at P^0, i.e., $\exp^\nabla(\xi_\bullet(P^0)\cdot z_\bullet(P))=P$: to be precise, first we set $e_0=\xi$, $e_\alpha=(\xi_\alpha+\xi_{\bar\alpha})/\sqrt{2}$, $e_{n+\alpha}=Je_\alpha=(\xi_{\bar\alpha}-\xi_\alpha)/\sqrt{-2}$, which together provide a ∇-parallel orthonormal frame $e_\bullet$ along the ∇-geodesics from P^0, next define the real ∇-normal coordinates $x=(x_0,x_1,\cdots,x_{2n})$ centered at P^0 by $\exp^\nabla(e_\bullet(P^0)\cdot x(P))=P$ and then put $z_0=x_0$, $z_\alpha=(x_\alpha+ix_{n+\alpha})/\sqrt{2}$, $z_{\bar\alpha}=\bar z_\alpha=(x_\alpha-ix_{n+\alpha})/\sqrt{2}$. We define the frames $(\partial/\partial z)=(\partial/\partial z_\bullet)=(\partial/\partial z_0,\partial/\partial z_1,\cdots,\partial/\partial z_{\bar 1},\cdots)$, $(dz)=(dz_\bullet)=(dz_0,dz_1,\cdots,dz_{\bar 1},\cdots)$ by

$$\partial/\partial z_0=\partial/\partial x_0,\ \partial/\partial z_\alpha=\frac{\partial/\partial x_\alpha-i\partial/\partial x_{n+\alpha}}{\sqrt{2}},\quad dz_0=dx_0,\ dz_\alpha=\frac{dx_\alpha+idx_{n+\alpha}}{\sqrt{2}}.$$

From now on, the unitary frames $\xi_\bullet$, $\theta^\bullet$ are always assumed to be ∇-parallel and the coordinates z are ∇-normal centered at P^0. So are the frames in the expression (0.2) of $e^{-t\Box_H}(P,P')=e^{-t\Box_H}(z,z')$ $(z:=z(P),\ z':=z(P'))$.

Theorem 2.3 *There is an asymptotic expansion*

$$(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\left(e^{-t\Box_H}\right)^{(I\bar K)(I'\bar K')}(P^0,P^0)\sim\sum_{m\ge-(|\mathbb{A}|_H+|\mathbb{A}'|_H)}t^{-(n+1)+m/2}\,a_{m/2}^{(I\bar K)(I'\bar K')}(P^0:\mathbb{A},\mathbb{A}')\tag{2.1}$$

when $t \to 0$, and the asymptotic coefficient $a_{m/2}^{(I\bar{K})(I'\bar{K}')}(P^0 : \mathbb{A}, \mathbb{A}')$ vanishes when m is odd. Here, for a multi-index $\mathbb{A} = (A_1, \ldots, A_{|\mathbb{A}|})$, we set $(\partial/\partial z)^{\mathbb{A}} = \partial/\partial z_{A_1} \cdots \partial/\partial z_{A_{|\mathbb{A}|}}$ and $|\mathbb{A}|_H = 2\#\{A_i = 0\} + \#\{A_i \neq 0\}$.

These two theorems are the generalizations of Stanton-Tartakoff [18, Theorems 1.1, 4.10 and 6.4].

Let us provide here an assertion, which will suggest that near P^0 the structure of M is approximated by the standard contact Riemannian structure the Heisenberg group has.

Proposition 2.4 (cf. Atiyah-Bott-Patodi [1, Appendix II]) *There exists a formal series expansion*

$$\omega^\alpha_\beta(\partial/\partial z_A)(z) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum z_{A_1} \cdots z_{A_\ell} \frac{\partial^{\ell-1} F(\nabla)^\alpha_\beta(\partial/\partial z_A, \partial/\partial z_{A_1})}{\partial z_{A_2} \cdots \partial z_{A_\ell}}(0), \tag{2.2}$$

and, by setting

$$\xi_A = \sum V_{BA}\, \partial/\partial z_B, \quad \theta^A = \sum V^{BA}\, dz_B, \quad \textit{hence } V_\bullet = {}^t(V^\bullet)^{-1}, \tag{2.3}$$

there exists a formal series expansion

$$\begin{aligned} V^{BA}(z) = \delta^{BA} &+ \sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum z_{A_1} \cdots z_{A_\ell} \frac{\partial^{\ell-1} T(\nabla)^A_{A_1}(\partial/\partial z_B)}{\partial z_{A_2} \cdots \partial z_{A_\ell}}(0) \\ &+ \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell+1)!} \sum z_{A_1} \cdots z_{A_\ell} \frac{\partial^{\ell-2} F(\nabla)^A_{A_1}(\partial/\partial z_{A_2}, \partial/\partial z_B)}{\partial z_{A_3} \cdots \partial z_{A_\ell}}(0), \end{aligned} \tag{2.4}$$

where we set $T(\nabla)(\xi_C, X) = \xi_A \cdot T(\nabla)^A_C(X)$.

Proof. The proof is similar to that in [1, Appendix II]. Set $\mathsf{R} = \sum x_i \partial/\partial x_i = \sum x_i e_i = \sum z_A \partial/\partial z_A = \sum z_A \xi_A$. Then we have $\mathcal{L}_\mathsf{R} dz_A = dz_A$, which yields $\mathcal{L}_\mathsf{R}\, \omega^\alpha_\beta = \sum \mathsf{R}\Big(\omega^\alpha_\beta(\partial/\partial z_A)\Big) dx_A + \omega^\alpha_\beta$. Since $\omega^\alpha_\beta(\mathsf{R}) = 0$, on the other hand we have

$$\begin{aligned} \mathcal{L}_\mathsf{R}\, \omega^\alpha_\beta = \mathsf{R} \vee d\,\omega^\alpha_\beta &= \sum z_{A'} \Big\{ \frac{\partial\, \omega^\alpha_\beta(\partial/\partial z_A)}{\partial z_{A'}} - \frac{\partial\, \omega^\alpha_\beta(\partial/\partial z_{A'})}{\partial z_A} \Big\}\, dz_A \\ &= \sum z_{A'}\, F(\nabla)^\alpha_\beta(\partial/\partial z_{A'}, \partial/\partial z_A)\, dz_A. \end{aligned}$$

Thus we obtain the equality

$$\sum \mathsf{R}\Big(\omega^\alpha_\beta(\partial/\partial z_A)\Big) dx_A + \omega^\alpha_\beta = \sum z_{A'}\, F(\nabla)^\alpha_\beta(\partial/\partial z_{A'}, \partial/\partial z_A)\, dz_A,$$

which says, by putting $\omega^\alpha_\beta(\partial/\partial z_A)=\sum\omega^\alpha_\beta(\partial/\partial z_A)[\ell]=\sum\frac{1}{\ell!}z_{A_1}\cdots z_{A_\ell}\frac{\partial^\ell\omega^\alpha_\beta(\partial/\partial z_A)}{\partial z_{A_1}\cdots\partial z_{A_\ell}}(0)$,

$$\sum(\ell+1)\,\omega^\alpha_\beta(\partial/\partial z_A)[\ell]=\sum z_{A'}\,F(\nabla)^\alpha_\beta(\partial/\partial z_{A'},\partial/\partial z_A).$$

We have $\omega^\alpha_\beta(\partial/\partial z_A)[0]=0$, so that (2.2) can be ascertained by induction. Next, let us prove (2.4). Consider the matrices $\omega=(\omega^A_B)$, $T=(T(\nabla)^A_B)$, $\mathcal{C}=\omega+T$ and set $\mathcal{C}(\theta^\bullet)=\omega(\theta^\bullet)+T(\theta^\bullet):=-{}^t\mathcal{C}=-{}^t\omega-{}^tT$. Then, referring to Lemma 1.2, we have

$$\mathcal{L}_{\mathsf{R}}\theta^\bullet=-z_\bullet\cdot\mathcal{C}(\theta^\bullet)+dz_\bullet \tag{2.5}$$

and

$$\begin{aligned}\mathcal{L}_{\mathsf{R}}\mathcal{C}(\theta^\bullet)&=\mathsf{R}\vee\sum\frac{\partial\,\mathcal{C}(\theta^\bullet)(\partial/\partial z_B)}{\partial z_{B'}}\,dz_{B'}\wedge dz_B+d(T(\theta^\bullet)(\mathsf{R}))\\&=\sum z_{B'}\Big\{F(\nabla,\theta^\bullet)(\partial/\partial z_{B'},\partial/\partial z_B)+\frac{\partial\,T(\theta^\bullet)(\partial/\partial z_B)}{\partial z_{B'}}\Big\}\,dz_B+T(\theta^\bullet),\end{aligned}$$

which yields, by setting $r^2=|z|^2\,(:=|z_\bullet|^2=|z_0|^2+\sum|z_\alpha|^2+\sum|z_{\bar\alpha}|^2=\sum x_j^2=|x|^2)$,

$$\begin{aligned}&r\mathcal{L}_{\mathsf{R}}\left(r^{-1}\mathcal{L}_{\mathsf{R}}\theta^\bullet\right)=-z_\bullet\cdot\mathcal{L}_{\mathsf{R}}\mathcal{C}(\theta^\bullet)\\&\quad=-z_\bullet\cdot T(\theta^\bullet)-\sum z_{B'}\,z_\bullet\cdot\Big\{F(\nabla,\theta^\bullet)(\partial/\partial z_{B'},\partial/\partial z_B)+\frac{\partial\,T(\theta^\bullet)(\partial/\partial z_B)}{\partial z_{B'}}\Big\}\,dz_B.\end{aligned}\tag{2.6}$$

Further we have

$$\begin{aligned}&\mathcal{L}_{\mathsf{R}}\theta^\bullet=\mathcal{L}_{\mathsf{R}}((dz_\bullet)\cdot V^\bullet)=(dz_\bullet)\cdot(\mathsf{R}V^\bullet+V^\bullet)\,,\\&r\mathcal{L}_{\mathsf{R}}\left(r^{-1}\mathcal{L}_{\mathsf{R}}\theta^\bullet\right)=r\mathcal{L}_{\mathsf{R}}\left\{(dz_\bullet)\cdot r^{-1}\left(\mathsf{R}V^\bullet+V^\bullet\right)\right\}=(dz_\bullet)\cdot\left(\mathsf{R}^2V^\bullet+\mathsf{R}V^\bullet\right),\end{aligned}$$

which, together with (2.5) and (2.6), yield

$$\begin{aligned}&\mathsf{R}V^{BA}+V^{BA}=\sum z_{A'}\,\mathcal{C}_{AA'}(\partial/\partial z_B)+\delta_{BA},\\&\mathsf{R}^2V^{BA}+\mathsf{R}V^{BA}=\sum z_{A'}\,T^A_{A'}(\partial/\partial z_B)\\&\qquad\qquad+\sum z_{B'}z_{A'}\left\{F(\nabla)^A_{A'}(\partial/\partial z_{B'},\partial/\partial z_B)+\frac{\partial\,T^A_{A'}(\partial/\partial z_B)}{\partial z_{B'}}\right\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(\ell+1)\,V^{BA}[\ell]=\delta_{BA}+\sum z_{A'}\,\mathcal{C}_{AA'}(\partial/\partial z_B)[\ell-1],\\&(\ell^2+\ell)\,V^{BA}[\ell]=\sum z_{A'}\,T^A_{A'}(\partial/\partial z_B)[\ell-1]\\&\qquad\qquad+\sum z_{B'}z_{A'}\left\{F(\nabla)^A_{A'}(\partial/\partial z_{B'},\partial/\partial z_B)+\frac{\partial\,T^A_{A'}(\partial/\partial z_B)}{\partial z_{B'}}\right\}[\ell-2].\end{aligned}$$

By induction we obtain (2.4). ∎

Proposition 2.4 says that the coefficients of the Taylor expansions of $\omega^\alpha_\beta(\partial/\partial z_A)$, V^{BA} and V_{BA} at $z=0$ are expressed as polynomials made of

$$(2.7)\qquad \begin{aligned} \mathcal{R}_{A_1A_2A_3A_4A_5\cdots A_\ell} &= \frac{\partial^{\ell-4} g(F(\nabla)((\partial/\partial z_{A_3}, \partial/\partial z_{A_4})\partial/\partial z_{A_2}, \partial/\partial z_{A_1})}{\partial z_{A_5}\cdots \partial z_{A_\ell}}(P^0),\\ \mathcal{T}_{A_1A_2A_3A_4\cdots A_\ell} &= \frac{\partial^{\ell-3} g(T(\nabla)(\partial/\partial z_{A_2}, \partial/\partial z_{A_3}), \partial/\partial z_{A_1})}{\partial z_{A_4}\cdots \partial z_{A_\ell}}(P^0) \end{aligned}$$

and their expressions can be described explicitly up to an arbitrarily high order. For example, we have

Corollary 2.5 *We have*

$$\begin{aligned} \theta &= dz_0 + dz_\beta\cdot z_{\bar\beta}\frac{-i}{2} + dz_{\bar\beta}\cdot z_\beta\frac{i}{2} + O(|z|^2),\\ \theta^\alpha &= dz_\alpha + dz_0\cdot z_{\bar\gamma}\frac{-\mathcal{T}_{\bar\alpha 0\bar\gamma}}{2} + dz_{\bar\beta}\cdot\left\{z_0\frac{\mathcal{T}_{\bar\alpha 0\bar\beta}}{2} + z_{\bar\gamma}\frac{\mathcal{T}_{\bar\alpha\bar\gamma\bar\beta}}{2}\right\} + O(|z|^2),\\ \xi &= \partial/\partial z_0 + \partial/\partial z_\alpha\cdot z_{\bar\gamma}\frac{\mathcal{T}_{\bar\alpha 0\bar\gamma}}{2} + \partial/\partial z_{\bar\alpha}\cdot z_\gamma\frac{\mathcal{T}_{\alpha 0\gamma}}{2} + O(|z|^2),\\ \xi_\beta &= \partial/\partial z_\beta + \partial/\partial z_0\cdot z_{\bar\beta}\frac{i}{2} + \partial/\partial z_{\bar\alpha}\cdot\left\{z_0\frac{-\mathcal{T}_{\alpha 0\beta}}{2} + z_\gamma\frac{-\mathcal{T}_{\alpha\gamma\beta}}{2}\right\} + O(|z|^2). \end{aligned}$$

The corollary asserts that the structure of M near P^0 is roughly approximated by that of the Heisenberg group $H_n = \mathbb{R}\times\mathbb{C}^n$ near the origin. Let us adjust the notation to check it. H_n is the Lie group, whose element is denoted by $z=(z_0,z_1,\dots,z_n)=(z_0,z_\blacktriangle)$, with the group action $zz' = (z_0+z_0'+\mathrm{Im}\sum z_\alpha z'_{\bar\alpha}, z_\blacktriangle + z'_\blacktriangle)$, and has a contact 1-form and the Reeb vector field

$$(2.8)\qquad \theta_H = dz_0 + dz_\beta\cdot z_{\bar\beta}\frac{-i}{2} + dz_{\bar\beta}\cdot z_\beta\frac{i}{2},\qquad \xi^H = \partial/\partial z_0.$$

We set

$$(2.9)\qquad \xi^H_\beta = \partial/\partial z_\beta + \partial/\partial z_0\cdot z_{\bar\beta}\frac{i}{2},$$

which satisfies $\theta_H(\xi^H_\beta)=0$. These vector fields canonically provide an almost complex structure J^H. Note that the dual frame of $\xi^H_\bullet$ is $\theta^\bullet_H = (\theta_H, dz_\blacktriangle, dz_{\bar\blacktriangle})$. These equipments, together with the Riemannian metric g^H defined by $g^H(X,Y)=\theta_H(X)\theta_H(Y) + d\theta_H(X, J^HY)$, provide a contact Riemannian structure to H_n, which, compared with the results in Corollary 2.5, certainly approximates the structure of M near P^0.

The structure J^H is integrable, hence $\mathcal{Q}=0$, so that the hermitian Tanno connection ∇^H coincides with the Tanaka-Webster connection. Further we have $\omega^\alpha_\beta=0$, $\tau=0$ and the Kohn-Rossi Laplacian is simplified to

$$\mathbf{L} = -\sum \xi^H_\alpha\xi^H_{\bar\alpha} - \sqrt{-1}\,q\,\xi^H \quad (\text{acting on } \Omega^{p,q}H_n).$$

Note that the left action of z' on H_n preserves $\xi_\bullet^H$ and also the Laplacian $\mathbf{L}$. Now, on the typical strictly pseudoconvex CR manifold H_n, the problem (0.1) has the fundamental solution, which can be described explicitly as follows.

Lemma 2.6 (Stanton [17]) *We suppose $-n < a < n$ and consider the smooth function*

$$r_t^a(z) = \frac{1}{(2\pi t)^{n+1}} \int_{-\infty}^{\infty} ds \Big(\frac{s}{\sinh s}\Big)^n \exp\Big(- i2s\frac{z_0}{t} - \frac{|z_\blacktriangle|^2 s}{t\tanh s} - as\Big)$$

on $(0,\infty)\times H_n$ $(\ni (t,z))$. Then we have:

(1) *The smooth function (with parameter t)*

$$\Phi_t^a(s, z_\blacktriangle) = \frac{1}{(2\pi t)^{n+1}} \Big(\frac{s}{\sinh s}\Big)^n \exp\Big(- \frac{|z_\blacktriangle|^2 s}{t\tanh s} - as\Big)$$

is rapidly decreasing and $r_t^a(z)$ is its Fourier transform relative to the variable s, i.e., $r_t^a(z) = \mathcal{F}_{(s\,:\,2z_0/t)}(\Phi_t^a(s,z_\blacktriangle)) := \int_{-\infty}^{\infty} ds\, e^{-is\cdot(2z_0/t)}\,\Phi_t^a(s,z_\blacktriangle)$, which is also rapidly decreasing. (Note that we have $\Phi_1^{n-2q}(s,0) = \Phi^{n-2q}(s)$: see (0.4).)

(2) *Assume $0<q<n$, i.e., $-n<n-2q<n$. Then the initial value problem on H_n*

$$\Big(\frac{\partial}{\partial t} + \mathbf{L}\Big)\phi = 0, \quad \lim_{t\to 0}\phi(t) = \varphi \quad (\varphi \in \Omega_0^{p,q} H_n) \tag{2.10}$$

has the unique fundamental solution

$$r_H(t,z,z') = \sum \theta_H^{I\bar K}(z) \boxtimes \theta_H^{\bar I K}(z') \cdot r_t^{n-2q}(z'^{-1}z),$$

which we call the ***Heisenberg kernel****. As to the initial condition, we have $\lim_{t\to 0} \int \bar\varphi(z)\wedge\star r_H(t,z,z') = \bar\varphi(z')$ as well.*

Proof. We wish to check the initial condition in (2.10). We have

$$\begin{aligned}
\int r_H(t,z,z')\wedge\star_{g^H}\varphi(z') &= \sum \theta_H^{I\bar K}(z)\cdot \int dV_{g^H}(z')\, r_t^{n-2q}(z'^{-1}z)\varphi^{I\bar K}(z') \\
&= \sum \theta_H^{I\bar K}(z)\cdot \int dV_{g^H}(z')\, r_1^{n-2q}(z')\varphi^{I\bar K}(z(-\iota_t(z'))), \\
\int dV_{g^H}(z')\, r_1^{n-2q}(z') &= 1,
\end{aligned} \tag{2.11}$$

where we set $\varphi = \sum \theta^{I\bar K}\cdot\varphi^{I\bar K}$ and $\iota_t(z_0,z_\blacktriangle) = (tz_0, t^{1/2}z_\blacktriangle)$. The two coordinates $z'^{-1}z$, $z-z'$ (near z') are related by $(z'^{-1}z)_\bullet = E(z)(z-z')_\bullet = E(z')(z-z')_\bullet$,

$$E(z) := \begin{pmatrix} 1 & z_{\bar\blacktriangle}\frac{-i}{2} & z_\blacktriangle \frac{i}{2} \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}, \tag{2.12}$$

and we have

$$\varphi^{I\bar{K}}(z(-\iota_t(z'))) = \varphi^{I\bar{K}}(z) + t^{1/2}\int_0^1 d\sigma\, \frac{\partial \varphi^{I\bar{K}}(z(-\iota_t(z')))}{\partial t^{1/2}}\Big|_{t^{1/2}\Rightarrow \sigma t^{1/2}}$$
$$= \varphi^{I\bar{K}}(z) - t^{1/2}\int_0^1 d\sigma\, \frac{\partial \varphi^{I\bar{K}}}{\partial z_B}(z(-\iota_{\sigma^2 t}(z')))E(-z)_{BB'}|B'|_H(\sigma^2 t)^{(|B'|_H-1)/2} z'_{B'},$$

where $E(-z)_{BB'}$ denotes the (B, B')-entry of the matrix $E(-z)$. Since $z = \iota_{\sigma^2 t}(z') + E(\iota_{\sigma^2 t}(z'))(z(-\iota_{\sigma^2 t}(z')))$, for each $k \in \mathbb{N}$ and multi-index $\mathbb{B}$ we can find a polynomial $\mathcal{P}(\cdots)$ such that

$$(1+|z|)^k(\partial/\partial z)^{\mathbb{B}}\Big(\varphi^{I\bar{K}}(z(-\iota_t(z'))) - \varphi^{I\bar{K}}(z)\Big)$$
$$= t^{1/2}\sum\int_0^1 d\sigma\, \Big((\partial/\partial z)^{\mathbb{C}}\varphi^{I\bar{K}}\Big)(z(-\iota_{\sigma^2 t}(z')))\,\mathcal{P}(\sigma t^{1/2}, z', z(-\iota_{\sigma^2 t}(z'))).$$

Hence, $\int r_H(t, z, z') \wedge \star_{g^H}\varphi(z') - \varphi(z)$ is rapidly decreasing. Choosing a semi-norm $\mathrm{sn}(\varphi)$ (in general, $\mathrm{sn}(\varphi) := \sup_{z\in H_n, |\mathbb{B}|\le \ell} C\left|(1+|z|)^k(\partial/\partial z)^{\mathbb{B}}\varphi(z)\right|_{g^H}$ with some $k, \ell \in \mathbb{N}$ and a constant $C > 0$), we have

$$\left\|\int r_H(t, z, z') \wedge \star_{g^H}\varphi(z') - \varphi(z)\right\|_{L^2_{g^H}(z)} \le t^{1/2}\,\mathrm{sn}(\varphi), \tag{2.13}$$

where $\|\cdot\|_{L^2_{g^H}(z)}$ denotes the L^2-norm with respect to the metric g^H and the variable z. ∎

In fact, the structure of H_n approximates that of M ever closer as follows: Let us take the **∇-normal coordinate system with respect to the ∇-parallel frame $\xi_\bullet$**

$$\Theta : U \times U \to H_n, \quad \exp^{\nabla}(\xi_\bullet(z') \cdot \Theta_\bullet(z', z)) = z_\bullet, \tag{2.14}$$

where U is a small neighborhood of P^0. Following the argument by Greiner-Stein [12, Proposition 4.3] (also Stanton-Tartakoff [18, (6.4)]), we know that Corollary 2.5 yields

$$\Theta_0(z', z) = (z'^{-1}z)_0 + \sum_{k+\ell=3}^{k>0,\ell>0} O(|z'^{-1}z|^k \cdot |z'|^\ell),$$
$$\Theta_\alpha(z', z) = (z'^{-1}z)_\alpha + (z'^{-1}z)_0 \cdot z'_{\bar\gamma}\frac{-\mathcal{T}_{\bar\alpha 0\bar\gamma}}{2} + (z'^{-1}z)_{\bar\beta}\cdot\left\{z'_0\frac{\mathcal{T}_{\bar\alpha 0\bar\beta}}{2} + z'_{\bar\gamma}\frac{\mathcal{T}_{\bar\alpha\bar\gamma\bar\beta}}{2}\right\} \tag{2.15}$$
$$+ \sum_{k+\ell=3}^{k>0,\ \ell>0} O(|z'^{-1}z|^k \cdot |z'|^\ell).$$

We could provide more detailed descriptions obtained by using the results in [13]. Now, even if J is not integrable, the argument by Folland-Stein [9, Theorem 14.1] is still valid because of (1.8). Setting $|z|_H = \{z_0^2 + |z_\blacktriangle|^4 + |z_{\bar\blacktriangle}|^4\}^{1/4}$, hence we have

Lemma 2.7 (cf. Folland-Stein [9, Theorem 14.1], Stanton-Tartakoff [18, Theorem 1.2, Corollary 1.3]) *The system* Θ *is admissible. Namely, setting* $O(\Theta)^k_H = O(|\Theta(z',z)|^k_H)$*, we have*

$$\xi_B = \begin{cases} \dfrac{\partial}{\partial\Theta_0} + \sum O(\Theta)^1_H \dfrac{\partial}{\partial\Theta_A} \quad (B=0), \\ \dfrac{\partial}{\partial\Theta_\beta} + \dfrac{\partial}{\partial\Theta_0}\cdot\Theta_{\bar\beta}\dfrac{i}{2} + \displaystyle\sum_{A\neq 0} O(\Theta)^1_H \dfrac{\partial}{\partial\Theta_A} + O(\Theta)^2_H \dfrac{\partial}{\partial\Theta_0} \quad (B=\beta). \end{cases}$$

Further, we have

$$\begin{aligned} \square_H = \mathbf{L}^\Theta &+ \sum_{A\neq 0, B\neq 0} O(\Theta)^1_H \frac{\partial}{\partial\Theta_A}\frac{\partial}{\partial\Theta_B} + \sum_{B\neq 0} O(\Theta)^2_H \frac{\partial}{\partial\Theta_0}\frac{\partial}{\partial\Theta_B} \\ &+ O(\Theta)^1_H O(\Theta)^2_H \frac{\partial}{\partial\Theta_0}\frac{\partial}{\partial\Theta_0} + \sum_{B\neq 0} O(\Theta)^0_H \frac{\partial}{\partial\Theta_B} + O(\Theta)^1_H \frac{\partial}{\partial\Theta_0} + O(\Theta)^0_H, \end{aligned}$$

where $\mathbf{L}^\Theta$ *denotes* $\mathbf{L}$ *calculated in the coordinates* $\Theta = \Theta(z',z)$.

Thus the structure of H_n approximates that of M closely enough for constructing the heat kernel by iteration method starting from the Heisenberg kernel (refer to [18, §1]).

As stated, almost all the arguments in this paper are devoted to the investigation on the asymptotic coefficients. As a step toward it, we will construct in §3 a contact Riemannian manifold denoted by $H_n(P^0)$, which is H_n but with a neighborhood of the origin replaced by that of P^0 in M naturally, and in §4 we will show that the initial value problem on $H_n(P^0)$ has a unique heat kernel and its every differential at the origin can be expanded asymptotically. The argument in §4 will be more than enough for the proofs of the theorems of this section.

Proof of Theorems 2.1 and 2.3. It owes to Theorem 5.3 that the coefficient in (2.1) vanishes when m is odd. The argument in §4 (the proofs of Theorems 3.4 and 3.5) will be readily altered so as to fit for Theorems 2.1 and 2.3. Some comments are in order here. (1) As to the first approximation (4.1): We cover M by a finite number of small open sets U_j centered at P^j. Each U_j is equipped with unitary frames $\xi^j_\bullet$, $\theta^\bullet_j$ which are ∇-parallel along the ∇-geodesics from P^j and the ∇-normal coordinate system Θ_j with respect to $\xi^j_\bullet$. Let ϕ_j be nonnegative C^∞ functions such that $\{\phi_j^2\}$ is a partition of unity subordinate to the cover $\{U_j\}$. We utilize

$$r(t,P,P') = \sum_j \sum_{I,K} \theta_j^{I\bar K}(P) \boxtimes \theta_j^{\bar I K}(P')\cdot\phi_j(P)\, r_t^{n-2q}\big(\Theta_j(P',P)\big)\,\phi_j(P')$$

as a first approximation ([18, §2]). (2) As to Lemma 4.1: It holds with no change. (3) As to Proposition 4.2: We set $\left|(\partial/\partial t)^m \xi_{\mathbb{A},P}\xi_{\mathbb{A}',P'} q^k(t,P,P')\right|_g = \sum_{U_j\ni P, U_{j'}\ni P'} \left|(\partial/\partial t)^m \xi^j_{\mathbb{A},P}\xi^{j'}_{\mathbb{A}',P'} q^k(t,P,P')\right|_g$, etc., and change the function $\delta(z',z) = \left|w(z')^{-1}w(z)\right|_H$ into

$$\delta(P',P) = \begin{cases} \min\limits_{j\,:\,P,P'\in U_j} \big(|\Theta_j(P',P)|_H, |\Theta_j(P,P')|_H\big) & ((P',P)\in U), \\ 1 \quad (\text{otherwise}), \end{cases}$$

where U is a small neighborhood of the diagonal set of $M\times M$ ([18, Lemma 5.2]). (4) As to Lemmas 4.3 and 4.4: On M, a kernel $k(t,P,P')$ is defined to be of type ℓ if it is described as $\sum \theta_j^{I\bar K}(P)\boxtimes\theta_j^{\bar I' K'}(P')\cdot k_j^{(I\bar K)(I'\bar K')}(t,P,P')$ and each $k_j^{(I\bar K)(I'\bar K')}(t,P,P')$ is a finite sum of functions $\mathcal{K}^b_{U_j}(t,P,P') = t^{-n-2+b/2}\rho_{U_j}(P',P)\,\mathcal{K}(\iota_{1/t}\Theta_j(P',P))$ $(b\ge\ell)$ (see (4.13)), whose supports are contained in $U_j\times U_j$ ([18, Definition 3.1]). The lemmas hold under the definition. (5) As to Proof of Theorem 3.4: There will be no need to set up such a paragraph because, on M, the convergences at (4.12) yield readily their convergences in the L^2-norm. ∎

3 Warped Heisenberg group $H_n(P^0)$

Let us construct carefully the Heisenberg group $H_n(P^0)$ which is warped near the origin. (Refer to the comment preceding the proof of Theorems 2.1 and 2.3.) We denote the standard Heisenberg group by $H_n = (H_n, w)$, whose standard structure is expressed as $(\tilde\theta_H, \tilde\xi^H, g^H, J^H, \nabla^H, \tilde{\mathbf{L}})$. (The symbol of variable was changed and $\tilde\theta_H$, etc., denote θ_H, etc., in the variable w: see (2.8) and (2.9).) Also we express the structure of M as $(\theta_M, \xi^M, g^M, J^M, \nabla^M, \Box_M)$. Now, we will identify a neighborhood $U^0 = \{P\in M \mid |\Theta(P^0,P)| < r_0'\}$ of P^0 with a neighborhood $U = \{w\in H_n \mid |w| < r_0'\}$ of $w=0$ by the diffeomorphism

$$\Theta_{P^0} : U^0 \to U, \quad P\mapsto w(P) = \Theta(P^0,P)\,(=z(P)).$$

Further let us fix a smooth function $\rho(s)$ on $[0,\infty)$ which satisfies $\rho(s)=1$ $(s\le 1/2)$, $\rho(s)=0$ $(s\ge 2/3)$, $0\le\rho(s)\le 1$, and, for every $r\in(0,r_0']$, set $\rho_0(w)=\rho(|w|/r)$, which is a smooth function on H_n. Then there exists a number $r_0\in(0,r_0']$ such that, for any $r\in(0,r_0]$, the 1-form on (H_n,w)

$$\theta = \rho_0\,\theta_M + (1-\rho_0)\,\tilde\theta_H$$

is nondegenerate. In fact, Corollary 2.5 yields

$$\theta \wedge (d\theta)^n = \{1 + \mathbb{O}(r)\}\, \tilde{\theta}_H \wedge (d\tilde{\theta}_H)^n,$$

which means: The remainder term $\mathbb{O}(r)$, which is determined for each $r \in (0, r_0']$, has support contained in $\{w \in H_n \mid |w| \le r\}$ ($r \in (0, r_0']$), and, for every multi-index $\mathbb{A}$ there exists a constant $C_{\mathbb{A}} > 0$ such that $\left|(\partial/\partial w)^{\mathbb{A}}\mathbb{O}(r)\right| \le C_{\mathbb{A}} r^{1-|\mathbb{A}|}$ ($w \in H_n$, $r \in (0, r_0']$). (Hence, the term $\mathbb{O}(r)$, regarded as a function of $(r, w) \in (0, r_0'] \times H_n$, can be extended continuously up to $r = 0$ by claiming $\mathbb{O}(r) = 0$ at $r = 0$.) By choosing $r_0 > 0$ so small that $C_{\emptyset} r_0 < 1$, θ is certainly nondegenerate when $0 < r \le r_0$. In general, we will use the symbol (or a function) $\mathbb{O}(r^k)$, which means that it satisfies $\left|(\partial/\partial w)^{\mathbb{A}}\mathbb{O}(r^k)\right| \le C_{\mathbb{A}} r^{k-|\mathbb{A}|}$ with the other parts unchanged. Note that $\rho_0 = \mathbb{O}(r^0) = \mathbb{O}(1)$ and, in general, $\partial\mathbb{O}(r^k)/\partial w_A = \mathbb{O}(r^{k-1})$ according to the notation.

We obtain thus a contact manifold (H_n, w, θ). Let us set $\theta^{\bullet}_{MH} = \rho_0\, \theta^{\bullet}_M + (1 - \rho_0)\, \tilde{\theta}^{\bullet}_H$ ($\theta^0_{MH} = \theta$) and denote its dual frame by $\xi^{MH}_{\bullet} = (\xi^{MH}_0, \xi^{MH}_{\triangle}) = (\xi^{MH}_0, \xi^{MH}_{\blacktriangle}, \xi^{MH}_{\bar{\blacktriangle}})$. Then $\xi^{MH}_{\triangle}$ is a frame of $HH_n \otimes \mathbb{C}$ and the Reeb vector field can be described as

$$\xi = \xi^{MH}_0 + \xi^{MH}_{\triangle} \cdot C_{\triangle} := \xi^{MH}_0 + \xi^{MH}_{\triangle} \cdot \Big(- d\theta(\xi^{MH}_{\triangle}, \xi^{MH}_{\triangle})\Big)^{-1} d\theta(\xi^{MH}_{\triangle}, \xi^{MH}_0),$$

where $C_{\triangle}$ is a column vector. Next, let us search for an associated pair (g, J) of a metric and an almost complex structure (refer to Blair [5, Theorem 4.4]). The frames

$$\hat{\theta}^{\bullet} = (\hat{\theta}^0, \hat{\theta}^{\triangle}) := (\theta,\ \theta^{\triangle}_{MH} - {}^t C_{\triangle}\theta^0_{MH}), \qquad \hat{\xi}_{\bullet} = (\hat{\xi}_0, \hat{\xi}_{\triangle}) := (\xi, \xi^{MH}_{\triangle})$$

are dual to each other. The metric $\hat{g} = \theta \otimes \theta + 2\sum \hat{\theta}^{\alpha}\, \hat{\theta}^{\bar{\alpha}}$ satisfies $\hat{g}(X, Y) = \theta(X)\theta(Y) + \hat{g}(-X + \theta(X)\xi, -Y + \theta(Y)\xi)$, hence, $\hat{g}(\xi, Y) = \theta(Y)$. We polarize the form $-d\theta$ with respect to $\hat{g}$-orthonormal frame $\hat{e}_{\triangle} = (e^{MH}_{\triangle}, e^{MH}_{n+\triangle})$ induced from the $\hat{g}$-unitary frame $\hat{\xi}_{\triangle}$. Namely, let us set $\mathcal{A} = -d\theta(\hat{e}_{\triangle}, \hat{e}_{\triangle}) = \theta([\hat{e}_{\triangle}, \hat{e}_{\triangle}])$, which is a $2n \times 2n$ nonsingular symmetric matrix, and decompose it into $\mathcal{A} = FG$, where F is an orthogonal matrix and G is a positive definite symmetric matrix. Then the pair (g, J) is defined as

$$\begin{aligned} g &:= \theta \otimes \theta + \sum G_{ij} \cdot \hat{e}^*_i \otimes \hat{e}^*_j = g^H + \mathbb{O}(r^2)\, \tilde{\theta}^0_H \otimes \tilde{\theta}^0_H + \sum_{(A,B) \ne (0,0)} \mathbb{O}(r)\, \tilde{\theta}^A_H \otimes \tilde{\theta}^B_H, \\ J\hat{e}_{\triangle} &:= \hat{e}_{\triangle} \cdot F, \end{aligned} \tag{3.1}$$

which coincides with (g^M, J^M) near the origin and with (g^H, J^H) apart from U. We obtain thus a contact Riemannian manifold $H_n(P^0) = (H_n, w, \theta, \xi, g, J)$. Referring to (1.3), (1.1) and (3.1), we know, by straightforward computation, that the hermitian

Tanno connection ∇, which also coincides with ∇^M near the origin and with ∇^H apart from U, provides

$$\begin{aligned}
\nabla_{\partial/\partial w_0}\partial/\partial w_A &= \partial/\partial w_0 \cdot \mathbb{O}(r) + \partial/\partial w_b \cdot \mathbb{O}(1), \\
\nabla_{\partial/\partial w_\gamma}\partial/\partial w_0 &= \partial/\partial w_0 \cdot \mathbb{O}(r) + \partial/\partial w_b \cdot \mathbb{O}(1), \\
\nabla_{\partial/\partial w_\gamma}\partial/\partial w_\alpha &= \partial/\partial w_B \cdot \mathbb{O}(1), \\
\nabla_{\partial/\partial w_\gamma}\partial/\partial w_{\bar\alpha} &= \partial/\partial w_0 \cdot \delta_{\alpha\gamma}\frac{i}{2} + \partial/\partial w_B \cdot \mathbb{O}(1).
\end{aligned} \tag{3.2}$$

Note that we have

$$\begin{aligned}
&\nabla^H_{\partial/\partial w_0}\partial/\partial w_A = \nabla^H_{\partial/\partial w_\gamma}\partial/\partial w_0 = \nabla^H_{\partial/\partial w_\gamma}\partial/\partial w_\alpha = 0, \\
&\nabla^H_{\partial/\partial w_\gamma}\partial/\partial w_{\bar\alpha} = \partial/\partial w_0 \cdot \delta_{\alpha\gamma}\frac{i}{2}.
\end{aligned} \tag{3.3}$$

Next, let us investigate the ∇-geodesics from the origin. Namely, we want to consider the curve $c(s) = {}^t(c_0(s), c_\blacktriangle(s)) = c(s,z;r)$ or $c_\bullet(s) = {}^t(c_0(s), c_\blacktriangle(s), c_{\bar\blacktriangle}(s))$ $(c_{\bar\blacktriangle}(s) = \overline{c_\blacktriangle(s)})$ satisfying

$$\begin{gathered}
\frac{d^2 c_A(s)}{ds^2} + dw_A(\nabla_{\partial/\partial w_C}\partial/\partial w_B)(c_\bullet(s))\,\frac{dc_B(s)}{ds}\frac{dc_C(s)}{ds} = 0, \\
c_\bullet(0) = 0, \quad \dot c_\bullet(0) := \frac{dc_\bullet}{ds}(0) = z_\bullet.
\end{gathered}$$

(Recall that we have set $|z|^2 := |z_\bullet|^2 = |z_0|^2 + |z_\blacktriangle|^2 + |z_{\bar\blacktriangle}|^2$.) It follows from (3.3) that, near $s = s_0$ with $|c(s_0,z;r)| \geq r$, we have

$$c(s,z;r) = c(s_0,z;r) + \dot c(s_0,z;r)(s - s_0). \tag{3.4}$$

We are, hence, mainly interested in its behavior when $|c(s,z;r)| \leq r$. We have $c(s,z;r) = sz$ $(|sz| \leq r/2)$, which gently curls off the line after that.

Lemma 3.1 *For a small $r_0 > 0$, there exists a constant $C > 0$ such that*

$$|c(s,z;r) - sz| \leq Cs^2|z|^2, \quad |\dot c(s,z;r) - z| \leq Cs|z|^2, \tag{3.5}$$

$$|(\partial/\partial z_D)(c(s,z;r) - sz)| \leq Cs^2|z|, \quad |(\partial/\partial z_D)(\dot c(s,z;r) - z)| \leq Cs|z| \tag{3.6}$$

when $0 < r \leq r_0$ and $|sz| \leq r_0$.

Proof. Let us set

$$\begin{aligned}
\Gamma(s,z) &= \sum \Gamma_{BC}(c(s))\,\dot c_B(s)\dot c_C(s) \\
&= \sum dw_\bullet((\nabla^H - \nabla)_{\partial/\partial w_C}\partial/\partial w_B)(c(s))\,\dot c_B(s)\dot c_C(s).
\end{aligned}$$

Then, obviously we have $\Gamma_{BC}(w) = \mathbb{O}(1)$ (by (3.2) and (3.3)), and

$$\dot{c}(s) - z = \int_0^s ds\, \ddot{c}(s) = \int_0^s ds\, \Gamma(s,z), \quad c(s) - sz = \int_0^s ds\, (\dot{c}(s) - z).$$

On the other hand, we have $|\dot{c}(s)| \le C|z|$ $(0 < r \le r_0)$. Indeed, we have $g(\dot{c}(s), \dot{c}(s)) = g(\dot{c}(0), \dot{c}(0)) = |z|^2$ and (by (3.2)) there exists $C > 0$ such that $|\dot{c}(s)|^2 \le Cg(\dot{c}(s), \dot{c}(s))$ $(0 < r \le r_0)$. Thus we obtain the inequalities (3.5), which hold actually without the assumption $|sz| \le r_0$. Next, let us show (3.6). We take $r_0 > 0$ sufficiently small and assume $0 < r \le r_0$, $|sz| \le r_0$, so that (by (3.5)) we have $C > 0$ satisfying

$$C^{-1}|sz| \le |c(s)| \le C|sz|, \quad C^{-1}|z| \le |\dot{c}(s)| \le C|z|. \tag{3.7}$$

Considering the equality

$$\begin{aligned}\frac{\partial \Gamma(s,z)}{\partial z_D} &= \Gamma_{BC}(c(s))\Big\{\frac{\partial(\dot{c}(s)-z)_B}{\partial z_D}\dot{c}_C(s) + \dot{c}_B(s)\frac{\partial(\dot{c}(s)-z)_C}{\partial z_D}\Big\} \\ &+ \frac{\partial(c(s)-sz)_E}{\partial z_D}\frac{\partial \Gamma_{BC}}{\partial w_E}(c(s))\,\dot{c}_B(s)\dot{c}_C(s) \\ &+ s\frac{\partial \Gamma_{BC}}{\partial w_D}(c(s))\,\dot{c}_B(s)\dot{c}_C(s) + \Gamma_{DC}(c(s))\dot{c}_C(s) + \Gamma_{BD}(c(s))\dot{c}_B(s)\end{aligned}$$

and the obvious estimate $\frac{\partial \Gamma_{BC}}{\partial w_E}(w) = \mathbb{O}(r^{-1})$, we know that there is $C_1 > 0$ such that

$$\begin{aligned}\left|\frac{\partial(\dot{c}(s)-z)}{\partial z_D}\right| &\le \int_0^s ds \left|\frac{\partial \Gamma(s,z)}{\partial z_D}\right| \\ &\le \frac{1}{2}\max_{0\le t\le s}\left|\frac{\partial(\dot{c}(t)-z)}{\partial z_D}\right| + C_1|z|\max_{0\le t\le s}\left|\frac{\partial(c(t)-tz)}{\partial z_D}\right| + C_1|sz|,\end{aligned}$$

which yields

$$\begin{aligned}\frac{1}{2}\left|\frac{\partial(\dot{c}(s)-z)}{\partial z_D}\right| &\le \max_{0\le t\le s}\left|\frac{\partial(\dot{c}(t)-z)}{\partial z_D}\right| - \frac{1}{2}\max_{0\le t\le s}\left|\frac{\partial(\dot{c}(t)-z)}{\partial z_D}\right| \\ &\le \max_{0\le t\le s}\left(\left|\frac{\partial(\dot{c}(t)-z)}{\partial z_D}\right| - \frac{1}{2}\max_{0\le t'\le t}\left|\frac{\partial(\dot{c}(t')-z)}{\partial z_D}\right|\right) \\ &\le C_1|z|\max_{0\le t\le s}\left|\frac{\partial(c(t)-tz)}{\partial z_D}\right| + C_1|sz|, \\ \left|\frac{\partial(c(s)-sz)}{\partial z_D}\right| &\le 2C_1\int_0^s ds\Big\{|z|\max_{0\le t\le s}\left|\frac{\partial(c(t)-tz)}{\partial z_D}\right| + |sz|\Big\} \\ &\le 2C_1|sz|\max_{0\le t\le s}\left|\frac{\partial(c(t)-tz)}{\partial z_D}\right| + C_1 s^2|z|.\end{aligned}$$

If we take the $r_0 > 0$ furthermore so small that $2C_1 r_0 < 1/2$, then the above estimates imply

$$\begin{aligned}&\frac{1}{2}\left|\frac{\partial(c(s)-sz)}{\partial z_D}\right| \le \max_{0\le t\le s}\left(\left|\frac{\partial(c(t)-tz)}{\partial z_D}\right| - \frac{1}{2}\max_{0\le t'\le t}\left|\frac{\partial(c(t')-t'z)}{\partial z_D}\right|\right) \le C_1 s^2|z|, \\ &\frac{1}{2}\left|\frac{\partial(\dot{c}(s)-z)}{\partial z_D}\right| \le C_1^2|sz|^2 + C_1|sz|.\end{aligned}$$

Thus we obtain (3.6). ∎

Proposition 3.2 *Suppose $r_0 > 0$ is sufficiently small and $0 < r \leq r_0$. Then the ∇-geodesics from the origin do not intersect with each other (except at the origin) and the ∇-normal coordinates centered at 0 can be taken globally,*

$$z : H_n(P^0) \cong H_n,\ w \mapsto z(w), \quad \exp^{\nabla}((\partial/\partial w_\bullet)_0 \cdot z_\bullet(w)) = w_\bullet. \tag{3.8}$$

Further, letting $z \mapsto w(z)$ be its inverse map, we have a constant $C > 0$ such that

$$|w(z) - z| \leq C|z| \min(|z|, r), \quad |\partial w_A(z)/\partial z_B - \delta_{AB}| \leq C \min(|z|, r), \tag{3.9}$$

$$|z(w) - w| \leq C|w| \min(|w|, r), \quad |\partial z_B(w)/\partial w_A - \delta_{BA}| \leq C \min(|w|, r), \tag{3.10}$$

$$C^{-1} \min(|z|, r) \leq \min(|w|, r) \leq C \min(|z|, r). \tag{3.11}$$

Remark In fact, we may take $r_0 > 0$ so small that, if $|w'| \leq r_0$, then the ∇-geodesics from w' do not intersect with each other and the ∇-normal coordinates centered at w' can be taken globally. We have similar inequalities as well.

Proof. We set $w(z) = c(1, z) = c(1, z; r)$. Lemma 3.1 implies

$$|w(z) - z| \leq C|z|^2, \quad |\partial w_A(z)/\partial z_B - \delta_{AB}| \leq C|z| \quad (0 < r \leq r_0,\ |z| \leq r_0). \tag{3.12}$$

Hence, via the inverse function theorem, for a small $r_0 > 0$, the map $z \mapsto w(z)$ provides an into diffeomorphism $w : \{z \in H_n \mid |z| \leq r_0\} \to H_n(P^0)$ parameterized smoothly by $r \in (0, r_0]$, whose image is a closed neighborhood of the origin, and (by (3.5), (3.7)), for each r, $|w(z)|$ increases as so does $|z|\,(\leq r_0)$. Let us take $r_1 \in (0, r_0]$ so small that $|w(z)| \geq r$ if $0 < r \leq r_1$ and $|z| = r_0$ (see (3.7)). Now, we assume $0 < r \leq r_1$ and want to study the behavior of the ray from the point $w(z) = c(1, z)$ with $|z| = r_0$ in the direction $\dot{c}(1, z)$, that is, the ∇-geodesic $w(sz) = c(s, z)$ $(s \geq 1)$ (see (3.4)). Let z^1, $z^2 \in \{z \in H_n \mid |z| = r_0\}$ be perpendicular to each other with respect to the standard metric $\langle\ ,\ \rangle$. We put $z(\theta) = z^1 \cos\theta + z^2 \sin\theta$ and project the point $c(1, z(\theta))$ onto the plane spanned by those two points and the origin. We want to show that the argument $\vartheta(\theta)$ of the image $c(1, z(\theta), z^1, z^2) = \sum \langle c(1, z(\theta)), \bar{z}^j/r_0 \rangle\, z^j/r_0$ from $z(0) = z^1$ increases as so does θ. It will be enough to check it near $\theta = 0$. Since Lemma 3.1 implies

$$\langle c(1, z(\theta)), \bar{z}^j/r_0 \rangle = \begin{cases} r_0(\cos\theta + O(r_0)) & (j = 1), \\ r_0(\sin\theta + O(r_0)) & (j = 2), \end{cases}$$

$$\frac{\partial}{\partial\theta} \langle c(1, z(\theta)), \bar{z}^j/r_0 \rangle = \begin{cases} r_0(-\sin\theta + O(r_0)) & (j = 1), \\ r_0(\cos\theta + O(r_0)) & (j = 2), \end{cases}$$

we have $\frac{\partial}{\partial\theta}\tan\vartheta(\theta) = \frac{1+O(r_0)}{(\cos\theta+O(r_0))^2}$, that is, if $r_0 > 0$ is sufficiently small then $\vartheta(\theta)$ certainly increases (near $\theta = 0$). Similarly we know that the argument of $\dot{c}(1, z(\theta), z^1, z^2) = \sum\langle\dot{c}(1, z(\theta)), \bar{z}^j/r_0\rangle\, z^j/r_0$ from $z(0) = z^1$ increases as so does θ. Thus, the rays from the points $c(1, z(\theta), z^1, z^2)$ in the directions $\dot{c}(1, z(\theta), z^1, z^2)$ do not intersect and so do not the ∇-geodesics $w(sz(\theta)) = c(s, z(\theta))$ $(s \geq 1)$. Consequently, if $r_0 > 0$ is sufficiently small and $0 < r \leq r_1$, then the ∇-geodesics from 0 do not intersect and we obtain the global ∇-normal coordinates (3.8). Let us show the remaining inequalities. We assume $0 < r \leq r_1$. If $|z| \leq r' := (r_0/r_1)r$ $(\leq r_0)$, then the inequality (3.9) follows from (3.12). If $|z| \geq r'$, hence, $|c(1, z)| \geq r$, then it follows from (3.4). With the use of the inverse function theorem, (3.10) will follow readily. As to (3.11): If $|z| \leq r_0$, then (3.7) yields $C^{-1}|w| \leq |z| \leq C|w|$, which implies (3.11). If $|z| \geq r_0$, then $|c(s)|$ increases. Hence we have $|w| \geq r$ and $\min(|z|, r) = r = \min(|w|, r)$. ∎

We will fix such a small number $r \in (0, r_0]$ and assume that the warped Heisenberg group $H_n(P^0)$ (associated with r) is equipped with the global ∇-normal coordinates z centered at the origin. Note that its Kohn-Rossi Laplacian $\Box_{H(P^0)}$ coincides with $\Box_M$ near the origin and with $\tilde{\mathbf{L}}$ apart enough from the origin. We will denote by $\xi_\bullet$, $\theta^\bullet$ the global ∇-parallel unitary frames and, further, we regard Lemma 1.2, Proposition 2.4, etc., as the assertions on $H_n(P^0)$. Added to the global (2.3), we set

$$\xi_{\bullet,z} = \tilde{\xi}^H_{\bullet,z} \cdot \mathcal{V}_\bullet(z), \quad \theta^\bullet_z = \tilde{\theta}^\bullet_{H,z} \cdot \mathcal{V}^\bullet(z), \quad \text{hence } \mathcal{V}_\bullet = {}^t(\mathcal{V}^\bullet)^{-1}.$$

($\tilde{\xi}^H_{\bullet,z}$ denotes $\tilde{\xi}^H_{\bullet,w(z)}$ calculated in the coordinates z.) Notice that we have

$$\mathcal{V}_\bullet(z) = E(w(z))\left(\frac{\partial w_\bullet}{\partial z_\bullet}(z)\right) V_\bullet(z),$$

where $E(w)$ is the matrix given at (2.12).

Lemma 3.3 *Suppose $|w(z)| \geq r$. Then $\mathcal{V}_\bullet(z)$ is unitary and $\overline{\mathcal{V}_{AB}} = \mathcal{V}_{\bar{A}\bar{B}}$. Further $\mathcal{V}_{00}$ is identically equal to 1 and $\mathcal{V}_{AB}$ $((A, B) \neq (0, 0))$ vanishes unless $A, B \in \{1, \ldots, n\}$ nor $A, B \in \{\bar{1}, \ldots, \bar{n}\}$. For each multi-index $\mathbb{B}$, there exists a constant $C_\mathbb{B} > 0$ such that*

$$\left|(\partial/\partial z)^\mathbb{B}\mathcal{V}_\bullet(z)\right| \leq C_\mathbb{B}\,|z|^{-|\mathbb{B}|}, \quad \left|(\partial/\partial z)^\mathbb{B} V_\bullet(z)\right| \leq C_\mathbb{B}\,|z|^{1-|\mathbb{B}|},$$
$$\left|(\partial/\partial z)^\mathbb{B}\det V_\bullet(z)\right| \leq C_\mathbb{B}\,|z|^{-|\mathbb{B}|}, \quad \left|(\partial/\partial z)^\mathbb{B}\omega^\alpha_\beta(\partial/\partial z_A)\right| \leq C_\mathbb{B}\,|z|^{-1-|\mathbb{B}|}$$

and so does also for $\mathcal{V}^\bullet(z)$, $V^\bullet(z)$, $\det V^\bullet(z)$.

Proof. The first half follows from $\xi_{0,z} = \tilde{\xi}^H_{0,z}$ and $g = g^H$ when $|w(z)| \geq r$. As to the second half: The point $I^H(z)$ on the ray sz $(0 \leq s < \infty)$ which satisfies $|I^H(z)|_H = 1$

is uniquely described as $I^H(z) = z/a(z) := z/\left(\left(z_0^2 + \sqrt{z_0^4 + 4|z_\blacktriangle|^4 + 4|z_{\bar\blacktriangle}|^4}\right)/2\right)^{1/2}$ and (by (3.4)) we have

$$w(z) = w(I^H(z)) + \sum \frac{\partial w}{\partial z_A}(I^H(z))\left(z_A - I^H(z)_A\right) \quad (|w(z)| \geq r).$$

Since $\left|(\partial/\partial z)^{\mathbb{B}} I^H(z)\right| \leq C_{\mathbb{B}}\,|z|^{-|\mathbb{B}|}$, hence we have

$$\left|(\partial/\partial z)^{\mathbb{B}} w(z)\right| \leq C_{\mathbb{B}}\,|z|^{1-|\mathbb{B}|} \quad (|w(z)| \geq r). \tag{3.13}$$

Further, since the two vector fields $\tilde\xi^H_\bullet|_{w(tz)} \cdot \mathcal{V}_\bullet(I^H(z))$, $\xi_{\bullet,tz} = \tilde\xi^H_\bullet|_{w(tz)} \cdot \mathcal{V}_\bullet(tz)$ along the ∇-geodesic $w(tz)$ $(a(z)^{-1} \leq t \leq 1)$ are ∇-parallel and coincide at $t = a(z)^{-1}$, these are the same. In particular, we have $\mathcal{V}_\bullet(z) = \mathcal{V}_\bullet(I^H(z))$. We obtain hence the inequalities. Remind that we have $\omega^\alpha_\beta(\partial/\partial z_A) = \sum \mathcal{V}^{C\alpha}\frac{\partial \mathcal{V}_{C\beta}}{\partial z_A}$ when $|w(z)| \geq r$. ∎

Let us introduce here some kinds of normal coordinate systems. On M the ∇^M-normal coordinate system Θ^M has been defined (see (2.14)). On the standard Heisenberg group $H_n = (H_n, w)$ we have the standard one $\Theta^{\tilde{\mathbf{L}}} : H_n \times H_n \to H_n$, $\exp^{\nabla^H}(\tilde\xi^H_{\bullet,w'} \cdot \Theta^{\tilde{\mathbf{L}}}_\bullet(w', w)) = w$, i.e., $\Theta^{\tilde{\mathbf{L}}}(w', w) = w'^{-1}w$, which induces two kinds of normal coordinate systems $\Theta^{\mathbf{L}}, \Theta^H : H_n(P^0) \times H_n(P^0) \to (H_n, z)$,

$$\begin{aligned}
&\exp^{\nabla^H}(\tilde\xi^H_{\bullet,z'} \cdot \Theta^{\mathbf{L}}_\bullet(z', z)) = z, \text{ i.e., } \Theta^{\mathbf{L}}(z', z) = \Theta^{\tilde{\mathbf{L}}}(w(z'), w(z)) = w(z')^{-1}w(z),\\
&\exp^{\nabla^H}(\xi_{\bullet,z'} \cdot \Theta^H_\bullet(z', z)) = z, \text{ i.e., } \Theta^H(z', z) = \mathcal{V}_\bullet(z')^{-1}\Theta^{\mathbf{L}}_\bullet(z, z').
\end{aligned}$$

It follows from Lemma 3.3 that we have

$$\begin{aligned}
&\Theta^H_0(z', z) = \Theta^{\mathbf{L}}_0(z, z'), \quad \left|\Theta^H_\blacktriangle(z', z)\right| = \left|\Theta^{\mathbf{L}}_\blacktriangle(z, z')\right|,\\
&r_t^{n-2q}(\Theta^H(z', z)) = r_t^{n-2q}(\Theta^{\mathbf{L}}(z', z)).
\end{aligned}$$

The ∇-normal coordinate system with respect to the ∇-parallel frame $\xi_\bullet$ defined on a neighborhood V of the diagonal set in $H_n(P^0) \times H_n(P^0)$

$$\Theta^\nabla : V \to H_n, \quad \exp^\nabla(\xi_\bullet(z') \cdot \Theta^\nabla_\bullet(z', z)) = z_\bullet$$

bears the relation

$$\Theta^\nabla(z', z) = \begin{cases} \Theta^M(z', z) & \text{on } \{(z', z) \in V \mid |w(z')| \leq r/3\},\\ \Theta^H(z', z) & \text{on } \{(z', z) \in V \mid |w(z')| \geq r\}. \end{cases} \tag{3.14}$$

In the next section, we will ascertain the following two theorems.

Theorem 3.4 (cf. Theorem 2.1) *By assuming $\varphi \in \Omega_0^{p,q} H_n(P^0)$, the initial value problem* (0.1) *on $H_n(P^0)$ has a unique heat kernel $e^{-t\Box_{H(P^0)}}(z,z')$. As to the initial condition, added to $\lim_{t\to 0}\int e^{-t\Box_{H(P^0)}}(z,z')\wedge\star\varphi(z') = \varphi(z)$, we have $\lim_{t\to 0}\int \bar\varphi(z)\wedge \star e^{-t\Box_{H(P^0)}}(z,z') = \bar\varphi(z')$.*

Remark Since $H_n(P^0)$ is not compact, it will be necessary to add even more condition: The form $\phi(t,z) := \int e^{-t\Box_{H(P^0)}}(z,z')\wedge\star\varphi(z')$ belongs to the domain of $\Box_{H(P^0)}$, the integral $\Phi(t) := \int dV_g(z)\,|\phi(t,z)|_g^2$ is differentiable and the equality $(\partial/\partial t)\Phi(t) = \int dV_g(z)(\partial/\partial t)\,|\phi(t,z)|_g^2$ holds.

Theorem 3.5 (cf. Theorem 2.3) *There is the asymptotic expansion*

$$
\begin{aligned}
(3.15)\qquad & (\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\left(e^{-t\Box_{H(P^0)}}\right)^{(I\bar K)(I'\bar K')}(0,0)\\
&\sim \sum_{m\geq -(|\mathbb{A}|_H+|\mathbb{A}'|_H)} t^{-(n+1)+m/2}\,a_{m/2}^{(I\bar K)(I'\bar K')}(P^0:\mathbb{A},\mathbb{A}')
\end{aligned}
$$

when $t\to 0$.

4 Construction of the heat kernel on $H_n(P^0)$ and the proofs of Theorems 3.4 and 3.5

On the basis of the work by Stanton-Tartakoff [18], we will construct the heat kernel on $H_n(P^0)$ to prove Theorems 3.4 and 3.5. Rather exhaustive calculation, some results of which are applied also to the proof of Theorem 5.3, will be required.

Let us set

$$
\begin{aligned}
r_M(t,z,z') &= \sum \theta^{I\bar K}(z)\boxtimes\theta^{\bar I K}(z')\cdot r_t^{n-2q}(\Theta^M(z',z))\\
&\qquad\qquad (\text{on } H_n(P^0)\times\{z'\in U^0 \mid |w(z')|<r_0\}),\\
r_{\mathbf{L}}(t,z,z') &= r_H(t,w(z),w(z'))\\
&= \sum \theta^{I\bar K}(z)\boxtimes\theta^{\bar I' K'}(z')\cdot\ \det(\mathcal{V}_{IJ}\mathcal{V}_{\bar K\bar L})(z)\det(\mathcal{V}_{\bar I'\bar J}\mathcal{V}_{K'L})(z')\,r_t^{n-2q}(\Theta^{\mathbf{L}}(z',z))\\
&\qquad\qquad (\text{on } H_n(P^0)\times H_n(P^0)),
\end{aligned}
$$

where $\Theta^M(z',z)$ denotes (not the original $\Theta^M(z',z)$ but) the system $\Theta^\nabla(z',z)$ restricted there. Notice that Θ^∇, which (by (3.14)) really coincides with Θ^M sufficiently near the origin, is certainly well-defined on the region because of Remark on Proposition 3.2. Let $\tilde\rho_M(w)$, $\tilde\rho_{\mathbf{L}}(w)$ be nonnegative C^∞ functions such that $\{\tilde\rho_M^2(w),\tilde\rho_{\mathbf{L}}^2(w)\}$ is a partition of unity subordinate to $\big\{\{w \in H_n(P^0) \mid |w| < 2r\}, \{w \in H_n(P^0) \mid |w| > r\}\big\}$ $(0 < 2r < r_0)$. Setting $\rho_M(z) = \tilde{\rho}_M(w(z))$, etc., as a first approximation of heat kernel we wish to choose

$$r(t, z, z') = \rho_M(z)\rho_M(z')\, r_M(t, z, z') + \rho_{\mathbf{L}}(z)\rho_{\mathbf{L}}(z')\, r_{\mathbf{L}}(t, z, z'). \tag{4.1}$$

Lemma 4.1 (cf. [18, Proposition 2.1(ii)]) *For any $\varphi \in \Omega_0^{p,q} H_n(P^0)$, we have*

$$\lim_{t\to 0} \int r(t, z, z') \wedge \star\varphi(z') = \varphi(z), \quad \lim_{t\to 0} \int \bar{\varphi}(z) \wedge \star r(t, z, z') = \bar{\varphi}(z')$$

in the $|\cdot|_g$-norm and in the L^i_g-norm $(i = 1, 2)$.

Proof. Let us prove the first convergence. It suffices to show

$$\lim_{t\to 0} \int \rho_M(z)\rho_M(z')\, r_M(t, z, z') \wedge \star\varphi(z') = \rho_M^2(z)\, \varphi(z), \tag{4.2}$$

$$\lim_{t\to 0} \int \tilde{\rho}_{\mathbf{L}}(w)\tilde{\rho}_{\mathbf{L}}(w')\, r_H(t, w, w') \wedge \star_{g^H} \tilde{\varphi}(w') = \tilde{\rho}_{\mathbf{L}}^2(w)\, \tilde{\varphi}(w), \tag{4.3}$$

where we set $\tilde{\varphi}(w) = \varphi(z(w))$. As to (4.2): The convergence in the $|\cdot|_g$-norm was shown in [18, Proposition 2.1(ii)]. Since ρ_M is compactly supported, it obviously implies the convergences in the other norms. One finds (4.3) valid in the three kinds of norms by referring to the proof of Lemma 2.6. ∎

Let us set $q(t, z, z') = (\frac{\partial}{\partial t} + \square_H) r(t, z, z')$ and $q^1 = q$, $q^2 = q\#q^1$, $q^3 = q\#q^2, \ldots$ inductively, where, in general, for double forms $h_i(t, z, z')$ $(i = 1, 2)$ we define the convolution $(h_1\#h_2)(t, z, z')$ by

$$(h_1\#h_2)(t, z, z') = \int_0^t ds \int h_1(t - s, z, z'') \wedge \star h_2(s, z'', z').$$

We put

$$p = \sum_{k=0}^{\infty} (-1)^k r\#q^k \quad (r\#q^0 := r), \quad R_{k_0}(p) = \sum_{k\geq k_0} (-1)^k r\#q^k,$$
$$q_\infty = \sum_{k=1}^{\infty} (-1)^k q^k, \quad R_{k_0}(q_\infty) = \sum_{k\geq k_0} (-1)^k q^k.$$

Proposition 4.2

(1) *The forms q^k, $r\#q^k$, $R_{k_0}(q_\infty)$, $R_{k_0}(p)$ are all well-defined and smooth on $(0, \infty) \times H_n(P^0) \times H_n(P^0)\, (\ni (t, z, z'))$. The last two forms are termwisely differentiable. For every integer $m \geq 0$ and multi-indices $\mathbb{A}$, $\mathbb{A}'$, there exist constants $b_k = b_{k,(m,\mathbb{A},\mathbb{A}')} > 0$, etc., such that, on $(0, T_0] \times H_n(P^0)\, (\ni (t, z'))$,*

$$\Big\| (\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'} q^k(t, z, z') \Big\|_{L^1(z)} \leq b_k\, t^{(k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m-1}, \tag{4.4}$$

$$(4.5)\qquad \left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(r\# q^k)(t,z,z')\right\|_{L^1(z)} \le c_k\, t^{(k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m},$$

$$(4.6)\qquad \left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\, R_{k_0}(q_\infty)(t,z,z')\right\|_{L^1(z)} \le b(k_0)\, t^{(k_0-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m-1},$$

$$(4.7)\qquad \left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\, R_{k_0}(p)(t,z,z')\right\|_{L^1(z)} \le c(k_0)\, t^{(k_0-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m},$$

and so are $\left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'} q^k(t,z,z')\right\|_{L^1(z')}$, *etc., where we set* $\xi_{\mathbb{A},z} = \xi_{A_1,z}\cdots\xi_{A_{|\mathbb{A}|},z}$, *etc.* (*The estimates for* $R_{k_0}(q_\infty)$, $R_{k_0}(p)$ *are the ones for the sum of the termwise* L^1-*norms of their termwise differentials.*) *Further, for every* $\ell = 0$ *or* $\ell \ge 2n+2$, *there exist constants* $B_k(\ell) = B_{k,(m,\mathbb{A},\mathbb{A}')}(\ell) > 0$, *etc., such that, on* $(0,T_0]\times H_n(P^0)\times H_n(P^0)$,

$$(4.8)\qquad \left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'} q^k(t,z,z')\right|_g \le B_k(\ell)\, t^{(k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+2)}\delta(z',z)^{-\ell},$$

$$(4.9)\qquad \left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(r\# q^k)(t,z,z')\right|_g \\ \le C_k(\ell)\, t^{(k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+1)}\delta(z',z)^{-\ell},$$

$$(4.10)\qquad \left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\, R_{k_0}(q_\infty)(t,z,z')\right|_g \\ \le B(k_0,\ell)\, t^{(k_0-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+2)}\delta(z',z)^{-\ell},$$

$$(4.11)\qquad \left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\, R_{k_0}(p)(t,z,z')\right|_g \\ \le C(k_0,\ell)\, t^{(k_0-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+1)}\delta(z',z)^{-\ell},$$

where we set $\delta(z',z) = \left|w(z')^{-1}w(z)\right|_H$. (*The last two estimates are the ones for the sum of the termwise* $|\cdot|_g$-*norms of their termwise differentials.*)

(2) *The convolutions* $r\# q_\infty$, $q\# q_\infty$ *are well-defined and smooth on* $(0,\infty)\times H_n(P^0)\times H_n(P^0)$ *and we have*

$$\Big(\frac{\partial}{\partial t} + \Box_{H(P^0)}\Big)(r\# q_\infty) = q_\infty + q\# q_\infty, \quad p = r + r\# q_\infty, \quad q\# q_\infty = -q - q_\infty,$$
$$\Big(\frac{\partial}{\partial t} + \Box_{H(P^0)}\Big)p(t,z,z') = 0.$$

Further, for any $\varphi \in \Omega_0^{p,q} H_n(P^0)$, *we have*

$$(4.12)\qquad \lim_{t\to 0}\int p(t,z,z')\wedge \star\varphi(z') = \varphi(z), \quad \lim_{t\to 0}\int \bar\varphi(z)\wedge\star p(t,z,z') = \bar\varphi(z')$$

in the $|\cdot|_g$-*norm and in the* L^1_g-*norm.*

Some preparatory arguments will be necessary before proving the proposition. In general, a smooth kernel $k(t,z,z') = \sum \theta^{I\bar K}(z)\boxtimes\theta^{\bar I' K'}(z')\cdot k^{(I\bar K)(I'\bar K')}(t,z,z')$ is said to be of type ℓ if each coefficient $k^{(I\bar K)(I'\bar K')}(t,z,z')$ is a finite sum of such smooth functions as

$$(4.13)\qquad \begin{aligned} \mathbf{K}^b_M(t,z,z') &= t^{-n-2+b/2}\rho_M(z',z)\,\mathcal{K}(\iota_{1/t}\Theta^M(z',z)) \\ \mathbf{K}^b_{\mathbf{L}}(t,z,z') &= t^{-n-2+b/2}\rho_{\mathbf{L}}(z',z)\,\mathcal{K}(\iota_{1/t}\Theta^{\mathbf{L}}(z',z)) \end{aligned} \qquad (b\ge\ell),$$

where $\mathcal{K}(\Theta)$ is a rapidly decreasing smooth function on $H_n\,(\ni \Theta)$ and $\rho_M(z',z)$, etc., are smooth functions with $\operatorname{supp}\rho_M \subset \{(z',z) \mid |w(z')| < 2r, |w(z)| < 2r\}$, $\operatorname{supp}\rho_{\mathbf{L}} \subset \{(z',z) \mid |w(z')| > r, |w(z)| > r\}$ on which $g = g^H$. Further we assume that, for every $\mathbb{A}$ and $\mathbb{A}'$, $\left|\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\rho_{\mathbf{L}}(z',z)\right|$ is bounded. The kernel whose coefficients consist of the second type of functions is equivalently interpreted in the variable w as follow: It is a smooth kernel $k_{\tilde{\mathbf{L}}}(t,w',w) = \sum \tilde{\theta}_H^{I\bar{K}}(w) \boxtimes \tilde{\theta}_H^{\bar{I}'K'}(w') \cdot k_{\tilde{\mathbf{L}}}^{(I\bar{K})(I'\bar{K}')}(t,w,w')$ each coefficient of which is a finite sum of such smooth functions as $\mathbf{K}^b_{\tilde{\mathbf{L}}}(t,w,w') = t^{-n-2+b/2}\rho_{\tilde{\mathbf{L}}}(w',w)\mathcal{K}(\iota_{1/t}\Theta^{\tilde{\mathbf{L}}}(w',w))$ $(b \geq \ell)$, where $\rho_{\tilde{\mathbf{L}}}(w',w)$ is a smooth function with $\operatorname{supp}\rho_{\tilde{\mathbf{L}}} \subset \{(w',w) \mid |w'| > r, |w| > r\}$ and $\left|\tilde{\xi}^H_{\mathbb{A},w}\tilde{\xi}^H_{\mathbb{A}',w'}\rho_{\tilde{\mathbf{L}}}(w',w)\right|$ is bounded for every $\mathbb{A}$ and $\mathbb{A}'$.

Lemma 4.3 (cf. [18, Propositions 3.2, 3.1, 3.3 and 3.4])

(1) *The kernel* $r(t,z,z')$ *is of type* 2 *and* $q(t,z,z')$ *is of type* 1.

(2) *For a kernel* $k(t,z,z')$ *of type* ℓ, $\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}k(t,z,z')$ *is a kernel of type* $\ell - |\mathbb{A}|_H - |\mathbb{A}'|_H$ *and* $(\partial/\partial t)k(t,z,z')$ *is of type* $\ell - 2$.

(3) *For a kernel* $k(t,z,z')$ *of type* ℓ, *there exists a constant* $C > 0$ *such that* $|k(t,z,z')|_g \leq Ct^{\ell/2-(n+2)}$, $\|k(t,z,z')\|_{L^1(z)} \leq Ct^{\ell/2-1}$, *etc., when* $0 < t \leq T_0$.

(4) *For a kernel* $k(t,z,z')$ *of type* ℓ, *we have*

$$\xi_{A,z}k(t,z,z') = \begin{cases} \displaystyle\sum_{B\neq 0} \xi_{B,z'}k_B(t,z,z') + k_*(t,z,z') & (A \neq 0), \\ \displaystyle\sum \xi_{B,z'}k_B(t,z,z') + k_*(t,z,z') & (A = 0), \end{cases}$$

where $k_B(t,z,z')$, $k_*(t,z,z')$ *are some kernels of type* ℓ: *One could set* $k_0 = -k$. *Also* $\xi_{A,z'}k(t,z,z')$ *can be described similarly.*

Proof. Recall that Θ^M and $\Theta^{\tilde{\mathbf{L}}}$ are both admissible. One finds the lemma valid for the kernels whose coefficients consist of the type of $\mathbf{K}^b_M$ by referring to [18] with some further argument. The lemma will be obvious for the kernels whose coefficients consist of the type of $\mathbf{K}^b_{\tilde{\mathbf{L}}}$. ∎

Lemma 4.4 (cf. [18, §4 and §5]) *Suppose that the kernels* $k_i(t,z,z')$ *are of types* $m_i\,(\geq 1)$. *Then* $(k_1\#\cdots\#k_j)(t,z,z')$ *is well-defined and smooth on* $(0,\infty) \times H_n(P^0) \times H_n(P^0)$, *and there exist constants* $b > 0$, $B(\ell) > 0$ $(\ell = 0$ *or* $\ell \geq 2n+2)$ *such that*

$$\left\|(\partial/\partial t)^m\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,z,z')\right\|_{L^1(z)} \leq b\,t^{(\sum m_i - |\mathbb{A}|_H - |\mathbb{A}'|_H)/2-m-1}, \tag{4.14}$$

$$\begin{aligned} &\left|(\partial/\partial t)^m\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,z,z')\right|_g \\ &\qquad \leq B(\ell)\,t^{(\sum m_i - |\mathbb{A}|_H - |\mathbb{A}'|_H)/2-m+\ell/2-(n+2)}\,\delta(z',z)^{-\ell}, \end{aligned} \tag{4.15}$$

etc., when $0 < t \le T_0$.

Proof. Lemma 4.3(2)(3) assert that $(4.14)^{j=1}$ and $(4.15)^{j=1}_{\ell=0}$ hold. The estimate $(4.15)^{j=1}_{\ell\ge 2n+2}$ (or, in fact, $(4.15)^{j=1}_{\ell>0}$) follows obviously from the fact that $\mathcal{K}(\Theta)$ is rapidly decreasing (refer to [18, (5.10)]). Thus the lemma in the case $j=1$ is valid. In general, it is verified by an induction relative to j on the basis of the following formula (cf. [18, (4.36), (5.3), etc.]) induced from Lemma 4.3(4) and integration by parts: Setting $k = k_2\# \cdots \# k_j$, we have

(4.16) $$(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,z,z')$$
$$= \sum_{m'+m''=m-1} C_{m'} \int ((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(\frac{t}{2},z,z'') \wedge \star((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(\frac{t}{2},z'',z')$$
$$+ \sum_{|\mathbb{B}'|_H \le |\mathbb{A}'|_H} \int_0^{t/2} ds \int ((\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{B}',z''}k_1)(t-s,z,z'') \wedge \star k_{2\mathbb{B}'}(s,z'',z')$$
$$+ \sum_{|\mathbb{B}|_H \le |\mathbb{A}|_H} \int_{t/2}^{t} ds \int k_{1\mathbb{B}}(t-s,z,z'') \wedge \star((\partial/\partial t)^m \xi_{\mathbb{B},z''}\xi_{\mathbb{A}',z'}k)(s,z'',z'),$$

where $k_{1\mathbb{B}}$ is a kernel of type m_1, $k_{2\mathbb{B}'}$ is a finite sum of convolutions of kernels of types m_i $(i=2,\cdots,j)$ and we put $C_{m'} = \binom{m-1}{m'}$. It implies

$$\begin{aligned}
&\left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,z,z')\right\|_{L^1(z)}\\
&\le \sum C_{m'} \max_{z''} \left\|((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(t/2,z,z'')\right\|_{L^1(z)}\\
&\qquad\qquad \times \left\|((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(t/2,z'',z')\right\|_{L^1(z'')}\\
&+\sum \int_0^{t/2} ds \max_{z''} \left\|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{B}',z''}k_1(t-s,z,z'')\right\|_{L^1(z)} \cdot \left\|k_{2\mathbb{B}'}(s,z'',z')\right\|_{L^1(z'')}\\
&+\sum \int_{t/2}^{t} ds \max_{z''} \left\|k_{1\mathbb{B}}(t-s,z,z'')\right\|_{L^1(z)} \cdot \left\|(\partial/\partial s)^m \xi_{\mathbb{B},z''}\xi_{\mathbb{A}',z'}k(s,z'',z')\right\|_{L^1(z'')}
\end{aligned}$$

and

$$\begin{aligned}
&\left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,z,z')\right|_g\\
&\le \sum C_{m'} \max_{z''} \left|((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(t/2,z,z'')\right|_g\\
&\qquad\qquad \times \left\|((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(t/2,z'',z')\right\|_{L^1(z'')}\\
&+\sum \int_0^{t/2} ds \max_{z''} \left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{B}',z''}k_1(t-s,z,z'')\right|_g \cdot \left\|k_{2\mathbb{B}'}(s,z'',z')\right\|_{L^1(z'')}\\
&+\sum \int_{t/2}^{t} ds \left\|k_{1\mathbb{B}}(t-s,z,z'')\right\|_{L^1(z'')} \cdot \max_{z''} \left|(\partial/\partial t)^m \xi_{\mathbb{B},z''}\xi_{\mathbb{A}',z'}k(s,z'',z')\right|_g .
\end{aligned}$$

Hence, (4.14) and $(4.15)_{\ell=0}$ can be shown inductively. Next, let us prove $(4.15)_{\ell\geq 2n+2}$. We know ([7, Proposition 1.7]) that there exists a constant $\gamma \geq 1$ such that the inequality $\delta(z,z') \leq \gamma(\delta(z,z'') + \delta(z'',z'))$ holds. Let us set $B_{\tilde{\delta}/2\gamma} = B_{\tilde{\delta}/2\gamma}(z) = \{z'' \in H_n \mid \delta(z,z'') < \tilde{\delta}/2\gamma\}$ ($\tilde{\delta} := \delta(z',z)$) and denote by $B^c_{\tilde{\delta}/2\gamma}$ its complement in $H_n(P^0)$. Then, referring to (4.16), we have

$$
\begin{aligned}
&\left|\int ((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(\frac{t}{2},z,z'') \wedge \star((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(\frac{t}{2},z'',z')\right|_g \\
&\leq \int_{B^c_{\tilde{\delta}/2\gamma}} \max_{z''}\left|((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(\frac{t}{2},z,z'')\right|_g \left\|((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(\frac{t}{2},z'',z')\right\|_{L^1(z'')} \\
&+ \int_{B_{\tilde{\delta}/2\gamma}} \left\|((\partial/\partial t)^{m'}\xi_{\mathbb{A},z}k_1)(\frac{t}{2},z,z'')\right\|_{L^1(z'')} \max_{z''}\left|((\partial/\partial t)^{m''}\xi_{\mathbb{A}',z'}k)(\frac{t}{2},z'',z')\right|_g,
\end{aligned}
$$

etc. Hence, inductively we can prove $(4.15)_{\ell\geq 2n+2}$ as well. The condition $\ell \geq 2n+2$ is required to guarantee the integrability of various integrals appearing in the inductive argument. ∎

Proof of Proposition 4.2. Lemma 4.4 implies that $r\#q^k$, q^k are smooth and the estimates (4.4), (4.5), (4.8), (4.9) hold. By exhaustive calculation such as that in [18, §4 and §5], we know that there exist constants $b > 0$, $c > 0$, $B(\ell) > 0$, $C(\ell) > 0$ ($\ell = 0$ or $\ell \geq 2n+2$) and a large integer k' such that, if $k \geq k'$, then the constants $b_k > 0$, $c_k > 0$, etc., appearing in (4.4), (4.5), etc., may be determined as

$$
\begin{aligned}
b_k &= \frac{b^k}{\Gamma((k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m)}, \quad c_k = \frac{c^k}{\Gamma((k+2-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m)},\\
B_k(\ell) &= \frac{B(\ell)^k}{\Gamma((k-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+1))},\\
C_k(\ell) &= \frac{C(\ell)^k}{\Gamma((k+2-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+1))}.
\end{aligned}
$$

Hence we obtain the estimates (4.6), (4.7), (4.10) and (4.11). As to (2): It is an easy consequence of (1) and Lemma 4.1 (cf. the proof of [18, Theorem 4.10]). ∎

Now, let us prove Theorems 3.4 and 3.5.

Proof of Theorem 3.4. Since we have already proved Proposition 4.2(2) and Lemma 4.1, it suffices to show

$$
\lim_{t\to 0}\int R_1(p)(t,z,z') \wedge \star\varphi(z') = 0, \quad \lim_{t\to 0}\int \bar{\varphi}(z) \wedge \star R_1(p)(t,z,z') = 0 \tag{4.17}
$$

in the L^2_g-norm. We will prove the first convergence. Let D' be a compact set which contains the support of φ and let N be a large integer. Then we have

$$
\left\|\int R_{k_0}(p)(t,z,z') \wedge \star\varphi(z')\right\|^2_{L^2(z)} = \left\|\int_{D'} R_{k_0}(p)(t,z,z') \wedge \star\varphi(z')\right\|^2_{L^2(z)} \tag{4.18}
$$

$$
\begin{aligned}
&\le \int dV_g(z)\Big(\int_{D'} dV_g(z')\,(1+|z'|^N)^{-1}\,\big|R_{k_0}(p)(t,z,z')\big|_g\,(1+|z'|^N)\,\big|\varphi(z')\big|_g\Big)^2\\
&\le \int dV_g(z)\Big(\int_{D'} dV_g(z')\,(1+|z'|^N)^{-1}\,\big|R_{k_0}(p)(t,z,z')\big|_g\Big)^2\cdot \mathrm{sn}(\varphi)^2\\
&\le \mathrm{sn}(\varphi)^2\int_{D'} dV_g(z')\,(1+|z'|^N)^{-1}\int dV_g(z)\,\big|R_{k_0}(p)(t,z,z')\big|_g\\
&\qquad\qquad \times\Big(\int_{D'} dV_g(z')\,(1+|z'|^N)^{-1}\Big)\cdot\max\big|R_{k_0}(p)(t,z,z')\big|_g\\
&\le \mathrm{sn}(\varphi)^2\Big(\int_{D'} dV_g(z')\,(1+|z'|^N)^{-1}\Big)^2\\
&\qquad \times\ \max\big\|R_{k_0}(p)(t,z,z')\big\|_{L^1(z)}\cdot\max\big|R_{k_0}(p)(t,z,z')\big|_g\,,
\end{aligned}
$$

where $\mathrm{sn}(\varphi)$ is a semi-norm of φ. Hence, by (4.7) and (4.11), if k_0 is sufficiently large, then we have $\big\|\int R_{k_0}(p)(t,z,z')\wedge\star\varphi(z')\big\|_{L^2(z)}\le t^{1/2}\,\mathrm{sn}(\varphi)$, where $\mathrm{sn}(\varphi)$ is a new one. There remains the estimation of $\int(r\#q^k)(t,z,z')\wedge\star\varphi(z')$ $(0<k<k_0)$. Let U be a relatively compact open set containing $D:=D'\cup\{z\mid |w(z)|\le 2r\}$ and let μ be a nonnegative C^∞ function such that $\mu=1$ on D and $\operatorname{supp}\mu\subset U$. Further, let V be a relatively compact open set containing $\bar U$ and let ν be a nonnegative C^∞ function such that $\nu=1$ on $\bar U$ and $\operatorname{supp}\nu\subset V$. In addition, let us set $\tilde q(t,z,z')=\mu(z)\mu(z')q(t,z,z')$, $\mu(z')r(t,z,z')=\nu(z)\mu(z')r(t,z,z')+(1-\nu(z))\mu(z')r(t,z,z')=r_0(t,z,z')+r_\infty(t,z,z')$. Then, recalling $\operatorname{supp}q(t)\subset\{(z,z')\mid |w(z)|\le 2r\}$, we have

$$
\begin{aligned}
\int(r\#q^k)(t,z,z')\wedge\star\varphi(z') &= \int(r\#\tilde q^k)(t,z,z')\wedge\star\varphi(z')\\
&= \int(r_0\#\tilde q^k)(t,z,z')\wedge\star\varphi(z')+\int(r_\infty\#\tilde q^k)(t,z,z')\wedge\star\varphi(z')
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\int(r_0\#\tilde q^k)(t,z,z')\wedge\star\varphi(z')\Big\|_{L^2(z)}\\
&\le\Big\{\int_V dV_g(z)\ \big\|(r_0\#\tilde q^k)(t,z,z')\big\|^2_{L^1(z')}\Big\}^{1/2}\max\big|\varphi(z')\big|_g\le t^{1/2}\,\mathrm{vol}(V)^{1/2}\mathrm{sn}(\varphi).
\end{aligned}
$$

On the other hand, since the supports of $r_\infty(t,z,z')$ relative to z, z' do not intersect with each other, the kernel r_∞ is of type ℓ for all ℓ. Hence, an estimation similar to (4.18) shows $\big\|\int(r_\infty\#\tilde q^k)(t,z,z')\wedge\star\varphi(z')\big\|_{L^2(z)}\le t^{1/2}\,\mathrm{sn}(\varphi)$. Thus we obtain the first convergence at (4.17). The second one follows similarly by setting $\tilde r(t,z,z')=\mu(z)\mu(z')r(t,z,z')$ and $\mu(z)q(t,z,z')=\mu(z)\nu(z')q(t,z,z')+\mu(z)(1-\nu(z'))q(t,z,z')=q_0(t,z,z')+q_\infty(t,z,z')$. Note that $(\tilde r\#\tilde q^{k-1})\#q=(\tilde r\#\tilde q^{k-1})\#q_0+(\tilde r\#\tilde q^{k-1})\#q_\infty$. ∎

Proof of Theorem 3.5 (also refer to Remark 6.2). Refer to the proof of [18, Theorem 6.4(i)]. We wish to show that there is an asymptotic expansion

$$(4.19)\qquad (\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\left(e^{-t\square_{H(P^0)}}\right)^{(I\bar{K})(I'\bar{K}')}(0,0) \sim \sum_{m\geq -(|\mathbb{A}|_H+|\mathbb{A}'|_H)} t^{-(n+1)+m/2}\, b_{m/2}^{(I\bar{K})(I'\bar{K}')}(P^0:\mathbb{A},\mathbb{A}').$$

Since the two differential operators $\square_{H(P^0)}$ and $\square_H$ coincide sufficiently near the point $0 = P^0$, it follows immediately from the Duhamel principle that $b_{m/2}^{(I\bar{K})(I'\bar{K}')}$ is equal to $a_{m/2}^{(I\bar{K})(I'\bar{K}')}$. To show (4.19) it will suffice to show that $\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,0,0)$ can be expanded similarly, where $k_1\#\cdots\#k_j$ is the convolution given in Lemma 4.4. We decompose it as

$$\begin{aligned}&\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}(k_1\#\cdots\#k_j)(t,0,0)\\ &=\sum_{p=1}^{j}\sum_{|\mathbb{B}'|_H\leq|\mathbb{A}'|_H}^{|\mathbb{B}|_H\leq|\mathbb{A}|_H}\left(k_{1\mathbb{B}}\#^{\sharp}\cdots\#^{\sharp}k_{p-1,\mathbb{B}}\#^{\sharp}(\xi_{\mathbb{B},z}\xi_{\mathbb{B}',z'}k_p)\#^{\flat}k_{p+1,\mathbb{B}'}\#\cdots\#k_{j\mathbb{B}'}\right)(t,0,0),\end{aligned}$$

where $k_{i\mathbb{B}}$, $k_{i'\mathbb{B}'}$ are kernels of types m_i, $m_{i'}$. Here, in general, we set

$$\begin{aligned}&\left(k_1\#^{\sharp}\cdots\#^{\sharp}k_{p-1}\#^{\sharp}k_p\#^{\flat}k_{p+1}\#\cdots\#k_j\right)(t,0,0)\\ &=\int_{t/2}^{t}ds^{(1)}\cdots\int_{s^{(p-2)}/2}^{s^{(s-2)}}ds^{(p-1)}\int_0^{s^{(p-1)}/2}ds^{(p)}\int_{z^{(1)}\in H_n(P^0)}\cdots\int_{z^{(p)}\in H_n(P^0)}\\ &\quad\cdot k_1(t-s^{(1)},0,z^{(1)})\wedge\star\cdots\\ &\qquad\wedge\star\, k_p(s^{(p-1)}-s^{(p)},z^{(p-1)},z^{(p)})\wedge\star(k_{p+1}\#\cdots\#k_j)(s^{(p)},z^{(p)},0).\end{aligned}$$

Thus it will be sufficient to show that this can be expanded similarly, under the assumption that the kernels k_i are of types m_i with $m_p\geq 1-|\mathbb{A}|_H-|\mathbb{A}'|_H$, $m_i\geq 1$ $(i\neq p)$ and, further, under the assumption that the supports of $k_i(t,z,z')$ $(i\geq 2)$ relative to z are contained in $\{z\in H_n(P^0)\mid |w(z)|\leq 2r\}$. Since the coefficients of each kernel k_i are expressed as finite sums of $\mathbf{K}_M^{b_i}$ or $\mathbf{K}_{\mathbf{L}}^{b_i}$ $(b_i\geq m_i)$, eventually we know that it suffices to examine, near $t^{1/2}=0$, the behavior of the function

$$\begin{aligned}(4.20)\qquad &\int_{s^{(0)}/2}^{s^{(0)}}ds^{(1)}\cdots\int_{s^{(p-2)}/2}^{s^{(s-2)}}ds^{(p-1)}\int_0^{s^{(p-1)}/2}ds^{(p)}\int_0^{s^{(p)}}ds^{(p+1)}\cdots\int_0^{s^{(j-2)}}ds^{(j-1)}\\ &\cdot\left(s^{(0)}-s^{(1)}\right)^{-(n+2)+b_1/2}\left(s^{(1)}-s^{(2)}\right)^{-(n+2)+b_2/2}\cdots\left(s^{(j-1)}-s^{(j)}\right)^{-(n+2)+b_j/2}\\ &\times\int_{H_n(P^0)}dV_g(z^{(1)})\cdots\int_{H_n(P^0)}dV_g(z^{(j-1)})\,\psi_1(z^{(1)},z^{(0)})\cdots\psi_j(z^{(j)},z^{(j-1)})\\ &\times\mathcal{K}_1(\iota_{1/(s^{(0)}-s^{(1)})}\Theta(z^{(1)},z^{(0)}))\cdots\mathcal{K}_j(\iota_{1/(s^{(j-1)}-s^{(j)})}\Theta(z^{(j)},z^{(j-1)})),\end{aligned}$$

where $\Theta=\Theta^M$ or $\Theta^{\mathbf{L}}$ and $s^{(0)}=t$, $s^{(j)}=0$, $z^{(0)}=z^{(j)}=0$. In fact, the domains of integrations relative to the variables $z^{(i)}$ may be reduced to $\{z^{(i)}\in H_n(P^0)\mid |w(z^{(i)})|\leq$

$2r\}$. Now let us use $u^{(i)} = \Theta(z^{(i)}, z^{(i-1)})$ $(i < p)$ and $u^{(i)} = \Theta(z^{(i+1)}, z^{(i)})$ $(i \geq p)$ as new coordinates. (We take the domains of the coordinate maps $\Theta(\cdot, z^{(i-1)}), \Theta(z^{(i+1)}, \cdot) : H_n(P^0) \to H_n$ large enough.) The function $u = u(u^{(1)}, \ldots, u^{(j-1)}) := \Theta(z^{(p)}, z^{(p-1)})$ is smooth and (4.20) is equal to

$$\begin{aligned}
&\int_{s^{(0)}/2}^{s^{(0)}} ds^{(1)} \cdots \int_{s^{(p-2)}/2}^{s^{(s-2)}} ds^{(p-1)} \int_0^{s^{(p-1)}/2} ds^{(p)} \int_0^{s^{(p)}} ds^{(p+1)} \cdots \int_0^{s^{(j-2)}} ds^{(j-1)} \\
&\cdot (s^{(0)} - s^{(1)})^{-(n+2)+b_1/2} (s^{(1)} - s^{(2)})^{-(n+2)+b_2/2} \cdots (s^{(j-1)} - s^{(j)})^{-(n+2)+b_j/2} \\
&\quad \times \int_{H_n \ni u^{(1)}} (\Theta(\cdot, z^{(0)})_* dV_g)(u^{(1)}) \cdots \int_{H_n \ni u^{(j-1)}} (\Theta(z^{(j)}, \cdot)_* dV_g)(u^{(j-1)}) \\
&\quad\quad \cdot \psi_1(z^{(1)}, z^{(0)})\, \psi_2(z^{(2)}, z^{(1)}) \cdots \psi_j(z^{(j)}, z^{(j-1)}) \\
&\quad\quad \times \mathcal{K}_1(\iota_{1/(s^{(0)}-s^{(1)})} u^{(1)}) \cdots \mathcal{K}_{p-1}(\iota_{1/(s^{(p-2)}-s^{(p-1)})} u^{(p-1)})\, \mathcal{K}_p(\iota_{1/(s^{(p-1)}-s^{(p)})} u) \\
&\quad\quad\quad\quad \times \mathcal{K}_{p+1}(\iota_{1/(s^{(p)}-s^{(p+1)})} u^{(p)}) \cdots \mathcal{K}_j(\iota_{1/(s^{(j-1)}-s^{(j)})} u^{(j-1)}) \\
&= \int_{s^{(0)}/2}^{s^{(0)}} ds^{(1)} \cdots \int_{s^{(p-2)}/2}^{s^{(s-2)}} ds^{(p-1)} \int_0^{s^{(p-1)}/2} ds^{(p)} \int_0^{s^{(p)}} ds^{(p+1)} \cdots \int_0^{s^{(j-2)}} ds^{(j-1)} \\
&\cdot (s^{(0)} - s^{(1)})^{-(n+2)+b_1/2} (s^{(1)} - s^{(2)})^{-(n+2)+b_2/2} \cdots (s^{(j-1)} - s^{(j)})^{-(n+2)+b_j/2} \\
&\quad \times \int_{H_n \ni u^{(1)}} dV_{g^H}(u^{(1)}) \cdots \int_{H_n \ni u^{(j-1)}} dV_{g^H}(u^{(j-1)})\, \psi(u^{(1)}, \cdots, u^{(j-1)}) \\
&\quad\quad \times \mathcal{K}_1(\iota_{1/(s^{(0)}-s^{(1)})} u^{(1)}) \cdots \mathcal{K}_{p-1}(\iota_{1/(s^{(p-2)}-s^{(p-1)})} u^{(p-1)})\, \mathcal{K}_p(\iota_{1/(s^{(p-1)}-s^{(p)})} u) \\
&\quad\quad\quad\quad \times \mathcal{K}_{p+1}(\iota_{1/(s^{(p)}-s^{(p+1)})} u^{(p)}) \cdots \mathcal{K}_j(\iota_{1/(s^{(j-1)}-s^{(j)})} u^{(j-1)}).
\end{aligned}$$

Here, the domain of integration relative to the set of variables $(u^{(1)}, \ldots, u^{(j-1)})$ may be reduced to a certain compact set as well and $\psi(u^{(1)}, \cdots, u^{(j-1)})$ is a smooth function on the domain. Further let us change the variables: We set $v^{(i)} = \iota_{1/(s^{(i-1)}-s^{(i)})} u^{(i)}$ $(i < p)$ and $v^{(i)} = \iota_{1/(s^{(i)}-s^{(i+1)})} u^{(i)}$ $(i \geq p)$, where $s^{(i)} := t\sigma^{(i)}$. Then the above is equal to

$$\begin{aligned}
(4.21)\ & t^{j-1} \int_{1/2}^{1} d\sigma^{(1)} \cdots \int_{\sigma^{(p-2)}/2}^{\sigma^{(s-2)}} d\sigma^{(p-1)} \int_0^{\sigma^{(p-1)}/2} d\sigma^{(p)} \int_0^{\sigma^{(p)}} d\sigma^{(p+1)} \cdots \int_0^{\sigma^{(j-2)}} d\sigma^{(j-1)} \\
&\cdot t^{-j(n+2)+\sum b_i/2} (1 - \sigma^{(1)})^{-(n+2)+b_1/2} \cdots (\sigma^{(j-1)})^{-(n+2)+b_j/2} \\
&\times t^{(j-1)(n+1)} (1 - \sigma^{(1)})^{n+1} \cdots \widehat{(\sigma^{(p-1)} - \sigma^{(p)})^{n+1}} \cdots (\sigma^{(j-1)})^{n+1} \\
&\times \int_{H_n \ni v^{(1)}} dV_{g^H}(v^{(1)}) \cdots \int_{H_n \ni v^{(j-1)}} dV_{g^H}(v^{(j-1)}) \\
&\quad \cdot \psi(\ldots, \iota_{t(\sigma^{(p-2)}-\sigma^{(p-1)})}(v^{(p-1)}), \iota_{t(\sigma^{(p)}-\sigma^{(p+1)})}(v^{(p)}), \ldots) \\
&\quad \times \mathcal{K}_1(v^{(1)}) \cdots \mathcal{K}_{p-1}(v^{(p-1)})\, \mathcal{K}_p(\iota_{1/t(\sigma^{(p-1)}-\sigma^{(p)})} u)\, \mathcal{K}_{p+1}(v^{(p)}) \cdots \mathcal{K}_j(v^{(j-1)}) \\
= {} & t^{-(n+2)+\sum b_i/2} \\
&\times \int_{1/2}^{1} d\sigma^{(1)} \cdots \int_{\sigma^{(p-2)}/2}^{\sigma^{(s-2)}} d\sigma^{(p-1)} \int_0^{\sigma^{(p-1)}/2} d\sigma^{(p)} \int_0^{\sigma^{(p)}} d\sigma^{(p+1)} \cdots \int_0^{\sigma^{(j-2)}} d\sigma^{(j-1)}
\end{aligned}$$

$$
\begin{aligned}
&\cdot (1-\sigma^{(1)})^{b_1/2-1}\cdots(\sigma^{(p-1)}-\sigma^{(p)})^{-(n+2)+b_p/2}\cdots(\sigma^{(j-1)})^{b_j/2-1}\\
&\times\int_{H_n\ni v^{(1)}} dV_{g^H}(v^{(1)})\cdots\int_{H_n\ni v^{(j-1)}} dV_{g^H}(v^{(j-1)})\\
&\cdot\psi(\ldots,\iota_{t(\sigma^{(p-2)}-\sigma^{(p-1)})}(v^{(p-1)}),\iota_{t(\sigma^{(p)}-\sigma^{(p+1)})}(v^{(p)}),\ldots)\\
&\times\mathcal{K}_1(v^{(1)})\cdots\mathcal{K}_{p-1}(v^{(p-1)})\,\mathcal{K}_{p+1}(v^{(p)})\cdots\mathcal{K}_j(v^{(j-1)})\,\mathcal{K}_p(\iota_{1/t(\sigma^{(p-1)}-\sigma^{(p)})}u).
\end{aligned}
$$

Here the function

$$
\begin{aligned}
\iota_{1/t(\sigma^{(p-1)}-\sigma^{(p)})}u &= \iota_{1/t(\sigma^{(p-1)}-\sigma^{(p)})}u(u^{(1)},\ldots,u^{(j-1)})\\
&= \iota_{1/t(\sigma^{(p-1)}-\sigma^{(p)})}u\Big(\ldots,\iota_{t(\sigma^{(p-2)}-\sigma^{(p-1)})}(v^{(p-1)}),\iota_{t(\sigma^{(p)}-\sigma^{(p+1)})}(v^{(p)}),\ldots\Big)
\end{aligned}
$$

is smooth up to $t^{1/2}=0$. Indeed, checking $\Theta_A(z^{(p)},z^{(p-1)})$ carefully (see (2.15)), we know that, if we expand $u_A\Big(\ldots,\iota_{t(\sigma^{(p-2)}-\sigma^{(p-1)})}(v^{(p-1)}),\iota_{t(\sigma^{(p)}-\sigma^{(p+1)})}(v^{(p)}),\ldots\Big)$ regarded as a function of $t^{1/2}$ into Taylor series at $t^{1/2}=0$, it starts from the term of order $|A|_H$. Thus (4.21) is asymptotically expanded as desired. It is easily examined that the integrations appearing in the coefficients of the expansion are all integrable. ∎

5 Adiabatic expansion of the Kohn-Rossi Laplacian and a formula for the asymptotic coefficients

We will introduce a new method of computing the asymptotic coefficients appearing in (2.1), i.e., (3.15), on the basis of the adiabatic expansion theory ([14]).

Let us consider the transformation of $H_n(P^0)$ defined by $z\mapsto\iota_\varepsilon(z)=(\varepsilon z_0,\varepsilon^{1/2}z_1,\ldots,\varepsilon^{1/2}z_n)$, $0<\varepsilon\le\varepsilon_0$ (see (2.11)), which induces a new contact Riemannian structure $(\theta^\bullet_{(\varepsilon)},\xi^{(\varepsilon)}_\bullet,g^{(\varepsilon)},J^{(\varepsilon)}):=(\iota^*_\varepsilon\theta^\bullet_\varepsilon,\iota^*_\varepsilon\xi^\varepsilon_\bullet,\iota^*_\varepsilon g^\varepsilon,\iota^*_\varepsilon J^\varepsilon)$ with

$$
\theta^A_\varepsilon:=\varepsilon^{-|A|_H/2}\theta^A,\quad \xi^\varepsilon_A:=\varepsilon^{|A|_H/2}\xi_A,\quad g^\varepsilon:=\sum\theta^A_\varepsilon\otimes\theta^{\bar A}_\varepsilon,\quad J^\varepsilon\xi^\varepsilon_\alpha:=i\xi^\varepsilon_\alpha.
$$

Obviously (2.3) produces

$$
\begin{aligned}
&\xi^{(\varepsilon)}_\bullet=(\partial/\partial z_\bullet)\cdot V^{(\varepsilon)}_\bullet,\quad V^{(\varepsilon)}_{BA}(z):=\varepsilon^{(|A|_H-|B|_H)/2}V_{BA}(\iota_\varepsilon(z)),\\
&\theta^\bullet_{(\varepsilon)}=(dz_\bullet)\cdot V^\bullet_{(\varepsilon)},\quad V^{BA}_{(\varepsilon)}(z):=\varepsilon^{(|B|_H-|A|_H)/2}V^{BA}(\iota_\varepsilon(z)).
\end{aligned}
\tag{5.1}
$$

Note that $\lim_{\varepsilon^{1/2}\to0}V^{(\varepsilon)}_\bullet(z)=E(-z)$, $\lim_{\varepsilon^{1/2}\to0}V^\bullet_{(\varepsilon)}(z)={}^tE(z)$ (see (2.12)). To the structure $(\theta^\bullet_\varepsilon,\xi^\varepsilon_\bullet,g^\varepsilon,J^\varepsilon)$ the Kohn-Rossi Laplacian $\Box^\varepsilon_{H(P^0)}:=\varepsilon\Box_{H(P^0)}$ and the hermitian Tanno connection $\nabla^\varepsilon:=\nabla$ are attached. Those for the structure $(\theta^\bullet_{(\varepsilon)},\xi^{(\varepsilon)}_\bullet,g^{(\varepsilon)},J^{(\varepsilon)})$ are $\Box^{(\varepsilon)}_{H(P^0)}:=\iota^*_\varepsilon\Box^\varepsilon_{H(P^0)}$, $\nabla^{(\varepsilon)}:=\iota^*_\varepsilon\nabla^\varepsilon$. The coordinates z are then the $\nabla^{(\varepsilon)}$-normal coordinates centered at 0 with $(\partial/\partial z_\bullet)_0 = \xi_\bullet^{(\varepsilon)}(0)$ and $\xi_\bullet^{(\varepsilon)}$ is $\nabla^{(\varepsilon)}$-parallel along the $\nabla^{(\varepsilon)}$-geodesics sz $(0 \le s < \infty)$ as well. The $\nabla^{(\varepsilon)}$-normal coordinate system with respect to $\xi_\bullet^{(\varepsilon)}$ is defined by

$$\Theta^{\nabla^{(\varepsilon)}} : V^{(\varepsilon)} := \{(z', z) \in H_n(P^0) \times H_n(P^0) \mid (\iota_\varepsilon(z'), \iota_\varepsilon(z)) \in V\} \to H_n,$$
$$\Theta^{\nabla^{(\varepsilon)}}(z', z) = \iota_{1/\varepsilon}\Theta^{\nabla}(\iota_\varepsilon(z'), \iota_\varepsilon(z)).$$

Further, obviously the initial value problem (0.1) on $(H_n(P^0), \theta_{(\varepsilon)})$ has a unique heat kernel $e^{-t\Box^{(\varepsilon)}_{H(P^0)}}(z, z')$, which is described as follows.

Lemma 5.1 *We have*

$$e^{-t\Box^{(\varepsilon)}_{H(P^0)}}(z, z') = \sum \theta^{I\bar{K}}_{(\varepsilon)}(z) \boxtimes \theta^{\bar{I}'K'}_{(\varepsilon)}(z') \cdot \varepsilon^{n+1}\left(e^{-t\varepsilon\Box_{H(P^0)}}\right)^{(I\bar{K})(I'\bar{K}')}(\iota_\varepsilon(z), \iota_\varepsilon(z')).$$

Proof. It follows immediately from the fact that the heat kernel on $(H_n(P^0), \theta_\varepsilon)$ is given as

$$e^{-t\Box^{\varepsilon}_{H(P^0)}}(z, z') = \sum \theta^{I\bar{K}}_{\varepsilon}(z) \boxtimes \theta^{\bar{I}'K'}_{\varepsilon}(z') \cdot \varepsilon^{n+1}\left(e^{-t\varepsilon\Box_{H(P^0)}}\right)^{(I\bar{K})(I'\bar{K}')}(z, z').$$

∎

Next, we consider the transformation

$$I_\varepsilon : \Omega^{p,q}H_n \cong \Omega^{p,q}(H_n(P^0), \theta_{(\varepsilon)}), \quad \sum \theta_H^{I\bar{K}} \cdot \varphi^{I\bar{K}} \mapsto \sum \theta_{(\varepsilon)}^{I\bar{K}} \cdot \varphi^{I\bar{K}},$$

which provides the Laplacian $\Box_{(\varepsilon)} = I_\varepsilon^* \Box^{(\varepsilon)}_{H(P^0)}$ $(:= I_\varepsilon^{-1} \circ \Box^{(\varepsilon)}_{H(P^0)} \circ I_\varepsilon)$ on the standard Heisenberg group $H_n = (H_n, z)$, which we call the **adiabatic Kohn-Rossi Laplacian at** P^0. (Refer to [14] for more information about (generalized) adiabatic operation.) Obviously, also the initial value problem (0.1) relative to $\Box_{(\varepsilon)}$ on H_n has a unique heat kernel $e^{-t\Box_{(\varepsilon)}}(z, z')$, which can be described as

$$e^{-t\Box_{(\varepsilon)}}(z, z') = \sum \theta^{I\bar{K}}_{H}(z) \boxtimes \theta^{\bar{I}'K'}_{H}(z') \cdot \varepsilon^{n+1}\left(e^{-t\varepsilon\Box_{H(P^0)}}\right)^{(I\bar{K})(I'\bar{K}')}(\iota_\varepsilon(z), \iota_\varepsilon(z')) \det V^\bullet(\iota_\varepsilon(z')) \tag{5.2}$$

because of Lemma 5.1 and $dV_{g^{(\varepsilon)}}(z') = dV_{g^H}(z') \det V^\bullet_{(\varepsilon)}(z') = dV_{g^H}(z') \det V^\bullet(\iota_\varepsilon(z'))$. In addition, by setting $\nabla^{(H,\varepsilon)} = I_\varepsilon^* \nabla^{(\varepsilon)}$, $\xi_\bullet^{(\varepsilon)} = I_\varepsilon^* \xi_\bullet^{(\varepsilon)}$, etc., Proposition 1.3 yields the **adiabatic Weitzenböck-type formula**

$$\Box_{(\varepsilon)} = -\sum\left(\nabla^{(H,\varepsilon)}_{\xi^{(\varepsilon)}_\alpha}\nabla^{(H,\varepsilon)}_{\xi^{(\varepsilon)}_{\bar\alpha}} - \nabla^{(H,\varepsilon)}_{\nabla^{(\varepsilon)}_{\xi^{(\varepsilon)}_\alpha}\xi^{(\varepsilon)}_{\bar\alpha}}\right) - \sqrt{-1}\, q\, \nabla^{(H,\varepsilon)}_{\xi^{(\varepsilon)}}$$
$$- \sum F(\nabla^{(\varepsilon)})^C_D(\xi^{(\varepsilon)}_{\bar\alpha}, \xi^{(\varepsilon)}_\beta) \cdot \theta_H^{\bar\alpha} \wedge \theta_H^{\bar\beta} \vee \theta_H^{\bar C} \wedge \theta_H^{\bar D} \vee \quad \text{(acting on } \Omega^{p,q}H_n). \tag{5.3}$$

Notice that we have

$$
(5.4)\qquad
\begin{aligned}
\nabla^{(H,\varepsilon)}_{\xi^{(\varepsilon)}_A} &= \xi^{(\varepsilon)}_A + \sum \varepsilon^{|A|_H/2}\omega^{\bar B}_{C}(\xi_A)(\iota_\varepsilon(z))\cdot\theta^B_H\wedge\theta^C_H\vee\,,\\
\nabla^{(\varepsilon)}_{\xi^{(\varepsilon)}_\alpha}\xi^{(\varepsilon)}_{\bar\alpha} &= \sum \varepsilon^{1/2}\omega^{\bar\beta}_{\bar\alpha}(\xi_\alpha)(\iota_\varepsilon(z))\,\xi^{(\varepsilon)}_{\bar\beta},\\
F(\nabla^{(\varepsilon)})^C_D(\xi^{(\varepsilon)}_{\bar\alpha},\xi^{(\varepsilon)}_\beta) &= \varepsilon^{2/2}F(\nabla)^C_D(\xi_{\bar\alpha},\xi_\beta)(\iota_\varepsilon(z)),
\end{aligned}
$$

which, together with (5.1) and Proposition 2.4, imply the following.

Proposition 5.2 *The differential operator $\square_{(\varepsilon)}$ can be extended smoothly up to $\varepsilon^{1/2}=0$. As to the formal series expansion*

$$
\square_{(\varepsilon)} = \sum_{m=0}^{\infty}\varepsilon^{m/2}\,\square_{m/2},\qquad \square_{0/2}=\mathbf{L}
$$

*which we call the **adiabatic expansion of** $\square_H$ **at** P^0, the coefficients are described as*

$$
(5.5)\qquad \square_{m/2} = \sum_{2+|\mathbb{C}|_H=|\mathbb{B}|_H+m}^{|\mathbb{B}|=0,1,2}\square_{m/2}(\mathbb{B},\mathbb{C})\cdot z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}}\quad (z^{\mathbb{C}}:=z_{C_1}\cdots z_{C_{|\mathbb{C}|}}),
$$

where each $\square_{m/2}(\mathbb{B},\mathbb{C})$ is a finite sum of operators which are the composites of such operators as $\theta^\alpha_H\wedge\theta^\beta_H\vee$, $\theta^{\bar\gamma}_H\wedge\theta^{\bar\delta}_H\vee$ multiplied by constants. If we express its action as

$$
\square_{m/2}(\mathbb{B},\mathbb{C})\,\theta^{I'\bar K'}_H = \sum_{|I|_H=|I'|_H}^{|K|_H=|K'|_H}\square^{(I\bar K)(I'\bar K')}_{m/2}(\mathbb{B},\mathbb{C})\cdot\theta^{I\bar K}_H,
$$

then the coefficients $\square^{(I\bar K)(I'\bar K')}_{m/2}(\mathbb{B},\mathbb{C})$ are all expressed as universal polynomials made of (2.7). Further, one can describe the polynomials explicitly up to an arbitrarily high order.

Proof. When $\varepsilon^{1/2}\to 0$, $\square_{(\varepsilon)}$ tends to

$$
\begin{aligned}
&-\sum\Big((\partial/\partial z_\bullet)\cdot E(-z)\Big)_\alpha\Big((\partial/\partial z_\bullet)\cdot E(-z)\Big)_{\bar\alpha} - \sqrt{-1}\,q\,\Big((\partial/\partial z_\bullet)\cdot E(-z)\Big)_0\\
&= -\sum\xi^H_\alpha\xi^H_{\bar\alpha} - \sqrt{-1}\,q\,\xi^H = \mathbf{L}.
\end{aligned}
$$

It follows from (5.3), (5.4) that we have

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial\varepsilon^{1/2}}\Big)^m\square_{(\varepsilon)} = \sum_{2+|\mathbb{C}|_H\geq|\mathbb{B}|_H+m}^{|\mathbb{B}|=0,1,2}\square^{(\mathbb{B},\mathbb{C})}_{m/2}(\varepsilon^{1/2},\iota_\varepsilon(z))\,z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}},\\
&\square^{(\mathbb{B},\mathbb{C})}_{m/2}(\varepsilon^{1/2},\iota_\varepsilon(z)) = \varepsilon^{(2+|\mathbb{C}|_H-|\mathbb{B}|_H-m)/2}\,\square^{(\mathbb{B},\mathbb{C})}_{m/2}(\iota_\varepsilon(z)),
\end{aligned}
$$

from which we draw the expression (5.5). ∎

The adiabatic series $\Box_{0/2}$, $\Box_{1/2}$, $\ldots$ will reveal various infinitesimal details of M at P^0 which are so condensed that we cannot perceive clearly. The following argument tells us that the series certainly reveal an infinitesimal behavior of the heat kernel at P^0.

Now, suggested by the formula $(\frac{\partial}{\partial t}+\Box_{(\varepsilon)})e^{-t\Box_{(\varepsilon)}}=0$, let us construct a formal power series

$$\mathfrak{p}_{(\varepsilon)}(t,z,z') = \sum_{m=0}^{\infty} \varepsilon^{m/2}\, \mathfrak{p}_{m/2}(t,z,z') \tag{5.6}$$

so as to satisfy $(\frac{\partial}{\partial t}+\Box_{(\varepsilon)})\mathfrak{p}_{(\varepsilon)}=0$. Namely, we define it inductively by

$$\mathfrak{p}_{0/2}(t,z,z') = r_H(t,z,z'), \tag{5.7}$$

$$\begin{aligned} \mathfrak{p}_{m/2}(t,z,z') &= -(\mathfrak{p}_{0/2}\# \sum_{m_1+m_2=m}^{m_1>0} \Box_{m_1/2}\mathfrak{p}_{m_2/2})(t,z,z') \\ &= \sum_{\sum m_\ell=m}^{m_1,\ldots,m_k>0} (-1)^k \Big(\mathfrak{p}_{0/2}\#\,\Box_{m_1/2}\mathfrak{p}_{0/2}\#\,\cdots\#\,\Box_{m_k/2}\mathfrak{p}_{0/2}\Big)(t,z,z') \quad (m>0), \end{aligned} \tag{5.8}$$

where we put $\# = \#_{g^H}$. Then it will be natural to expect (5.6) is a formal series expansion of the heat kernel (5.2). Thus, by setting

$$\mathcal{P}_{(\varepsilon)}(t,z,z') := \mathfrak{p}_{(\varepsilon)}(t,z,z') \det V_\bullet(\iota_\varepsilon(z')) = \sum_{m=0}^{\infty} \varepsilon^{m/2}\, \mathcal{P}_{m/2}(t,z,z')$$

and $\mathcal{P}_{m/2}(t,z,z') = \sum \theta_H^{I\bar K}(z)\boxtimes \theta_H^{\bar I' K'}(z')\cdot \mathcal{P}_{m/2}^{(I\bar K)(I'\bar K')}(t,z,z')$, it will be expected that

$$\varepsilon^{n+1}\Big(e^{-t\varepsilon\Box_{H(P^0)}}\Big)^{(I\bar K)(I'\bar K')}(\iota_\varepsilon(z),\iota_\varepsilon(z')) = \sum_{m=0}^{\infty} \varepsilon^{m/2}\, \mathcal{P}_{m/2}^{(I\bar K)(I'\bar K')}(t,z,z'). \tag{5.9}$$

If it is valid, then we have the asymptotic expansion

$$\Big(e^{-t\Box_{H(P^0)}}\Big)^{(I\bar K)(I'\bar K')}(0,0) \sim \sum_{m=0}^{\infty} t^{-(n+1)+m/2}\, \mathcal{P}_{m/2}^{(I\bar K)(I'\bar K')}(1,0,0), \tag{5.10}$$

that is,

$$a_{m/2}^{(I\bar K)(I'\bar K')}(P^0:\emptyset,\emptyset) = \mathcal{P}_{m/2}^{(I\bar K)(I'\bar K')}(1,0,0) = \mathfrak{p}_{m/2}^{(I\bar K)(I'\bar K')}(1,0,0). \tag{5.11}$$

Hence, the following formulas must be valid:

$$a_{0/2}^{(I\bar K)(I'\bar K')}(P^0:\emptyset,\emptyset) = \delta_{(I\bar K)(I'\bar K')}\,\mathbf{r}_H(1,0,0) = \delta_{(I\bar K)(I'\bar K')}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s), \tag{5.12}$$

$$\begin{aligned} &a_{m/2}^{(I\bar K)(I'\bar K')}(P^0:\emptyset,\emptyset) \\ &= \sum_{\sum m_\ell=m\,(>0)}^{m_1,\ldots,m_k>0} (-1)^k \sum \prod_{\ell=1}^{k} \Box_{m_\ell/2}^{(I^{(\ell-1)}\bar K^{(\ell-1)})(I^{(\ell)}\bar K^{(\ell)})}(\mathbb{B}^{(\ell)},\mathbb{C}^{(\ell)}) \\ &\qquad \times \Big(\mathbf{r}_H\,\#\,z^{\mathbb{C}^{(1)}}(\partial/\partial z)^{\mathbb{B}^{(1)}}\mathbf{r}_H\,\#\,\cdots\#\,z^{\mathbb{C}^{(k)}}(\partial/\partial z)^{\mathbb{B}^{(k)}}\mathbf{r}_H\Big)(1,0,0), \end{aligned} \tag{5.13}$$

where we set $\mathbf{r}_H = \mathbf{r}_H(t,z,z') = r_t^{n-2q}(z'^{-1}z)$. Here, the summation $\sum_{\sum m_\ell = m}^{m_1,\ldots,m_k>0}$ means to sum up all the terms with indices $(m_1,\ldots,m_k)$ satisfying the condition, and the next $\sum$ means, for each $(m_1,\ldots,m_k)$, to sum up all the terms determined by the indices $(\mathbb{B}^{(\ell)},\mathbb{C}^{(\ell)})$ and the sequences of indices $(I\bar{K}) = (I^{(0)}\bar{K}^{(0)})$, $(I^{(1)}\bar{K}^{(1)}),\ldots,(I^{(k)}\bar{K}^{(k)}) = (I'\bar{K}')$ appearing in $\square_{m_\ell/2}$ $(1 \le \ell \le k)$. The term appearing in the third line of (5.13) is the value at $(t,z,z') = (1,0,0)$ of the convolution of the functions $\mathbf{r}_H(t,z,z')$, $z^{\mathbb{C}^{(1)}}(\partial/\partial z)^{\mathbb{B}^{(1)}}\mathbf{r}_H(t,z,z'),\ldots$ with respect to the metric g^H.

Further, if the differentials of the left hand side of (5.9) can be formally expanded into the series of termwise differentials of the right hand side, that is, if

$$\varepsilon^{(n+1)+(|\mathbb{A}|_H+|\mathbb{A}'|_H)/2}\Big((\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\Big(e^{-t\varepsilon\square_{H(P^0)}}\Big)^{(I\bar{K})(I'\bar{K}')}\Big)(\iota_\varepsilon(z),\iota_\varepsilon(z'))$$
$$= \sum_{m=0}^{\infty}\varepsilon^{m/2}\,(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\mathcal{P}_{m/2}^{(I\bar{K})(I'\bar{K}')}(t,z,z'),$$

then, by setting $\mathcal{P}_{m/2}^{(I\bar{K})(I'\bar{K}')}(t,z,z' : \mathbb{A},\mathbb{A}') = (\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\mathcal{P}_{m/2}^{(I\bar{K})(I'\bar{K}')}(t,z,z')$, the formula (5.11) is generalized as follows.

Theorem 5.3 *We have*

$$a_{m/2}^{(I\bar{K})(I'\bar{K}')}(P^0 : \mathbb{A},\mathbb{A}') = \mathcal{P}_{(m+|\mathbb{A}|_H+|\mathbb{A}'|_H)/2}^{(I\bar{K})(I'\bar{K}')}(1,0,0 : \mathbb{A},\mathbb{A}'), \tag{5.14}$$

which vanishes when m is odd. Moreover, this is expressed as a universal polynomial made of (2.7)*, which can be described explicitly by using only a basic knowledge of calculus.*

6 Proof of Theorem 5.3

We wish to prove the following assertion in this section.

Proposition 6.1 *The double form $p_{(\varepsilon)}(t,z,z') := e^{-t\square_{(\varepsilon)}}(z,z')$ can be extended smoothly up to $\varepsilon^{1/2} = 0$. As to the Taylor expansion*

$$p_{(\varepsilon)}(t,z,z') = \sum_{0\le m<m_*}\varepsilon^{m/2}\,p_{m/2}(t,z,z') + \varepsilon^{m_*/2}\,p_{m_*/2}(\varepsilon^{1/2},t,z,z'),$$

we have

$$p_{m/2}(t,z,z') = \mathfrak{p}_{m/2}(t,z,z') \quad (0 \le m < m_*). \tag{6.1}$$

If this is valid, then certainly we have the formal series expansion (5.9). Further, the formula (5.14) holds because the proposition asserts that (5.9) is termwisely differentiable.

Remark 6.2 *Consequently, Proposition* 6.1 *provides the asymptotic expansion* (5.10) *and also that of every differential. Namely, it ascertains Theorems* 3.5 *and* 2.3 *as well.*

We start our discussion with some preparations needed for its proof. We set $\# = \#_{g^H}$, $dV(z) = dV_{g^H}(z)$, $|\cdot| = |\cdot|_{g^H}$, $|\cdot|_{L^i} = |\cdot|_{L^i(z)} = |\cdot|_{L^i_{g^H}(z)}$, etc., if no confusion occurs.

6.1 Standard kernels on H_n

The argument in §4 holds good for the standard (H_n, z) because it may be regarded naturally as a warped Heisenberg group. A kernel on H_n whose coefficients consist of $t^{-n-2+b/2}\rho(z', z)\,\mathcal{K}(\iota_{1/t}(z'^{-1}z))$ will be called a standard kernel, where $|\xi^H_{\mathbb{A},z}\xi^H_{\mathbb{A}',z'}\rho(z', z)|$ is assumed to be bounded for any $(\mathbb{A}, \mathbb{A}')$ (see (4.13)). Obviously Lemma 4.4 holds also for the standard kernels on H_n and we have:

Lemma 6.3 *Let k_i be standard kernels of types $m_i\,(\geq 1)$. Then the convolution $(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j)(t, z, z')$ is well-defined and smooth on $(0,\infty)\times H_n\times H_n$, and there exist constants $b' > 0$, $B'(\ell) > 0$ ($\ell = 0$ or $\ell \geq 2n+2$) and an integer $N > 0$ such that*

$$(6.2)\quad \left\|(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(t, z, z')\right\|_{L^1(z)} \leq b't^{\sum m_i/2-N-d-1}\sum|z'^{\mathbb{C}'}|,$$

$$(6.3)\quad \left|(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(t, z, z')\right| \leq B'(\ell)\,t^{\sum m_i/2-N-d+\ell/2-(n+2)}|z'^{-1}z|_H^{-\ell}\sum|z'^{\mathbb{C}'}|$$

on $(0, T_0]\times H_n\times H_n$, where $\sum|z'^{\mathbb{C}'}|$ which is a finite sum depends on $(\mathbb{A}, \mathbb{A}')$ and $(\mathbb{C}_i, \mathbb{B}_i)$ $(1 \leq i \leq j)$ and so do the constants $b' > 0$, etc., and the integer N. Moreover, for every $\varphi \in \Omega^{p,q}_0 H_n$, the integral $\int(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j)(t, z, z')\wedge\star\varphi(z')$ is well-defined and rapidly decreasing, and there exists a semi-norm $\mathrm{sn}(\cdot)$ *such that, for any $\varphi \in \Omega^{p,q}_0 H_n$, we have*

$$\left|\int\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(t, z, z')\wedge\star\varphi(z')\right| \leq t^{\sum m_i/2-1}\mathrm{sn}(\varphi),$$

$$(6.4)\quad \left\|\int\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(t, z, z')\wedge\star\varphi(z')\right\|_{L^k} \leq t^{\sum m_i/2-1}\mathrm{sn}(\varphi),$$

where $k = 1, 2$.

Proof. Lemma 4.3(4) for H_n implies

$$
\begin{aligned}
(6.5)\qquad & z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}} z'^{\mathbb{C}'}(\partial/\partial z')^{\mathbb{B}'} t^{-n-2+b/2}\mathcal{K}(\iota_{1/t}(z'^{-1}z)) \\
&= \sum_{|\tilde{\mathbb{B}}|\le|(\mathbb{B},\mathbb{B}')|} z^{\tilde{\mathbb{C}}}(\partial/\partial z)^{\tilde{\mathbb{B}}}\, t^{-n-2+b/2+\ell/2}\mathcal{K}(\iota_{1/t}(z'^{-1}z)) \\
&= \sum_{|\tilde{\mathbb{B}}'|\le|(\mathbb{B},\mathbb{B}')|} z'^{\tilde{\mathbb{C}}'}(\partial/\partial z')^{\tilde{\mathbb{B}}'}\, t^{-n-2+b/2+\ell'/2}\mathcal{K}(\iota_{1/t}(z'^{-1}z)),
\end{aligned}
$$

where the rapidly decreasing functions $\mathcal{K}(\Theta)$ appearing in the second and third lines, which differ from that in the first line, depend on the respective indices $(\tilde{\mathbb{B}}, \tilde{\mathbb{C}}, \ell)$, etc. It follows from (6.5) and integration by parts that we have

$$
\begin{aligned}
&(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(t,z,z') \\
&= \sum_{|\mathbb{B}'|\le|(\mathbb{A},\mathbb{A}')|+\sum|\mathbb{B}_i|} (\partial/\partial t)^d z^{\mathbb{C}'}(\partial/\partial z')^{\mathbb{B}'}\Big(k_1\#\cdots\#k_j\Big)(t,z,z') \\
&= \sum_{|\mathbb{B}''|\le|(\mathbb{A},\mathbb{A}')|+\sum|\mathbb{B}_i|} (\partial/\partial t)^d z^{\mathbb{C}''}\xi^H_{\mathbb{B}'',z'}\Big(k_1\#\cdots\#k_j\Big)(t,z,z'),
\end{aligned}
$$

where, again, the kernels k_i appearing in the second and third lines, which differ from that in the first line, depend on the respective indices, but are of the same types as those of the original k_i. Hence, with the use of Lemma 4.4 for H_n and the argument in the proof of Lemma 2.6, we obtain the lemma. ∎

From now on, as above, all the rapidly decreasing functions are usually expressed as $\mathcal{K}(\Theta)$ with no distinction to simplify the description.

6.2 Rough estimation of remainder term

Let us set $r(t,z,z') = \sum \theta^{I\bar{K}}(z)\boxtimes\theta^{\bar{I}'K'}(z')\cdot r^{(I\bar{K})(I'\bar{K}')}(t,z,z') = \sum \theta^{I\bar{K}}(z)\boxtimes\theta^{\bar{I}'K'}(z')\cdot \rho_\circ(z',z)\, r_t^{n-2q}(\Theta^\circ(z',z))$ ($\circ = M$ or $\mathbf{L}$) (see (4.1)) and

$$
\begin{aligned}
r_{(\varepsilon)}(t,z,z') &= \sum \theta_H^{I\bar{K}}(z)\boxtimes\theta_H^{\bar{I}'K'}(z')\cdot\varepsilon^{n+1}r^{(I\bar{K})(I'\bar{K}')}(t\varepsilon,\iota_\varepsilon(z),\iota_\varepsilon(z'))\det V^\bullet(\iota_\varepsilon(z')) \\
&= \sum \theta_H^{I\bar{K}}(z)\boxtimes\theta_H^{\bar{I}'K'}(z')\cdot\rho_\circ(\iota_\varepsilon(z'),\iota_\varepsilon(z))\, r_t^{n-2q}(\Theta^{\circ(\varepsilon)}(z',z))\det V^\bullet(\iota_\varepsilon(z')),
\end{aligned}
$$

where we put $\Theta^{\circ(\varepsilon)}(z',z) = \iota_{1/\varepsilon}\Theta^\circ(\iota_\varepsilon(z'),\iota_\varepsilon(z))$. Further, let us set $q_{(\varepsilon)}(t,z,z') = (\frac{\partial}{\partial t} + \square_{(\varepsilon)})r_{(\varepsilon)}(t,z,z')$ and $q^1_{(\varepsilon)} = q_{(\varepsilon)}$, $q^2_{(\varepsilon)} = q_{(\varepsilon)}\#q^1_{(\varepsilon)}$, $q^3_{(\varepsilon)} = q_{(\varepsilon)}\#q^2_{(\varepsilon)}, \ldots$ inductively. Then we have

$$
(6.6)\qquad p_{(\varepsilon)}(t,z,z') = \sum_{k=0}^{\infty}(-1)^k(r_{(\varepsilon)}\#q^k_{(\varepsilon)})(t,z,z') \quad (r_{(\varepsilon)}\#q^0_{(\varepsilon)} := r_{(\varepsilon)}).
$$

Notice that the coefficient of the remainder term $R_{k_0}(p_{(\varepsilon)}) := \sum_{k\geq k_0}(-1)^k r_{(\varepsilon)}\#q^k_{(\varepsilon)}$ can be described as

$$(6.7)\quad R_{k_0}(p_{(\varepsilon)})^{(I\bar{K})(I'\bar{K}')}(t,z,z') = \varepsilon^{n+1}R_{k_0}(p)^{(I\bar{K})(I'\bar{K}')}(t\varepsilon,\iota_\varepsilon(z),\iota_\varepsilon(z'))\det V^\bullet(\iota_\varepsilon(z')).$$

Lemma 6.4 *There exists a constant $C'(k_0,\ell)>0$ ($\ell=0$ or $\ell\geq 2n+2$) such that*

$$\left|(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}(\frac{\partial}{\partial\varepsilon^{1/2}})^m R_{k_0}(p_{(\varepsilon)})(t,z,z')\right|$$
$$\leq \varepsilon^{k_0/2-m/2+\ell/2}C'(k_0,\ell)\,t^{k_0/2-d-|(\mathbb{A},\mathbb{A}')|_H/2-m+\ell/2-(n+1)}\delta^{(\varepsilon)}(z',z)^{-\ell}\sum|z^{\mathbb{C}}z'^{\mathbb{C}'}|$$

on $(0,\varepsilon_0^{1/2}]\times(0,T_0]\times H_n\times H_n$, where we set $\delta^{(\varepsilon)}(z',z)=\iota_{1/\varepsilon}\delta(\iota_\varepsilon(z'),\iota_\varepsilon(z))$. Here the finite sum $\sum|z^{\mathbb{C}}z'^{\mathbb{C}'}|$ depends on the choice of $(d,\mathbb{A},\mathbb{A}')$.

Proof. The differential of the right hand side of (6.7) but with ε^{n+1} removed by the differentiation $(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}(\frac{\partial}{\partial\varepsilon^{1/2}})^{m'}$ can be described as

$$\sum_{d'+|(\mathbb{B},\mathbb{B}')|_H/2\leq d+|(\mathbb{A},\mathbb{A}')|_H/2+m'}\varepsilon^{d'+|(\mathbb{B},\mathbb{B}')|_H/2-m'/2}h(\varepsilon^{1/2},t,z,z')\,\mathcal{B}(\iota_\varepsilon(z'),\iota_\varepsilon(z))$$
$$\times\Big((\partial/\partial t)^{d'}\xi_{\mathbb{B},z}\xi_{\mathbb{B}',z'}R_{k_0}(p)\Big)^{(I\bar{K})(I'\bar{K}')}(t\varepsilon,\iota_\varepsilon(z),\iota_\varepsilon(z')),$$

where $h(\varepsilon^{1/2},t,z,z')$ is a polynomial and the absolute value of the function $\mathcal{B}(w',w)$ is bounded on $H_n\times H_n$ (refer to Lemma 3.3). Hence, (4.11) implies the lemma. ∎

6.3 Detailed investigation of the term $(-1)^k(r_{(\varepsilon)}\#_{g^H}q^k_{(\varepsilon)})(t,z,z')$

We wish to investigate each term $(-1)^k r_{(\varepsilon)}\#_{g^H}q^k_{(\varepsilon)}$ appearing in (6.6) closely.

Lemma 6.5 *The system $\Theta^{M(\varepsilon)}(z',z)$ ($=\Theta^{\nabla^{(\varepsilon)}}(z',z)$) can be extended smoothly up to the domain $\operatorname{dom}\Theta^{M(\bullet)} := \{(\varepsilon^{1/2},z',z)\in[0,\varepsilon_0^{1/2}]\times H_n\times H_n \mid (z',z)\in V^{(\varepsilon)}\}$ and so can be the system $\Theta^{\mathbf{L}(\varepsilon)}(z',z)$ up to the domain $\operatorname{dom}\Theta^{\mathbf{L}(\bullet)} := [0,\varepsilon_0^{1/2}]\times H_n\times H_n$. The extended ones provide*

$$(6.8)\quad \Theta^{\circ(\varepsilon)}_B\Big|_{\varepsilon^{1/2}=0} = (z'^{-1}z)_B,$$

$$(6.9)\quad (\frac{\partial}{\partial\varepsilon^{1/2}})^m\Theta^{\circ(\varepsilon)}_B = \sum^{\text{(finite)}}_{|(\mathbb{C}',\mathbb{D})|_H\geq m+|B|_H,\ |\mathbb{D}|>0} z'^{\mathbb{C}'}(z'^{-1}z)^{\mathbb{D}}\,\Theta^{\circ,m/2}_B(\varepsilon^{1/2},\iota_\varepsilon(z'),\iota_\varepsilon(z)),$$
$$\Theta^{\circ,m/2}_B(\varepsilon^{1/2},\iota_\varepsilon(z'),\iota_\varepsilon(z)) = \varepsilon^{(|(\mathbb{C}',\mathbb{D})|_H-m-|B|_H)/2}\,\Theta^{\circ,m/2}_B(\iota_\varepsilon(z'),\iota_\varepsilon(z)),$$

where the coefficients $\Theta^{\circ,m/2}_B(\varepsilon^{1/2},\iota_\varepsilon(z'),\iota_\varepsilon(z))(=\Theta^{\circ,m/2}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(z'),\iota_\varepsilon(z)))$ are smooth on $\operatorname{dom}\Theta^{\circ(\bullet)}$ and bounded in the sense: In the case $\Theta^{\circ(\varepsilon)}=\Theta^{M(\varepsilon)}$, their differentials by every (high order) differentiation relative to the variables (z',z) are bounded on

$\mathrm{dom}_r\,\Theta^{M(\bullet)} := \{(\varepsilon^{1/2}, z', z) \in \mathrm{dom}\,\Theta^{M(\bullet)} \mid |w(\iota_\varepsilon(z'))| \le 2r, |w(\iota_\varepsilon(z))| \le 2r\}$ *and, in the case* $\Theta^{\circ(\varepsilon)} = \Theta^{\mathbf{L}(\varepsilon)}$*, so are their differentials on* $\mathrm{dom}\,\Theta^{\mathbf{L}(\bullet)}$*.*

Proof. The function $\Theta^\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))$ is obviously smooth on $\mathrm{dom}\,\Theta^{\circ(\bullet)}$. As $\varepsilon^{1/2} \to 0$, $\Theta^{M(\varepsilon)}(z', z)$ tends to $z'^{-1}z$ (by (2.15)) and also $\Theta^{\mathbf{L}(\varepsilon)}(z', z) = \iota_{1/\varepsilon}(w(\iota_\varepsilon(z'))^{-1}w(\iota_\varepsilon(z)))$ tends to $z'^{-1}z$. Thus $\Theta^{\circ(\varepsilon)}(z', z)$ can be extended smoothly up to $\mathrm{dom}\,\Theta^{\circ(\bullet)}$ and (6.8) holds. It follows readily from the differentiation rule and the property $\Theta^{\circ(\varepsilon)}_B(z', z') = 0$ that (6.9) but with the boundedness condition ignored holds. In the case $\Theta^{\circ(\varepsilon)} = \Theta^{\mathbf{L}(\varepsilon)}$, we will need to further expand the coefficients into finite Taylor series (with remainder terms) so as to satisfy the boundedness condition (see (3.13)). ∎

The lemma asserts that one can express $\Theta^{\circ(\varepsilon)}_B$ as a finite sum of functions of the variables $(z', z'^{-1}z)$, and, conversely, express $(z'^{-1}z)_D$ as a finite sum of functions of the variables $(z', \Theta^{\circ(\varepsilon)})$ as follows:

$$\begin{aligned}
&\Theta^{\circ(\varepsilon)}_B = \sum_{|(\mathbb{C}',\mathbb{D})|_H \ge |B|_H,\ |\mathbb{D}|>0} z'^{\mathbb{C}'}(z'^{-1}z)^{\mathbb{D}}\,\mathcal{B}_B(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z)),\\
&\mathcal{B}_B(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z)) = \varepsilon^{(|(\mathbb{C}',\mathbb{D})|_H - |B|_H)/2}\mathcal{B}_B(\iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z)),\\
&(z'^{-1}z)_D = \sum_{|(\mathbb{C}',\mathbb{B})|_H \ge |D|_H,\ |\mathbb{B}|>0} z'^{\mathbb{C}'}(\Theta^{\circ(\varepsilon)})^{\mathbb{B}}\,\mathcal{B}_D(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}),\\
&\mathcal{B}_D(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) = \varepsilon^{(|(\mathbb{C}',\mathbb{B})|_H - |D|_H)/2}\mathcal{B}_D(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}).
\end{aligned} \tag{6.10}$$

Here, the coefficients $\mathcal{B}_B(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z))\ (= \mathcal{B}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z)))$ are smooth on $\mathrm{dom}\,\Theta^{\circ(\bullet)}$ and quasi-bounded in the sense: Their differentials by every (high order) differentiation relative to the variables $(\varepsilon^{1/2}, z', z)$ are described as finite sum of such functions as $z'^{\mathbb{C}''}(z'^{-1}z)^{\mathbb{D}'}\tilde{\mathcal{B}}(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z))$, where the functions $\tilde{\mathcal{B}}(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z))$ are bounded on $\mathrm{dom}_r\,\Theta^{M(\bullet)}$ when $\Theta^{\circ(\varepsilon)} = \Theta^{M(\varepsilon)}$, and bounded on $\mathrm{dom}\,\Theta^{\mathbf{L}(\bullet)}$ when $\Theta^{\circ(\varepsilon)} = \Theta^{\mathbf{L}(\varepsilon)}$. Also the coefficients $\mathcal{B}_D(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)})$ are smooth on $\mathrm{dom}\,\Theta^{\circ(\bullet)}$ and quasi-bounded in similar sense.

In general, if we regard a quasi-bounded function $\mathcal{B}(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon(z'^{-1}z))$ naturally as a function of $(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)})$, a function of $(\varepsilon^{1/2}, \iota_\varepsilon(z), \iota_\varepsilon(z^{-1}z'))$ and a function of $(\varepsilon^{1/2}, \iota_\varepsilon(z), \iota_\varepsilon\Psi^{\circ(\varepsilon)})$ $(\Psi^{\circ(\varepsilon)}(z, z') := -\Theta^{\circ(\varepsilon)}(z', z))$, then the respective functions are quasi-bounded in the respective senses. Similar assertions are valid also for the other quasi-bounded functions $\mathcal{B}(\varepsilon^{1/2}, \iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)})$, etc. In the following we may express a quasi-bounded function simply as $\mathcal{B}(\varepsilon^{1/2})$ if no confusion occurs.

Now, for a kernel $k(t,z,z') = \sum \theta^{I\bar{K}}(z) \boxtimes \theta^{\bar{I}'K'}(z') \cdot \mathbf{K}^b_\circ(t,z,z')$ (see (4.13)), we set $k_{(\varepsilon)}(t,z,z') = \sum \theta^{I\bar{K}}_H(z) \boxtimes \theta^{\bar{I}'K'}_H(z') \cdot \mathbf{K}^b_{\circ(\varepsilon)}(t,z,z')$ with

$$\mathbf{K}^b_{\circ(\varepsilon)}(t,z,z') = t^{-n-2+b/2} \rho_\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z',z)),$$

which we call an (ε)-kernel of type b. Note that by Lemma 3.3 the kernel $r(t,z,z')$ multiplied by $\det V^\bullet(z')$ is still of type 2, so that $r_{(\varepsilon)}(t,z,z')$ is an (ε)-kernel of type 2.

Lemma 6.6

(1) *The function $\mathbf{K}^b_{\circ(\varepsilon)}(t,z,z')$ can be extended smoothly up to $\varepsilon^{1/2} = 0$ and has a Taylor expansion*

$$(6.11)\qquad \begin{aligned} \mathbf{K}^b_{\circ(\varepsilon)} &= \sum_{0 \le m < m_*} \varepsilon^{m/2}\, \mathbf{K}^b_{\circ,m/2} + \varepsilon^{m_*/2}\, \mathbf{K}^b_{\circ,m_*/2}(\varepsilon^{1/2}), \\ \mathbf{K}^b_{M,0/2} &= t^{-n-2+b/2} \rho_M(0,0)\, \mathcal{K}(\iota_{1/t}(z'^{-1}z)), \quad \mathbf{K}^b_{\mathbf{L},0/2} = 0. \end{aligned}$$

Further, there exist finite sum expressions ($\ell \ge 0$)

$$(6.12)\qquad \mathbf{K}^b_{M,m/2} = \sum z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}}\, t^{-n-2+b/2+\ell/2} \mathcal{K}(\iota_{1/t}(z'^{-1}z)), \quad \mathbf{K}^b_{\mathbf{L},m/2} = 0,$$

$$(6.13)\qquad \mathbf{K}^b_{\circ,m_*/2}(\varepsilon^{1/2}) = \sum z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}}\, t^{-n-2+b/2+\ell/2} \Big[\mathcal{B}(\varepsilon^{1/2})\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}.$$

Here, in general, we set $[f(\delta,\ldots)]^\delta_m = \int_0^1 d\sigma_1 \cdots \int_0^{\sigma_{m-1}} d\sigma_m f(\sigma_m \delta, \ldots)\mathrm{poly}(\sigma_m)$, *where* $\mathrm{poly}(\sigma_m)$ *is a polynomial of* σ_m. *(At* (6.13) *we may set* $\mathrm{poly}(\sigma_{m_*}) = 1$.*) The functions* $\mathcal{K}(\Theta)\,(= \mathcal{K}_{(\ell,\mathbb{C},\mathbb{B})}(\Theta))$ *are rapidly decreasing and* $\mathcal{B}(\varepsilon^{1/2})\,(= \mathcal{B}_{(\ell,\mathbb{C},\mathbb{B})}(\varepsilon^{1/2}))$ *are quasi-bounded.*

(2) (cf. (6.5)) *We have*

$$(6.14)\qquad \begin{aligned} & z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} z'^{\mathbb{C}'} (\partial/\partial z')^{\mathbb{B}'} (\partial/\partial \varepsilon^{1/2})^m\, t^{-n-2+b/2} \Big[\mathcal{B}(\varepsilon^{1/2})\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*} \\ &= \sum_{|\tilde{\mathbb{B}}| \le |(\mathbb{B},\mathbb{B}')| + m} z^{\tilde{\mathbb{C}}} (\partial/\partial z)^{\tilde{\mathbb{B}}}\, t^{-n-2+b/2+\ell/2} \Big[\mathcal{B}(\varepsilon^{1/2})\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*} \\ &= \sum_{|\tilde{\mathbb{B}}'| \le |(\mathbb{B},\mathbb{B}')| + m} z'^{\tilde{\mathbb{C}}'} (\partial/\partial z')^{\tilde{\mathbb{B}}'}\, t^{-n-2+b/2+\ell'/2} \Big[\mathcal{B}(\varepsilon^{1/2})\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}, \end{aligned}$$

where $\mathcal{B}(\varepsilon^{1/2})$, $\mathcal{K}(\Theta)$ appearing in the second and third lines depend on the respective indices.

Proof. By Lemma 6.5, certainly $\mathbf{K}^b_{\circ(\varepsilon)}$ can be extended smoothly. Further, by recalling the location of $\operatorname{supp}\rho_\circ$, the second line of (6.11) and the second identity at (6.12) will be obvious. Let us show the first one at (6.12). We assume $m > 0$. Lemma 6.5 says that $\mathbf{K}^b_{M,m/2}$ can be expressed as a finite sum of such functions as $t^{-n-2+b/2}z'^{\mathbb{C}_1}(z'^{-1}z)^{\mathbb{D}}(\partial/\partial w)^{\mathbb{B}}\mathcal{K}(\iota_{1/t}(w))$ $(w := z'^{-1}z)$. We can alter the function $z'^{\mathbb{C}_1}(z'^{-1}z)^{\mathbb{D}}(\partial/\partial w)^{\mathbb{B}}\mathcal{K}(\iota_{1/t}(w))$ successively as follow:

$$
\begin{aligned}
&z'^{\mathbb{C}_1}(z'^{-1}z)^{\mathbb{D}}(\partial/\partial w)^{\mathbb{B}}\mathcal{K}(\iota_{1/t}(w)) \Rightarrow z'^{\mathbb{C}_1}(\partial/\partial w)^{\mathbb{B}_1}\Big(w^{\mathbb{D}_1}\mathcal{K}_1(\iota_{1/t}(w))\Big)\\
&\Rightarrow t^{|\mathbb{D}_1|_H/2}z'^{\mathbb{C}_1}(\partial/\partial w)^{\mathbb{B}_1}\mathcal{K}_2(\iota_{1/t}(w)) \Rightarrow t^{|\mathbb{D}_1|_H/2}z'^{\mathbb{C}_2}(\partial/\partial z)^{\mathbb{B}_2}\mathcal{K}_2(\iota_{1/t}(z'^{-1}z))\\
&\Rightarrow t^{|\mathbb{D}_1|_H/2}(\partial/\partial z)^{\mathbb{B}_2}\Big(z^{\mathbb{C}_3}(z^{-1}z')^{\mathbb{D}_3}\mathcal{K}_2(\iota_{1/t}(z'^{-1}z))\Big)\\
&\Rightarrow t^{|(\mathbb{D}_1,\mathbb{D}_3)|_H/2}z^{\mathbb{C}_4}(\partial/\partial z)^{\mathbb{B}_3}\mathcal{K}_4(\iota_{1/t}(z'^{-1}z)).
\end{aligned}
\tag{6.15}
$$

Thus we obtain (6.12). Next, let us show (6.13). Taylor's integral formula implies that the remainder term $\mathbf{K}^b_{\circ,m_*/2}(\varepsilon^{1/2})$ can be expressed as a finite sum of such functions as $t^{-n-2+b/2}\Big[z'^{\mathbb{C}'}(\Theta^{\circ(\varepsilon)})^{\mathbb{A}}\mathcal{B}(\varepsilon^{1/2})(\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{B}}\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}$. In the successive alterations at (6.15), the change of variables $(z', z'^{-1}z) \Leftrightarrow (z', z)$ was used. Here, using the changes of variables $(z', \Theta^{\circ(\varepsilon)}) \Leftrightarrow (z', z'^{-1}z) \Leftrightarrow (z', z)$ (see (6.10)), similarly we obtain (6.13). As to (2): By Lemma 4.3(4) (for $\xi^{(\varepsilon)}_{A,z}$), we have

$$
\begin{aligned}
&z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}}\,t^{-n-2+b/2}\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\\
&\quad= \sum_{|\tilde{\mathbb{B}}'|\le|\mathbb{B}|} z'^{\tilde{\mathbb{C}}'}(\partial/\partial z')^{\tilde{\mathbb{B}}'}\,t^{-n-2+b/2+\ell'/2}\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}).
\end{aligned}
$$

In addition, obviously we have

$$
(\partial/\partial\varepsilon^{1/2})\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*} = \Big[(\partial/\partial\varepsilon^{1/2})\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}.
$$

Thus, recalling the action of $\partial/\partial\varepsilon^{1/2}$ on $\mathcal{B}(\varepsilon^{1/2})$ and $\Theta^{\circ(\varepsilon)}$ (Lemma 6.5 and (6.10)), we obtain (6.14). ∎

Lemma 6.7 *Let us set*

$$
\begin{aligned}
\mathbf{k}_i &= \sum \theta^{I\bar{K}}_H(z)\boxtimes\theta^{\bar{I}'K'}_H(z')\cdot t^{-n-2+m_i/2}\mathcal{K}(\iota_{1/t}(z'^{-1}z)),\\
\mathbf{k}_i(\varepsilon^{1/2}) &= \sum \theta^{I\bar{K}}_H(z)\boxtimes\theta^{\bar{I}'K'}_H(z')\cdot t^{-n-2+m_i/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{n_i},
\end{aligned}
\tag{6.16}
$$

where $m_i \ge 1$ and $n_i \ge 0$. Then, Lemma 6.3 still holds even if we change each standard kernel k_i into $\mathbf{k}_i$ or $\mathbf{k}_i(\varepsilon^{1/2})$ arbitrarily. Further, the estimates (6.2), (6.3) can be generalized to

$$
\begin{aligned}
&\Big\|(\partial/\partial t)^d(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}(\partial/\partial\varepsilon^{1/2})^m\\
&\qquad\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}k_1\#\cdots\#z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j}k_j\Big)(\varepsilon^{1/2},t,z,z')\Big\|_{L^1}
\end{aligned}
$$

$$\leq b' t^{\sum m_i/2 - N - d - 1} \sum |z'^{\mathbb{C}'}|,$$

$$\begin{aligned}(6.17)\qquad &\Big|(\partial/\partial t)^d (\partial/\partial z)^{\mathbb{A}} (\partial/\partial z')^{\mathbb{A}'} (\partial/\partial \varepsilon^{1/2})^m \\ &\Big(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k_1 \# \cdots \# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k_j\Big)(\varepsilon^{1/2}, t, z, z')\Big| \\ &\leq B'(\ell)\, t^{\sum m_i/2 - N - d + \ell/2 - (n+2)} \delta^{(\bullet)}(z', z))^{-\ell} \sum |z'^{\mathbb{C}'}|,\end{aligned}$$

where $k_i = \mathbf{k}_i$ *or* $\mathbf{k}_i(\varepsilon^{1/2})$*, and* $\delta^{(\bullet)}(z', z) := \min_{\varepsilon^{1/2} \in [0, \varepsilon_0^{1/2}]} \delta^{(\varepsilon)}(z', z)$*.*

Proof. Added to (6.5), we have (6.14). Hence the lemma will be proved in the same way as Lemma 6.3. ∎

Lemma 6.8 *Suppose* $k_{i,(\varepsilon)}$ *are* (ε)*-kernels of types* b_i (≥ 1)*. Then, the convolution* $k_{1,(\varepsilon)} \# \cdots \# k_{j,(\varepsilon)}$ *can be extended smoothly up to* $\varepsilon^{1/2} = 0$*.*

Proof. By Lemma 6.6, each $k_{i,(\varepsilon)}$ is extended smoothly up to $\varepsilon^{1/2} = 0$. Let us denote its expansion by $k_{i,(\varepsilon)}$ $(= k_{i,0/2}(\varepsilon^{1/2})) = \sum_{0 \leq m < m_*} \varepsilon^{m/2}\, k_{i,m/2} + \varepsilon^{m_*/2}\, k_{i,m_*/2}(\varepsilon^{1/2})$, where $k_{i,m/2}$, $k_{i,m_*/2}(\varepsilon^{1/2})$ are expressed as (6.16) with (m_i, n_i) replaced by $(b_i + \ell_i, m_*)$ $(\ell_i \geq 0)$. Then, $k_{\#,(\varepsilon)} := k_{1,(\varepsilon)} \# \cdots \# k_{j,(\varepsilon)}$ is described as

$$\begin{aligned} k_{\#,(\varepsilon)} &= \sum_{0 \leq m < m_*} \varepsilon^{m/2} k_{\#,m/2} + \varepsilon^{m_*/2} k_{\#,m_*/2}(\varepsilon^{1/2}) \\ &= \sum_{0 \leq m < m_*} \varepsilon^{m/2} \sum^{\sum m_i = m} k_{1,m_1/2} \# \cdots \# k_{j,m_j/2} \\ &\quad + \varepsilon^{m_*/2} \sum_{m_i > 0 = m_{i+1} = \cdots = m_j}^{\sum m_i = m_*} k_{1,m_1/2} \# \cdots \# k_{i-1,m_{i-1}/2} \# k_{i,m_i/2}(\varepsilon^{1/2}) \# \cdots \# k_{j,m_j/2}(\varepsilon^{1/2}), \end{aligned}$$

where, by Lemma 6.3, $k_{\#,m/2}(t, z, z')$ is well-defined and smooth on $(0, \infty) \times H_n \times H_n$. Further, it follows from (6.17) with $\ell = 0$ that also $k_{\#,m_*/2}(\varepsilon^{1/2}, t, z, z')$ is well-defined and smooth on $(0, \varepsilon_0^{1/2}] \times (0, \infty) \times H_n \times H_n$, and is estimated as

$$\begin{aligned} &\Big|(\partial/\partial t)^d (\partial/\partial z)^{\mathbb{A}} (\partial/\partial z')^{\mathbb{A}'} (\partial/\partial \varepsilon^{1/2})^m k_{\#,m_*/2}(\varepsilon^{1/2}, t, z, z')\Big| \\ &\qquad \leq B'\, t^{\sum b_i/2 - N - d - (n+2)} \sum |z'^{\mathbb{C}'}| \end{aligned}$$

on $(0, \varepsilon_0^{1/2}] \times (0, T_0] \times H_n \times H_n$. Hence, the term $\varepsilon^{m_*/2} k_{\#,m_*/2}(\varepsilon^{1/2})$ can be extended up to $\varepsilon^{1/2} = 0$ so as to be of class $C^{m_* - 1}$ by claiming that its differentials up to the order $m_* - 1$ relative to the variables $(\varepsilon^{1/2}, t, z, z')$ are equal to 0 at $\varepsilon^{1/2} = 0$. Namely, $k_{\#,(\varepsilon)}$ can be extended up to $\varepsilon^{1/2} = 0$ so as to be of class $C^{m_* - 1}$. Since m_* can be chosen arbitrarily large, certainly it can be extended smoothly up to $\varepsilon^{1/2} = 0$. ∎

Now we can show the desired assertion.

Lemma 6.9 *Each term $(-1)^k(r_{(\varepsilon)}\#q^k_{(\varepsilon)})(t,z,z')$ can be extended smoothly up to $\varepsilon^{1/2}=0$ and has a series expansion*

$$(6.18)\qquad (-1)^k r_{(\varepsilon)}\#q^k_{(\varepsilon)} = \sum_{k\le m<m_*} \varepsilon^{m/2} p^k_{m/2} + \varepsilon^{m_*/2} p^k_{m_*/2}(\varepsilon^{1/2}), \quad p^0_{0/2} = r_H.$$

Further, for every $\varphi \in \Omega^{p,q}_0 H_n$, the integrals $\int p^k_{m/2}(t,z,z')\wedge\star\varphi(z')$, $\int p^k_{m_*/2}(\varepsilon^{1/2},t,z,z') \wedge \star\,\varphi(z')$ are well-defined and smooth on $[0,\varepsilon_0^{1/2}]\times[0,\infty)\times H_n$ $(\ni(\varepsilon^{1/2},t^{1/2},z))$, and there exists a semi-norm $\mathrm{sn}(\cdot)$ such that, for any $\varphi\in\Omega^{p,q}_0 H_n$, we have

$$(6.19)\qquad \left\|\int p^0_{0/2}(t,z,z')\wedge\star\varphi(z') - \varphi(z)\right\|_{L^2} \le t^{1/2}\,\mathrm{sn}(\varphi),$$

$$(6.20)\qquad \left\|\int p^0_{m/2}(t,z,z')\wedge\star\varphi(z')\right\|_{L^2} \le t^{1/2}\,\mathrm{sn}(\varphi) \quad (m>0),$$

$$(6.21)\qquad \left\|\int p^k_{m/2}(t,z,z')\wedge\star\varphi(z')\right\|_{L^2} \le t^{k/2}\,\mathrm{sn}(\varphi) \quad (k>0)$$

when $0\le t^{1/2}\le T_0^{1/2}$.

Proof. Note that $r_{(\varepsilon)}$ is an (ε)-kernel of type 2 and $q_{(\varepsilon)}$ has a finite sum expression $q_{(\varepsilon)} = \sum \varepsilon^{b/2}\,\tilde q_{b,(\varepsilon)}$, where each $\tilde q_{b,(\varepsilon)}$ is an (ε)-kernel of type $b\,(\ge 1)$. Thus we have

$$(-1)^k\, r_{(\varepsilon)}\#q^k_{(\varepsilon)} = (-1)^k \sum_{b_i\ge 1} \varepsilon^{\sum b_i/2}\, r_{(\varepsilon)}\#\tilde q_{b_1,(\varepsilon)}\#\cdots\#\tilde q_{b_k,(\varepsilon)},$$

which, together with Lemma 6.8, ascertains the first half of the lemma. Next, let us examine the integrals $\int p^0_{m/2}(t,z,z')\wedge\star\varphi(z')$ $(m\ge 0)$. It suffices to consider (6.18) with $k=0$, i.e., $r_{(\varepsilon)} = \sum_{0\le m<m_*}\varepsilon^{m/2} r_{m/2} + \varepsilon^{m_*/2} r_{m_*/2}(\varepsilon^{1/2})$. The coefficients of $r_{m/2}$, $r_{m_*/2}(\varepsilon^{1/2})$ consist of such functions as $\mathbf{K}^2_{\circ,m/2}$, $\mathbf{K}^2_{\circ,m_*/2}(\varepsilon^{1/2})$ (see Lemma 6.6(1)), respectively. Hence, with reference to (6.5) and Lemma 6.6(2), using integration by parts and then changing the variables in the same way as (2.11), we obtain the finite sum expressions

$$\begin{aligned}
&\int dV(z')\, r^{(I\bar K)(I'\bar K')}_{m/2}(t,z,z')\,\varphi^{I'\bar K'}(z') \\
&\quad = \sum_{\ell\ge 0} t^{\ell/2}\int dV(z')\,\mathcal{K}(z')\,(\partial/\partial w)^{\mathbb{B}'}\Big(w^{\mathbb{C}'}\varphi^{I'\bar K'}(w)\Big)\Big|_{w=z(-\iota_t(z'))}, \\
&\int dV(z')\, r^{(I\bar K)(I'\bar K')}_{m_*/2}(\varepsilon^{1/2},t,z,z')\,\varphi^{I'\bar K'}(z') = \sum_{\ell\ge 0} t^{\ell/2} \\
&\quad \times \int dV(z')\Big[\mathcal{B}(\varepsilon^{1/2})\mathcal{K}(z')\,(\partial/\partial w)^{\mathbb{B}'}\Big(w^{\mathbb{C}'}\varphi^{I'\bar K'}(w)\Big)\Big|_{w=\Theta^{(\varepsilon)}(\cdot,z)^{-1}(\iota_t(z'))}\Big]^{\varepsilon^{1/2}}_{m_*}.
\end{aligned}$$

The argument similar to the proof of Lemma 2.6 claims that these are smooth on $[0,\varepsilon_0^{1/2}]\times[0,\infty)\times H_n$ and, setting $\varphi_{m/2}(z)=\lim_{t^{1/2}\to 0}\int r_{m/2}(t,z,z')\wedge\star\varphi(z')$, we have

$$\left\|\int r_{m/2}(t,z,z')\wedge\star\varphi(z')-\varphi_{m/2}(z)\right\|_{L^2}\le t^{1/2}\,\mathrm{sn}(\varphi)$$

when $0\le t^{1/2}\le T_0^{1/2}$. Further we have $\varphi_{0/2}(z)=\varphi(z)$ (see (2.13)) and $\varphi_{m/2}(z)=0$ ($m>0$). Indeed, since Lemma 4.1 implies $\lim_{t^{1/2}\to 0}\int r_{(\varepsilon)}(t,z,z')\wedge\star\varphi(z')=\varphi(z)$ for every $(\varepsilon^{1/2},z)\in[0,\varepsilon_0^{1/2}]\times H_n$, setting $\varphi_{m_*/2}(\varepsilon^{1/2},z)=\lim_{t^{1/2}\to 0}\int r_{m_*/2}(\varepsilon^{1/2},t,z,z')\wedge\star\varphi(z')$ as well, we know that the form $\sum_{0\le m<m_*}\varepsilon^{m/2}\varphi_{m/2}(z)+\varepsilon^{m_*/2}\varphi_{m_*/2}(\varepsilon^{1/2},z)$ on $[0,\varepsilon_0^{1/2}]\times H_n$ is identically equal to $\varphi(z)$. Thus (6.19) and (6.20) were proved. Similarly the form $\int p^k_{m/2}(t,z,z')\wedge\star\varphi(z')$ ($k>0$) is smooth on $[0,\infty)\times H_n$ and (6.4) implies the estimate (6.21). ∎

6.4 The proof of Proposition 6.1

Now, let us prove Proposition 6.1.

Lemma 6.9 says that $\sum_{0\le k<k_0}(-1)^k r_{(\varepsilon)}\#q^k_{(\varepsilon)}$ can be extended smoothly up to the domain $[0,\varepsilon_0^{1/2}]\times(0,\infty)\times H_n\times H_n$, and Lemma 6.4 with $\ell=0$ says that $R_{k_0}(p_{(\varepsilon)})$ can be extended up to the domain so as to be of class C^{k_0-1} (by claiming that its differentials up to the order k_0-1 relative to the variables $(\varepsilon^{1/2},t,z,z')$ are equal to 0 at $\varepsilon^{1/2}=0$). Since k_0 can be chosen arbitrarily large, certainly $p_{(\varepsilon)}(t,z,z')$ is extended smoothly up to $\varepsilon^{1/2}=0$ and we have

$$p_{m/2}(t,z,z')=\sum_{0\le k\le m}p^k_{m/2}(t,z,z'). \tag{6.22}$$

Let us show (6.1) by induction. When $m=0$, it is valid because of (6.18) and (5.7). We fix $m'>0$ and assume that it is valid when $m<m'$. Then, certainly we have

$$\Big(\frac{\partial}{\partial t}+\Box_{0/2}\Big)(p_{m'/2}-\mathfrak{p}_{m'/2})=0.$$

Further, since $\left\|\int p_{m'/2}(t,z,z')\wedge\star\varphi(z')\right\|_{L^2}\le t^{1/2}\mathrm{sn}(\varphi)$ (by (6.22), (6.20), (6.21)) and $\left\|\int \mathfrak{p}_{m'/2}(t,z,z')\wedge\star\varphi(z')\right\|_{L^2}\le t^{2/2}\mathrm{sn}(\varphi)$ (by (5.8), (6.4)), we have

$$\lim_{t\to 0}\left\|\int\Big(p_{m'/2}(t,z,z')-\mathfrak{p}_{m'/2}(t,z,z')\Big)\wedge\star\varphi(z')\right\|_{L^2}=0\quad(\varphi\in\Omega_0^{p,q}H_n).$$

Hence, by the uniqueness of the solution of the initial value problem relative to $\Box_{0/2}=\mathbf{L}$ (refer to Lemma 2.6), (6.1) with $m=m'$ is valid.

6.5 (5.14) vanishes when m is odd

For its proof, it will suffice to show the following.

Lemma 6.10 *For $z=(z_0,z_\blacktriangle)$, we set $\tilde z=(z_0,-z_\blacktriangle)$. Then we have*

$$\mathcal{P}^{(I\bar K)(I'\bar K')}_{m/2}(t,z,z':\mathbb{A},\mathbb{A}')=(-1)^{m+|\mathbb{A}|_H+|\mathbb{A}'|_H}\mathcal{P}^{(I\bar K)(I'\bar K')}_{m/2}(t,\tilde z,\tilde z':\mathbb{A},\mathbb{A}'). \tag{6.23}$$

Proof. Let us expand $\det V_\bullet(\iota_\varepsilon(z'))$ into the series $\sum_{m\geq 0}\varepsilon^{m/2}\mathrm{det}_{m/2}(z')$. Then we have $(\partial/\partial z')^{\mathbb{A}'}\mathrm{det}_{m/2}(z')=(-1)^{m+|\mathbb{A}'|_H}(\partial/\partial z')^{\mathbb{A}'}\mathrm{det}_{m/2}(z')|_{z'=\tilde z'}$. Hence, it will suffice to ascertain the formula (6.23) with $\mathcal{P}$ replaced by $\mathfrak{p}$. With the use of the notation at (5.13), further it will suffice to show

$$\begin{aligned}(6.24)&\Big((\partial/\partial z)^{\mathbb{A}}\mathbf{r}_H\# z^{\mathbb{C}^{(1)}}(\partial/\partial z)^{\mathbb{B}^{(1)}}\mathbf{r}_H\#\cdots\# z^{\mathbb{C}^{(k)}}(\partial/\partial z)^{\mathbb{B}^{(k)}}(\partial/\partial z')^{\mathbb{A}'}\mathbf{r}_H\Big)(t,z,z')\\ &=(-1)^{\sum m_\ell+|\mathbb{A}|_H+|\mathbb{A}'|_H}\\ &\quad\times\Big((\partial/\partial z)^{\mathbb{A}}\mathbf{r}_H\# z^{\mathbb{C}^{(1)}}(\partial/\partial z)^{\mathbb{B}^{(1)}}\mathbf{r}_H\#\cdots\# z^{\mathbb{C}^{(k)}}(\partial/\partial z)^{\mathbb{B}^{(k)}}(\partial/\partial z')^{\mathbb{A}'}\mathbf{r}_H\Big)(t,\tilde z,\tilde z').\end{aligned}$$

Now, since $2+|\mathbb{C}^{(\ell)}|_H=|\mathbb{B}^{(\ell)}|_H+m_\ell$ (refer to (5.5)), we have

$$\begin{aligned}(\partial/\partial z)^{\mathbb{A}}\mathbf{r}_H(t,z,z')&=(\partial/\partial z)^{\mathbb{A}}\mathbf{r}_H(t,\tilde z,\tilde z')=(-1)^{|\mathbb{A}|_H}(\partial/\partial\tilde z)^{\mathbb{A}}\mathbf{r}_H(t,\tilde z,\tilde z'),\\ z^{\mathbb{C}^{(\ell)}}(\partial/\partial z)^{\mathbb{B}^{(\ell)}}\mathbf{r}_H(t,z,z')&=(-1)^{|(\mathbb{B}^{(\ell)},\mathbb{C}^{(\ell)})|_H}\tilde z^{\mathbb{C}^{(\ell)}}(\partial/\partial\tilde z)^{\mathbb{B}^{(\ell)}}\mathbf{r}_H(t,\tilde z,\tilde z')\\ &=(-1)^{m_\ell}\tilde z^{\mathbb{C}^{(\ell)}}(\partial/\partial\tilde z)^{\mathbb{B}^{(\ell)}}\mathbf{r}_H(t,\tilde z,\tilde z'),\end{aligned}$$

etc. In addition, we have $dV(z)=dV(\tilde z)$. Thus we obtain the equality (6.24). ∎

7 The proof of the formula (0.6)

In this section, we assume that J is integrable, that is, M is a strictly pseudo-convex CR manifold. Hence, the hermitian Tanno connection ∇ coincides with the Tanaka-Webster connection and the asymptotic coefficients $a^{(I\bar K)(I'\bar K')}_{m/2}(P^0:\mathbb{A},\mathbb{A}')$ can be described as universal polynomials made of

$$\begin{aligned}\mathcal{R}_{A_1A_2A_3A_4A_5\cdots A_\ell}&=\frac{\partial^{\ell-4}g(F(\nabla)((\partial/\partial z_{A_3},\partial/\partial z_{A_4})\partial/\partial z_{A_2},\partial/\partial z_{A_1})}{\partial z_{A_5}\cdots\partial z_{A_\ell}}(P^0),\\ \mathbf{T}_{A_1A_3A_4\cdots A_\ell}=\mathcal{T}_{A_10A_3A_4\cdots A_\ell}&=\frac{\partial^{\ell-3}g(T(\nabla)(\partial/\partial z_0,\partial/\partial z_{A_3}),\partial/\partial z_{A_1})}{\partial z_{A_4}\cdots\partial z_{A_\ell}}(P^0),\end{aligned}$$

(cf. (2.7)). The purpose in this section is to show the formula (0.6) for $a_1(P^0)=\sum_{I,K}a^{(I\bar K)(I\bar K)}_{2/2}(P^0):=\sum_{I,K}a^{(I\bar K)(I\bar K)}_{2/2}(P^0:\emptyset,\emptyset)$.

Proposition 7.1 *We have*

$$
\begin{aligned}
(7.1)\qquad & a^{(I\bar{K})(I\bar{K})}_{2/2}(P^0) \\
&= \sum_{\alpha\in K}^{\beta\notin I} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} + \Big\{ \sum_{\alpha\in I} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} - \sum_{\alpha\in K} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} \Big\} \Big\{ \frac{1}{2} + \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s)\Phi_1(s) \Big\} \\
&\quad + \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s) \Big\{ -\frac{1}{12} + \frac{4}{3}\Phi_2(s) \Big\}.
\end{aligned}
$$

If this is valid, then we have

$$
\begin{aligned}
a_1(P^0) &= \binom{n-1}{q-1}\binom{n-1}{p} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} \\
&+ \Big\{ \binom{n-1}{p-1} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} - \binom{n-1}{q-1} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} \Big\} \Big\{ \frac{1}{2} + \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s)\Phi_1(s) \Big\} \\
&+ \binom{n}{q}\binom{n}{p} \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} \int_{-\infty}^{\infty} ds\, \Phi^{n-2q}(s) \Big\{ -\frac{1}{12} + \frac{4}{3}\Phi_2(s) \Big\}.
\end{aligned}
$$

Thus we obtain the formula (0.6). Hence, the purpose in the following is to prove Proposition 7.1. First, advancing such a calculation as in Corollary 2.5, we get

Lemma 7.2 *We have*

$$
\begin{aligned}
\theta &= dz_0 \cdot \Big\{ 1 + z_\gamma z_\delta \frac{-i\mathbf{T}_{\gamma\delta}}{6} + z_{\bar\gamma} z_{\bar\delta} \frac{i\mathbf{T}_{\bar\gamma\bar\delta}}{6} + \mathcal{O}(|z|^3) \Big\} \\
&+ dz_\beta \cdot \Big\{ z_{\bar\beta} \frac{-i}{2} + z_0 z_\gamma \frac{i\mathbf{T}_{\beta\gamma}}{6} + z_{\bar\beta} z_\gamma z_\delta \frac{-\mathbf{T}_{\gamma\delta}}{24} + z_{\bar\beta} z_{\bar\gamma} z_{\bar\delta} \frac{\mathbf{T}_{\bar\gamma\bar\delta}}{24} \\
&\qquad + z_0 z_0 z_{\bar\gamma} \frac{-i\mathbf{T}_{\beta\delta}\mathbf{T}_{\bar\delta\bar\gamma}}{24} + z_0 z_\gamma z_A \frac{i\mathbf{T}_{\beta\gamma A}}{12} + z_{\bar\gamma} z_\delta z_A \frac{i\mathcal{R}_{\bar\gamma\delta\beta A}}{12} + \mathcal{O}(|z|^4) \Big\} \\
&+ dz_{\bar\beta} \cdot \Big\{ z_\beta \frac{i}{2} + z_0 z_{\bar\gamma} \frac{-i\mathbf{T}_{\bar\beta\bar\gamma}}{6} + z_\beta z_{\bar\gamma} z_{\bar\delta} \frac{-\mathbf{T}_{\bar\gamma\bar\delta}}{24} + z_\beta z_\gamma z_\delta \frac{\mathbf{T}_{\gamma\delta}}{24} \\
&\qquad + z_0 z_0 z_\gamma \frac{i\mathbf{T}_{\bar\beta\bar\delta}\mathbf{T}_{\delta\gamma}}{24} + z_0 z_{\bar\gamma} z_A \frac{-i\mathbf{T}_{\bar\beta\bar\gamma A}}{12} + z_\gamma z_{\bar\delta} z_A \frac{-i\mathcal{R}_{\gamma\bar\delta\bar\beta A}}{12} + \mathcal{O}(|z|^4) \Big\}, \\
\theta^\alpha &= dz_0 \cdot \Big\{ z_{\bar\gamma} \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}}{2} + z_0 z_\gamma \frac{-\mathbf{T}_{\bar\alpha\bar\delta}\mathbf{T}_{\gamma\delta}}{6} + z_{\bar\gamma} z_A \frac{-\mathbf{T}_{\bar\alpha\bar\gamma A}}{3} + z_\gamma z_A \frac{-\mathcal{R}_{\bar\alpha\gamma 0A}}{6} \Big\} \\
&+ dz_\beta \cdot \Big\{ \delta_{\beta\alpha} + z_{\bar\beta} z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_0 z_0 \frac{\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\beta\gamma}}{6} + z_\gamma z_A \frac{-\mathcal{R}_{\bar\alpha\gamma\beta A}}{6} \Big\} \\
&+ dz_{\bar\beta} \cdot \Big\{ z_0 \frac{\mathbf{T}_{\bar\alpha\bar\beta}}{2} + z_\beta z_{\bar\gamma} \frac{-i\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_0 z_A \frac{\mathbf{T}_{\bar\alpha\bar\beta A}}{3} + z_\gamma z_A \frac{-\mathcal{R}_{\bar\alpha\gamma\bar\beta A}}{6} \Big\} + \mathcal{O}(|z|^3)
\end{aligned}
$$

and

$$
\begin{aligned}
\xi &= \partial/\partial z_0 \cdot \Big\{ 1 + z_\gamma z_\delta \frac{-i\mathbf{T}_{\gamma\delta}}{12} + z_{\bar\gamma} z_{\bar\delta} \frac{i\mathbf{T}_{\bar\gamma\bar\delta}}{12} \Big\} \\
&+ \partial/\partial z_\alpha \cdot \Big\{ z_{\bar\gamma} \frac{\mathbf{T}_{\bar\alpha\bar\gamma}}{2} + z_0 z_\gamma \frac{-\mathbf{T}_{\bar\alpha\bar\delta}\mathbf{T}_{\gamma\delta}}{12} + z_{\bar\gamma} z_A \frac{\mathbf{T}_{\bar\alpha\bar\gamma A}}{3} + z_\gamma z_A \frac{\mathcal{R}_{\bar\alpha\gamma 0A}}{6} \Big\} \\
&+ \partial/\partial z_{\bar\alpha} \cdot \Big\{ z_\gamma \frac{\mathbf{T}_{\alpha\gamma}}{2} + z_0 z_{\bar\gamma} \frac{-\mathbf{T}_{\alpha\delta}\mathbf{T}_{\bar\gamma\bar\delta}}{12} + z_\gamma z_A \frac{\mathbf{T}_{\alpha\gamma A}}{3} + z_{\bar\gamma} z_A \frac{\mathcal{R}_{\alpha\bar\gamma 0A}}{6} \Big\} + \mathcal{O}(|z|^3),
\end{aligned}
$$

$$\xi_\beta = \partial/\partial z_0 \cdot \Big\{ z_{\bar\beta}\frac{i}{2} + z_0 z_\gamma \frac{i\mathbf{T}_{\beta\gamma}}{12} + z_0 z_\gamma z_A \frac{i\mathbf{T}_{\beta\gamma A}}{12} + z_{\bar\gamma} z_\delta z_A \frac{i\mathcal{R}_{\bar\gamma\delta\beta A}}{12} + \mathcal{O}(|z|^4)\Big\}$$
$$+ \partial/\partial z_\alpha \cdot \Big\{ \delta_{\alpha\beta} + z_{\bar\beta} z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{12} + z_0 z_0 \frac{\mathbf{T}_{\bar\alpha\bar\gamma}\mathbf{T}_{\beta\gamma}}{12} + z_\gamma z_A \frac{\mathcal{R}_{\bar\alpha\gamma\beta A}}{6} + \mathcal{O}(|z|^3)\Big\}$$
$$+ \partial/\partial z_{\bar\alpha} \cdot \Big\{ z_0 \frac{-\mathbf{T}_{\alpha\beta}}{2} + z_{\bar\beta} z_\gamma \frac{i\mathbf{T}_{\alpha\gamma}}{12} + z_0 z_A \frac{-\mathbf{T}_{\alpha\beta A}}{3} + z_{\bar\gamma} z_A \frac{\mathcal{R}_{\alpha\bar\gamma\beta A}}{6} + \mathcal{O}(|z|^3)\Big\}.$$

In addition, we have

$$\omega^\alpha_\beta(\partial/\partial z_A) = z_{A_1}\frac{-\mathcal{R}_{\bar\alpha\beta A A_1}}{2} + \mathcal{O}(|z|^2), \quad \mathcal{R}_{\bar\alpha\beta\bar\gamma\bar\delta} = -i\mathbf{T}_{\bar\alpha\bar\gamma}\,\delta_{\delta\beta} + i\mathbf{T}_{\bar\alpha\bar\delta}\,\delta_{\gamma\beta}.$$

Corollary 7.3 *We have*

(7.2) $\quad \Box_{1/2} = 0,$

(7.3)
$$\Box_{2/2} = \Big\{ z_0 \frac{\mathbf{T}_{\bar\alpha\bar\beta}}{2} + z_\gamma z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\alpha\bar\beta}}{6} + z_\alpha z_{\bar\gamma} \frac{-i\mathbf{T}_{\bar\beta\bar\gamma}}{12} + z_\gamma z_\delta \frac{-\mathcal{R}_{\bar\alpha\gamma\bar\beta\delta}}{6}\Big\}\partial/\partial z_\alpha \partial/\partial z_\beta$$
$$+ \Big\{ z_0 \frac{\mathbf{T}_{\alpha\beta}}{2} + z_\gamma z_{\bar\gamma} \frac{-i\mathbf{T}_{\alpha\beta}}{6} + z_{\bar\alpha} z_\gamma \frac{i\mathbf{T}_{\beta\gamma}}{12} + z_{\bar\gamma} z_{\bar\delta} \frac{-\mathcal{R}_{\bar\gamma\alpha\bar\delta\beta}}{6}\Big\}\partial/\partial z_{\bar\alpha} \partial/\partial z_{\bar\beta}$$
$$+ \Big\{ z_{\bar\beta} z_{\bar\gamma} \frac{-i\mathbf{T}_{\bar\alpha\bar\gamma}}{4} + z_\alpha z_\gamma \frac{i\mathbf{T}_{\beta\gamma}}{4}$$
$$+ \delta_{\alpha\beta}\cdot z_\gamma z_\delta \frac{-i\mathbf{T}_{\gamma\delta}}{6} + \delta_{\alpha\beta}\cdot z_{\bar\gamma} z_{\bar\delta} \frac{i\mathbf{T}_{\bar\gamma\bar\delta}}{6} + z_{\bar\gamma} z_\delta \frac{\mathcal{R}_{\bar\alpha\beta\bar\gamma\delta}}{3}\Big\}\partial/\partial z_\alpha \partial/\partial z_{\bar\beta}$$
$$+ \Big\{ z_0 z_{\bar\gamma} \frac{i\mathbf{T}_{\bar\alpha\bar\gamma}}{3} + z_{\bar\gamma} z_\delta z_{\bar\delta} \frac{-\mathbf{T}_{\bar\alpha\bar\gamma}}{6} + z_\alpha z_{\bar\gamma} z_{\bar\delta} \frac{\mathbf{T}_{\bar\gamma\bar\delta}}{12} + z_\gamma z_{\bar\delta} z_\rho \frac{-i\mathcal{R}_{\bar\alpha\gamma\bar\delta\rho}}{4}\Big\}\partial/\partial z_0 \partial/\partial z_\alpha$$
$$+ \Big\{ z_0 z_\gamma \frac{-i\mathbf{T}_{\alpha\gamma}}{3} + z_\gamma z_\delta z_{\bar\delta} \frac{-\mathbf{T}_{\alpha\gamma}}{6} + z_{\bar\alpha} z_\gamma z_\delta \frac{\mathbf{T}_{\gamma\delta}}{12} + z_{\bar\gamma} z_{\bar\delta} z_\rho \frac{i\mathcal{R}_{\bar\gamma\alpha\bar\delta\rho}}{4}\Big\}\partial/\partial z_0 \partial/\partial z_{\bar\alpha}$$
$$+ \Big\{ z_0 z_\gamma z_\delta \frac{-\mathbf{T}_{\gamma\delta}}{24} + z_0 z_{\bar\gamma} z_{\bar\delta} \frac{-\mathbf{T}_{\bar\gamma\bar\delta}}{24} + z_{\bar\gamma} z_\delta z_{\bar\rho} z_\sigma \frac{\mathcal{R}_{\bar\gamma\delta\bar\rho\sigma}}{12}\Big\}\partial/\partial z_0 \partial/\partial z_0$$
$$+ \Big\{ z_{\bar\gamma} \frac{i(-n+5-6q)\mathbf{T}_{\bar\alpha\bar\gamma}}{12} + z_\gamma \frac{-\mathcal{R}_{\bar\alpha\gamma\bar\delta\delta}}{3}$$
$$+ \Big(z_{\bar\mu} \frac{-i\mathbf{T}_{\bar\alpha\bar\nu}}{2} + \delta_{\alpha\mu}\cdot z_{\bar\delta} \frac{i\mathbf{T}_{\bar\delta\bar\nu}}{2} + z_\delta \frac{\mathcal{R}_{\bar\alpha\delta\bar\nu\mu}}{2}\Big)\theta^{\bar\nu}_H \wedge \theta^{\bar\mu}_H \vee$$
$$+ \Big(z_{\bar\nu} \frac{i\mathbf{T}_{\bar\alpha\bar\mu}}{2} + \delta_{\alpha\nu}\cdot z_{\bar\delta} \frac{-i\mathbf{T}_{\bar\delta\bar\mu}}{2} + z_\delta \frac{-\mathcal{R}_{\bar\alpha\delta\bar\mu\nu}}{2}\Big)\theta^{\nu}_H \wedge \theta^{\mu}_H \vee \Big\}\partial/\partial z_\alpha$$
$$+ \Big\{ z_\gamma \frac{i(7n-5-6q)\mathbf{T}_{\alpha\gamma}}{12} + z_{\bar\gamma} \frac{-\mathcal{R}_{\bar\gamma\alpha\bar\delta\delta}}{3}$$
$$+ \Big(z_\nu \frac{-i\mathbf{T}_{\alpha\mu}}{2} + \delta_{\alpha\nu}\cdot z_\delta \frac{i\mathbf{T}_{\delta\mu}}{2} + z_{\bar\delta} \frac{-\mathcal{R}_{\bar\delta\alpha\bar\nu\mu}}{2}\Big)\theta^{\bar\nu}_H \wedge \theta^{\bar\mu}_H \vee$$
$$+ \Big(z_\mu \frac{i\mathbf{T}_{\alpha\nu}}{2} + \delta_{\alpha\mu}\cdot z_\delta \frac{-i\mathbf{T}_{\delta\nu}}{2} + z_{\bar\delta} \frac{\mathcal{R}_{\bar\delta\alpha\bar\mu\nu}}{2}\Big)\theta^{\nu}_H \wedge \theta^{\mu}_H \vee \Big\}\partial/\partial z_{\bar\alpha}$$
$$+ \Big\{ z_\gamma z_\delta \frac{(2n-2-q)\mathbf{T}_{\gamma\delta}}{12} + z_{\bar\gamma} z_{\bar\delta} \frac{(n-2+q)\mathbf{T}_{\bar\gamma\bar\delta}}{12}$$
$$+ z_{\bar\gamma} z_\delta \frac{i\mathcal{R}_{\bar\gamma\delta\bar\nu\mu}}{2}\,\theta^{\bar\nu}_H \wedge \theta^{\bar\mu}_H \vee + z_{\bar\gamma} z_\delta \frac{-i\mathcal{R}_{\bar\gamma\delta\bar\mu\nu}}{2}\,\theta^{\nu}_H \wedge \theta^{\mu}_H \vee \Big\}\partial/\partial z_0$$
$$+ \Big\{ \frac{\mathcal{R}_{\bar\gamma\gamma\bar\nu\mu}}{2}\,\theta^{\bar\nu}_H \wedge \theta^{\bar\mu}_H \vee + \frac{-\mathcal{R}_{\bar\gamma\gamma\bar\mu\nu}}{2}\,\theta^{\nu}_H \wedge \theta^{\mu}_H \vee$$
$$- \mathcal{R}_{\bar\alpha\beta\bar\nu\mu}\,\theta^{\bar\alpha}_H \wedge \theta^{\bar\beta}_H \vee \theta^{\bar\nu}_H \wedge \theta^{\bar\mu}_H \vee + \mathcal{R}_{\bar\alpha\beta\bar\mu\nu}\,\theta^{\bar\alpha}_H \wedge \theta^{\bar\beta}_H \vee \theta^{\nu}_H \wedge \theta^{\mu}_H \vee \Big\}.$$

By (7.2), the formula (5.13) with $m=2$ is reduced to

$$
\begin{aligned}
(7.4) \qquad & a_{2/2}^{(I\bar{K})(I\bar{K})}(P^0) = -\sum \square_{2/2}^{(I\bar{K})(I\bar{K})}(P^0 : \mathbb{C}, \mathbb{B})\, \mathbf{r}_H(\mathbb{C} : \mathbb{B}), \\
& \mathbf{r}_H(\mathbb{C} : \mathbb{B}) := \Big(\mathbf{r}_H \# (z)^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} \mathbf{r}_H\Big)(1,0,0) \\
& \qquad = \int_0^1 dt \int_{H_n} dV(z)\, r_{1-t}^a(-z)\,(z)^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} r_t^a(z) \quad (a = n-2q).
\end{aligned}
$$

We will investigate these closely. Referring to Lemma 2.6(1), we set

$$
r_t^a(z) = \mathcal{F}_{(s\,:\,2z_0/t)}\Big\{c(t,s)\,\mathbf{e}(T, z_\blacktriangle)\Big\},
$$

$$
\begin{cases}
c(t,s) = \dfrac{e^{-as}}{2\pi t}\Big(\dfrac{1}{\cosh s}\Big)^n, \quad T = T(t,s) = \dfrac{t \tanh s}{2s}, \\
\mathbf{e}(T, z_\blacktriangle) = \dfrac{1}{(4\pi T)^n} \exp\Big(-\dfrac{|z_\blacktriangle|^2}{2T}\Big).
\end{cases}
$$

Note that $\mathbf{e}(T, z_\blacktriangle)$ is the Gaussian kernel on $(\mathbb{C}^n, z_\blacktriangle)$ $(|z_\blacktriangle|^2 = \sum x_i^2/2)$ and

$$
\begin{aligned}
& (z)^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} r_t^a(z) = \mathcal{F}_{(s\,:\,2z_0/t)}\Big\{P(\mathbb{C}:\mathbb{B})\, c(t,s)\mathbf{e}(T, z_\blacktriangle - z'_\blacktriangle)\Big\}\Big|_{z'_\blacktriangle = 0}, \\
& P(\mathbb{C}:\mathbb{B}) = P((\mathbb{C}:\mathbb{B}) : t, s, \partial/\partial s : z'_\blacktriangle, \partial/\partial z'_\blacktriangle) \\
& \quad := (\partial/\partial s)^{|\mathbb{C}^0|} \circ (t/2i)^{|\mathbb{C}^0| - |\mathbb{B}^0|} (-s)^{|\mathbb{B}^0|} (-\partial/\partial z')^{\mathbb{B}^H} (2T\partial/\partial z' + \bar{z}')^{\overline{\mathbb{C}^H}},
\end{aligned}
$$

where $\mathbb{C} = \mathbb{C}^0 \sqcup \mathbb{C}^H$ with $\mathbb{C}^H = \{C_j \in \mathbb{C} \mid C_j \neq 0\}$, etc. Hence, the Parseval formula implies that $\mathbf{r}_H(\mathbb{C}:\mathbb{B})$ is equal to the value when $z'_\blacktriangle = 0$ of

$$
\begin{aligned}
& \int_0^1 dt \int_{H_n} dV(z)\, \overline{\mathcal{F}_{(s_1\,:\,2z_0/(1-t))}\Big\{c(1-t, s_1)\,\mathbf{e}(T(1-t,s_1), z_\blacktriangle)\Big\}} \\
& \qquad\qquad \times \mathcal{F}_{(s_2\,:\,2z_0/t)}\Big\{P((\mathbb{C}:\mathbb{B}) : t, s_2, \partial/\partial s_2 :)\, c(t,s_2)\mathbf{e}(T(t,s_2), z_\blacktriangle - z'_\blacktriangle)\Big\} \\
& = \int_0^1 dt \frac{(1-t)t}{4} \int_{\mathbb{C}^n} dV(z_\blacktriangle) \\
& \quad \cdot \int_{-\infty}^{\infty} dz_0\, \overline{\mathcal{F}_{(u_1\,:\,z_0)}\Big\{c(1-t, \frac{(1-t)u_1}{2})\,\mathbf{e}(T(1-t, \frac{(1-t)u_1}{2}), z_\blacktriangle)\Big\}} \\
& \qquad \times \mathcal{F}_{(u_2\,:\,z_0)}\Big\{P((\mathbb{C}:\mathbb{B}) : t, \frac{tu_2}{2}, \frac{2}{t}\partial/\partial u_2 :)\, c(t, \frac{tu_2}{2})\mathbf{e}(T(t, \frac{tu_2}{2}), z_\blacktriangle - z'_\blacktriangle)\Big\} \\
& = \int_0^1 dt\, \frac{(1-t)t}{4} \int_{\mathbb{C}^n} dV(z_\blacktriangle)\, 2\pi \int_{-\infty}^{\infty} du\, \overline{c(1-t, \frac{(1-t)u}{2})\,\mathbf{e}(T(1-t, \frac{(1-t)u}{2}), z_\blacktriangle)} \\
& \qquad \times P((\mathbb{C}:\mathbb{B}) : t, \frac{tu}{2}, \frac{2}{t}\partial/\partial u :)\, c(t, \frac{tu}{2})\mathbf{e}(T(t, \frac{tu}{2}), z_\blacktriangle - z'_\blacktriangle) \\
& = 4\pi \int_0^1 dt\, \frac{(1-t)t}{4} \int_{\mathbb{C}^n} dV(z_\blacktriangle) \int_{-\infty}^{\infty} ds\, c(1-t, (1-t)s)\,\mathbf{e}(T(1-t, (1-t)s), z_\blacktriangle) \\
& \qquad \times P((\mathbb{C}:\mathbb{B}) : t, ts, \frac{1}{t}\partial/\partial s :)\, c(t, ts)\mathbf{e}(T(t, ts), z_\blacktriangle - z'_\blacktriangle)
\end{aligned}
$$

$$
\begin{aligned}
&= 4\pi \int_0^1 dt \int_{\mathbb{C}^n} dV(z_\blacktriangle) \\
&\quad \cdot \int_{-\infty}^{\infty} ds\, D(1-t,s)\mathbf{e}(S(1-t,s),z_\blacktriangle)\, \tilde{P}(\mathbb{C}:\mathbb{B})\, D(t,s)\mathbf{e}(S(t,s),z_\blacktriangle - z'_\blacktriangle)
\end{aligned}
$$

with

$$
\begin{aligned}
D(t,s) &= \frac{tc(t,ts)}{2} = \frac{e^{-ats}}{4\pi}\Big(\frac{1}{\cosh ts}\Big)^n, \\
\tilde{P}(\mathbb{C}:\mathbb{B}) &= P((\mathbb{C}:\mathbb{B}) : t, ts, \frac{1}{t}\partial/\partial s : z'_\blacktriangle, \partial/\partial z'_\blacktriangle) \\
&= (\partial/\partial s)^{|\mathbb{C}^0|} \circ s^{|\mathbb{B}^0|}(-1)^{|\mathbb{B}|}(2i)^{|\mathbb{B}^0|-|\mathbb{C}^0|}(\partial/\partial z')^{\mathbb{B}^H}(2S(t,s)\partial/\partial z' + \bar{z}')^{\overline{\mathbb{C}^H}}.
\end{aligned}
$$

In particular, if $|\mathbb{C}^0| = 0$ then we have

$$
\begin{aligned}
&\tilde{P}(\mathbb{C}:\mathbb{B}) = \tilde{P}((\mathbb{C}:\mathbb{B}) : t, s : z_\blacktriangle, \partial/\partial z_\blacktriangle) \\
&\quad = s^{|\mathbb{B}^0|}(-1)^{|\mathbb{B}|}(2i)^{|\mathbb{B}^0|}(\partial/\partial z)^{\mathbb{B}^H}(2S(t,s)\partial/\partial z + \bar{z})^{\overline{\mathbb{C}^H}}, \\
&\mathbf{r}_H(\mathbb{C}:\mathbb{B}) = 4\pi \int_0^1 dt \int_{-\infty}^{\infty} ds\, D(1-t,s)D(t,s) \\
&\qquad\qquad \times\ \tilde{P}(\mathbb{C}:\mathbb{B})\Big|_{z_\blacktriangle=0} \int_{\mathbb{C}^n} dV(z'_\blacktriangle)\, \mathbf{e}(S(1-t,s),z'_\blacktriangle)\, \mathbf{e}(S(t,s),z'_\blacktriangle - z_\blacktriangle) \\
&\quad = 4\pi \int_0^1 dt \int_{-\infty}^{\infty} ds\, D(1-t,s)D(t,s)\, \tilde{P}(\mathbb{C}:\mathbb{B})\Big|_{z_\blacktriangle=0} \mathbf{e}(S(1-t,s)+S(t,s),z_\blacktriangle)
\end{aligned}
\tag{7.5}
$$

and, if $|\mathbb{C}^0| = 1$ then we have

$$
\begin{aligned}
&\tilde{P}(\mathbb{C}:\mathbb{B}) = (\partial/\partial s) \circ \tilde{P}_0(\mathbb{C}:\mathbb{B}) \\
&\quad = (\partial/\partial s) \circ s^{|\mathbb{B}^0|}(-1)^{|\mathbb{B}|}(2i)^{|\mathbb{B}^0|-1}(\partial/\partial z)^{\mathbb{B}^H}(2S(t,s)\partial/\partial z + \bar{z})^{\overline{\mathbb{C}^H}}, \\
&\mathbf{r}_H(\mathbb{C}:\mathbb{B}) = 4\pi \int_0^1 dt \int_{\mathbb{C}^n} dV(z'_\blacktriangle) \int_{-\infty}^{\infty} ds\, D(1-t,s)\mathbf{e}(S(1-t,s),z'_\blacktriangle) \\
&\qquad\qquad \times\ \Big\{(\partial/\partial s)\Big(\tilde{P}_0(\mathbb{C}:\mathbb{B})D(t,s)\Big)\mathbf{e}(S(t,s),z'_\blacktriangle - z_\blacktriangle) \\
&\qquad\qquad\qquad + \tilde{P}_0(\mathbb{C}:\mathbb{B})D(t,s)(\partial/\partial s)\mathbf{e}(S(t,s),z'_\blacktriangle - z_\blacktriangle)\Big\}\Big|_{z_\blacktriangle=0} \\
&= 4\pi \int_0^1 dt \int_{-\infty}^{\infty} ds\, D(1-t,s)\frac{\partial D(t,s)\tilde{P}_0(\mathbb{C}:\mathbb{B})}{\partial s}\Big|_{z_\blacktriangle=0} \mathbf{e}(S(1-t,s)+S(t,s),z_\blacktriangle) \\
&\quad + 4\pi \int_0^1 dt \int_{-\infty}^{\infty} ds\, D(1-t,s)D(t,s)\, 2\frac{\partial S(t,s)}{\partial s}\tilde{P}_0(\mathbb{C}:\mathbb{B})(\partial/\partial z_\alpha)(\partial/\partial z_{\bar{\alpha}})\Big|_{z_\blacktriangle=0} \\
&\qquad\qquad \times \int_{\mathbb{C}^n} dV(z'_\blacktriangle)\, \mathbf{e}(S(1-t,s),z'_\blacktriangle)\, \mathbf{e}(S(t,s),z'_\blacktriangle - z_\blacktriangle) \\
&= 4\pi \int_0^1 dt \int_{-\infty}^{\infty} ds\, D(1-t,s)\Big\{\frac{\partial D(t,s)\tilde{P}_0(\mathbb{C}:\mathbb{B})}{\partial s} \\
&\quad + 2D(t,s)\frac{\partial S(t,s)}{\partial s}\, \tilde{P}_0(\mathbb{C}:\mathbb{B})\, (\partial/\partial z_\alpha)(\partial/\partial z_{\bar{\alpha}})\Big\}\Big|_{z_\blacktriangle=0} \mathbf{e}(S(1-t,s)+S(t,s),z_\blacktriangle).
\end{aligned}
$$

Notice that (by (7.3)) only the multi-indices $\mathbb{C}$ with $|\mathbb{C}^0| = 0, 1$ appear in (7.4), and, from the above researches, for example we know the following and so forth.

Lemma 7.4 *As to $\mathbf{r}_H(\mathbb{C} : \mathbb{B})$ appearing in (7.4), we have: (1) it vanishes if there exists a number $k > 0$ such that $\#\{A \in \overline{\mathbb{C}} \sqcup \mathbb{B} \mid A = k\} \neq \#\{A \in \overline{\mathbb{C}} \sqcup \mathbb{B} \mid A = \bar{k}\}$. (2) We have $\mathbf{r}_H(\mathbb{C} : \mathbb{B}) = \mathbf{r}_H(\overline{\mathbb{C}} : \overline{\mathbb{B}})$, $\overline{\mathbf{r}_H(\mathbb{C} : \mathbb{B})} = (-1)^{|\mathbb{C}^0|+|\mathbb{B}^0|}\mathbf{r}_H(\mathbb{C} : \mathbb{B})$.*

Consequently, (7.4) is reduced to

$$
\begin{aligned}
(7.6)\quad a_{2/2}^{(I\bar{K})(I\bar{K})}(P^0) &= \mathcal{R}_{\bar{\alpha}\alpha\bar{\alpha}\alpha}\Big\{\frac{-\mathbf{r}_H(11\bar{1}\bar{1} : 00)}{12} + \frac{\mathbf{r}_H(1\bar{1}2\bar{2} : 00)}{6}\Big\} \\
&+\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\{\frac{\mathbf{r}_H(1 : 1)}{3} + \frac{-\mathbf{r}_H(1\bar{1}2\bar{2} : 00)}{6}\Big\} + \Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\}\frac{i\mathbf{r}_H(1\bar{1} : 0)}{2} \\
&+\frac{1}{2}\Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\} + \sum_{\alpha\in K}^{\beta\notin I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}.
\end{aligned}
$$

Proof of Proposition 7.1. We set $S(t) = S(t, s)$ for short. Only the multi-indices $\mathbb{C}$ with $|\mathbb{C}^0| = 0$ appear in (7.6) and, referring to (7.5), we have

$$
\begin{aligned}
(7.7)\quad & a_{2/2}^{(I\bar{K})(I\bar{K})}(P^0) - \frac{1}{2}\Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\} - \sum_{\alpha\in K}^{\beta\notin I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} \\
&= \int_{-\infty}^{\infty} ds \int_0^1 dt\, 4\pi D(1-t, s)D(t, s)\bigg(\mathcal{R}_{\bar{\alpha}\alpha\bar{\alpha}\alpha}\Big\{\frac{-\tilde{P}(11\bar{1}\bar{1} : 00)}{12} + \frac{\tilde{P}(1\bar{1}2\bar{2} : 00)}{6}\Big\} \\
&\qquad +\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\{\frac{\tilde{P}(1 : 1)}{3} + \frac{-\tilde{P}(1\bar{1}2\bar{2} : 00)}{6}\Big\} \\
&\qquad +\Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar{\alpha}\alpha\bar{\beta}\beta}\Big\}\frac{i\tilde{P}(1\bar{1} : 0)}{2}\bigg)\mathbf{e}(S(1-t) + S(t), z_\blacktriangle)\Big|_{z_\blacktriangle=0}.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
&\tilde{P}(1 : 1)\Big|_{z_\blacktriangle=0}\mathbf{e}(S(1-t) + S(t), z_\blacktriangle) \\
&\quad = \Big\{-2S(t)(\partial/\partial z_1)(\partial/\partial z_{\bar{1}}) - 1\Big\}\mathbf{e}(S(1-t) + S(t), z_\blacktriangle)\Big|_{z_\blacktriangle=0} \\
&\quad = -\Big(\frac{1}{4\pi(S(1-t) + S(t))}\Big)^n \frac{S(1-t)}{S(1-t) + S(t)}, \\
&i\tilde{P}(1\bar{1} : 0)\Big|_{z_\blacktriangle=0}\mathbf{e}(S(1-t) + S(t), z_\blacktriangle) \\
&\quad = 4sS(t)(\partial/\partial z_1)(2S(t)\partial/\partial z_{\bar{1}} + z_1)\mathbf{e}(S(1-t, s) + S(t), z_\blacktriangle)\Big|_{z_\blacktriangle=0} \\
&\quad = 4s\Big(\frac{1}{4\pi(S(1-t) + S(t))}\Big)^n \frac{S(1-t)S(t)}{S(1-t) + S(t)}, \\
&\tilde{P}(1\bar{1}2\bar{2} : 00)\Big|_{z_\blacktriangle=0}\mathbf{e}(S(1-t) + S(t), z_\blacktriangle)
\end{aligned}
$$

$$
\begin{aligned}
&= -16s^2 S(t)^2(\partial/\partial z_1)(2S(t)\partial/\partial z_{\bar 1} + z_1) \\
&\qquad\qquad \cdot (\partial/\partial z_2)(2S(t)\partial/\partial z_{\bar 2} + z_2)\mathbf{e}(S(1-t)+S(t), z_\blacktriangle)\Big|_{z_\blacktriangle = 0} \\
&= -16s^2\Big(\frac{1}{4\pi(S(1-t)+S(t))}\Big)^n\Big(\frac{S(1-t)S(t)}{S(1-t)+S(t)}\Big)^2,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde P(1\bar 1 1\bar 1 : 00)\Big|_{z_\blacktriangle = 0}\mathbf{e}(S(1-t)+S(t), z_\blacktriangle) \\
&= -16s^2 S(t)^2(\partial/\partial z_1)(\partial/\partial z_1) z_1^2\Big(\frac{S(1-t)}{S(1-t)+S(t)}\Big)^2\mathbf{e}(S(1-t)+S(t), z_\blacktriangle)\Big|_{z_\blacktriangle = 0} \\
&= -32s^2\Big(\frac{1}{4\pi(S(1-t)+S(t))}\Big)^n\Big(\frac{S(1-t)S(t)}{S(1-t)+S(t)}\Big)^2
\end{aligned}
$$

and, added to (0.5), we have

$$
\int_0^1 dt\,\frac{S(1-t)}{S(1-t)+S(t)} = \frac{1}{2}, \qquad \frac{4\pi D(1-t,s)D(t,s)}{(4\pi(S(1-t)+S(t)))^n} = \frac{1}{2}\Phi^{n-2q}(s).
$$

Hence, (7.7) is equal to

$$
\begin{aligned}
&\int_{-\infty}^{\infty} ds\int_0^1 dt\,\frac{4\pi D(1-t,s)D(t,s)}{(4\pi(S(1-t)+S(t)))^n} \\
&\qquad \times \Bigg\{\mathcal{R}_{\bar\alpha\alpha\bar\alpha\alpha}\Big\{\frac{8}{3}\Big(\frac{s\,S(1-t)S(t)}{S(1-t)+S(t)}\Big)^2 - \frac{8}{3}\Big(\frac{s\,S(1-t)S(t)}{S(1-t)+S(t)}\Big)^2\Big\} \\
&\qquad\qquad + \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\{-\frac{1}{3}\frac{S(1-t)}{S(1-t)+S(t)} + \frac{8}{3}\Big(\frac{s\,S(1-t)S(t)}{S(1-t)+S(t)}\Big)^2\Big\} \\
&\qquad\qquad + \Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\}\frac{2s\,S(1-t)S(t)}{S(1-t)+S(t)}\Bigg\} \\
&= \mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Big\{-\frac{1}{12} + \frac{4}{3}\Phi_2(s)\Big\} \\
&\qquad + \Big\{\sum_{\alpha\in I}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta} - \sum_{\alpha\in K}\mathcal{R}_{\bar\alpha\alpha\bar\beta\beta}\Big\}\int_{-\infty}^{\infty} ds\,\Phi^{n-2q}(s)\Phi_1(s).
\end{aligned}
$$

Thus we obtain (7.1). ∎

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