A formula for the heat kernel coefficients on Riemannian manifolds

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Abstract

Based on the idea of adiabatic expansion theory, we will present a new formula for the asymptotic expansion coefficients of every derivative of the heat kernel on a compact Riemannian manifold. It will be very useful for having systematic understanding of the coefficients, and, furthermore, by using only a basic knowledge of calculus added to the formula, one can describe them explicitly up to an arbitrarily high order.

Keywords: heat kernel; asymptotic expansion; adiabatic expansion

1 Introduction

Let (M, g) be an n-dimensional compact oriented Riemannian manifold and $\Delta = \delta d + d\delta$ be the Laplacian acting on q-forms, where δ is the formal adjoint of the exterior differentiation d. In terms of the Hodge operator $\star_g$ it is given by $\delta = (-1)^{nq+n+1} \star_g d \star_g$ on $\Omega^q M := \Gamma(\wedge^q T^* M)$. The initial value problem for the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right)\phi = 0, \qquad \lim_{t\to 0} \phi(t) = \varphi \quad (\varphi \in \Omega^q M) \tag{1.1}$$

has a unique fundamental solution or heat kernel $e^{-t\Delta}(P, P')$, where the convergence is in the L^2-norm. Near a point P^0 we will take a positively oriented orthonormal frame $e_\bullet = (e_1, \ldots, e_n)$ of TM and its dual frame $e^\bullet = (e^1, \ldots, e^n)$ which are parallel along the

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geodesics from P^0. In addition, let us take normal coordinates $x = (x_1, \ldots, x_n)$ centered at P^0 with $(\partial/\partial x_i)_0 = e_i(P^0)$ for any i, i.e., $\exp^\nabla(e_\bullet(P^0) \cdot x(P)) = P$, where ∇ is the Levi-Civita connection, and consider the local expression

$$e^{-t\Delta}(x, x') = \sum e^I(x) \boxtimes e^{I'}(x') \cdot (e^{-t\Delta})^{II'}(x, x'),$$

where we set $I = (i_1 < i_2 < \cdots < i_q)$ and $e^I = e^{i_1} \wedge \cdots \wedge e^{i_q}$. (In this paper we adopt such a notation as $e^I(X_1, \ldots, X_q) = \det(e^{i_k}(X_\ell))$.) Then, every differential of the coefficient at the point $(0,0) = (P^0, P^0)$ can be asymptotically expanded as

$$(1.2) \quad (\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}(e^{-t\Delta})^{II'}(P^0, P^0) \;\sim \sum_{m \geq -(|\mathbb{A}|+|\mathbb{A}'|)} t^{-n/2+m/2} a^{II'}_{m/2}(P^0 : \mathbb{A}, \mathbb{A}')$$

when $t \to 0$, where for a multi-index $\mathbb{A} = (A_1, \ldots, A_{|\mathbb{A}|})$ $(1 \leq A_j \leq n)$ we set $(\partial/\partial x)^{\mathbb{A}} = \partial/\partial x_{A_1} \cdots \partial/\partial x_{A_{|\mathbb{A}|}}$.

In this paper, we wish to present a new formula (1.24) for the asymptotic expansion coefficients. It will be very useful for having systematic understanding of them, and, furthermore, using only a basic knowledge of calculus added to the formula, one can describe them explicitly up to an arbitrarily high order. Compare our calculation of $a_{2/2}(P^0) := \sum a^{II}_{2/2}(P^0 : \emptyset, \emptyset)$ following Theorem 1.1 with those by Gilkey ([4], [5, Theorem 4.8.18 (b)]) and by Branson-Gilkey ([2], [6, Theorem 4.1.7 (b)]).

Only familiar sources are required for inducing the formula. First, due to Atiyah-Bott-Patodi [1, Proposition 3.7 and Appendix II], the connection coefficients $\omega^{i_1}_{i_2}(\partial/\partial x_j) := g(\nabla_{\partial/\partial x_j} e_{i_2}, e_{i_1})$ are formally expanded as

$$(1.3) \quad \omega^{i_1}_{i_2}(\partial/\partial x_j)(x) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F(\nabla)^{i_1}_{i_2}(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0),$$

where we set $F(\nabla)^{i_1}_{i_2}(\partial/\partial x_j, \partial/\partial x_{j_1}) = g(F(\nabla)(\partial/\partial x_j, \partial/\partial x_{j_1})e_{i_2}, e_{i_1})$ $(F(\nabla)(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]})$. Second, consider the frames $(\partial/\partial x_\bullet) = (\partial/\partial x_1, \ldots)$, $(dx_\bullet) = (dx_1, \ldots)$ and set

$$(1.4) \qquad e_\bullet = (\partial/\partial x_\bullet) \cdot V_\bullet(x) \quad (\text{i.e., } e_i = \textstyle\sum V_{ji}(x)\, \partial/\partial x_j), \qquad e^\bullet = (dx_\bullet) \cdot V^\bullet(x).$$

Then, [1, Proposition 2.11 and Appendix II] says that the transition functions V^{ji} are formally expanded as

$$(1.5) \qquad V^{ji}(x) = \delta_{ji} - \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-2} F(\nabla)^{i}_{j_1}(\partial/\partial x_j, \partial/\partial x_{j_2})}{\partial x_{j_3} \cdots \partial x_{j_\ell}}(0).$$

Hence, the coefficients of the Taylor expansions of $\omega^{i_1}_{i_2}(\partial/\partial x_j)$, V^{ji}, V_{ji} are all expressed as universal polynomials made of

$$(1.6) \qquad R_{j_1 j_2 j_3 j_4 j_5 \cdots j_\ell}(P^0) = \frac{\partial^{\ell-4} g(F(\nabla)(\partial/\partial x_{j_3}, \partial/\partial x_{j_4})\partial/\partial x_{j_2}, \partial/\partial x_{j_1})}{\partial x_{j_5} \cdots \partial x_{j_\ell}}(P^0),$$

which can be concretely described easily. For example we have

$$
\begin{aligned}
&\omega^{i_1}_{i_2}(\partial/\partial x_j)(x) = -x_{j_1}\frac{1}{2}R_{i_1 i_2 j j_1}(P^0) + O(|x|^2),\\
(1.7)\qquad &V^{ji}(x) = \delta_{ji} - x_{j_1}x_{j_2}\frac{1}{6}R_{i j_1 j j_2}(P^0) + O(|x|^3),\\
&V_{ji}(x) = \delta_{ji} + x_{j_1}x_{j_2}\frac{1}{6}R_{i j_1 j j_2}(P^0) + O(|x|^3).
\end{aligned}
$$

Here the symbol $\sum$ is omitted and so may be also in the following. What is required last is the Weitzenböck formula (e.g. Wu [10, Chap. 2])

$$
(1.8)\qquad \Delta = -\sum\left(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}\right) - \sum F(\nabla)^{i_1}_{i_2}(e_i, e_j)\cdot e^i\wedge\, e^j\vee e^{i_1}\wedge\, e^{i_2}\vee\,,
$$

where $e^i\wedge$, $e^i\vee(=\iota_{e_i} = e_i\lrcorner)$ denote the exterior, interior products, respectively. Note that $\nabla_{e_i} = e_i + \sum\omega^{i_1}_{i_2}(e_i)\cdot e^{i_1}\wedge\, e^{i_2}\vee$.

Our formula is derived from them by applying the adiabatic expansion theory developed in [8]. In the following we will roughly explain how it is obtained. (Refer to [9], in which similar sources and idea provide a similar formula for the Kohn-Rossi heat kernel on contact Riemannian manifolds.)

For the sake of distinction, here we denote the metric, etc., on M by g^M, etc. Let us consider the n-dimensional Euclidean space $E^n = (E^n, y)$ with the standard metric g^E, and identify a small neighborhood U of the origin with a small neighborhood $U^0\,(\subset M)$ of P^0 via the coordinate map $U^0 \ni P \mapsto y = x(P) \in U$. We will take and fix a metric g on E^n which coincides with g^M near $0\,(= P^0)$ and with g^E a little apart from 0. The space E^n equipped with the metric g, denoted by $E^n(P^0)$ and called a Euclidean space warped near the origin, may be assumed to satisfy (refer to Lemma 2.1): The normal coordinates centered at the origin are globally defined, i.e.,

$$
(1.9)\qquad x : E^n(P^0) \cong (E^n, x), \quad y \mapsto x = x(y), \qquad \exp^{\nabla}((\partial/\partial y)_0\cdot x(y)) = y,
$$

where ∇ is the associated Levi-Civita connection, which coincides with ∇^M sufficiently near 0. Thus $E^n(P^0)$ has two kinds of global coordinates, y and x. The parallel frames $e_\bullet$, $e^\bullet$ are assumed to be given also globally. The problem (1.1) relative to the Laplacian $\Delta_{E(P^0)}$ on $E^n(P^0)$ with $\varphi \in \Omega^q_0 E(P^0)$ (that is, φ is compactly supported) has a unique fundamental solution (e.g. Dodziuk [3])

$$
(1.10)\qquad e^{-t\Delta_{E(P^0)}}(x, x') = \sum e^I(x)\boxtimes e^{I'}(x')\cdot(e^{-t\Delta_{E(P^0)}})^{II'}(x, x')
$$

and, by Duhamel's principle, we have the same asymptotic expansion as in (1.2), i.e.,

$$
(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}(e^{-t\Delta_{E(P^0)}})^{II'}(0,0) \ \sim \sum_{m\geq -(|\mathbb{A}|+|\mathbb{A}'|)} t^{-n/2+m/2}a^{II'}_{m/2}(P^0 : \mathbb{A}, \mathbb{A}')
$$

(refer to Proposition 2.3 and the comment following Proposition 3.1). Thus, it suffices to investigate the heat kernel on $E^n(P^0)$.

Now, let us set $e^\varepsilon_\bullet = \varepsilon^{1/2} e_\bullet$, $e^\bullet_\varepsilon = \varepsilon^{-1/2} e^\bullet$ $(0 < \varepsilon \leq \varepsilon_0)$ and consider the transformation $\iota_\varepsilon : x \mapsto \iota_\varepsilon(x) := \varepsilon^{1/2} x$ of $E^n(P^0)$, which induces the global frames $e^{(\varepsilon)}_\bullet := \iota^*_\varepsilon e^\varepsilon_\bullet$, $e^\bullet_{(\varepsilon)} := \iota^*_\varepsilon e^\bullet_\varepsilon$ on $E^n(P^0)$. Obviously (1.4) gives the relation

$$e^{(\varepsilon)}_\bullet = (\partial/\partial x_\bullet) \cdot V_\bullet(\iota_\varepsilon(x)), \qquad e^\bullet_{(\varepsilon)} = (dx_\bullet) \cdot V^\bullet(\iota_\varepsilon(x)). \tag{1.11}$$

To the metric $g^\varepsilon := \sum e^i_\varepsilon \otimes e^i_\varepsilon$, the Levi-Civita connection $\nabla^\varepsilon := \nabla$ and the Laplacian $\Delta^\varepsilon_{E(P^0)} := \varepsilon \Delta_{E(P^0)}$ are attached. Those for the metric $g^{(\varepsilon)} := \sum e^i_{(\varepsilon)} \otimes e^i_{(\varepsilon)}$ are $\nabla^{(\varepsilon)} := \iota^*_\varepsilon \nabla^\varepsilon$, $\Delta^{(\varepsilon)}_{E(P^0)} := \iota^*_\varepsilon \Delta^\varepsilon_{E(P^0)}$. The coordinates x are then the $g^{(\varepsilon)}$-normal coordinates centered at the origin with $(\partial/\partial x)_0 = e^{(\varepsilon)}_\bullet(0)$ and $e^{(\varepsilon)}_\bullet$, $e^\bullet_{(\varepsilon)}$ are $g^{(\varepsilon)}$-parallel along the $g^{(\varepsilon)}$-geodesics from the origin. Also there exist unique heat kernels

$$\begin{aligned}
&e^{-t\Delta^\varepsilon_{E(P^0)}}(x,x') = \sum e^I_\varepsilon(x) \boxtimes e^{I'}_\varepsilon(x') \cdot \varepsilon^{n/2} \Big(e^{-t\varepsilon \Delta_{E(P^0)}}\Big)^{II'}(x,x'),\\
&e^{-t\Delta^{(\varepsilon)}_{E(P^0)}}(x,x') = \sum e^I_{(\varepsilon)}(x) \boxtimes e^{I'}_{(\varepsilon)}(x') \cdot \varepsilon^{n/2} \Big(e^{-t\varepsilon \Delta_{E(P^0)}}\Big)^{II'}(\iota_\varepsilon(x), \iota_\varepsilon(x')).
\end{aligned} \tag{1.12}$$

Next, we consider the transformation $I_\varepsilon : \Omega^q E^n \cong \Omega^q(E^n(P^0), g^{(\varepsilon)})$, $\sum (dx)^I \cdot \varphi^I \mapsto \sum e^I_{(\varepsilon)} \cdot \varphi^I$ $((dx)^I := dx_{i_1} \wedge \cdots \wedge dx_{i_q})$, which provides the Laplacian $\Delta_{(\varepsilon)} := I^*_\varepsilon \Delta^{(\varepsilon)}_{E(P^0)}$ $(:= I^{-1}_\varepsilon \circ \Delta^{(\varepsilon)}_{E(P^0)} \circ I_\varepsilon)$ on the standard Euclidean space $E^n = (E^n, x)$, called the **adiabatic Laplacian at** P^0. Obviously, the problem (1.1) relative to $\Delta_{(\varepsilon)}$ also has a unique fundamental solution, which is described as

$$\begin{aligned}
e^{-t\Delta_{(\varepsilon)}}(x,x') = \sum &(dx)^I(x) \boxtimes (dx)^{I'}(x')\\
&\cdot \varepsilon^{n/2} \Big(e^{-t\varepsilon \Delta_{E(P^0)}}\Big)^{II'}(\iota_\varepsilon(x), \iota_\varepsilon(x')) \det V^\bullet(\iota_\varepsilon(x'))
\end{aligned} \tag{1.13}$$

because of (1.12) and $dV_{g^{(\varepsilon)}}(x') = dV_{g^E}(x') \cdot \det V^\bullet(\iota_\varepsilon(x'))$. Here $dV_{g^{(\varepsilon)}}$, etc., denote the volume elements with respect to $g^{(\varepsilon)}$, etc., and g^E denotes the standard metric in the coordinates x (not in y). In addition, by setting $\nabla^{(E,\varepsilon)} = I^*_\varepsilon \nabla^{(\varepsilon)}$ and $e^{(\varepsilon)}_\bullet = I^*_\varepsilon e^{(\varepsilon)}_\bullet$, the formula (1.8) provides the **adiabatic Weitzenböck formula**

$$\begin{aligned}
\Delta_{(\varepsilon)} = &-\sum \Big(\nabla^{(E,\varepsilon)}_{e^{(\varepsilon)}_i} \nabla^{(E,\varepsilon)}_{e^{(\varepsilon)}_i} - \nabla^{(E,\varepsilon)}_{\nabla^{(\varepsilon)}_{e^{(\varepsilon)}_i} e^{(\varepsilon)}_i}\Big)\\
&- \sum F(\nabla^{(\varepsilon)})^{i_1}_{i_2}(e^{(\varepsilon)}_i, e^{(\varepsilon)}_j) \cdot dx_i \wedge\, dx_j \vee dx_{i_1} \wedge\, dx_{i_2} \vee .
\end{aligned} \tag{1.14}$$

Notice that we have

$$\begin{aligned}
&\nabla^{(E,\varepsilon)}_{e^{(\varepsilon)}_i} = e^{(\varepsilon)}_i + \sum \varepsilon^{1/2} \omega^{i_1}_{i_2}(e_i)(\iota_\varepsilon(x)) \cdot dx_{i_1} \wedge\, dx_{i_2} \vee ,\\
&\nabla^{(\varepsilon)}_{e^{(\varepsilon)}_i} e^{(\varepsilon)}_i = \sum \varepsilon^{1/2} \omega^j_i(e_i)(\iota_\varepsilon(x))\, e^{(\varepsilon)}_j,\\
&F(\nabla^{(\varepsilon)})^{i_1}_{i_2}(e^{(\varepsilon)}_i, e^{(\varepsilon)}_j) = \varepsilon^{2/2} F(\nabla)^{i_1}_{i_2}(e_i, e_j)(\iota_\varepsilon(x)),
\end{aligned} \tag{1.15}$$

which, together with (1.11), (1.3) and (1.5), imply that the differential operator $\Delta_{(\varepsilon)}$ can be extended smoothly up to $\varepsilon^{1/2}=0$. As to the formal power series expansion

$$(1.16)\qquad \Delta_{(\varepsilon)}=\sum_{m=0}^{\infty}\varepsilon^{m/2}\,\Delta_{m/2},\qquad \Delta_{0/2}=\Delta_E:=-\sum\partial/\partial x_i\,\partial/\partial x_i$$

which we call the **adiabatic expansion of Δ at** P^0, the coefficients can be described explicitly up to an arbitrarily high order. Indeed, for example, by (1.7) we have

$$\begin{aligned}(1.17)\qquad \Delta_{(\varepsilon)}=\Delta_E+\varepsilon^{2/2}\Big\{&-x_{j_1}x_{j_2}\frac{1}{3}R_{ij_1jj_2}(P^0)\,\partial/\partial x_i\,\partial/\partial x_j\\ &+x_{j_1}\frac{2}{3}R_{ijij_1}(P^0)\,\partial/\partial x_j+x_{j_1}R_{i_1i_2ij_1}(P^0)\,\partial/\partial x_i\cdot dx_{i_1}\wedge\,dx_{i_2}\vee\\ &-R_{i_1i_2i_3i_4}(P^0)\cdot dx_{i_1}\wedge\,dx_{i_2}\vee\,dx_{i_3}\wedge\,dx_{i_4}\vee\Big\}+O(\varepsilon^{3/2}).\end{aligned}$$

Suggested by the equality $(\frac{\partial}{\partial t}+\Delta_{(\varepsilon)})e^{-t\Delta_{(\varepsilon)}}=0$, let us construct now a formal power series

$$(1.18)\qquad \mathfrak{p}_{(\varepsilon)}(t,x,x')=\sum_{m=0}^{\infty}\varepsilon^{m/2}\,\mathfrak{p}_{m/2}(t,x,x')$$

so as to satisfy $(\frac{\partial}{\partial t}+\Delta_{(\varepsilon)})\mathfrak{p}_{(\varepsilon)}=0$. Namely, we define it inductively by

$$\begin{aligned}(1.19)\quad \mathfrak{p}_{0/2}(t,x,x')&=\mathfrak{r}_E(t,x,x')\\ &:=\sum(dx)^I(x)\boxtimes(dx)^I(x')\cdot r_t(x-x'),\quad r_t(x):=\frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}},\end{aligned}$$

$$\begin{aligned}(1.20)\quad \mathfrak{p}_{m/2}(t,x,x')&=-\Big(\mathfrak{p}_{0/2}\#\sum_{m_1+m_2=m}^{m_1>0}\Delta_{m_1/2}\mathfrak{p}_{m_2/2}\Big)(t,x,x')\\ &=\sum_{\sum m_\ell=m}^{m_1,\dots,m_k>0}(-1)^k\Big(\mathfrak{p}_{0/2}\#\,\Delta_{m_1/2}\mathfrak{p}_{0/2}\#\,\cdots\#\,\Delta_{m_k/2}\mathfrak{p}_{0/2}\Big)(t,x,x')\quad(m>0),\end{aligned}$$

where, in general, for double forms $h_i(t,x,x')$ $(i=1,2)$ on E^n, we define the convolution $h_1\#h_2$ by $(h_1\#h_2)(t,x,x')=\int_0^t ds\int_{E^n}h_1(t-s,x,x'')\wedge\star h_2(s,x'',x')$ $(\star=\star_{g^E},\ \#=\#_{g^E})$. It follows from Lemmas 3.2 and 3.9 that (1.20) is well-defined. Then it will be natural to expect (1.18) is the formal power series expansion of the heat kernel (1.13). Thus, setting

$$\mathcal{P}_{(\varepsilon)}(t,x,x'):=\mathfrak{p}_{(\varepsilon)}(t,x,x')\det V_\bullet(\iota_\varepsilon(x'))=\sum_{m=0}^{\infty}\varepsilon^{m/2}\,\mathcal{P}_{m/2}(t,x,x')$$

and $\mathcal{P}_{m/2}(t,x,x')=\sum(dx)^I(x)\boxtimes(dx)^{I'}(x')\cdot\mathcal{P}_{m/2}^{II'}(t,x,x')$, we must have

$$(1.21)\qquad \varepsilon^{n/2}\Big(e^{-t\varepsilon\Delta_{E(P^0)}}\Big)^{II'}(\iota_\varepsilon(x),\iota_\varepsilon(x'))=\sum_{m=0}^{\infty}\varepsilon^{m/2}\,\mathcal{P}_{m/2}^{II'}(t,x,x').$$

If this is valid, then we have the asymptotic expansion

$$\left(e^{-t\Delta_{E(P^0)}}\right)^{II'}(0,0) \;\sim \sum_{m=0}^{\infty} t^{-n/2+m/2}\,\mathcal{P}^{II'}_{m/2}(1,0,0),$$

that is,

$$a^{II'}_{m/2}(P^0:\emptyset,\emptyset) = \mathcal{P}^{II'}_{m/2}(1,0,0) = \mathfrak{p}^{II'}_{m/2}(1,0,0). \tag{1.22}$$

Further, if the differentials of the left hand side of (1.21) can be formally expanded into the series of termwise differentials of the right hand side, that is, if

$$\begin{aligned}\varepsilon^{n/2+(|\mathbb{A}|+|\mathbb{A}'|)/2}&\left((\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}\left(e^{-t\varepsilon\Delta_{E(P^0)}}\right)^{II'}\right)(\iota_\varepsilon(x),\iota_\varepsilon(x'))\\ &= \sum_{m=0}^{\infty}\varepsilon^{m/2}\,(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}\mathcal{P}^{II'}_{m/2}(t,x,x'),\end{aligned} \tag{1.23}$$

then, by setting $\mathcal{P}^{II'}_{m/2}(t,x,x':\mathbb{A},\mathbb{A}') = (\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}\mathcal{P}^{II'}_{m/2}(t,x,x')$, the formula (1.22) is generalized as follows:

Theorem 1.1 *We have*

$$a^{II'}_{m/2}(P^0:\mathbb{A},\mathbb{A}') = \mathcal{P}^{II'}_{(m+|\mathbb{A}|+|\mathbb{A}'|)/2}(1,0,0:\mathbb{A},\mathbb{A}'), \tag{1.24}$$

which vanishes when m is odd. Moreover, this is expressed as a universal polynomial made of (1.6)*, which can be described explicitly by using only a basic knowledge of calculus.*

Remark: Assume that M has boundary and a certain boundary condition is assigned to the Laplacian. Then we have also an asymptotic expansion at a point of the boundary and it is easy to induce a similar formula, which will be discussed closely elsewhere. Notice that some of the integrals over E^n having appeared in the convolutions at (1.20) will be replaced by those over $E^{n-1}\times[0,\infty)$, so that the asymptotic expansion coefficient may not vanish even if m is odd (refer to §.3.5).

Using (1.24) (or (1.22)), let us try to calculate some of the asymptotic expansion coefficients of the trace of the heat kernel. Obviously we have

$$a_0(P^0) := \sum a^{II}_0(P^0:\emptyset,\emptyset) = \sum \mathfrak{p}^{II}_{0/2}(1,0,0) = \frac{1}{(4\pi)^{n/2}}\binom{n}{q}$$

and the next coefficient $a_1(P^0) := \sum a^{II}_1(P^0:\emptyset,\emptyset)$ is calculated as follows: (1.17) implies

$$\begin{aligned}&\Delta_{2/2}\mathfrak{p}_{0/2}(s,x,0)\\ &=\Big\{-x_{k_1}x_{k_2}\frac{1}{6s}R_{j_1k_1j_1k_2} - R_{i_1i_2i_3i_4}\cdot dx_{i_1}\wedge dx_{i_2}\vee\, dx_{i_3}\wedge dx_{i_4}\vee\Big\}\mathfrak{p}_{0/2}(s,x,0)\\ &=\Big\{-\frac{2s}{3}R_{j_1k_1j_1k_2}\partial/\partial x_{k_1}\partial/\partial x_{k_2} - \frac{1}{3}R_{j_1k_1j_1k_1}\\ &\qquad\qquad -R_{i_1i_2i_3i_4}\cdot dx_{i_1}\wedge dx_{i_2}\vee\, dx_{i_3}\wedge dx_{i_4}\vee\Big\}\mathfrak{p}_{0/2}(s,x,0)\end{aligned}$$

and, hence,

$$\begin{aligned}
\mathfrak{p}_{2/2}(1,0,0) &= -\int_0^1 ds \int \mathfrak{p}_{0/2}(1-s,0,x') \wedge \star\Delta_{2/2}\mathfrak{p}_{0/2}(s,x',0) \\
&= \int_0^1 ds \int \Big\{ \frac{2s}{3} R_{j_1k_1j_1k_2}\partial/\partial x_{k_1}\partial/\partial x_{k_2} + \frac{1}{3}R_{j_1k_1j_1k_1} \\
&\qquad + R_{i_1i_2i_3i_4}\cdot dx_{i_1}\wedge dx_{i_2}\vee\, dx_{i_3}\wedge dx_{i_4}\vee \Big\}\mathfrak{p}_{0/2}(1-s,x,x')\wedge\star\mathfrak{p}_{0/2}(s,x',0)\Big|_{x=0} \\
&= \Big\{\frac{1}{6}R_{j_1j_2j_1j_2} + R_{i_1i_2i_3i_4}\cdot dx_{i_1}\wedge dx_{i_2}\vee\, dx_{i_3}\wedge dx_{i_4}\vee\Big\}\mathfrak{p}_{0/2}(1,0,0),
\end{aligned}$$

where we put $R_{j_1k_1j_1k_2} = R_{j_1k_1j_1k_2}(P^0)$, $\star = \star_{g^E}$ for short. Thus we obtain

$$\begin{aligned}
a_1^{II}(P^0:\emptyset,\emptyset) &= \frac{1}{(4\pi)^{n/2}}\Big\{\frac{1}{6}R_{j_1j_2j_1j_2}(P^0) + \sum_{i_1\in I\not\ni i_2} R_{i_1i_2i_2i_1}(P^0)\Big\}, \\
a_1(P^0) &= \frac{1}{(4\pi)^{n/2}}\Big\{\frac{1}{6}\binom{n}{q} - \binom{n-2}{q-1}\Big\}R_{j_1j_2j_1j_2}(P^0).
\end{aligned}$$

Similarly those of higher orders can be calculated easily (with the aid of Mathematica).

2 The warped Euclidean space $E^n(P^0)$ and the heat kernel

In this section we will discuss some of the property of the space $E^n(P^0)$ and present some estimates on the heat kernel (1.10) to be used in the proof of Theorem 1.1.

First, let us construct the space $E^n(P^0)$ carefully. We set $U^0 = \{P \in M \mid |x(P)| < r_0'\}$, which is identified with $U = \{y \in E^n \mid |y| < r_0'\}$ via the map $P \mapsto y = x(P)$, and fix a smooth function $\rho(s)$ on $[0,\infty)$ which satisfies $\rho(s) = 1$ $(s \le 1/2)$, $\rho(s) = 0$ $(s \ge 2/3)$ and $0 \le \rho(s) \le 1$. For each $r \in (0, r_0']$, setting $\rho_0(y) = \rho(|y|/r)$, we consider the metric

$$g\,(= g^r) := \rho_0 g^M + (1-\rho_0)g^E = \{1 + \mathbb{O}(r^2)\}g^E \tag{2.1}$$

on E^n. Note that (1.7) induces the second description. Here, in general, $\mathbb{O}(r^k) = \mathbb{O}(y;r^k)$ $(k \in \mathbb{Z})$ is defined to be a smooth function on $E^n \times (0, r_0']\,(\ni (y,r))$ satisfying: as a function of y, it has support contained in $\{y \in E^n \mid |y| \le r\}$ for each r and, for every multi-index $\mathbb{A}$, there exists a constant $C_{\mathbb{A}} > 0$ such that $|(\partial/\partial y)^{\mathbb{A}}\mathbb{O}(r^k)| \le C_{\mathbb{A}} r^{k-|\mathbb{A}|}$ on $E^n \times (0, r_0']$. Certainly we have $\rho_0 = \mathbb{O}(1)\,(= \mathbb{O}(r^0))$, $\partial\mathbb{O}(r^k)/\partial y_A = \mathbb{O}(r^{k-1})$. The associated connection $\nabla\,(= \nabla^r)$ is thus described as

$$\nabla_{\partial/\partial y_k}\partial/\partial y_j = \Gamma^i_{jk}(y;r)\cdot\partial/\partial y_i, \quad \Gamma^i_{jk}(y;r) = \mathbb{O}(r). \tag{2.2}$$

Hence, $\Gamma^i_{jk}(y;r)$ can be extended continuously up to $r = 0$ by putting $\Gamma^i_{jk}(y;0) = 0$ and the extended one satisfies the Lipschitz condition, so that, as for the geodesic from the origin, that is, the curve $c(s) = c(s,x;r)$ satisfying $\ddot{c}_i(s) + \Gamma^i_{jk}(c(s);r)\,\dot{c}_j(s)\dot{c}_k(s) = 0$, $c(0) = 0$ and $\dot{c}(0) = x$, both $c(s,x;r)$ and $\dot{c}(s,x;r)$ are continuous on $[0,\infty)\times E^n\times[0,r_0']$.

Lemma 2.1 *Suppose $r_0 > 0$ is sufficiently small and $0 < r \le r_0$. Then the g-geodesics from the origin do not intersect with each other (except at the origin), that is, we have the global normal coordinates* (1.9).

Remark: In fact, we may take $r_0 > 0$ so small that, if $0 < r \le r_0$ and $|y'| \le r_0$, then the g-geodesics from y' do not intersect with each other.

Proof. There exist a small $r_0 > 0$ and a constant $C > 0$ such that

$$|c(1, x; r) - x| \le Cr|x|^2, \quad |(\partial/\partial x_D)(c(1, x; r) - x)| \le Cr|x| \tag{2.3}$$

if $0 \le r \le r_0$ and $|x| \le r_0$. The proof is similar to [9, Lemma 3.1]. (In [9], $\Gamma^A_{BC} = \mathbb{O}(1)$ is the counterpart of $\Gamma^i_{jk} = \mathbb{O}(r)$ of this paper. The estimates in [9, Lemma 3.1] are, hence, weaker than (2.3).) For instance, we have

$$\dot c(s) - x = \int_0^s ds\, \ddot c(s) = \int_0^s ds\, \Gamma(s, x; r), \quad c(s) - sx = \int_0^s ds\, (\dot c(s) - x)$$
$$\Gamma(s, x; r) := -\sum \Gamma_{jk}(c(s); r)\, \dot c_j(s) \dot c_k(s)$$

and, by (2.1) and (2.2), there are a small $r_0 > 0$ and a constant $C > 0$ such that $C^{-1}|x| \le |\dot c(s)| \le C|x|$ and, hence, $|\Gamma(s, x; r)| \le Cr|x|^2$ if $0 \le r \le r_0$. Thus we obtain the first estimate at (2.3). Now, let us fix such a number $r_0 > 0$. Then (2.3) and the inverse function theorem imply that there exist constants $\delta_i > 0$ ($i = 1, 2$) such that the map $c(1, \cdot\,; r) : \{x \in E^n \mid |x| \le \delta_1\} \to E^n$ is an into diffeomorphism and satisfies $c(1, \{x \in E^n \mid |x| \le \delta_1\}; r) \supset \{y \in E^n \mid |y| < \delta_2\}$ for any $r \in [0, r_0]$. Hence, if $0 \le r \le r_1 := \min(r_0, \delta_2)$, the g^r-geodesic $c(s, x; r) = c(1, sx : r)$ ($|x| = \delta_1$, $s \ge 0$) is just a ray when $s \ge 1$, at least for a while, from $c(1, x : r)$ in the direction $\dot c(1, x; r)$. Since $(c(1, x : r), \dot c(1, x : r))$ ($|x| = \delta_1$) varies continuously with $r \in [0, r_1]$ and $(c(1, x : 0), \dot c(1, x : 0)) = (x, x)$, the g^r-geodesics from 0 do not intersect with each other if $r > 0$ is sufficiently small. ∎

Thus we obtain the desired space $E^n(P^0)$. Apart from the origin (precisely, if $|y| \ge 2r/3$) the global coordinates y, x are described as

$$\begin{aligned} y(x) &= y(\frac{x}{|x|}) + \sum \frac{\partial y}{\partial x_B}(\frac{x}{|x|})(x_B - \frac{x_b}{|x|}), \quad \left|(\partial/\partial x)^{\mathbb{B}} y(x)\right| \le C_{\mathbb{B}}\, |x|^{1-|\mathbb{B}|}, \\ x(y) &= x(\frac{y}{|y|}) + \sum \frac{\partial x}{\partial y_A}(\frac{y}{|y|})(y_A - \frac{y_A}{|y|}), \quad \left|(\partial/\partial y)^{\mathbb{A}} x(y)\right| \le C_{\mathbb{A}}\, |y|^{1-|\mathbb{A}|}. \end{aligned} \tag{2.4}$$

Added to the canonical frames $(\partial/\partial x_\bullet)$, $(dx_\bullet)$, we will consider the canonical ones $(\partial/\partial y_\bullet)$, $(dy_\bullet)$ and set

$$e_\bullet(x) = (\partial/\partial y_\bullet)(x) \cdot \mathcal{V}_\bullet(x), \qquad e^\bullet(x) = (dy_\bullet)(x) \cdot \mathcal{V}^\bullet(x),$$

where $(\partial/\partial y_\bullet)(x)$, etc., denote $(\partial/\partial y_\bullet)(y(x))$, etc., calculated in the coordinates x, i.e., $(\partial/\partial y_\bullet)(x) = (\partial/\partial x_\bullet)(x) \cdot \frac{\partial x_\bullet}{\partial y_\bullet}(y(x))$, etc. Hence we have $\mathcal{V}_\bullet(x) = \Big(\frac{\partial y_\bullet}{\partial x_\bullet}(x)\Big) V_\bullet(x)$.

Lemma 2.2 *Apart from the origin, $\mathcal{V}_\bullet(x)$ is an orthogonal matrix and, for each multi-index $\mathbb{B}$, there is a constant $C_{\mathbb{B}} > 0$ such that*

$$\Big|(\partial/\partial x)^{\mathbb{B}}\mathcal{V}_\bullet(x)\Big| \le C_{\mathbb{B}}\, |x|^{-|\mathbb{B}|}, \quad \Big|(\partial/\partial x)^{\mathbb{B}} V_\bullet(x)\Big| \le C_{\mathbb{B}}|x|^{-|\mathbb{B}|},$$
$$\Big|(\partial/\partial x)^{\mathbb{B}} \det V_\bullet(x)\Big| \le C_{\mathbb{B}}|x|^{-|\mathbb{B}|}, \quad \Big|(\partial/\partial x)^{\mathbb{B}}\omega^\alpha_\beta(\partial/\partial x_A)\Big| \le C_{\mathbb{B}}|x|^{-1-|\mathbb{B}|}.$$

Also $\mathcal{V}^\bullet(x)$, $V^\bullet(x)$, $\det V^\bullet(x)$ are estimated similarly.

Proof. $\mathcal{V}_\bullet(x)$ is orthogonal because, apart from the origin, $e_\bullet(x)$ and $(\partial/\partial y_\bullet)(x)$ are both g^E-orthonormal. Since the vector fields $(\partial/\partial y_\bullet)(sx) \cdot \mathcal{V}_\bullet(x/|x|)$ and $e_\bullet(sx) = (\partial/\partial y_\bullet)(sx) \cdot \mathcal{V}_\bullet(sx)$ along the geodesic $y(sx)$ ($|x|^{-1} \le s \le 1$) are parallel and coincide when $s = |x|^{-1}$, they are the same ones, hence, we have $\mathcal{V}_\bullet(x) = \mathcal{V}_\bullet(x/|x|)$, which provides the estimates on $\mathcal{V}_\bullet$, $\mathcal{V}^\bullet$, $\omega^{i_1}_{i_2}(\partial/\partial x_j) = \sum \mathcal{V}^{Ci_1}\frac{\partial \mathcal{V}_{Ci_2}}{\partial x_j}$. By using (2.4), also the remaining estimates will be shown. ∎

Next, following the argument by McKean-Singer [7], we will actually construct the heat kernel (1.10) by applying Levi's iteration method. Let us take a ∇-normal coordinate system $\mathcal{E}^M : W \to E^n$, i.e., $\exp^\nabla(e_\bullet(x')\cdot\mathcal{E}^M(x',x)) = x$, where W is a neighborhood of the diagonal set in $E^n(P^0) \times E^n(P^0)$. On (E^n, y) we consider the ∇^E-normal coordinate system $\tilde{\mathcal{E}}^E : E^n(P^0) \times E^n(P^0) \to E^n$, $\tilde{\mathcal{E}}^E(y', y) = y - y'$, i.e., $\exp^{\nabla^E}((\partial/\partial y_\bullet)(y') \cdot \tilde{\mathcal{E}}^E(y', y)) = y$, and set $\mathcal{E}^E(x', x) = \tilde{\mathcal{E}}^E(y(x'), y(x))$, i.e., $\exp^{\nabla^E}((\partial/\partial y_\bullet)(x')\cdot\mathcal{E}^E(x', x)) = x$. Then we put

$$\begin{aligned} r_M(t, x, x') &= \sum e^I(x) \boxtimes e^I(x') \cdot r_t(\mathcal{E}^M(x', x)) \\ &\qquad\qquad (\text{on } E^n(P^0) \times \{x' \in U^0 \mid |y(x')| < r_0\}), \\ r_E(t, x, x') &= \mathfrak{r}_E(t, y, y')\Big|_{(y,y')=(y(x),y(x'))} \\ &= \sum e^I(x) \boxtimes e^{I'}(x') \cdot\ \det \mathcal{V}_{IJ}(x) \det \mathcal{V}_{I'J}(x')\, r_t(\mathcal{E}^E(x', x)) \\ &\qquad\qquad (\text{on } E^n(P^0) \times E^n(P^0)). \end{aligned}$$

Here, $r_M(t, x, x')$ is well-defined on the region (that is, W can be assumed to contain it) because of Remark to Lemma 2.1. Note that $r_E(t, x, x')$ is $\mathfrak{r}_E(t, y, y')$ (on (E^n, y)) calculated in the coordinates (x, x'), and $\mathcal{V}_{IJ}$ is the matrix $(\mathcal{V}_{ij})_{(i,j)\in I\times J}$ (refer to Lemma 2.2). In addition, we will take non-negative smooth functions $\tilde\rho_M(y)$, $\tilde\rho_E(y)$ such that $\{\tilde\rho^2_M(y), \tilde\rho^2_E(y)\}$ is a partition of unity subordinated to the cover $\{\{y \in E^n(P^0) \mid |y| < 2r\}, \{y \in E^n(P^0) \mid |y| > r\}\}$, where we assume that $r > 0$ has been taken so small that

$0 < 2r \le r_0$. We define the first approximation to the heat kernel by

$$r(t,x,x') = \rho_M(x)\rho_M(x')\, r_M(t,x,x') + \rho_E(x)\rho_E(x')\, r_E(t,x,x'),$$

where we set $\rho_M(x) = \tilde{\rho}_M(y(x))$, etc. This is more geometric than that in [7] (given by freezing the coefficients of the principal part of Δ at a point) and certainly satisfies: For any $\varphi \in \Omega_0^q E^n(P^0)$,

$$\lim_{t\to 0}\int r(t,x,x') \wedge \star\varphi(x') = \varphi(x), \qquad \lim_{t\to 0}\int \varphi(x) \wedge \star r(t,x,x') = \varphi(x') \tag{2.5}$$

in the $|\cdot|_g$-norm (the pointwise norm with respect to the metric g) and also in the L^2_g-norm, where we put $\star = \star_g$. Let us set $q(t,x,x') = (\frac{\partial}{\partial t} + \Delta_{E(P^0)})r(t,x,x')$ and $q^1 = q$, $q^2 = q\#q^1$, $q^3 = q\#q^2, \ldots$ $(\# = \#_g)$. Then we define

$$p = \sum_{k=0}^{\infty}(-1)^k r\#q^k \quad (r\#q^0 := r), \qquad R_{k_0}(p) = \sum_{k\ge k_0}(-1)^k r\#q^k,$$
$$q_\infty = \sum_{k=1}^{\infty}(-1)^k q^k, \qquad R_{k_0}(q_\infty) = \sum_{k\ge k_0}(-1)^k q^k.$$

Proposition 2.3 *We have $e^{-t\Delta_{E(P^0)}}(x,x') = p(t,x,x')$. To be precise:*

(1) *The forms q^k, $r\#q^k$, $R_{k_0}(q_\infty)$ and $R_{k_0}(p)$ are all well-defined and smooth on $(0,\infty) \times E^n(P^0) \times E^n(P^0)\, (\ni (t,x,x'))$. The last two forms are termwisely differentiable. For every integer $m \ge 0$ and multi-indices $\mathbb{A}$, $\mathbb{A}'$, there exist constants $B_k\, (= B_{(k,m,\mathbb{A},\mathbb{A}')}) > 0$, etc., and exponentially decaying functions $K_k(\mathcal{E})\, (= K_{(k,m,\mathbb{A},\mathbb{A}')}(\mathcal{E}))$ on $E^n\, (\ni \mathcal{E})$, i.e., $C_1\exp(-C_2|\mathcal{E}|^2)$ with some $C_i > 0$ $(i = 1,2)$, such that, on $(0,T_0] \times E^n(P^0) \times E^n(P^0)$,*

$$\Big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} q^k(t,x,x')\Big|_g \le B_k t^{(k-|\mathbb{A}|-|\mathbb{A}'|)/2-m-(n+2)/2} K_k(\iota_{1/t}(x-x')), \tag{2.6}$$

$$\Big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} (r\#q^k)(t,x,x')\Big|_g \le C_k t^{(k-|\mathbb{A}|-|\mathbb{A}'|)/2-m-n/2} K_k(\iota_{1/t}(x-x')), \tag{2.7}$$

$$\begin{aligned}&\big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'}\, R_{k_0}(q_\infty)(t,x,x')\big|_g \\ &\qquad \le B(k_0)\, t^{(k_0-|\mathbb{A}|-|\mathbb{A}'|)/2-m-(n+2)/2} K_{k_0}(\iota_{1/t}(x-x')),\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'}\, R_{k_0}(p)(t,x,x')\big|_g \\ &\qquad \le C(k_0)\, t^{(k_0-|\mathbb{A}|-|\mathbb{A}'|)/2-m-n/2} K_{k_0}(\iota_{1/t}(x-x')),\end{aligned} \tag{2.9}$$

where we set $e_{\mathbb{A},x} = e_{A_1,x}\cdots e_{A_{|\mathbb{A}|},x}$, etc. The last two are the estimates of sums of the $|\cdot|_g$-norms of their termwise differentials.

(2) *The form $p(t,x,x')$ is smooth on $(0,\infty) \times E^n(P^0) \times E^n(P^0)$ and satisfies*

$$\Big(\frac{\partial}{\partial t} + \Delta_{E(P^0)}\Big)p(t,x,x') = 0. \tag{2.10}$$

In addition, for every $\varphi \in \Omega_0^q E^n(P^0)$,

$$\lim_{t\to 0} \int p(t,x,x') \wedge \star\varphi(x') = \varphi(x), \qquad \lim_{t\to 0} \int \varphi(x) \wedge \star p(t,x,x') = \varphi(x') \tag{2.11}$$

in the $|\cdot|_g$-norm and in the L_g^2-norm. Further, $\phi(t,x) := \int p(t,x,x') \wedge \star\varphi(x')$ belongs to the domain of $\Delta_{E(P^0)}$, the integral $\Phi(t) := \int dV_g(x)\, |\phi(t,x)|_g^2$ is differentiable and the equality $(\partial/\partial t)\Phi(t) = \int dV_g(x)(\partial/\partial t)\, |\phi(t,x)|_g^2$ holds.

This is a generalization of the assertions in [7, §.3] and is shown similarly (see also [9, §.4]). The proof will follow some preparatory arguments.

In general, a smooth kernel $k(t,x,x') = \sum e^I(x) \boxtimes e^{I'}(x) \cdot k^{II'}(t,x,x')$ on $E^n(P^0) \times E^n(P^0)$ is said to be of type ℓ if each coefficient $k^{II'}$ is a finite sum of such functions as

$$\begin{aligned} \mathbf{K}_M^b(t,x,x') &= t^{-(n+2)/2+b/2} \rho_M(x',x)\, \mathcal{K}(\iota_{1/t}\mathcal{E}^M(x',x)) \\ \mathbf{K}_E^b(t,x,x') &= t^{-(n+2)/2+b/2} \rho_E(x',x)\, \mathcal{K}(\iota_{1/t}\mathcal{E}^E(x',x)) \end{aligned} \qquad (b \geq \ell). \tag{2.12}$$

Here $\mathcal{K}(\mathcal{E})$ is an exponentially decreasing function, i.e., $\mathcal{K}(\mathcal{E}) = C_1 \mathcal{E}^{\mathbb{A}} \exp(-C_2|\mathcal{E}|^2)$ ($\mathcal{E}^{\mathbb{A}} := \mathcal{E}_{A_1} \cdots \mathcal{E}_{A_{|\mathbb{A}|}}$) with some $\mathbb{A}$ and $C_i > 0$ ($i = 1,2$), and $\rho_M(x',x)$, $\rho_E(x',x)$ are smooth functions such that $\operatorname{supp} \rho_M \subset \{(x',x) \mid |y(x')| < 2r, |y(x)| < 2r\}$, $\operatorname{supp} \rho_E \subset \{(x',x) \mid |y(x')| > r, |y(x)| > r\}$ and, for every $\mathbb{A}$ and $\mathbb{A}'$, $\left|e_{\mathbb{A},x} e_{\mathbb{A}',x'} \rho_E(x',x)\right|$ is bounded. A kernel whose coefficients consist of the second type of functions is equivalently interpreted in the variable y as $\tilde{k}_E(t,y',y) = \sum (dy)^I(y) \boxtimes (dy)^{I'}(y') \cdot \tilde{k}_E^{II'}(t,y,y')$ whose coefficients are finite sums of such functions as $t^{-(n+2)/2+b/2} \tilde{\rho}_E(y',y) \mathcal{K}(\iota_{1/t}\tilde{\mathcal{E}}^E(y',y))$ ($b \geq \ell$), where $\tilde{\rho}_E(y',y)$ is a smooth function with $\operatorname{supp} \tilde{\rho}_E \subset \{(y',y) \mid |y'| > r, |y| > r\}$ and, for every $\mathbb{A}$, $\mathbb{A}'$, $\left|(\partial/\partial y)^{\mathbb{A}} (\partial/\partial y')^{\mathbb{A}'} \tilde{\rho}_E(y',y)\right|$ is bounded.

Lemma 2.4

(1) *The kernel $r(t,x,x')$ is of type 2 and $q(t,x,x')$ is of type 1. The support of the latter with respect to the variable x is contained in $U_{<2r} := \{x \in E^n(P^0) \mid |x| < 2r\}$.*

(2) *For a kernel $k(t,x,x')$ of type ℓ, $e_{\mathbb{A},x} e_{\mathbb{A}',x'} k(t,x,x')$ is a kernel of type $\ell - |\mathbb{A}| - |\mathbb{A}'|$ and $(\partial/\partial t) k(t,x,x')$ is of type $\ell - 2$.*

(3) *For a kernel $k(t,x,x')$ of type ℓ, there exist a constant $C > 0$ and an exponentially decaying function $K(\mathcal{E})$ such that, when $0 < t \leq T_0$,*

$$\left|k(t,x,x')\right|_g \leq C t^{\ell/2-(n+2)/2} K(\iota_{1/t}(x - x')), \quad \left\|k(t,x,x')\right\|_{L_g^1(x)} \leq C t^{\ell/2-1},$$

where $\|\cdot\|_{L^1(x)}$ is the L_g^1-norm with respect to the variable x. Also $\|k(t,x,x')\|_{L_g^1(x')}$ is estimated similarly.

(4) *For a kernel $k(t,x,x')$ of type ℓ, we have $e_{A,x} k(t,x,x') = -e_{A,x'} k(t,x,x') + k_*(t,x,x')$, where $k_*(t,x,x')$ is a kernel of type ℓ.*

Proof. In the coordinates $\mathcal{E}^M = \mathcal{E}^M(x', x)$, we have

$$e_{A,x} = \frac{\partial}{\partial \mathcal{E}_A^M} + \sum O(|\mathcal{E}^M|)\frac{\partial}{\partial \mathcal{E}_B^M}, \qquad e_{A,x'} = -\frac{\partial}{\partial \mathcal{E}_A^M} + \sum O(|\mathcal{E}^M|)\frac{\partial}{\partial \mathcal{E}_B^M},$$
$$\Delta_x = -\sum \frac{\partial}{\partial \mathcal{E}_i^M}\frac{\partial}{\partial \mathcal{E}_i^M} + \sum O(|\mathcal{E}^M|)\frac{\partial}{\partial \mathcal{E}_{j_1}^M}\frac{\partial}{\partial \mathcal{E}_{j_2}^M} + \sum O(|\mathcal{E}^M|^0)\frac{\partial}{\partial \mathcal{E}_j^M} + O(|\mathcal{E}^M|^0).$$

Hence, (1), (2), (4) are valid for the kernels k_M consisting of $\mathbf{K}_M$. We know immediately that these hold for another type of kernels k_E by examining them in the coordinates y. As for (3): It is obvious for k_M in turn. We have

$$\big|k_E(t,x,x')\big|_g \le C\, t^{\ell/2-(n+2)/2} K(\iota_{1/t}\mathcal{E}^E(x',x)), \tag{2.13}$$

$$\mathcal{E}^E(x',x) = y(x) - y(x') = \sum (x_B - x'_B)\int_0^1 ds\, \frac{\partial y}{\partial x_B}((1-s)x' + sx). \tag{2.14}$$

It follows from (2.4) and the argument in the proof of Lemma 2.1 that the matrix $(\partial y_A/\partial x_B)(x)$ $(x \in E^n(P^0))$ stays sufficiently near the identity matrix. Hence one may replace $\mathcal{E}^E(x',x)$ by $x - x'$ at (2.13). We obtain thus the first estimate for k_E, which certainly implies the second. ∎

Lemma 2.5 *Let $k_i(t,x,x')$ be kernels of types m_i (≥ 1). Then $(k_1\#\cdots\#k_j)(t,x,x')$ is well-defined and smooth. In addition, there exist a constant C $(= C_{(m,\mathbb{A},\mathbb{A}')}) > 0$ and an exponentially decaying function $K(\mathcal{E})$ $(= K_{(m,\mathbb{A},\mathbb{A}')}(\mathcal{E}))$ such that, when $0 < t \le T_0$,*

$$\begin{aligned}\big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'}(k_1\#\cdots\#k_j)(t,x,x')\big|_g \\ \le C\, t^{(\sum m_i - |\mathbb{A}| - |\mathbb{A}'|)/2 - m - (n+2)/2} K(\iota_{1/t}(x-x')).\end{aligned} \tag{2.15}$$

Proof. Let us prove the lemma by induction. It holds when $j = 1$ because of Lemma 2.4(2)(3). Assume $j > 1$ and set $k = k_2\#\cdots\#k_j$. Then, by Lemma 2.4(4), etc.,

$$\begin{aligned}&(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'}(k_1\#\cdots\#k_j)(t,x,x') \\ &= \sum_{m'+m''=m-1}\binom{m-1}{m'}\int ((\partial/\partial t)^{m'} e_{\mathbb{A},x}k_1)(\frac{t}{2},x,x'') \\ &\qquad\qquad \wedge \star\,((\partial/\partial t)^{m''} e_{\mathbb{A}',x'}k)(\frac{t}{2},x'',x') \\ &+ \sum_{|\mathbb{B}'|\le|\mathbb{A}'|}\int_0^{t/2} ds \int ((\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{B}',x''}k_1)(t-s,x,x'') \wedge \star k_{2\mathbb{B}'}(s,x'',x') \\ &+ \sum_{|\mathbb{B}|\le|\mathbb{A}|}\int_{t/2}^t ds \int k_{1\mathbb{B}}(t-s,x,x'') \wedge \star\,((\partial/\partial t)^m e_{\mathbb{B},x''} e_{\mathbb{A}',x'}k)(s,x'',x'),\end{aligned} \tag{2.16}$$

where $k_{1\mathbb{B}}$ is a kernel of type m_1 and $k_{2\mathbb{B}'}$ is a finite sum of convolutions of kernels of types m_i $(i = 2,\cdots,j)$. We suppose (2.15) holds for k, $k_{2\mathbb{B}'}$, etc. Then we have

$$\Big|\int ((\partial/\partial t)^{m'} e_{\mathbb{A},x}k_1)(\frac{t}{2},x,x'') \wedge \star((\partial/\partial t)^{m''} e_{\mathbb{A}',x'}k)(\frac{t}{2},x'',x')\Big|_g$$

$$
\begin{aligned}
&\le \int dV_g(x) \left| ((\partial/\partial t)^{m'} e_{\mathbb{A},x} k_1)(\frac{t}{2}, x, x'') \right|_g \left| ((\partial/\partial t)^{m''} e_{\mathbb{A}',x'} k)(\frac{t}{2}, x'', x') \right|_g \\
&\le K_0(\iota_{1/t}(x-x')) \max_{x''} \left| C t^{(m_1 - |\mathbb{A}|)/2 - m' - (n+2)/2} \tilde{K}_1(\iota_{1/t}(x-x'')) \right| \\
&\qquad \times \int dV_g(x'') \left| C\, t^{(\sum_{i>1} m_i - |\mathbb{A}'|)/2 - m'' - (n+2)/2} \tilde{K}(\iota_{1/t}(x''-x')) \right| \\
&\le K_0(\iota_{1/t}(x-x'))\, C' t^{(\sum m_i - |\mathbb{A}| - |\mathbb{A}'|)/2 - m - (n+2)/2},
\end{aligned}
$$

where K_0, $\tilde{K}_1$, $\tilde{K}$ are exponentially decaying. Further we have

$$
\begin{aligned}
&\left| \int_0^{t/2} ds \int ((\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{B}',x''} k_1)(t-s, x, x'') \wedge \star k_{2\mathbb{B}'}(s, x'', x') \right|_g \\
&\le \int_0^{t/2} ds \int dV_g(x'') \left| ((\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{B}',x''} k_1)(t-s, x, x'') \right|_g \left| k_{2\mathbb{B}'}(s, x'', x') \right|_g \\
&\le K_0(\iota_{1/t}(x-x')) \\
&\quad \times \int_0^{t/2} ds \max_{x''} \left| C\, (t-s)^{(m_1 - |\mathbb{A}| - |\mathbb{B}'|)/2 - m - (n+2)/2} \tilde{K}_1(\iota_{1/(t-s)}(x-x'')) \right| \\
&\qquad \times \int dV_g(x'') \left| C\, s^{\sum_{i>1} m_i/2 - (n+2)/2} \tilde{K}_{2\mathbb{B}'}(\iota_{1/s}(x''-x')) \right| \\
&\le K_0(\iota_{1/t}(x-x')) \int_0^{t/2} ds\, C'(t-s)^{(m_1 - |\mathbb{A}| - |\mathbb{B}'|)/2 - m - (n+2)/2} s^{\sum_{i>1} m_i/2 - 1} \\
&\le K_0(\iota_{1/t}(x-x'))\, t^{(\sum m_i - |\mathbb{A}| - |\mathbb{A}'|)/2 - m - (n+2)/2} \\
&\qquad \times \int_0^{1/2} d\sigma\, C'(1-\sigma)^{(m_1 - |\mathbb{A}| - |\mathbb{B}'|)/2 - m - (n+2)/2} \sigma^{\sum_{i>1} m_i/2 - 1}
\end{aligned}
$$

and the third term is estimated similarly. ∎

Proof of Proposition 2.3. Lemma 2.5 implies (2.6) and (2.7). Let us prove (2.9). Assume that $k_0 > 0$ is sufficiently large. Then there exists an exponentially decaying function $K_{k_0}(\mathcal{E})\,(= K_{(k_0,m,\mathbb{A},\mathbb{A}')}(\mathcal{E}))$ such that, for any $\ell \ge 1$,

$$
\begin{aligned}
&\left| \Big(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} q^{\ell k_0}(t, x, x') \right|_g \\
&\quad \le t^{\ell(k_0 - n)/2 - (|\mathbb{A}| + |\mathbb{A}'|)/2 - m - 1} K_{k_0}(\iota_{1/t}(x-x')) \\
&\qquad \times B_{k_0}^{\ell} \mathrm{vol}(U_{<2r})^{\ell-1} \frac{\Gamma((k_0 - n - |\mathbb{A}|)/2 - m)^{\ell-1} \Gamma((k_0 - n - |\mathbb{A}'|)/2)}{\Gamma(\ell(k_0 - n)/2 - (|\mathbb{A}| + |\mathbb{A}'|)/2 - m)}.
\end{aligned}
\tag{2.17}
$$

Indeed, we have

$$
\begin{aligned}
&\left| \Big(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} q^{2k_0}(t, x, x') \right|_g = \left| \Big((\partial/\partial t)^m e_{\mathbb{A},x} q^{k_0} \# e_{\mathbb{A}',x'} q^{k_0} \Big)(t, x, x') \right|_g \\
&\quad = \left| \int_0^t ds \int_{U_{<2r}} (\partial/\partial t)^m e_{\mathbb{A},x} q^{k_0}(t-s, x, x'') \wedge \star e_{\mathbb{A}',x'} q^{k_0}(s, x'', x') \right|_g \\
&\quad \le K_{k_0}(\iota_{1/t}(x-x'))\, B_{k_0}^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \int_0^t ds \int_{U_{<2r}} dV_g(x'')\,(t-s)^{(k_0-n-2)/2-|\mathbb{A}|/2-m} s^{(k_0-n-2)/2-|\mathbb{A}'|/2} \\
&\le t^{(k_0-n-2)/2+1-(|\mathbb{A}|+|\mathbb{A}'|)/2-m} K_{k_0}(\iota_{1/t}(x-x'))\, B_{k_0}^2 \operatorname{vol}(U_{<2r}) \\
&\quad \times \int_0^1 d\sigma\,(1-\sigma)^{(k_0-n-2)/2-|\mathbb{A}|/2-m} \sigma^{(k_0-n-2)/2-|\mathbb{A}'|/2} \\
&= t^{(k_0-n-2)/2+1-(|\mathbb{A}|+|\mathbb{A}'|)/2-m} K_{k_0}(\iota_{1/t}(x-x'))\, B_{k_0}^2 \operatorname{vol}(U_{<2r}) \\
&\quad \times \frac{\Gamma((k_0-n-2)/2+1-|\mathbb{A}|/2-m)\Gamma((k_0-n-2)/2+1-|\mathbb{A}'|/2)}{\Gamma(2(k_0-n-2)/2+2-(|\mathbb{A}|+|\mathbb{A}'|)/2-m)},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\Big((\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} q^{3k_0}(t,x,x')\Big|_g = \Big|\Big(((\partial/\partial t)^m e_{\mathbb{A},x} q^{k_0} \# e_{\mathbb{A}',x'} q^{2k_0}\Big)(t,x,x')\Big|_g \\
&\le K_{k_0}(\iota_{1/t}(x-x'))\, B_{k_0}^3 \operatorname{vol}(U_{<2r}) \\
&\quad \times \frac{\Gamma((k_0-n-2)/2+1-|\mathbb{A}|/2-m)\Gamma((k_0-n-2)/2+1-|\mathbb{A}'|/2)}{\Gamma(2(k_0-n-2)/2+2-|\mathbb{A}'|/2)} \\
&\quad \times \int_0^t ds \int_{U_{<2r}} dV_g(x'')\,(t-s)^{(k_0-n-2)/2-|\mathbb{A}|/2-m} s^{2(k_0-n-2)/2+1-|\mathbb{A}'|/2} \\
&\le t^{3(k_0-n-2)/2+2-(|\mathbb{A}|+|\mathbb{A}'|)/2-m} K_{k_0}(\iota_{1/t}(x-x'))\, B_{k_0}^3 \operatorname{vol}(U_{<2r})^2 \\
&\quad \times \frac{\Gamma((k_0-n-2)/2+1-|\mathbb{A}|/2-m)^2\Gamma((k_0-n-2)/2+1-|\mathbb{A}'|/2)}{\Gamma(3(k_0-n-2)/2+3-(|\mathbb{A}|+|\mathbb{A}'|)/2-m)},
\end{aligned}
$$

etc. Assuming $0 \le k < k_0$, in a way similar to (2.16) we write

$$
\begin{aligned}
&(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} (r \# q^{k+\ell k_0})(t,x,x') \\
&= \sum \binom{m-1}{m'} \int ((\partial/\partial t)^{m'} e_{\mathbb{A},x} (r \# q^k))(\frac{t}{2}, x, x'') \\
&\qquad\qquad\qquad \wedge \star\, ((\partial/\partial t)^{m''} e_{\mathbb{A}',x'} q^{\ell k_0})(\frac{t}{2}, x'', x') \\
&+ \int_0^{t/2} ds \int ((\partial/\partial t)^m e_{\mathbb{A},x} (r \# q^k))(t-s,x,x'') \wedge \star e_{\mathbb{A}',x'} q^{\ell k_0})(s,x'',x') \\
&+ \sum_{|\mathbb{B}| \le |\mathbb{A}|} \int_{t/2}^t ds \int (r \# q^k)_{1\mathbb{B}}(t-s,x,x'') \wedge \star\, ((\partial/\partial t)^m e_{\mathbb{B},x''} e_{\mathbb{A}',x'} q^{\ell k_0})(s,x'',x')
\end{aligned}
$$

and, using (2.7), (2.17), etc., we obtain

$$
\begin{aligned}
&\Big|(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} (r \# q^{k+\ell k_0})(t,x,x')\Big|_g \\
&\qquad \le K_{(k_0,k)}(\iota_{1/t}(x-x'))\, \frac{t^{(k_0-|\mathbb{A}|-|\mathbb{A}'|)/2-m-n/2}\, C^{\ell}_{(k_0)}}{\Gamma(\ell(k_0-n)/2-(|\mathbb{A}|+|\mathbb{A}'|)/2-m)}.
\end{aligned}
$$

Hence, $(\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} R_{k_0}(p) = \sum_{\ell=1,2,\ldots}^{0\le k<k_0} (\partial/\partial t)^m e_{\mathbb{A},x} e_{\mathbb{A}',x'} (r \# q^{k+\ell k_0})$ is estimated as at (2.9). Similarly (2.8) can be shown. Next, let us show (2). By (1), the convolutions $r \# q_\infty$, $q \# q_\infty$ are well-defined and smooth and

$$
\Big(\frac{\partial}{\partial t} + \Delta_{E(P^0)}\Big)(r \# q_\infty) = q_\infty + q \# q_\infty, \quad p = r + r \# q_\infty, \quad q \# q_\infty = -q - q_\infty.
$$

Thus (2) certainly holds up to (2.10). As to the first convergence at (2.11): By (2.9), for any $\ell \geq 0$,

$$
\begin{aligned}
&(2.18)\left|(1+|x|^{\ell}) \int R_{k_0}(p)(t,x,x') \wedge \star\varphi(x')\right|_g \\
&\quad \leq \int dV_g(x')\,(1+|t^{1/2}\iota_{1/t}(x-x')+x'|^{\ell}) C(k_0) t^{(k_0-n)/2} K_{k_0}(\iota_{1/t}(x-x'))\,\left|\varphi(x')\right|_g \\
&\quad \leq\ \tilde{C}(k_0) t^{k_0/2} \sum_{\ell' \leq \ell} \int dV_g(x')\, t^{-n/2} \tilde{K}_{k_0}(\iota_{1/t}(x-x'))(1+|x'|^{\ell'})\,\left|\varphi(x')\right|_g \\
&\quad \leq t^{k_0/2}\,\mathrm{sn}(\varphi),
\end{aligned}
$$

where $\mathrm{sn}(\varphi)$ is a semi-norm of $\varphi \in \Omega_0^q E^n(P^0)$. (In general, we set $\mathrm{sn}(\varphi) = \sup_{x \in E^n, |\mathbb{B}| \leq k} C\left|(1+|x|)^{\ell}(\partial/\partial x)^{\mathbb{B}}\varphi(x)\right|$ with some $\ell, k \in \mathbb{N}$ and a constant $C > 0$.) Hence, the integral $\int R_1(p)(t,x,x') \wedge \star\varphi(x')$ is bounded in the $|\cdot|_g$-norm and in the L_g^2 norm, and $\lim_{t \to 0} \int R_1(p)(t,x,x') \wedge \star\varphi(x') = 0$ in both norms, which, together with (2.5), implies the first convergence at (2.11). The remaining assertions will be obvious now. ∎

3 The proof of Theorem 1.1

In the section, we will prove:

Proposition 3.1 *The heat kernel $p_{(\varepsilon)}(t,x,x') := e^{-t\Delta_{(\varepsilon)}}(x,x')$ can be extended smoothly up to $[0,\varepsilon_0^{1/2}] \times (0,T_0] \times E^n \times E^n$ $(\ni (\varepsilon^{1/2},t,x,x'))$. As to the Taylor expansion*

$$
p_{(\varepsilon)}(t,x,x') = \sum_{0 \leq m < m_*} \varepsilon^{m/2}\, p_{m/2}(t,x,x') + \varepsilon^{m_*/2}\, p_{m_*/2}(\varepsilon^{1/2},t,x,x'),
$$

we have

$$
p_{m/2}(t,x,x') = \mathfrak{p}_{m/2}(t,x,x') \quad (0 \leq m < m_*). \tag{3.1}
$$

If this is valid, then we have the formal power series expansion (1.21) and also (1.23). Consequently we obtain the formula (1.24). (Note that thus Proposition 3.1 also implies that every differential of (1.10) can be asymptotically expanded at $(0,0)$.) Let us start our discussion with some preparations needed for the proof. We set $\# = \#_{g^E}$, $dV(x) = dV_{g^E}(x)$, $|\cdot| = |\cdot|_{g^E}$, $\|\cdot\| = \|\cdot\|_{L^2_{g^E}(x)}$, etc., for short.

3.1 Standard kernels on E^n

The argument in §.2 holds good for the standard $E^n = (E^n, x)$ because it may be regarded as a warped one. A kernel on E^n whose coefficients consist of $t^{-(n+2)/2+b/2}$

$\rho(x',x)\,\mathcal{K}(\iota_{1/t}(x-x'))$ will be called a standard one, where $|(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}\rho(x',x)|$ is assumed to be bounded for any $(\mathbb{A},\mathbb{A}')$ (see (2.12) around). Obviously Lemma 2.5 holds also for the standard kernels on E^n and we have:

Lemma 3.2 *Let k_i be standard kernels of types $m_i\,(\geq 1)$. Then the convolution $(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(t,x,x')$ is well-defined and smooth on $(0,\infty)\times E^n\times E^n$, and there exist a constant $B'>0$, an integer $N>0$ and an exponentially decaying function $K(\mathcal{E})$ such that, when $0<t\leq T_0$,*

$$\left|(\partial/\partial t)^d(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(t,x,x')\right| \leq B't^{\sum m_i/2-N-d-(n+2)/2}K(\iota_{1/t}(x-x'))\sum|x'^{\mathbb{C}'}|. \tag{3.2}$$

Here $B'>0$, $N>0$ and $\sum|x'^{\mathbb{C}'}|$, which is a finite sum, depend only on $(\mathbb{C}_i,\mathbb{B}_i)$, $(\mathbb{A},\mathbb{A}')$. In addition, for every $\varphi\in\Omega_0^qE^n$, the integral $\int(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)$ $(t,x,x')\wedge\star\varphi(x')$ is well-defined and rapidly decreasing, and there exists a semi-norm $\mathrm{sn}(\cdot)$ such that, when $0<t\leq T_0$, for any $\varphi\in\Omega_0^qE^n$,

$$\left|\int(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(t,x,x')\wedge\star\varphi(x')\right|\leq t^{\sum m_i/2-1}\mathrm{sn}(\varphi),$$

$$\left\|\int(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(t,x,x')\wedge\star\varphi(x')\right\|\leq t^{\sum m_i/2-1}\mathrm{sn}(\varphi). \tag{3.3}$$

Proof. By Lemma 2.4(4) for E^n or by direct calculation,

$$\begin{aligned}
&x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}x'^{\mathbb{C}'}(\partial/\partial x')^{\mathbb{B}'}t^{-(n+2)/2+b/2}\mathcal{K}(\iota_{1/t}(x-x'))\\
&=\sum_{|\tilde{\mathbb{B}}|\leq|\mathbb{B}|+|\mathbb{B}'|}x^{\tilde{\mathbb{C}}}(\partial/\partial x)^{\tilde{\mathbb{B}}}\,t^{-(n+2)/2+b/2+\ell/2}\mathcal{K}(\iota_{1/t}(x-x'))\\
&=\sum_{|\tilde{\mathbb{B}}'|\leq|\mathbb{B}|+|\mathbb{B}'|}x'^{\tilde{\mathbb{C}}'}(\partial/\partial x')^{\tilde{\mathbb{B}}'}\,t^{-(n+2)/2+b/2+\ell'/2}\mathcal{K}(\iota_{1/t}(x-x')),
\end{aligned} \tag{3.4}$$

where the exponentially decreasing functions $\mathcal{K}$ appearing in the second and third lines, which differ from that in the first line, depend on the respective indices $(\tilde{\mathbb{B}},\tilde{\mathbb{C}})$, etc. By integration by parts, hence we have

$$\begin{aligned}
&(\partial/\partial t)^d(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(t,x,x')\\
&=\sum_{|\mathbb{B}|\leq|\mathbb{A}|+|\mathbb{A}'|+\sum|\mathbb{B}_i|}(\partial/\partial t)^dx'^{\mathbb{C}}(\partial/\partial x')^{\mathbb{B}}(k_1\#\cdots\#k_j)(t,x,x'),
\end{aligned}$$

where, again, the kernels k_i appearing in the second line, which differ from those in the first line, depend on the respective indices, but are of the same types as those of the original k_i. Hence, Lemma 2.5 for E^n implies (3.2) and the others can be shown by integration by parts. ∎

3.2 Rough estimation of remainder term

Putting $r(t,x,x') = \sum e^I(x) \boxtimes e^{I'}(x') \cdot \rho_\circ(x',x)\, r_t(\mathcal{E}^\circ(x',x))$ ($\circ = M$ or E), let us define

$$r_{(\varepsilon)}(t,x,x') = \sum (dx)^I(x) \boxtimes (dx)^{I'}(x') \\ \cdot \rho_\circ(\iota_\varepsilon(x'), \iota_\varepsilon(x))\, r_t(\mathcal{E}^{\circ(\varepsilon)}(x',x)) \det V^\bullet(\iota_\varepsilon(x'))$$

with $\mathcal{E}^{\circ(\varepsilon)}(x',x) := \iota_{1/\varepsilon}\mathcal{E}^\circ(\iota_\varepsilon(x'), \iota_\varepsilon(x))$, and $q_{(\varepsilon)}(t,x,x') = (\frac{\partial}{\partial t} + \Delta_{(\varepsilon)}) r_{(\varepsilon)}(t,x,x')$. Then we have

$$p_{(\varepsilon)}(t,x,x') = \sum_{k=0}^{\infty} (-1)^k (r_{(\varepsilon)} \# q_{(\varepsilon)}^k)(t,x,x') \qquad (r_{(\varepsilon)} \# q^0_{(\varepsilon)} := r_{(\varepsilon)}) \tag{3.5}$$

(cf. (1.13)). The coefficient of the remainder term $R_{k_0}(p_{(\varepsilon)}) := \sum_{k \ge k_0} (-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)}$ is described as

$$R_{k_0}(p_{(\varepsilon)})^{II'}(t,x,x') = \varepsilon^{n/2} R_{k_0}(p)^{II'}(t\varepsilon, \iota_\varepsilon(x), \iota_\varepsilon(x')) \det V^\bullet(\iota_\varepsilon(x')).$$

Lemma 3.3 *There exist a constant $C'(k_0) > 0$ and an exponentially decaying function $K(\mathcal{E})$ such that, on $(0, \varepsilon_0^{1/2}] \times (0, T_0] \times E^n \times E^n$,*

$$\left| (\partial/\partial t)^d (\partial/\partial x)^{\mathbb{A}} (\partial/\partial x')^{\mathbb{A}'} (\partial/\partial \varepsilon^{1/2})^m\, R_{k_0}(p_{(\varepsilon)})(t,x,x') \right| \\ \le C'(k_0)\, \varepsilon^{(k_0-m)/2}\, t^{(k_0 - |\mathbb{A}| - |\mathbb{A}'|)/2 - d - m - n/2} K(\iota_{1/t}(x-x')) \sum |x'^{\mathbb{C}'}|.$$

Proof. The differential of $\varepsilon^{-n/2} R_{k_0}(p_{(\varepsilon)})^{II'}(t,x,x')$ by $(\partial/\partial t)^d (\partial/\partial x)^{\mathbb{A}} (\partial/\partial x')^{\mathbb{A}'}$ $(\partial/\partial \varepsilon^{1/2})^{m'}$ is described as

$$\sum_{d' + (|\mathbb{B}| + |\mathbb{B}'|)/2 \le d + (|\mathbb{A}| + |\mathbb{A}'|)/2 + m'} \varepsilon^{d' + (|\mathbb{B}| + |\mathbb{B}'|)/2 - m'/2} h(\varepsilon^{1/2}, t, x, x')\, \mathcal{B}(\iota_\varepsilon(x'), \iota_\varepsilon(x)) \\ \times \Big((\partial/\partial t)^{d'} e_{\mathbb{B},x} e_{\mathbb{B}',x'} R_{k_0}(p) \Big)^{II'} (t\varepsilon, \iota_\varepsilon(x), \iota_\varepsilon(x')),$$

where $h(\varepsilon^{1/2}, t, x, x')$ is a polynomial and $|\mathcal{B}(y', y)|$ is bounded on $E^n \times E^n$ (refer to Lemma 2.2). Hence (2.9) implies the lemma. ∎

3.3 Detailed investigation of the term $(-1)^k (r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t,x,x')$

We will investigate closely the terms $(-1)^k (r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t,x,x')$ appearing in (3.5). First, let us consider the coordinate transformation $(x', x - x) \leftrightarrow (x', \mathcal{E}^{\circ(\varepsilon)})$.

Lemma 3.4 *The coordinate system $\mathcal{E}^{M(\varepsilon)}(x',x)$ can be extended smoothly up to the domain* $\operatorname{dom} \mathcal{E}^{M(\bullet)} := \{ (\varepsilon^{1/2}, x', x) \in [0, \varepsilon_0^{1/2}] \times E^n \times E^n \mid (\iota_\varepsilon(x'), \iota_\varepsilon(x)) \in W \}$ *and so* *can be the system* $\mathcal{E}^{E(\varepsilon)}(x',x)$ *up to* $\operatorname{dom}\mathcal{E}^{E(\bullet)} := [0,\varepsilon_0^{1/2}] \times E^n \times E^n$. *The coordinate transformation is then extended smoothly up to* $\varepsilon^{1/2}=0$ *and*

$$\begin{aligned}
\mathcal{E}_B^{\circ(\varepsilon)}(x',x) &= \sum_{|\mathbb{C}'|+|\mathbb{D}|=1,2}^{|\mathbb{D}|>0} x'^{\mathbb{C}'}(x-x')^{\mathbb{D}}\,\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x')) \\
&= (x-x')_B + \sum_{|\mathbb{C}'|+|\mathbb{D}|=2}^{|\mathbb{D}|>0} x'^{\mathbb{C}'}(x-x')^{\mathbb{D}}\,\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x')),
\end{aligned}$$

$$\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x')) = \varepsilon^{(|\mathbb{C}'|+|\mathbb{D}|-1)/2}\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\iota_\varepsilon(x'),\iota_\varepsilon(x-x')),$$

$$\begin{aligned}
(x-x')_D &= \sum_{|\mathbb{C}'|+|\mathbb{B}|=1,2}^{|\mathbb{B}|>0} x'^{\mathbb{C}'}(\mathcal{E}^{\circ(\varepsilon)})^{\mathbb{B}}\,\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{B});D}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)}) \\
&= \mathcal{E}_D^{\circ(\varepsilon)} + \sum_{|\mathbb{C}'|+|\mathbb{B}|=2}^{|\mathbb{B}|>0} x'^{\mathbb{C}'}(\mathcal{E}^{\circ(\varepsilon)})^{\mathbb{B}}\,\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{B});D}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)}),
\end{aligned}$$

$$\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)}) = \varepsilon^{(|\mathbb{C}'|+|\mathbb{D}|-1)/2}\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)}).$$

Here, each coefficient $\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{D});B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x'))$ *is smooth on* $\operatorname{dom}\mathcal{E}^{\circ(\bullet)}$ *and quasi-bounded in the sense: Every (high order) differential relative to the variables* $(\varepsilon^{1/2},x',x-x')$ *is described as a finite sum of such functions as* $x'^{\mathbb{C}''}(x-x')^{\mathbb{D}'}\mathcal{B}^{\circ}_{(\mathbb{C}'',\mathbb{D}')}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x'))$, *where* $|\mathcal{B}^{\circ}_{(\mathbb{C}'',\mathbb{D}')}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x'))|$ *are bounded on* $\operatorname{dom}_r\mathcal{E}^{M(\bullet)} := \{(\varepsilon^{1/2},x',x)\in\operatorname{dom}\mathcal{E}^{M(\bullet)} \mid |y(\iota_\varepsilon(x'))|\le 2r, |y(\iota_\varepsilon(x))|\le 2r\}$ *if* $\circ=M$, *and on* $\operatorname{dom}\mathcal{E}^{E(\bullet)}$ *if* $\circ=E$. *Also each coefficient* $\mathcal{B}^{\circ}_{(\mathbb{C}',\mathbb{B});D}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)})$ *is smooth and quasi-bounded in a similar sense.*

Proof. The convergence $\lim_{\varepsilon^{1/2}\to 0}\mathcal{E}_B^{\circ(\varepsilon)}(x',x) = (x-x')_B$ will be obvious. Quasi-boundedness in the case $\circ = E$ will come from (2.4) and such an expansion as at (2.14). Note that also $x-x' = (\mathcal{E}^E_{x'})^{-1}(y) - (\mathcal{E}^E_{x'})^{-1}(0) = x(x',\mathcal{E}^E) - x(x',0)$ ($y=\mathcal{E}^E(x',x) = \mathcal{E}^E_{x'}(x)$) can be expanded similarly. ∎

In general, if we regard a quasi-bounded function $\mathcal{B}(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon(x-x'))$ naturally as a function of $(\varepsilon^{1/2},\iota_\varepsilon(x'),\iota_\varepsilon\mathcal{E}^{\circ(\varepsilon)})$ then it is quasi-bounded, and the converse is also true. Accordingly one may express a quasi-bounded function simply as $\mathcal{B}(\varepsilon^{1/2})$ in the following.

Now, for a kernel $k(t,x,x') = \sum e^I(x)\boxtimes e^{I'}(x')\cdot\mathbf{K}^b_\circ(t,x,x')$ (of type ℓ) (see (2.12)), we set $k_{(\varepsilon)}(t,x,x') = \sum (dx)^I(x)\boxtimes(dx)^{I'}(x')\cdot\mathbf{K}^b_{\circ(\varepsilon)}(t,x,x')$ with

$$\mathbf{K}^b_{\circ(\varepsilon)}(t,x,x') = t^{-(n+2)/2+b/2}\rho_\circ(\iota_\varepsilon(x'),\iota_\varepsilon(x))\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)}(x',x)),$$

which we call an (ε)-kernel (of type ℓ). For example, since the kernel $r(t,x,x')\det V^\bullet(x')$ is of type 2 (by Lemma 2.2), $r_{(\varepsilon)}(t,x,x')$ is an (ε)-kernel of type 2.

Lemma 3.5

(1) *The function* $\mathbf{K}^b_{\circ(\varepsilon)}(t,x,x')$ *can be extended smoothly up to* $\varepsilon^{1/2}=0$ *and has a Taylor expansion*

$$\mathbf{K}^b_{\circ(\varepsilon)} = \sum_{0\le m<m_*} \varepsilon^{m/2}\,\mathbf{K}^b_{\circ,m/2} + \varepsilon^{m_*/2}\,\mathbf{K}^b_{\circ,m_*/2}(\varepsilon^{1/2}),$$
$$\mathbf{K}^b_{M,0/2} = t^{-(n+2)/2+b/2}\rho_M(0,0)\,\mathcal{K}(\iota_{1/t}(x-x')),\quad \mathbf{K}^b_{E,0/2}=0.$$

Further there exist finite sum expressions $(\ell\ge 0)$

$$\mathbf{K}^b_{M,m/2} = \sum x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}\,t^{-(n+2)/2+b/2+\ell/2}\mathcal{K}(\iota_{1/t}(x-x')),\quad \mathbf{K}^b_{E,m/2}=0, \tag{3.6}$$

$$\mathbf{K}^b_{\circ,m_*/2}(\varepsilon^{1/2}) = \sum x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}\,t^{-(n+2)/2+b/2+\ell/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}. \tag{3.7}$$

Here, in general, we set $[f(\delta,\ldots)]^{\delta}_m = \int_0^1 d\sigma_1\cdots\int_0^{\sigma_{m-1}} d\sigma_m f(\sigma_m\delta,\ldots)\mathrm{poly}(\sigma_m)$, *where* $\mathrm{poly}(\sigma_m)$ *is a polynomial of* σ_m. *(At* (3.7) *we may set* $\mathrm{poly}(\sigma_{m_*})=1$.*) The functions* $\mathcal{K}(\mathcal{E})$ *are exponentially decreasing and* $\mathcal{B}(\varepsilon^{1/2})$ *are quasi-bounded.*

(2)(cf. (3.4)) *We have*

$$\begin{aligned}
&x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}x'^{\mathbb{C}'}(\partial/\partial x')^{\mathbb{B}'}(\partial/\partial\varepsilon^{1/2})^m t^{-(n+2)/2+b/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}\\
&= \sum_{|\tilde{\mathbb{B}}|\le|\mathbb{B}|+|\mathbb{B}'|+m} x^{\tilde{\mathbb{C}}}(\partial/\partial x)^{\tilde{\mathbb{B}}}\,t^{-(n+2)/2+b/2+\ell/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*}\\
&= \sum_{|\tilde{\mathbb{B}}'|\le|\mathbb{B}|+|\mathbb{B}'|+m} x'^{\tilde{\mathbb{C}}'}(\partial/\partial x')^{\tilde{\mathbb{B}}'}\,t^{-(n+2)/2+b/2+\ell'/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]^{\varepsilon^{1/2}}_{m_*},
\end{aligned}\tag{3.8}$$

where $\mathcal{B}(\varepsilon^{1/2})$, $\mathcal{K}(\mathcal{E})$ *appearing in the second and third lines depend on the respective indices.*

Proof. Lemma 3.4 implies that $\mathbf{K}^b_{\circ(\varepsilon)}$ is extended smoothly and the coefficient $\mathbf{K}_{M,m/2}$ $(m>0)$ can be described as a finite sum of such functions as $t^{-(n+2)/2+b/2}$ $x'^{\mathbb{C}_1}(x-x')^{\mathbb{D}}(\partial/\partial y)^{\mathbb{B}}\mathcal{K}(\iota_{1/t}(y))\Big|_{y=x-x'}$. We can alter the function $x'^{\mathbb{C}_1}(x-x')^{\mathbb{D}}(\partial/\partial y)^{\mathbb{B}}$ $\mathcal{K}(\iota_{1/t}(y))$ successively as follows:

$$\begin{aligned}
&x'^{\mathbb{C}_1}(x-x')^{\mathbb{D}}(\partial/\partial y)^{\mathbb{B}}\mathcal{K}(\iota_{1/t}(y)) \Rightarrow x'^{\mathbb{C}_1}\cdot(\partial/\partial y)^{\mathbb{B}_1}\Big(y^{\mathbb{D}_1}\mathcal{K}_1(\iota_{1/t}(y))\Big)\\
&\Rightarrow t^{|\mathbb{D}_1|/2}x'^{\mathbb{C}_1}(\partial/\partial y)^{\mathbb{B}_1}\mathcal{K}_2(\iota_{1/t}(y)) \Rightarrow t^{|\mathbb{D}_1|/2}x'^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}\mathcal{K}_2(\iota_{1/t}(x-x'))\\
&\Rightarrow t^{|\mathbb{D}_1|/2}(\partial/\partial x)^{\mathbb{B}_2}\Big(x^{\mathbb{C}_3}(x-x')^{\mathbb{D}_3}\mathcal{K}_3(\iota_{1/t}(x-x'))\Big)\\
&\Rightarrow t^{(|\mathbb{D}_1|+|\mathbb{D}_3|)/2}x^{\mathbb{C}_4}(\partial/\partial x)^{\mathbb{B}_3}\mathcal{K}_5(\iota_{1/t}(x-x')).
\end{aligned}\tag{3.9}$$

Thus we obtain the first formula at (3.6). Obviously we have $\mathbf{K}_{E,m/2}=0$. Taylor's integration formula yields that the remainder term $\mathbf{K}_{\circ,m_*/2}(\varepsilon^{1/2})$ can be expressed as a finite sum of such functions as $t^{-(n+2)/2+b/2}\Big[x'^{\mathbb{C}_1}(\mathcal{E}^{\circ(\varepsilon)})^{\mathbb{A}}\mathcal{B}(\varepsilon^{1/2})(\partial/\partial\mathcal{E}^{\circ(\varepsilon)})^{\mathbb{B}}\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]_{m_*}^{\varepsilon^{1/2}}$. In the successive alterations at (3.9), the change of variables $(x', x-x') \leftrightarrow (x', x)$ was used. Here, using the changes of variables $(x', \mathcal{E}^{\circ(\varepsilon)}) \leftrightarrow (x', x-x') \leftrightarrow (x', x)$ (see Lemma 3.4), similarly we obtain (3.7). Next, let us show (2). By Lemma 2.4(4) (for $e^{(\varepsilon)}_{A,x}$), we have

$$x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}\Big(t^{-(n+2)/2+b/2}\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big)$$
$$=\sum x'^{\mathbb{C}'_1}(\partial/\partial x')^{\mathbb{B}_1}\Big(t^{-(n+2)/2+b/2+\ell/2}\mathcal{B}_1(\varepsilon^{1/2})\,\mathcal{K}_1(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big).$$

In addition, obviously we have

$$(\partial/\partial\varepsilon^{1/2})\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]_{m_*}^{\varepsilon^{1/2}}=\Big[(\partial/\partial\varepsilon^{1/2})\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]_{m_*}^{\varepsilon^{1/2}}.$$

Recalling the actions of $\partial/\partial\varepsilon^{1/2}$ on $\mathcal{B}(\varepsilon^{1/2})$ and $\mathcal{E}^{\circ(\varepsilon)}$ (see Lemma 3.4), we obtain (3.8). ∎

Lemma 3.6 *Let us set*

$$(3.10)\quad \begin{aligned}\mathbf{k}_i &= \sum (dx)^I(x)\boxtimes(dx)^{I'}(x')\cdot t^{-(n+2)/2+m_i/2}\mathcal{K}(\iota_{1/t}(x-x')),\\ \mathbf{k}_i(\varepsilon^{1/2}) &= \sum (dx)^I(x)\boxtimes(dx)^{I'}(x')\cdot t^{-(n+2)/2+m_i/2}\Big[\mathcal{B}(\varepsilon^{1/2})\,\mathcal{K}(\iota_{1/t}\mathcal{E}^{\circ(\varepsilon)})\Big]_{n_i}^{\varepsilon^{1/2}},\end{aligned}$$

where $m_i \geq 1$, $n_i \geq 0$. Even if we change each standard kernel k_i into $\mathbf{k}_i$ or $\mathbf{k}_i(\varepsilon^{1/2})$ arbitrarily, Lemma 3.2 still holds and (3.2) can be generalized into

$$\Big|(\partial/\partial t)^d(\partial/\partial x)^{\mathbb{A}}(\partial/\partial x')^{\mathbb{A}'}(\partial/\partial\varepsilon^{1/2})^m$$
$$(x^{\mathbb{C}_1}(\partial/\partial x)^{\mathbb{B}_1}k_1\#\cdots\#x^{\mathbb{C}_j}(\partial/\partial x)^{\mathbb{B}_j}k_j)(\varepsilon^{1/2},t,x,x')\Big|$$
$$\leq B't^{\sum m_i/2-N-d-(n+2)/2}K(\iota_{1/t}(x-x'))\sum|x'^{\mathbb{C}'}|$$

on $(0,\varepsilon_0^{1/2}]\times(0,T_0]\times E^n\times E^n$.

Proof. Added to (3.4), we have (3.8). The lemma will be proved in the same way as Lemma 3.2. ∎

Lemma 3.7 *If $k_{i,(\varepsilon)}$ are (ε)-kernels of types $b_i\,(\geq 1)$, then the convolution $(k_{1,(\varepsilon)}\#\cdots\#k_{j,(\varepsilon)})(t,x,x')$ can be extended smoothly up to $\varepsilon^{1/2}=0$.*

Proof. Lemma 3.5 asserts that each $k_{i,(\varepsilon)}$ is extended smoothly up to $\varepsilon^{1/2}=0$. Let us denote its expansion by $k_{i,(\varepsilon)}\,(=\,k_{i,0/2}(\varepsilon^{1/2}))=\sum_{0\leq m<m_*}\varepsilon^{m/2}\,k_{i,m/2}+$

$\varepsilon^{m_*/2}\, k_{i,m_*/2}(\varepsilon^{1/2})$, where $k_{i,m/2}$, $k_{i,m_*/2}(\varepsilon^{1/2})$ are such forms as at (3.10) with (m_i, n_i) replaced by $(b_i + \ell_i, m_*)$ $(\ell_i \geq 0)$. Then, $k_{\#,(\varepsilon)} := k_{1,(\varepsilon)} \# \cdots \# k_{j,(\varepsilon)}$ is described as

$$
\begin{aligned}
k_{\#,(\varepsilon)} &= \sum_{0 \leq m < m_*} \varepsilon^{m/2} k_{\#,m/2} + \varepsilon^{m_*/2} k_{\#,m_*/2}(\varepsilon^{1/2}) \\
&= \sum_{0 \leq m < m_*} \varepsilon^{m/2} \sum^{\sum m_i = m} k_{1,m_1/2} \# \cdots \# k_{j,m_j/2} \\
&\quad + \varepsilon^{m_*/2} \sum_{m_i > 0 = m_{i+1} = \cdots = m_j}^{\sum m_i = m_*} k_{1,m_1/2} \# \cdots \# k_{i-1,m_{i-1}/2} \# k_{i,m_i/2}(\varepsilon^{1/2}) \# \cdots \# k_{j,m_j/2}(\varepsilon^{1/2}),
\end{aligned}
$$

where, by Lemma 3.6, $k_{\#,m/2}(t,x,x')$ and $k_{\#,m_*/2}(\varepsilon^{1/2},t,x,x')$ are well-defined and smooth on $(0,\varepsilon_0^{1/2}] \times (0,T_0] \times E^n \times E^n$. Further, there exist a constant $B' > 0$, an integer $N > 0$ and an exponentially decaying function $K(\mathcal{E})$ such that

$$
\begin{aligned}
&\left| (\partial/\partial t)^d (\partial/\partial x)^{\mathbb{A}} (\partial/\partial x')^{\mathbb{A}'} (\partial/\partial \varepsilon^{1/2})^m k_{\#,m_*/2}(\varepsilon^{1/2},t,x,x') \right| \\
&\qquad \leq B'\, t^{\sum b_i/2 - N - d - (n+2)/2} K(\iota_{1/t}(x - x')) \sum |x'^{\mathbb{C}'}|.
\end{aligned}
$$

Hence, the term $\varepsilon^{m_*/2} k_{\#,m_*/2}(\varepsilon^{1/2})$ can be extended up to $\varepsilon^{1/2} = 0$ so as to be of class C^{m_*-1} by claiming that its differentials up to the order $m_* - 1$ relative to the variables $(\varepsilon^{1/2}, t, x, x')$ are equal to 0 at $\varepsilon^{1/2} = 0$. Namely, $k_{\#,(\varepsilon)}$ can be extended up to $\varepsilon^{1/2} = 0$ so as to be of class C^{m_*-1}. Since m_* can be chosen arbitrarily large, certainly it can be extended smoothly up to $\varepsilon^{1/2} = 0$. ∎

Now we will show:

Lemma 3.8 *Each term $(-1)^k (r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t,x,x')$ can be extended smoothly up to $\varepsilon^{1/2} = 0$ and has a series expansion*

$$
(3.11) \qquad (-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)} = \sum_{k \leq m < m_*} \varepsilon^{m/2} p^k_{m/2} + \varepsilon^{m_*/2} p^k_{m_*/2}(\varepsilon^{1/2}), \quad p^0_{0/2} = \mathfrak{r}_E.
$$

For every $\varphi \in \Omega^q_0 E^n$, the integrals $\int p^k_{m/2}(t,x,x') \wedge \star\varphi(x')$, $\int p^k_{m_*/2}(\varepsilon^{1/2},t,x,x') \wedge \star\varphi(x')$ are well-defined and smooth on $[0,\varepsilon_0^{1/2}] \times [0,T_0^{1/2}] \times E^n$. In addition, there exists a semi-norm $\operatorname{sn}(\cdot)$ such that, on $[0,T_0^{1/2}] \times E^n$, for any $\varphi \in \Omega^q_0 E^n$,

$$
(3.12) \qquad \left\| \int p^0_{0/2}(t,x,x') \wedge \star\varphi(x') - \varphi(x) \right\| \leq t^{1/2} \operatorname{sn}(\varphi),
$$

$$
(3.13) \qquad \left\| \int p^0_{m/2}(t,x,x') \wedge \star\varphi(x') \right\| \leq t^{1/2} \operatorname{sn}(\varphi) \quad (m > 0),
$$

$$
(3.14) \qquad \left\| \int p^k_{m/2}(t,x,x') \wedge \star\varphi(x') \right\| \leq t^{k/2} \operatorname{sn}(\varphi) \quad (k > 0).
$$

Proof. Note that $r_{(\varepsilon)}$ is an (ε)-kernel of type 2 and $q_{(\varepsilon)}$ has a finite sum expression $q_{(\varepsilon)} = \sum \varepsilon^{b/2}\tilde{q}_{b,(\varepsilon)}$, where each $\tilde{q}_{b,(\varepsilon)}$ is an (ε)-kernel of type $b\,(\geq 1)$. Thus we have

$$(-1)^k\, r_{(\varepsilon)}\#q^k_{(\varepsilon)} = (-1)^k \sum_{b_i\geq 1} \varepsilon^{\sum b_i/2}\, r_{(\varepsilon)}\#\tilde{q}_{b_1,(\varepsilon)}\#\cdots\#\tilde{q}_{b_k,(\varepsilon)},$$

which, together with Lemma 3.7, ascertains the first half of the lemma. Next, considering (3.11) with $k=0$, i.e., $r_{(\varepsilon)} = \sum_{0\leq m<m_*} \varepsilon^{m/2} r_{m/2} + \varepsilon^{m_*/2} r_{m_*/2}(\varepsilon^{1/2})$, we examine the integrals $\int p^0_{m/2}(t,x,x') \wedge \star\varphi(x') = \int r_{m/2}(t,x,x')\wedge\star\varphi(x')$. The coefficients of $r_{m/2}$, $r_{m_*/2}(\varepsilon^{1/2})$ consist of such functions as $\mathbf{K}^2_{m/2} = \sum \mathbf{K}^2_{\circ,m/2}$, etc., (see Lemma 3.5(1)). Hence, with reference to (3.4) and (3.8), using integration by parts and then changing the variables, we obtain finite sum expressions

$$\begin{aligned}
&\int dV(x')\, r^{II'}_{m/2}(t,x,x')\,\varphi^{I'}(x')\\
&\quad= \sum_{\ell\geq 0} t^{\ell/2}\int dV(x')\,\mathcal{K}(x')\,(\partial/\partial y)^{\mathbb{B}'}\Big(y^{\mathbb{C}'}\varphi^{I'}(y)\Big)\Big|_{y=x-t^{1/2}x'},\\
&\int dV(x')\, r^{II'}_{m_*/2}(\varepsilon^{1/2},t,x,x')\,\varphi^{I'}(x')\\
&\quad= \sum_{\ell\geq 0} t^{\ell/2}\int dV(x')\Big[\mathcal{B}(\varepsilon^{1/2})\mathcal{K}(x')\,(\partial/\partial y)^{\mathbb{B}'}\Big(y^{\mathbb{C}'}\varphi^{I'}(y)\Big)\Big|_{y=\mathcal{E}^{(\varepsilon)}(\cdot,x)^{-1}(t^{1/2}x')}\Big]^{\varepsilon^{1/2}}_{m_*}.
\end{aligned}$$

These are smooth on $[0,\varepsilon_0^{1/2}]\times[0,T_0^{1/2}]\times E^n$ and, setting $\varphi_{m/2}(x) = \lim_{t^{1/2}\to 0}\int r_{m/2}(t, x,x')\wedge\star\varphi(x')$, we know

$$\left\|\int r_{m/2}(t,x,x')\wedge\star\varphi(x') - \varphi_{m/2}(x)\right\| \leq t^{1/2}\,\mathrm{sn}(\varphi)$$

(refer to the argument at (2.18)). Further we have $\varphi_{0/2}(x) = \varphi(x)$ and $\varphi_{m/2}(x) = 0$ $(m>0)$. Indeed, since (2.5) implies $\lim_{t^{1/2}\to 0}\int r_{(\varepsilon)}(t,x,x')\wedge\star\varphi(x') = \varphi(x)$ for every $(\varepsilon^{1/2},x)\in[0,\varepsilon_0^{1/2}]\times E^n$, setting $\varphi_{m_*/2}(\varepsilon^{1/2},x) = \lim_{t^{1/2}\to 0}\int r_{m_*/2}(\varepsilon^{1/2},t,x,x')\wedge\star\varphi(x')$ as well, we know that the form $\sum_{0\leq m<m_*}\varepsilon^{m/2}\varphi_{m/2}(x) + \varepsilon^{m_*/2}\varphi_{m_*/2}(\varepsilon^{1/2},x)$ on $[0,\varepsilon_0^{1/2}]\times E^n$ is identically equal to $\varphi(x)$. The estimates (3.12) and (3.13) are thus proved. Similarly the form $\int p^k_{m/2}(t,x,x')\wedge\star\varphi(x')$ $(k>0)$ is smooth on $[0,\infty)\times E^n$ and (3.3) implies the estimate (3.14). ∎

3.4 The proof of Proposition 3.1

Lemma 3.8 says that $\sum_{0\leq k<k_0}(-1)^k r_{(\varepsilon)}\#q^k_{(\varepsilon)}$ can be extended smoothly up to the domain $[0,\varepsilon_0^{1/2}]\times(0,\infty)\times E^n\times E^n$, and Lemma 3.3 says that, by taking $k_0>0$ large, $R_{k_0}(p_{(\varepsilon)})$ can be extended up to the domain so as to be of class C^{k_0-1} (by claiming that its differentials up to the order k_0-1 relative to the variables $(\varepsilon^{1/2},t,x,x')$ are equal to

0 at $\varepsilon^{1/2}=0$). Since k_0 can be chosen arbitrarily large, certainly $p_{(\varepsilon)}(t,x,x')$ is extended smoothly up to $\varepsilon^{1/2}=0$ and we have

$$p_{m/2}(t,x,x') = \sum_{0\le k\le m} p^k_{m/2}(t,x,x'). \tag{3.15}$$

Let us show (3.1) by induction. When $m=0$, it is valid because of (3.11) and (1.19). We fix $m'>0$ and assume that it is valid for m smaller than m'. Then, certainly we have

$$\Big(\frac{\partial}{\partial t}+\Delta_{0/2}\Big)(p_{m'/2}-\mathfrak{p}_{m'/2})=0.$$

In addition, since $\big\|\int p_{m'/2}(t,x,x')\wedge\star\varphi(x')\big\|\le t^{1/2}\mathrm{sn}(\varphi)$ (by (3.15), (3.13), (3.14)) and $\big\|\int \mathfrak{p}_{m'/2}(t,x,x')\wedge\star\varphi(x')\big\|\le t^{2/2}\mathrm{sn}(\varphi)$ (by (1.20), (3.3)), we have

$$\lim_{t\to 0}\bigg\|\int\Big(p_{m'/2}(t,x,x')-\mathfrak{p}_{m'/2}(t,x,x')\Big)\wedge\star\varphi(x')\bigg\|=0.$$

Hence, by the uniqueness of the solution of the initial value problem relative to $\Delta_{0/2}=\Delta_E$, (3.1) with $m=m'$ is valid.

3.5 (1.24) vanishes if m is odd

Lemma 3.9 *The coefficients in the expansion* (1.16) *are described as*

$$\Delta_{m/2}=\sum_{2+|\mathbb{C}|=|\mathbb{B}|+m}^{|\mathbb{B}|=0,1,2}\Delta_{m/2}(\mathbb{B},\mathbb{C})\cdot x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}}, \tag{3.16}$$

where each $\Delta_{m/2}(\mathbb{B},\mathbb{C})$ is a finite sum of operators which are the composites of such operators as $dx_i\wedge dx_j\vee$ multiplied by constants. Thus, we have $\Delta_{m/2}(\mathbb{B},\mathbb{C})\,(dx)^{I'}=\sum_{|I|=|I'|}\Delta^{II'}_{m/2}(\mathbb{B},\mathbb{C})\cdot(dx)^I$.

Proof. Since (1.14) and (1.15) yield

$$\Big(\frac{\partial}{\partial\varepsilon^{1/2}}\Big)^m\Delta_{(\varepsilon)}=\sum_{2+|\mathbb{C}|\ge|\mathbb{B}|+m}^{|\mathbb{B}|=0,1,2}\Delta_{m/2}(\mathbb{B},\mathbb{C}:\varepsilon^{1/2},\iota_\varepsilon(x))\cdot x^{\mathbb{C}}(\partial/\partial x)^{\mathbb{B}},$$
$$\Delta_{m/2}(\mathbb{B},\mathbb{C}:\varepsilon^{1/2},\iota_\varepsilon(x))=\varepsilon^{(2+|\mathbb{C}|-|\mathbb{B}|-m)/2}\,\Delta_{m/2}(\mathbb{B},\mathbb{C}:\iota_\varepsilon(x)),$$

we obtain the expression (3.16). ∎

Recalling the definition of $\mathfrak{p}_{m/2}(t,x,x':\mathbb{A},\mathbb{A}')$ $(m>0)$, we have

$$\mathfrak{p}^{II'}_{m/2}(t,x,x':\mathbb{A},\mathbb{A}')=\sum_{\sum_{\ell=1}^k m_\ell=m}^{m_1,\dots,m_k>0}(-1)^k\sum\prod_{\ell=1}^k\Delta^{I^{(\ell-1)}I^{(\ell)}}_{m_\ell/2}(\mathbb{B}^{(\ell)},\mathbb{C}^{(\ell)}) \tag{3.17}$$
$$\times\Big((\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E\#x^{\mathbb{C}^{(1)}}(\partial/\partial x)^{\mathbb{B}^{(1)}}\mathbf{r}_E\#\cdots\#x^{\mathbb{C}^{(k)}}(\partial/\partial x)^{\mathbb{B}^{(k)}}(\partial/\partial x')^{\mathbb{A}'}\mathbf{r}_E\Big)(t,x,x'),$$

where we set $\mathbf{r}_E(t,x,x') = r_t(x-x')$. Here, the second summation $\sum$ means, for each $(m_1,\ldots,m_k)$, to sum up all the terms determined by the indices $(\mathbb{B}^{(\ell)},\mathbb{C}^{(\ell)})$ and the sequences of indices $I = I^{(0)}$, $I^{(1)}$, $\cdots$, $I^{(k)} = I'$ appearing in $\Delta_{m_\ell/2}$ $(1 \le \ell \le k)$. The term appearing in the second line is the value at (t,x,x') of the convolution of the functions $(\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E(t,x,x')$, $x^{\mathbb{C}^{(1)}}(\partial/\partial x)^{\mathbb{B}^{(1)}}\mathbf{r}_E(t,x,x'),\ldots$.

To prove that (1.24) vanishes if m is odd, it will suffice to show:

Lemma 3.10 *We have*

$$\mathcal{P}^{II'}_{m/2}(t,x,x' : \mathbb{A},\mathbb{A}') = (-1)^{m+|\mathbb{A}|+\mathbb{A}'|}\,\mathcal{P}^{II'}_{m/2}(t,-x,-x' : \mathbb{A},\mathbb{A}'). \tag{3.18}$$

Proof. We put $\tilde{x} = -x$. If we expand $\det V_\bullet(\iota_\varepsilon(x'))$ into $\sum_{m=0}^{\infty}\varepsilon^{m/2}\mathrm{det}_{m/2}(x')$, then $(\partial/\partial x')^{\mathbb{A}'}\mathrm{det}_{m/2}(x') = (-1)^{m+|\mathbb{A}'|}(\partial/\partial\tilde{x}')^{\mathbb{A}'}\mathrm{det}_{m/2}(\tilde{x}')$. Hence, it suffices to ascertain (3.18) with $\mathcal{P}$ replaced by $\mathfrak{p}$. With the use of the notation at (3.17), further it will suffice to show

$$\begin{aligned}
&\Big((\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E \# x^{\mathbb{C}^{(1)}}(\partial/\partial x)^{\mathbb{B}^{(1)}}\mathbf{r}_E \# \cdots \# x^{\mathbb{C}^{(k)}}(\partial/\partial x)^{\mathbb{B}^{(k)}}(\partial/\partial x')^{\mathbb{A}'}\mathbf{r}_E\Big)(t,x,x')\\
&= (-1)^{\sum m_\ell + |\mathbb{A}|+|\mathbb{A}'|}\\
&\quad\times\Big((\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E \# x^{\mathbb{C}^{(1)}}(\partial/\partial x)^{\mathbb{B}^{(1)}}\mathbf{r}_E \# \cdots \# x^{\mathbb{C}^{(k)}}(\partial/\partial x)^{\mathbb{B}^{(k)}}(\partial/\partial x')^{\mathbb{A}'}\mathbf{r}_E\Big)(t,\tilde{x},\tilde{x}').
\end{aligned}$$

Since $2 + |\mathbb{C}^{(\ell)}| = |\mathbb{B}^{(\ell)}| + m_\ell$ (see (3.16)) and $\mathbf{r}_E(t,x,x') = \mathbf{r}_E(t,\tilde{x},\tilde{x}')$, we have

$$\begin{aligned}
(\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E(t,x,x') &= (\partial/\partial x)^{\mathbb{A}}\mathbf{r}_E(t,\tilde{x},\tilde{x}') = (-1)^{|\mathbb{A}|}(\partial/\partial\tilde{x})^{\mathbb{A}}\mathbf{r}_E(t,\tilde{x},\tilde{x}'),\\
x^{\mathbb{C}^{(\ell)}}(\partial/\partial x)^{\mathbb{B}^{(\ell)}}\mathbf{r}_E(t,x,x') &= (-1)^{|\mathbb{B}^{(\ell)}|+|\mathbb{C}^{(\ell)}|}\tilde{x}^{\mathbb{C}^{(\ell)}}(\partial/\partial\tilde{x})^{\mathbb{B}^{(\ell)}}\mathbf{r}_E(t,\tilde{x},\tilde{x}')\\
&= (-1)^{m_\ell}\tilde{x}^{\mathbb{C}^{(\ell)}}(\partial/\partial\tilde{x})^{\mathbb{B}^{(\ell)}}\mathbf{r}_E(t,\tilde{x},\tilde{x}'),
\end{aligned}$$

etc. In consideration of the change of orientation, we obtain the equality. ∎

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