ON THE DEFORMATION WITH CONSTANT MILNOR NUMBER AND NEWTON POLYHEDRON

OULD M ABDERRAHMANE

Abstract- We show that every μ-constant family of isolated hypersurface singularities satisfying a non-degeneracy condition in the sense of Kouchnirenko, is topologically trivial, also is equimultiple.

Let $f\colon (\mathbf{C}^n,0) \to (\mathbf{C},0)$ be the germ of a holomorphic function with an isolated singularity. The Milnor number $\mu(f)$ is by definition $\dim_{\mathbf{C}} \mathbf{C}\{z_1,\dots,z_n\}/(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n})$ and the multiplicity $m(f)$ is the lowest degree in the power series expansion of f at $0 \in \mathbf{C}^n$. Let $F\colon (\mathbf{C}^n \times \mathbf{C},0) \to (\mathbf{C},0)$ be the deformation of f given by $F(z,t) = f(z) + \sum c_\nu(t) z^\nu$, where $c_\nu\colon (\mathbf{C},0) \to (\mathbf{C},0)$ are germs of holomorphic functions. We use the notation $F_t(z) = F(z,t)$ when t is fixed. Let m_t denote the multiplicity and μ_t denote the Milnor number of F_t at the origin. The deformation F is equimultiple (resp. μ-constant) if $m_0 = m_t$ (resp. $\mu_0 = \mu_t$) for small t. It is well-known by the result of Lê-Ramanujam [8] that for $n \neq 3$, the topological type of the family F_t is constant under μ-constant deformations. The question is still open for $n = 3$. However, under some additional assumption, positive answers have been given. For example, if F_t is non-degenerate in the sense of Kouchnirenko [6] and the Newton boundary $\Gamma(F_t)$ of F_t is independent of t, i.e., $\Gamma(F_t) = \Gamma(f)$, it follows that $\mu^*(F_t)$ is constant, and hence F_t is topologically trivial (see [11, 14] for details). Motivated by the Briançon-Speder μ-constant family $F_t(z) = z_1^5 + z_2 z_3^7 + z_2^{15} + t z_1 z_3^6$, which is topologically trivial but not μ^*-constant, M. Oka [12] shows that any non-degenerate family of type $F(z,t) = f(z) + t z^A$, is topologically trivial, under the assumption of μ-constancy. Our purpose of this paper is to generalize this result, more precisely, we show that every μ-constant non-degenerate family F_t with not necessarily Newton boundary $\Gamma(F_t)$ independent of t, is topologically trivial. Moreover, we show that F is equimultiple, which gives a positive answer to a question of Zariski [16] for a non-degenerate family.

To prove the main result (Theorem 1 below), we shall use the notion of (c)-regularity in the stratification theory, introduced by K. Bekka in [3], which is weaker than the Whitney regularity, never the less (c)-regularity condition implies topological triviality. First, we give a characterization of (c)-regularity (Theorem 3 below). By using it, we can show that the μ-constancy condition for a non-degenerate family implies Bekka's (c)-regularity condition and then obtain the topological triviality as a corollary.

1. Newton polyhedron, main results

First we recall some basic notions about the Newton polyhedron (see [6, 11] for details), and state the main result.

Let $f\colon (\mathbf{C}^n,0) \to (\mathbf{C},0)$ be an analytic function defined by a convergent power series $\sum_\nu c_\nu x^\nu$. Also, let $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n,$ each $x_i \geq 0,\ i = 1,\dots,n\}$. The Newton polyhedron of f, $\Gamma_+(f) \subset \mathbb{R}^n$ is defined by the convex hull of $\{\nu + \mathbb{R}^n_+ \,|\, c_\nu \neq 0\}$, and

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let $\Gamma(f)$ be the Newton boundary, i.e., the union of the compact faces of $\Gamma_+(f)$. For a face γ of $\Gamma(f)$, we write $f_\gamma(z) := \sum_{\nu\in\gamma} c_\nu x^\nu$. We say that f is non-degenerate if, for any face γ of $\Gamma(f)$, the equations $\frac{\partial f_\gamma}{\partial x_1} = \cdots = \frac{\partial f_\gamma}{\partial x_n} = 0$ have no common solution on $x_1 \cdots x_n \neq 0$. The power series f is said to be convenient if $\Gamma_+(f)$ meets each of the coordinate axes. We let $\Gamma_-(f)$ denote the compact polyhedron which is the cone over $\Gamma(f)$ with the origin as a vertex. When f is convenient, the Newton number $\nu(f)$ is defined as $\nu(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n$, where V_n is the n-dimensional volumes of $\Gamma_-(f)$ and for $1 \leq k \leq n-1$, V_k is the sum of the k-dimensional volumes of the intersection of $\Gamma_-(f)$ with the coordinate planes of dimension k. The Newton number may also be defined for non-convenient analytic function (see [6]). Finally, we define the Newton vertices of f as $\mathrm{ver}(f) = \{\, \alpha \;:\; \alpha \text{ is a vertex of } \Gamma(f)\,\}$.

Now we can state the main result

Theorem 1. *Let $F\colon (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ be a one parameter deformation of a holomorphic germ $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate. Then F_t is topologically trivial, and moreover, F is equimultiple.*

Remark 2. *In the above theorem, we do not require the independence of t for the Newton boundary $\Gamma(F_t)$.*

2. A criterion for (c)-regularity

Let M be a smooth manifold, and let X, Y be smooth submanifolds of M such that $Y \subseteq \overline{X}$ and $X \cap Y = \emptyset$.

(i) (Whitney (a)-regularity)
(X, Y) is (a)-regular at $y_0 \in Y$ if:
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{T_{x_i}X\}$ tends in the Grassman space of (dim X)-planes to some plane τ, then $T_{y_0}Y \subset \tau$. We say (X, Y) is (a)-regular if it is (a)-regular at any point $y_0 \in Y$.

(ii) (Bekka (c)-regularity)
Let ρ be a smooth non-negative function such that $\rho^{-1}(0) = Y$. (X, Y) is (c)-regular at $y_0 \in Y$ for the control function ρ if:
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{Kerd\rho(x_i) \cap T_{x_i}X\}$ tends in the Grassman space of (dim $X-1$)-planes to some plane τ, then $T_{y_0}Y \subset \tau$. (X, Y) is (c)-regular at y_0 if it is (c)-regular for some control function ρ. We say (X, Y) is (c)-regular if it is (c)-regular at any point $y_0 \in Y$.

Let $F\colon (\mathbf{C}^n \times \mathbf{C}, \{0\} \times \mathbf{C}) \to (\mathbf{C}, 0)$ be a deformation of an analytic function f. We denote by $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbf{C},\ \{0\} \times \mathbf{C}\}$ the canonical stratification of the germ variety V_F of the zero locus of F. We may assume that f is convenient, this is not a restriction when it defines an isolated singularity, in fact, by adding z_i^N for a sufficiently large N for which the isomorphism class of F_t does not change. Hereafter, we will assume that f is convenient,

$$X = F^{-1}(0) - \{0\} \times \mathbf{C},\ Y = \{0\} \times \mathbf{C} \ \text{ and } \ \rho(z) = \sum_{\alpha\in \mathrm{ver}(F_t)} z^\alpha \bar{z}^\alpha.$$

Here $\mathrm{ver}(F_t)$ denotes the Newton vertices of F_t when $t \neq 0$.

Note that by the convenience assumption on f, $\rho^{-1}(0) = Y$.

We also let

$$\partial\rho = \sum_{i=1}^{n} \frac{\partial\rho}{\partial z_i}\frac{\partial}{\partial z_i} + \frac{\partial\rho}{\partial\bar{z}_i}\frac{\partial}{\partial\bar{z}_i} = \partial_z\rho + \partial_{\bar{z}}\rho$$

and

$$\partial F = \sum_{i=1}^{n} \frac{\partial F}{\partial z_i}\frac{\partial}{\partial z_i} + \frac{\partial F}{\partial t} = \partial_z F + \partial_t F.$$

Calculation of the map $\partial_z\rho_{|X}$

First of all we remark that $\partial_z\rho = \partial_z\rho_{|X} + \partial_z\rho_{|N}$ (where N denotes the normal space to X). Since N is generated by the gradient of F, we have that $\partial_z\rho = \partial_z\rho_{|X} + \eta\partial F$. On the other hand, $\langle\partial_z\rho_{|X}, \partial F\rangle = 0$, so we get $\eta = \frac{\langle\partial_z\rho, \partial F\rangle}{|\partial F|^2}$. It follows that

$$\partial_z\rho_{|X} = \partial_z\rho - \frac{\langle\partial_z\rho, \partial F\rangle}{|\partial F|^2}\partial F = (\partial_z\rho_{|X})_z + (\partial_z\rho_{|X})_t, \tag{2.1}$$

where

$$(\partial_z\rho_{|X})_z = \partial_z\rho - \frac{\langle\partial_z\rho, \partial F\rangle}{|\partial F|^2}\partial_z F, \quad (\partial_z\rho_{|X})_t = -\frac{\langle\partial_z\rho, \partial F\rangle}{|\partial F|^2}\partial_t F$$

and

$$|\partial_z\rho_{|X}|^2 = \frac{|\partial F|^2\,|\partial_z\rho|^2 - \left|\langle\partial_z\rho, \partial F\rangle\right|^2}{|\partial F|^2} = \frac{\|\partial F \wedge \partial_z\rho\|^2}{|\partial F|^2}.$$

Then we can characterize the (c)-regularity as follows:

Theorem 3. *Consider X and Y as above. The following conditions are equivalent*

(i) *(X, Y) is (c)-regular for the the control function ρ.*

(ii) *(X, Y) is (a)-regular and $|(\partial_z\rho_{|X})_t| \ll |\partial_z\rho_{|X}|$ as $(z, t) \in X$ and $(z, t) \to Y$.*

(iii) *$|\partial_t F| \ll \frac{\|\partial F\wedge\partial_z\rho\|}{|\partial_z\rho|}$ as $(z, t) \in X$ and $(z, t) \to Y$.*

Proof. Since $(i) \Leftrightarrow (ii)$ is proved in ([1], Theorem 1), and $(iii) \Rightarrow (ii)$ is trivial, it is enough to see $(ii) \Rightarrow (iii)$.

To show that $(ii) \Rightarrow (iii)$, it suffices to show this on any analytic curves $\lambda(s) = (z(s), t(s)) \in X$ and $\lambda(s) \to Y$. Indeed, we have to distinguish two cases:

First case, we suppose that along λ, $|\langle\partial_z\rho, \partial F\rangle| \sim |\partial_z\rho|\,|\partial F|$, hence by (2.1) and (ii), we have

$$|(\partial_z\rho_{|X})_t| = |\,\frac{\langle\partial_z\rho, \partial F\rangle}{|\partial F|^2}\partial_t F\,| \ll \frac{\|\partial F \wedge \partial_z\rho\|}{|\partial F|}.$$

But this clearly implies

$$|\partial_t F| \ll \frac{\|\partial F \wedge \partial_z\rho\|}{|\partial_z\rho|} \quad \text{along the curve } \lambda(s),$$

where $|\langle\partial_z\rho, \partial F\rangle| \sim |\partial_z\rho|\,|\partial F|$.

Second case, we suppose that along λ, $|\langle\partial_z\rho, \partial F\rangle| \ll |\partial_z\rho|\,|\partial F|$, thus

$$\|\partial F \wedge \partial_z\rho\| \sim |\partial_z\rho|\,|\partial F| \text{ along the curve } \lambda(s).$$

On the other hand, by the Whitney (a)-regularity in (ii) we get

$$|\partial_t F| \ll |\partial F|.$$

Therefore, $|\partial_t F| \ll |\partial F| \sim \frac{\|\partial F\wedge\partial_z\rho\|}{|\partial_z\rho|}$ along the curve $\lambda(s)$. The Theorem 3 is proved. □

3. **Proof of the theorem 1**

Before starting the proofs, we will recall some important results on the Newton number and the geometric characterization of μ-constancy.

Theorem 4 (A. G. Kouchnirenko [6]). *Let $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function with an isolated singularity, then the Milnor number $\mu(f) \geq \nu(f)$. Moreover, the equality holds if f is non-degenerate.*

As an immediate corollary we have

Corollary 5 (M. Furuya[5]). *Let f, $g\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be two germs of holomorphic functions with $\Gamma_+(g) \subset \Gamma_+(f)$. Then $\nu(g) \geq \nu(f)$.*

On the other hand, concerning the μ-constancy, we have

Theorem 6 (Lê-Saito [9], Teissier [14]). *Let $F\colon (\mathbf{C}^n \times \mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ be the deformation of a holomorphic $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with isolated singularity. The following statement are equivalent.*

1. *F is a μ-constant deformation of f*
2. *$\frac{\partial F}{\partial t_j} \in \overline{J(F_t)}$, where $\overline{J(F_t)}$ denotes the integral closure of the Jacobian ideal of F_t generated by the partial derivatives of F with respect to the variables $z_1, \ldots, z_n$.*
3. *The deformation $F(z,t) = F_t(z)$ is a Thom map, that is,*
$$\sum_{j=1}^{m} \Big|\frac{\partial F}{\partial t_j}\Big| \ll \|\partial F\| \text{ as } (z,t) \to (0,0).$$
4. *The polar curve of F with respect to $\{t = 0\}$ does not splits, that is,*
$$\{(z,t) \in C^n \times \mathbf{C}^m \mid \partial_z F(z,t) = 0\} = \{0\} \times \mathbf{C}^m \quad \text{near } (0,0).$$

We now want to prove the theorem 1, in fact, let $F\colon (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ be a deformation of a holomorphic germ $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate. Then, by theorem 4, we have

$$\mu(f) = \nu(f) = \mu(F_t) = \nu(F_t). \tag{3.1}$$

Consider the deformation $\tilde{F}$ of f given by
$$\tilde{F}(z,t,\lambda) = F_t(z) + \sum_{\alpha \in \mathrm{ver}(F_t)} \lambda_\alpha z^\alpha.$$
From the upper semi-continuity of Milnor number [10], we obtain

$$\mu(f) \geq \mu(\tilde{F}_{t,\lambda}) \quad \text{for } (t,\lambda) \text{ near } (0,0). \tag{3.2}$$

By Theorem 4 and Corollary 5 therefore
$$\mu(\tilde{F}_{t,\lambda}) \geq \nu(\tilde{F}_{t,\lambda}) \geq \nu(F_t).$$
It follows from (3.1) and (3.2) that the deformation $\tilde{F}$ is μ-constant, and hence, by Theorem 6 we get

$$|\partial_t F| + \sum_{\alpha \in \mathrm{ver}(F_t)} |z^\alpha| \ll \Big|\partial_z F + \sum_{\alpha \in \mathrm{ver}(F_t)} \lambda_\alpha \partial_z z^\alpha\Big| \text{ as } (z,t,\lambda) \to (0,0,0). \tag{3.3}$$

Therefore, for all $\alpha \in \mathrm{ver}(F_t)$ we have $|z^\alpha| \ll |\partial_z f|$, and so $m(z^\alpha) \geq m(f)$. Hence the equality $m(F_t) = m(f)$ follows. In other word, F is equimultiple.

We also show that condition (3.3), in fact, implies Bekka's (c)-regularity, hence, this deformation is topologically trivial. For this purpose, we need the following lemma (see [13]).

Lemma 7. *Suppose F_t is a deformation as above, then we have*

$$\sum_{\alpha \in ver(F_t)} |z^\alpha| \ll \inf_{\eta \in \mathbf{C}} \{ \big|\partial F + \sum_{\alpha \in ver(F_t)} \eta\, \bar{z}^\alpha \partial_z z^\alpha \big| \} \text{ as } (z,t) \to (0,0), F(z,t) = 0. \tag{3.4}$$

Proof. Suppose (3.4) does not hold. Then by the curve selection lemma, there exists an analytic curve $p(s) = (z(s), t(s))$ and an analytic function $\eta(s)$, $s \in [0, \epsilon)$, such that:

(a) $p(0) = 0$,
(b) $F(p(s)) \equiv 0$, and hence $d\,F(p(s))\frac{dp}{ds} \equiv 0$,
(c) along the curve $p(s)$ we have

$$\sum_{\alpha \in \mathrm{ver}(F_t)} |z^\alpha| \gtrsim \big|\partial F + \sum_{\alpha \in \mathrm{ver}(F_t)} \eta(s)\, \bar{z}^\alpha \partial_z z^\alpha \big|.$$

Set

$$g(z.\bar{z}) = \Big(\sum_{\alpha \in \mathrm{ver}(F_t)} \bar{z}^\alpha z^\alpha \Big)^{\frac{1}{2}} \text{ and } \gamma(s) = \eta(s) g(z(s), \bar{z}(s)). \tag{3.5}$$

First suppose that $\gamma(s) \to 0$. Since $|\bar{z}^\alpha| \leq g$, we have,

$$\lambda_\alpha(s) = \frac{\gamma(s)\bar{z}^\alpha(s)}{g(z(s), \bar{z}(s))} \to 0, \quad \forall \alpha \in \mathrm{ver}(F_t).$$

Next, using (3.3) and (3.5) it follows

$$\sum_{\alpha \in \mathrm{ver}(F_t)} |z^\alpha(s)| \ll \big|\partial F(p(s)) + \sum_{\alpha \in \mathrm{ver}(F_t)} \eta(s)\, \bar{z}^\alpha(s) \partial_z z^\alpha(s) \big| \text{ as } s \to 0,$$

which contradicts (c).

Suppose now that the limit of $\gamma(s)$ is not zero (i.e., $|\gamma(s)| \gtrsim 1$). Since $p(0) = 0$ and $g(z(0), \bar{z}(0)) = 0$, we have, asymptotically as $s \to 0$,

$$s \big| \frac{dp}{ds}(s) \big| \sim |p(s)| \text{ and } s \frac{d}{ds} g(z(s), \bar{z}(s)) \sim g(z(s), \bar{z}(s)). \tag{3.6}$$

But

$$\frac{d}{ds} g(z(s), \bar{z}(s)) = \sum_{\alpha \in \mathrm{ver}(F_t)} \frac{1}{2g(z(s), \bar{z}(s))} \Big(\bar{z}^\alpha dz^\alpha \frac{dz}{ds} + z^\alpha d\bar{z}^\alpha \frac{d\bar{z}}{ds} \Big). \tag{3.7}$$

We have $\bar{z}^\alpha dz^\alpha \frac{dz}{ds} = \overline{z^\alpha d\bar{z}^\alpha \frac{d\bar{z}}{ds}}$ and $1 \lesssim |\gamma(s)|$. Thus,

$$\big| \frac{d}{ds} g(z(s), \bar{z}(s)) \big| \lesssim \Big| \sum_{\alpha \in \mathrm{ver}(F_t)} \frac{\gamma(s)}{g(z(s), \bar{z}(s))} \bar{z}^\alpha dz^\alpha \frac{dz}{ds} \Big|. \tag{3.8}$$

This together with (3.6), (3.5) and (b) gives

$$g(z(s), \bar{z}(s)) \sim \big| s \frac{d}{ds} g(z(s), \bar{z}(s)) \big| \lesssim s \Big| \sum_{\alpha \in \mathrm{ver}(F_t)} \eta(s) \bar{z}^\alpha dz^\alpha \frac{dz}{ds} + d\,F(p(s)) \frac{dp}{ds} \Big|.$$

Hence

$$g(z(s),\bar{z}(s)) \lesssim s\left|\frac{dp}{ds}(s)\right| \left| \sum_{\alpha\in \mathrm{ver}(F_t)} \eta(s)\bar{z}^{\alpha}\partial z^{\alpha} + \partial F(p(s)) \right|,$$

which contradicts (c). This ends the proof of Lemma. □

We shall complete the proof of Theorem 1. Since $\Gamma_+(\partial_t F) \subset \Gamma_+(F_t)$. Then, by an argument, based again on the curve selection lemma, we get the following inequality

$$\left|\partial_t F\right| \lesssim \sum_{\alpha\in \mathrm{ver}(F_t)} |z^{\alpha}|. \tag{3.9}$$

Then, by the above Lemma 7, we obtain

$$\left|\partial_t F\right| \ll \inf_{\eta\in\mathbf{C}} \left\{\left|\partial F + \eta\,\partial_z\rho\right|\right\} \text{ as } (z,t)\to(0,0), F(z,t)=0,$$

we recall that

$$\rho(z) = \sum_{\alpha\in \mathrm{ver}(F_t)} z^{\alpha}\bar{z}^{\alpha}.$$

But

$$\inf_{\eta\in\mathbf{C}} \left\{\left|\partial F + \eta\,\partial_z\rho\right|\right\}^2 = \frac{|\partial F|^2\,|\partial_z\rho|^2 - \left|\langle\partial_z\rho,\partial F\rangle\right|^2}{|\partial_z\rho|^2} = \frac{\|\partial F\wedge\partial_z\rho\|^2}{|\partial_z\rho|^2}.$$

Therefore, by Theorem 3, we see that the canonical stratification $\Sigma(V_F)$ is (c)-regular for the control function ρ, then F is a topologically trivial deformation (see[3]).

This completes the proof of Theorem 1.

Remark 8. *We should mention that our arguments still hold for any μ-constant deformation F of weighted homogeneous polynomial f with isolated singularity. Indeed, we can find from Varchenko's theorem* [15] *that $\mu(f)=\nu(f)=\mu(F_t)=\nu(F_t)$. Thus, the above proof can be applied.*

Unfortunately this approach does not work, if we only suppose that f is non-degenerate. For consider the example of Altman [2] *defined by*

$$F_t(x,y,z) = x^5 + y^6 + z^5 + y^3z^2 + 2tx^2y^2z + t^2x^4y,$$

which is a μ-constant degenerate deformation of the non-degenerate polynomial $f(x,y,z) = x^5+y^6+z^5+y^3z^2$. He showed that this family has a weak simultaneous resolution. Thus, by Laufer's theorem [7]*, F is a topologically trivial deformation. But we cannot apply the above proof because $\mu(f)=\nu(f)=\mu(F_t)=68$ and $\nu(F_t)=67$ for $t\neq 0$.*

We conclude with several examples.

Example 9. *Consider the family given by*

$$F_t(x,y,z) = x^{13} + y^{20} + zx^6y^5 + tx^6y^8 + t^2x^{10}y^3 + z^l,\ \ l\geq 7.$$

It is not hard to see that this family is non-degenerate. Moreover, by using the formula for the computation of Newton number we get $\mu(F_t)=\nu(F_t)=153\,l+32$. Thus, by theorem 1, we have that F_t is topologically trivial. We remark that this deformation is not μ^*-constant, in fact, the Milnor numbers of the generic hyperplane sections $\{z=0\}$ of F_0 and F_t (for $t\neq 0$) are 260 and 189 respectively.

Example 10. *Let*

$$F_t(x,y,z) = x^{10} + x^3y^4z + y^l + z^l + t^3x^4y^5 + t^5x^4y^5$$

where $l \geq 6$. *Since* $\mu(F_t) = 2l^2 + 32l + 9$ *and* F_t *is a non-degenerate family, it follows from Theorem 1 that* F *is a topologically trivial deformation.*

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Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Urawa, 338-8570 Japan

E-mail address: vould@rimath.saitama-u.ac.jp