RESEARCH REPORT

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The main area of my research is the equisingularity theory of analytic functions, in particular decidability criteria for topological triviality. My starting point was the *theorem of Kuiper-Kuo*. Let me recall it briefly. Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function. If it has an isolated singularity at the origin, then there exist $r > 1$ and $\epsilon > 0$ such that we have the following Łojasiewicz Inequality with exponent r:

$$||grad\, f(x)|| \geq \epsilon ||x||^{r-1},\ \ x \text{ near the origin.}$$

As a consequence of this inequality the r-th jet of f is C^0-sufficient, that is, all perturbations of f by terms $o(||x||^r)$ are topologically constant.

Subsequently, T.-C. Kuo proposed to generalize this criterion, to take into account a given system of weights of the coordinates, or more generally, relatively to a given Newton polyhedron. L. Paunescu has obtained the first generalization by replacing the Euclidean distance by a weighted one. The Newton polyhedron case was the subject of my thesis. I introduced a pseudo-metric adapted to the Newton polyhedron Γ which allows us to define the gradient of f with respect to Γ. Using this construction I obtained versions relative to the Newton filtration of the Łojasiewicz Inequality and the Kuiper-Kuo-Paunescu theorem ([1], Part I), thus giving an answer to the Kuo problem. I have shown that this result is optimal: if the Łojasiewicz Inequality with exponent r is not satisfied for f, then the r-jet of f with respect to the Newton filtration is not C^0-sufficient ([1], Part II). In the homogeneous case this result is known as the Bochnak-Łojasiewicz Theorem.

In my thesis I also studied the families of germs $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of analytic functions under the assumption that the leading terms of f_t (with respect to a Newton filtration) satisfy the *uniform Łojasiewicz Inequality*. I showed that the family f_t is blow-analytically equisingular via a toric modification, that is, f_t is topologically trivial and analytically trivial after some toric modification of $\mathbb{R}^n$ ([1], Part III). This result implies in particular the criteria for blow-analytic triviality due to Kuo, Fukui-Paunescu, and Fukui-Yoshinaga.

During my stay in Japan I have continued my work on the equisingularity problem, in particular the investigation of the zero sets of map germs $: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. More precisely, I formulated certain criteria, which allow us to tell when we can *truncate the Taylor expansion with respect to the Newton filtration of an analytic map*, without affecting the local topological picture determined by it. These criteria can be used to understand the lists of V.I Arnold and C.T. Wall, of simple and unimodular germs [2].

My work in this area has also led me to study the stratification theory from the Newton polyhedron perspective. Recall that the stratification theory was introduced by R. Thom and H. Whitney; it deals with the *Equisingularity Problem* from a topologist's viewpoint.

A stratification of a variety V is an expression of V as a disjoint union of a locally finite set of connected analytic manifolds, called strata, such that the frontier of each stratum is the union of a set of lower-dimensional strata. The most important notion in stratification theory is the regularity condition between strata. In [3], I introduced the regularity condition (Verdier w-regularity condition) with respect to a Newton filtration, and I have given several criteria of the regularity conditions relative to a Newton filtration in terms of the defining equations of the strata. Moreover, I used one criterion of the regularity conditions to clarify the J. Damon and T. Gaffney condition in [7].

Another problem concerns the classification of real singularities via "*blow-analytic*" equivalence. The analytic classification of real analytic function germs has already moduli near non-simple germs in the sense of V. I. Arnold. For instance the Whitney family $W_t(x, y) = xy(x - y)(x - ty)$, $t > 1$, has infinitely many different C^1-types. Motived by this family, T.-C. Kuo [10] suggests classifying the analytic functions by topological equivalence which becomes analytic equivalence after some blowing-ups (blow-analytic equivalence). I have obtained a simple classification of the two variable quasi-homogeneous polynomials with isolated singularity by the blow-analytic equivalence, more precisely, I showed that *the weight is a blow-analytic invariant for quasi-homogeneous polynomials in two variables with isolated singularity* [4]. This gives a positive answer to T. Fukui's problem (Compositio Math, 1997) for the two variables case. The main ingredient in proving this was the motivic zeta function introduced by A. Parusiński and S. Koike [9].

Substantial effort was made in deducing various topological invariants of an analytic function germ f at 0 from its Newton polyhedron $\Gamma(f)$. For example, B. Lichtin [12], found an estimate in the 2-variable case of the degree of C^0-sufficiency, under a non-degeneracy condition stronger than A. G. Kouchnirenko's condition. In [5] I extended this to include the n-variable case. Moreover, I gave an estimation of the Łojasiewicz exponent of the gradient of a holomorphic function under Kouchnirenko's nondegeneracy condition, using information from the Newton polyhedron.

Whether you are a topologist, an algebraist or a geometer, one of the fundamental goals in your research is to find a necessary and sufficient condition for two given objects to be isomorphic in the category of interest. In the theory of isolated hypersurface singularities, a fundamental problem is as follows :

Problem. Let $(V, 0)$ and $(W, 0)$ be two isolated hypersurfaces singularities in $\mathbf{C}^n$. Give a simple algebraic criterion for $(V, 0)$ to be homeomorphic to $(W, 0)$.

The progress on this problem was not as fast as one have desired, despite the fact that many well known mathematicians, including Milnor and Zarski, worked on this problem. Actually even the Zariski multiplicity problem, whether or not the multiplicity of isolated hypersurface singularities is an invariant of topological type was solved only for the case $n = 2$. Even special cases of Zarisk's problem have proved to be extremely difficult. For instance, a necessary condition that the hypersurfaces in a family of isolated singularities are all homeomorphic, is that the family is μ-constant. It is natural to ask, is any μ-constant family equimultiple? Only Greuel and O'Shea proved (independently) that the μ-constant type isolated quasi-homogeneous singularities are equimultiple. In order to do this they used previous work of Varchenko [13], which described the μ-constant stratum of quasi-homogeneous singularities in terms of the mixed Hodge structures.

In [6], I studied the deformations with constant Milnor number and Newton polyhedron. More generally, I showed that every μ-constant family of isolated hypersurface singularities satisfying the non-degeneracy condition in the sense of Kouchnirenko [11], is topologically trivial, it is also equimultiple; this gives a partial result to Zariski's multiplicity conjecture [14].

In conclusion, I am writing five papers [2, 3, 4, 5, 6], during the period of my stay in Japan.

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