CHARACTERIZATION OF V-SUFFICIENCY FROM THE NEWTON FILTRATION

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Abstract- The aim of this paper is to study the germ mappings : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, more precisely, we formulate certain criteria for the v-sufficiency related to the Newton polyhedron. Our result implies in particular the criteria for v-sufficiency due to Kuo and Paunescu.

Let $\mathcal{E}_{[k]}(n,p)$ be the set of C^k-function germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. Given an element f of $\mathcal{E}_{[k]}(n,p)$, it is natural to ask when we can truncate f without effecting the local topological picture determined by f. This problem concerns the property of sufficiency of jet. In [8], T.-C. Kuo presented a beautiful criterion for v-sufficiency of jet, where necessary and sufficient conditions for v-sufficiency are given, a jet is v-sufficient if and only if some Łojasiewicz inequality is satisfied in the horn-neighborhood of the zero set. For the weighted filtration, L. Paunescu [9], extends the previous result. He obtained a weighted version of Kuo's criterion for v-sufficiency.

The purpose of this paper is to give a characterization of Kuo's v-sufficiency from the viewpoint of the Newton filtration. First, we define the compensation factors, following [1, 2], which compensate the effect of differentiation on the polyhedron and then reduce the problem to the procedure of Kuo-Paunescu. Namely, having defined the compensation factors, we construct a Riemannian metric associated with the filtration given by the Newton polyhedron. This allow us to consider the gradient associated with this metric and to show a Newton polyhedron analog of Kuo-Paunescu Theorems (Theorems 2 and 4 below). Moreover, we shall use Bekka's (c)-regularity in stratification theory to clarify the Łojasiewcz inequality relative to a Newton filtration (Theorem 6 below). This gives a generalization of Bekka-Koike's Theorem [4].

Let us denote by $E(n,p)$ the set of all germs of functions : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ which are C^2 in a punctured neighborhood of the origin. Let f, $g \in E(n,p)$. These functions are said to have the same (local) v-type at 0 (where v stands for variety), if the germs at 0 of $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic.

1. **Newton filtration.**

Let us recall some basic definitions and properties of the Newton filtration (see [1, 2, 6] for details).

The Newton polyhedron, $\Gamma_+(\mathcal{A})$, where $\mathcal{A} \subset \mathbb{Q}^n_+$ is the convex hull of $\{a + \mathbb{R}^n_+ \mid a \in \mathcal{A}\}$. The Newton boundary of $\mathcal{A}$, $\Gamma(\mathcal{A})$ is the union of the compact faces of $\Gamma_+(\mathcal{A})$. We let $\mathcal{F}(\mathcal{A})$ denote the union of the top dimensional faces of $\Gamma(\mathcal{A})$. The Newton vertex $Ver(\mathcal{A})$ is defined by $\{\alpha : \alpha \text{ is vertex of } \Gamma(A)\}$. $\mathcal{A}$ is called convenient if the intersection of $\Gamma_+(\mathcal{A})$ with each coordinate axis is non-empty. Throughout, we suppose that $\mathcal{A}$ is convenient.

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From the Newton polyhedron, we construct the Newton filtration. We first observe that by the convenience assumption on $\mathcal{A}$, any face $F \in \mathcal{F}(\mathcal{A})$, $\dim F = n-1$. So let w^F be the unique vector of $\mathbb{Q}^n_+$ such that $F = \{\, b \in \Gamma_+(\mathcal{A}) \ : \ \langle b\,,\, w^F\rangle = 1\,\}$. We can suppose that the vertices of $\mathcal{A}$ are sufficiently close to the origin so that all the $w^F \in \mathbb{Z}^n_+$. We will suppose henceforth that $\mathcal{A}$ satisfies this property. Then, we construct the following map $\phi\colon \mathbb{R}^n_+ \to \mathbb{R}_+$. The restriction of ϕ to each cone $C(F)$(where $C(F)$ denotes the cone of half-rays emanating from 0 and passing through F) is defined as follows:

$$\phi_{|C(F)}(\alpha) = \langle \alpha\,,\, w^F\rangle, \qquad \text{for all } \alpha \in C(F).$$

We extend this map to $\mathbb{R}^n_+$ as follows:

$$\phi(\alpha) = \min\left\{\langle\, \alpha\,,\, w^F\rangle \ : \ F \in \mathcal{F}(\mathcal{A})\,\right\}, \qquad \text{for all } \alpha \in \mathbb{R}^n_+. \tag{1.1}$$

The map ϕ is linear on each cone $C(F)$ (where $F \in \mathcal{F}(\mathcal{A})$), and the value of ϕ along each point over $\Gamma(\mathcal{A})$ is equal to 1 and $\phi(\mathbb{Z}^n_+) \subset \mathbb{Z}_+$. This is called the Newton filtration induced by $\mathcal{A}$.

Now we introduce the control functions associated to $\mathcal{A}$ as follows:

$$\rho(x) = \left(\sum_{\alpha\in Ver(\mathcal{A})} x^{2p\alpha}\right)^{\frac{1}{2p}} \tag{1.2}$$

where p a positive integer. Moreover if p is big enough (it suffices, for example, that $p\alpha \in \mathbb{Z}^n_+$).

1.1. **Compensation factor.** Let L_j denote the x_j-axis. We then put $\alpha^j = L_j \cap \Gamma(\mathcal{A})$ for $j = 1, \ldots, n$ (the axial vertices of $\Gamma(\mathcal{A})$). We define the weight of the variable x_i, $\mathcal{A}(i) = \mathcal{A}(x_i) = \max\{\, w^F_i \ : \ F \in \mathcal{F}(\mathcal{A})\}$. We may introduce the compensation factors associated with $\mathcal{A}$ as follows:

$$\rho_i(x) = \left(x_i^{\frac{2p}{\mathcal{A}(i)}} + \sum_{\alpha\in Ver(\mathcal{A})\setminus\{\alpha^i\}} x^{2p\,\alpha}\right)^{\frac{\mathcal{A}(i)}{2p}}, \quad i = 1, \ldots, n$$

(for more details about these see [1, 2]).

Now let us introduce a singular Riemannian metric on $\mathbb{R}^n$ by the following bilinear form

$$\langle \rho_i \frac{\partial}{\partial x_i}\,,\, \rho_j \frac{\partial}{\partial x_j}\rangle = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{1.3}$$

We will denote by $\nabla_{\mathcal{A}}$, $\|\ \|_{\mathcal{A}}$, the corresponding gradient and norm associated with this Riemannian metric.

In order to state the version relative to the Newton filtration of the Kuo and Paunescu Theorems ([8, 9]) we need to introduce the Newton horn-neighborhood, of degree d and width $c > 0$, of a variety $f^{-1}(0)$, $f \in E(n,p)$. This is by definition

$$H_d(f, c) = \{\, x \in \mathbb{R}^n \ : \ |\, f(x)\,| \le c\,\rho^d\,\}.$$

Definition 1. *We say that $f, g \in E(n,p)$ are d-equivalent with respect to $\mathcal{A}$ or simply d-equivalent, if there exist $a > 0$ and a neighborhood U of the origin such that*

(i) $\left|\, f_j(x) - g_j(x)\,\right| \le a\rho^d, \quad 1 \le j \le p$

(ii) $\left|\, \rho_i(\frac{\partial f_j}{\partial x_i}(x) - \frac{\partial g_j}{\partial x_i}(x))\,\right| \le a\rho^d, \quad 1 \le j \le p, 1 \le i \le n$

(these f_j, g_j are the components of f and g respectively).

It is easy to see that this is an equivalence relation.

2. **The results.**

We call a given germ $f \in E(n,p)$ $v^{\mathcal{A}}$-sufficient at degree d (where "v"stands for variety and $\mathcal{A}$ stands for the Newton filtration associated with $\mathcal{A}$), or simply d-sufficient if for any $P \in E(n,p)$ such that f and $f+P$ are d-equivalent then f and $f+P$ have the same v-type at 0. We first remark that if f is d-sufficient then f is d_1-sufficient for any $d_1 \geq d$.

These are clearly version relative to the Newton filtration of the corresponding weighted or homogeneous cases (see for instance [8, 9]). For any $f \in E(n,p)$, let V_x be the subspace spanned by the $\{\nabla_{\mathcal{A}} f_1(x), \dots, \nabla_{\mathcal{A}} f_p(x)\}$. Let us consider now $N(f_j, \mathcal{A}, x)$, or simply $N(f_j, x)$, to be the vector $\nabla_{\mathcal{A}} f_j(x) - p_j(x)$, $1 \leq j \leq p$, where $p_j(x)$ is the projection of $\nabla_{\mathcal{A}} f_j(x)$, with respect to our metric, to the subspace $V_{x,j}$ spanned by the $\nabla_{\mathcal{A}} f_k(x)$, $k \neq j$. Hence, $\|N(f_j,x)\|_{\mathcal{A}} = \{$ distance of $\nabla_{\mathcal{A}} f_j(x)$ to $V_{x,j}\}$. Finally, we will denote by $d_{\mathcal{A}}(\nabla_{\mathcal{A}} f_1(x), \dots, \nabla_{\mathcal{A}} f_p(x))$ the minimum of $\|N(f_j,x)\|_{\mathcal{A}}$, $1 \leq j \leq p$.

Now using the above construction, we can announce our results.

Theorem 2. *If for any $g \in E(n,p)$ d-equivalent to f, there are positive numbers c, ϵ, δ, and a neighborhood U of 0, all depending on g, such that the following Łojasiewicz inequality related to the Newton polyhedron*

$$d_{\mathcal{A}}(\nabla_{\mathcal{A}} f_1(x), \dots, \nabla_{\mathcal{A}} f_p(x)) \geq \epsilon\, \rho^{d-\delta} \tag{2.1}$$

holds for $x \in H_d(f,c) \cap U$, then f is d-sufficient.

Corollary 3. *A sufficient condition for $f \in E(n,p)$ to be d-sufficient is that there exist $\epsilon > 0$, $c > 0$ and $\delta > 0$ for which*

$$d_{\mathcal{A}}(\nabla_{\mathcal{A}} f_1(x), \dots, \nabla_{\mathcal{A}} f_p(x)) \geq \epsilon\, \rho^{d-\delta}$$

is satisfied for all x in $H_{d-\delta}(f,c)$, near 0.

This corollary follows from the observation that for any $g \in E(n,p)$ with g d-equivalent to f, we have $H_d(g,c) \subset H_{d-\delta}(f,c)$ in a sufficiently small neighborhood of 0.

In the case when $f \in \mathcal{E}_{[w]}(n,p)$ (i.e., f is analytic) we have the following theorem.

Theorem 4. *For a given $f \in \mathcal{E}_{[w]}(n,p)$, and $d \geq 3\max\{\mathcal{A}(1), \dots, \mathcal{A}(n)\}$, the following are equivalent:*

- (i) *f is d-sufficient.*
- (ii) *The hypothesis of Theorem 2 hold.*
- (iii) *for any $g \in E(n,p)$, g d-equivalent to f, the variety $g^{-1}(0)$ admits 0 as a topologically isolated singularity ($\nabla g_j(x)$, $1 \leq j \leq p$, $x \in g^{-1}(0)$, are linearly independent near 0, $x \neq 0$).*

Remark 5. *We should note that in the case where $\#\mathcal{F}(\mathcal{A}) = 1$ i.e., weighted filtration associated with $w = (\mathcal{A}(1), \dots, \mathcal{A}(n))$, we find that $\rho_i(x) = \rho^{\mathcal{A}(i)}(x)$ for $i = 1, \dots, n$, hence our Theorems 2 and 4 reduce to Paunescu's Theorems (see [9], Theorem A and B).*

Another characterization of the v-sufficiency is given by the following theorem.

Theorem 6. *Suppose $f \in \mathcal{E}_{[w]}(n,p)$ which satisfy the hypothesis of Theorem 2, then for any $g \in \mathcal{E}_{[w]}(n,p)$, g is d-equivalent to f, the deformation variety*

$$F(x,t) \equiv f(x) + t[f(x) + g(x)] = 0$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, is (c)-regular over the t-axis in the sense introduced by Bekka (see [3]*).*

Remark 7. *We should note that in the usual filtration, i.e., $\#\mathcal{F}(\mathcal{A}) = 1$ and $w^F = (1, \dots, 1)$, Kuo* [8] *proves that the Lojasiewicz inequality in Theorem 2 implies that the stratification $\Sigma(V_F)$ is Whitney (b)-regular. Here, V_F denotes the variety of the zero locus of F, and $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{R},\ \{0\} \times \mathbb{R}\}$. Furthermore, if we suppose that $f, g \in \mathcal{E}_{[d]}(n,p)$, Bekka and Koike* [4] *prove that the stratification $\Sigma(V_F)$ is (c)-regular.*

We can also prove a component-wise variant of our Theorem 2. We will do this considering instead of the positive number d, a positive p-tuple $\underline{d} = (d_1, \dots, d_p)$. We say that $f,\ g \in E(n,p)$ are $\underline{d}$-equivalent if there exists a neighborhood U of the origin such that

(i) $f_j(x) - g_j(x) = o(\rho^{d_j})$,

(ii) $\| \nabla_{\mathcal{A}} f_j - \nabla_{\mathcal{A}} g_j \|_{\mathcal{A}} = o(\rho_j^d), \quad 1 \le j \le p, \quad x \in U.$

Also we can introduce the corresponding horn-neighborhood $H_{\underline{d}}(f,c) = \{ x \in \mathbb{R}^n : |\, f_j(x)\,| \le c\, \rho^{d_j} \}$, and the corresponding notion of $\underline{d}$-sufficiency. Then, we can state the following theorem.

Theorem 8. *Suppose $f \in E(n,p)$ such that there exist positive numbers ϵ, c and*

$$\|N(f_j, x)\|_{\mathcal{A}} \ge \epsilon\, \rho^{d_j}, \quad j = 1, \dots, p, \quad x \in H_{\underline{d}}(f,c).$$

Then f is $\underline{d}$-sufficient.

The proof is similar to the proof of Theorem 2 and it will be omitted.

Example 9. *Consider the map $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ defined by*

$$f(x,y) = (xy\,,\ x^{10} - x^2y^4 - y^{10}).$$

We set $\mathcal{A} = \{\, (1,0)\,,\ (\frac{1}{5}, \frac{2}{5})\,,\ (0,1)\,\}$, it not hard to see the following inequalities :

$$\|N(f_1,x)\|_{\mathcal{A}}^2 = \frac{\rho_1^2\rho_2^2 \begin{vmatrix} y & x \\ 10x^9 - 2xy^4 & -4x^2y^3 - 10y^9 \end{vmatrix}^2}{\|\nabla_{\mathcal{A}} f_2\|_{\mathcal{A}}^2} \ge \epsilon\, \rho^{10},$$

$$\|N(f_2,x)\|_{\mathcal{A}}^2 = \frac{\rho_1^2\rho_2^2 \begin{vmatrix} y & x \\ 10x^9 - 2xy^4 & -4x^2y^3 - 10y^9 \end{vmatrix}^2}{\|\nabla_{\mathcal{A}} f_1\|_{\mathcal{A}}^2} \ge \epsilon\, \rho^{20}.$$

Therefore, we may conclude that f is $(5,10)$-sufficient with respect to this Newton filtration induced by $\mathcal{A}$.

Example 10. *Let $f\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be the map defined by*

$$f(x,y,z) = (xy\,,\ x^6 + xz + y^2 + z^2).$$

We let $\mathcal{A} = \{\, (1,0,0)\,,\ (\frac{1}{6}, 0, \frac{1}{6})\,,\ (0, \frac{1}{3}, 0)\,,\ (0,0,\frac{1}{3})\}$. By standard argument, based on the curve selection lemma, we can see that

$$d_{\mathcal{A}}(\nabla_{\mathcal{A}} f_1(x), \nabla_{\mathcal{A}} f_2(x)) \ge \epsilon\, \rho^6, \quad x \in H_6(f,c),$$

where ϵ and c are sufficiently small. Therefore, f is $(6,6)$-sufficient with respect to this Newton filtration induced by $\mathcal{A}$.

3. **Proofs of Theorem 2, 4 and 6.**

3.1. **Proof of Theorem 2.**

In order to show this Theorem 2 we need the following lemma.

Lemma 11. $\|\nabla_{\mathcal{A}}\rho\|_{\mathcal{A}} \leq c\,\rho(x), \quad c > 0$ *constant.*

Proof. We first recall that:

$$\|\nabla_{\mathcal{A}}\rho\|_{\mathcal{A}}^2 = \sum_{i=1}^{n}\left(\rho_i \frac{\partial \rho}{\partial x_i}(x)\right)^2.$$

Therefore, our lemma is a simple consequence of the construction of the compensation factors and the control function (see [1, 2] for details). □

The proof of Theorem 2 will follow the proof given by Kuo and Paunescu in the quasi-homogeneous case ([8, 9]). Take any $P \in E(n,p)$ with the property that f and $f+P$ are d-equivalent, and let $F(x,t) = f(x) + t\,P(x)$, where $t \in \mathbb{R}$. By addition to the bilinear form in (1.3), we define a new metric by

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\rangle = 0, \quad i = 1,\ldots,n, \quad \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle = 1.$$

By elementary calculation, we can express the gradient vector field of the function F_j with respect to this metric singular Riemannian metric as follows:

$$\nabla_{\mathcal{A}}F_j(x,t) == \sum_{i=1}^{n}\rho_i\left(\frac{\partial f_j}{\partial x_i}(x) + t\frac{\partial P_j}{\partial x_i}(x)\right)\rho_i\frac{\partial}{\partial x_i} + P_j(x)\frac{\partial}{\partial t}$$

(here f_j, P_j are the corresponding components of f, P respectively).

We shall show that any $t_0 \in \mathbb{R}$ has a neighborhood T such that for any $t_1, t_2 \in T$, the germs of $F(x,t_1) = 0$ and $F(x,t_2) = 0$ are homeomorphic, and due to the fact that $I = [0,1]$ is compact it will follow that the germs of $F(x,0) = 0$ and $F(x,1) = 0$ are homeomorphic. Therefore f is d-sufficient.

Let $(0,t_0)$ be a given point of the t-axis. Consider the map $g(x) = f(x) + t_0P(x)$, then $|F_j(x,t) - g_j(x)| = |t - t_0|\,|P_j(x)|$, $j = 1,\ldots,p$. Since f and $f+P$ are d-equivalent, we can choose a sufficiently small neighborhood T of t_0 and a neighborhood U of $0 \in \mathbb{R}^n$, such that $|F(x,t) - g(x)| \leq c\,\rho^d$.

Thus, the variety $F(x,t) = 0$ for $(x,t) \in U \times T$, is contained in the $H_d(g,c) \times T$. This is one of the reasons for introducing this set. We have the following lemma.

Lemma 12. $\|N(F_j,(x,t))\|_{\mathcal{A}} \geq \frac{\epsilon}{2}\,\rho^{d-\delta}, \quad (x,t) \in H_d(g,c) \times T, \quad j = 1,\ldots,p.$

Proof. We note that if $\#\mathcal{F}(\mathcal{A}) = 1$, one finds Lemma 1 of Paunescu [9]. Moreover, the proof of this lemma is similar to that of Lemma 1 in [9] (we omit the details). □

Corollary 13. *For (x,t) as above, $x \neq 0$, the vectors $\nabla_{\mathcal{A}}F_j(x,t)$, $1 \leq j \leq p$, are linearly independent.*

Proof. Obvious. Use Lemma 12. □

Now we can introduce the Kuo vector field as follows:

$$K(x,t) = \frac{\partial}{\partial t} - \sum_{j=1}^{p}\frac{P_j}{\|N_j\|_A}N_j \quad \text{if} \quad x \neq 0 \quad \text{and} \quad K(0,t) = \frac{\partial}{\partial t},$$

where $N_j = N(F_j,(x,t)), \quad j=1,\dots,p$. We see K is tangent to the level of $F=0$, whenever $x\neq 0$. Furthermore, from the construction of the Kuo vector field, K is C^1 outside $x=0$ and continuous everywhere in $H_d(g,c)\times T$.

One can write $N_j = \sum_{i=1}^n \rho_i C_{ij}(x,t)\rho_i \frac{\partial}{\partial x_i} + L_j(x,t)\frac{\partial}{\partial t}$, and K can be written as :

$$K(x,t) = \left(1 - \sum_{j=1}^p \frac{L_j P_j}{\|N_j\|^2_{\mathcal{A}}}\right)\frac{\partial}{\partial t} - \sum_{i=1}^n \left(\sum_{j=1}^p \frac{P_j C_{ij}}{\|N_j\|^2_{\mathcal{A}}}\right)\rho_i^2 \frac{\partial}{\partial x_i}$$
$$= X\frac{\partial}{\partial t} - \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}.$$

Recall that $\|N_j\|^2_{\mathcal{A}} = \sum_{i=1}^n (C_{ij}\rho_i)^2 + L_j^2$ and $|P_j|\leq a\rho^d$. It follows from Lemma 12 that X tends to 1 as x tends to 0, and X_i tends to 0 when x tends to 0. In other words, we have the following inequalities

$$\frac{|P_j|}{\|N_j\|_{\mathcal{A}}} \leq \frac{a\rho^d}{\frac{\epsilon}{2}\,\rho^{d-\delta}} \quad \text{and} \quad |X_i| \leq \sum_{j=1}^p \frac{|P_j|}{\|N_j\|_{\mathcal{A}}}\frac{|C_{ij}\rho_i|}{\|N_j\|_{\mathcal{A}}}\rho_i \leq c_i\rho_i \tag{3.1}$$

in a small Newton horn-neighborhood of 0, $c_i>0,\ 1\leq i\leq n,\ j=1,\dots,p$.

Now consider two Liapunov functions $U(x,t) = e^{2Lt}\rho^2$ and $V(x,t)=e^{-2Lt}\rho^2$. In order to show that the integration of this vector field gives us the homeomorphism, it is enough to show that $\nabla U\cdot K>0$ and $\nabla V\cdot K<0$ for $x\neq 0$ (see [7, 8]). Indeed, by a simple computation yields

$$\nabla U\cdot K = 2e^{2Lt}\rho\left(L\rho X + \sum_{i=1}^n \frac{\partial\rho}{\partial x_i}X_i\right)$$
$$\geq e^{2Lt}\rho\left(L\rho X - \sum_{i=1}^n \left|\frac{\partial\rho}{\partial x_i}\right||X_i|\right)$$
$$\geq e^{2Lt}\rho\left(L\rho X - \sum_{i=1}^n \left|\frac{\partial\rho}{\partial x_i}\right|c_i\rho_i\right)$$

(here we have used the second inequality in (3.1)).

According to the lemma 11, we have $\rho_i|\frac{\partial\rho}{\partial x_i}|\leq M\rho$, some $M>0$. Thus, we can choose L big enough such that $\nabla U\cdot K>0$, $x\neq 0$. In a similar way we can prove the other inequality. This completes the proof of Theorem 2.

3.2. **Proof of Theorem 4.** Since (ii) $\Rightarrow$ (i) have already been obtained in Theorem 2, and (ii) $\Rightarrow$ (iii) is proved in exactly the same way as Theorem B in [9], we have only to prove (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii).

Suppose now that (ii) is fails. In order to prove that (iii) is fails we will construct a function $\tilde{f}\in E(n,P)$ such that $\tilde{f}$ and f are d-equivalent but $\nabla\tilde{f}_j,\ j=1,\dots,p$, are linearly dependent along an analytic arc in $\tilde{f}^{-1}(0)$.

The following proof will be similar to that given in [8, 9], so we will point out only the differences.

Let $g\in\mathcal{E}_{[w]}(n,p)$ d-equivalent with f and such that for any $c,\ \epsilon,\ \delta$ and any neighborhood U of 0, the Łojasiewicz inequality related to the Newton polyhedron (2.1) fails. Let E be the following semi-analytic set

$$E = \{\, u\in H_d(g,1)\,|\, d_{\mathcal{A}}(\nabla_{\mathcal{A}}f_1(u),\dots,\nabla_{\mathcal{A}}f_p(u)\,) = \min_{\substack{\rho(u)=\rho(x)\\ x\in H_d(g,1)}} d_{\mathcal{A}}(\nabla_{\mathcal{A}}f_1(x),\dots,\nabla_{\mathcal{A}}f_p(x)\,)\,\}$$

Because E is a semi-analytic set. Applying the curve selection lemma, one can find an analytic arc $\beta\colon [0\,,\epsilon] \to E$ such that $\beta(0) = 0$ and $\beta(t) \neq 0$ for $t > 0$. Modulo a permutation, we can choose this arc such that along β,

$$\begin{aligned} d_{\mathcal{A}}(\nabla_{\mathcal{A}} f_1(\beta(t)), \dots, \nabla_{\mathcal{A}} f_p(\beta(t))\,) &= \|N(f_1, \beta(t))\|_{\mathcal{A}} \\ &= \|(\nabla_{\mathcal{A}} f_1(\beta(t)) - \sum_{k=2}^{p} \lambda_k(t)\, \nabla_{\mathcal{A}} f_k(\beta(t))\|_{\mathcal{A}}, \end{aligned}$$

where λ_k are analytic for $k = 2, \dots, p$.

Let r and μ be the numbers such that $\rho(\beta(t)) \sim t^r$ and $\|N(f_1, \beta(t))\|_{\mathcal{A}} \sim t^{\mu}$. Here $A(t) \sim B(t)$ means that $\frac{A}{B}$ lies between two positive constants, for $t > 0$ and t small. Then due to the fact that (ii) fails we have $\frac{\mu}{r} \geq d$. Actually if $\beta_i(t) \sim t^{s_i}$, $\rho_i(\beta(t)) \sim t^{q_i}$ and $|\frac{\partial f_1}{\partial x_i}(\beta(t) - \sum_{k=2}^{p} \lambda_k(t) \frac{\partial f_k}{\partial x_i}(\beta(t)| \sim t^{\mu_i}$ for $i = 1, \dots, n$, then we can see

(3.2) $\quad \mu = \min\{\mu_i + q_i\}, \quad r = \min\{< \alpha, s > \mid \alpha \in Ver(\mathcal{A})\}$ and $q_i = \min\{rA(i)\,,\, s_i\}.$

Modulo a permutation we may assume $s_1 = \min\{s_i\}$ and $\beta_1(t) = t^{s_1}$.

Let $l(\mathcal{A})$ denote the distance from the origin to the Newton polyhedron $\Gamma_+(\mathcal{A})$ by defining

$$l(\mathcal{A}) = \min\left\{ \sum_{i=1}^{n} \alpha_i \mid \alpha \in Ver(\mathcal{A}) \right\}.$$

From the definition of the weight $\mathcal{A}(i)$, it is easy to check that

$$l(\mathcal{A}) \max\{\mathcal{A}(1), \dots, \mathcal{A}(n)\} \geq 1. \tag{3.3}$$

Let us put $M(\mathcal{A}) = \max\{\mathcal{A}(1), \dots, \mathcal{A}(n)\}$. Since $\mu \geq rd$, it follows from (3.2) that for $1 \leq i \leq n$ we have $\mu_i + q_i \geq \mu \geq rd$ and $q_i \leq rM(\mathcal{A})$ so $\mu_i \geq r(d - M(\mathcal{A}))$. Also from (3.2) we have that $r \geq s_1 l(\mathcal{A})$, hence

$$\frac{\mu_i}{s_1} \geq l(\mathcal{A})(d - M(\mathcal{A})) \quad \text{for } i = 1, \dots, n. \tag{3.4}$$

Now let us consider the following functions

$$Q_i(x_1) = \left(\frac{\partial f_1}{\partial x_i}(\beta(|x_1|^{1/s_1})) - \sum_{k=2}^{p} \lambda_k(|x_1|^{1/s_1}) \frac{\partial f_k}{\partial x_i}(\beta(|x_1|^{1/s_1})) \right)$$

for $i = 1, \dots, n$ and

$$P(x) = f(\beta(|x_1|^{1/s_1})) + \sum_{i=2}^{n} Q_i(x_1)(x_i - \beta_i(|x_1|^{1/s_1})).$$

Also we define $\tilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by

$$\begin{aligned} \tilde{f}_1(x) &= f_1(x) - P(x) \\ \tilde{f}_k(x) &= f_k(x) - f_k(\beta(|x_1|^{1/s_1}), \quad k = 2 \dots, p. \end{aligned}$$

Using our assumption about d, $\;d \geq 3M(\mathcal{A})$, it follows from (3.4) and $l(\mathcal{A})M(\mathcal{A}) \geq 1$ that the order of $Q_i(x_1)$ is $\frac{\mu_i}{s_1} \geq l(\mathcal{A})(d - M(\mathcal{A})) \geq 2$ for $i = 1, \dots, n$, thus our P is a C^2 function, hence $\tilde{f} \in E(n,p)$ and moreover f and $\tilde{f}$ are d-equivalent. This follows from the construction of $\tilde{f}$ and the fact that f is analytic. We can check that our representative of the class of f has the following property :

$$\tilde{f}(\beta(t)) = 0 \text{ and } \nabla \tilde{f}_1(\beta(t)) - \sum_{k=2}^{p} \lambda_k(t) \nabla \tilde{f}_k(\beta(t)) = 0.$$

Moreover, using this $\tilde{f}$ one can prove (just as in [8, 9]) that non (ii) $\Rightarrow$ non (i). This completes the proof of Theorem 4.

3.3. **Proof of Theorem 6.** Let us put $X = F^{-1} - \{0\} \times \mathbb{R}$ and $Y = \{0\} \times \mathbb{R}$. Now from the Theorem 5 in [1], it is enough to show that (X, Y) is $(w^{\mathcal{A}})$-regular, that is,

$$\|d^x F\|_{\mathcal{A}} \lesssim \|d_x F\|_{\mathcal{A}} \text{ holds on } X \text{ near } Y. \tag{3.5}$$

Where

$$\|d_x F\|^2_{\mathcal{A}} = \sum_{1 \le i_1 < \cdots < i_p \le n} \left(\rho_{i_1} \cdots \rho_{i_p} \left| \frac{\partial F}{\partial(x_{i_1}, \ldots, x_{i_p})} \right| \right)^2,$$

$$\|d^x F\|^2_{\mathcal{A}} = \sum_{1 \le i_1 < \cdots < i_{p-1} \le n} \left(\rho_{i_1} \cdots \rho_{i_{p-1}} \left| \frac{\partial F}{\partial(x_{i_1}, \ldots, x_{i_{p-1}}, t)} \right| \right)^2 \text{ and}$$

$$\|dF\|^2_{\mathcal{A}} = \|d_x F\|^2_{\mathcal{A}} + \|d^x F\|^2_{\mathcal{A}}.$$

Here $A \lesssim B$ means there is some positive constant C with $A \le CB$.

For $j = 1, \ldots, p$, we let $F^{[j]} = (F_1, \ldots, \widehat{F_j}, \ldots, F_p)$ and $X^{[j]} \equiv F^{[j]} = 0$. Here, $\widehat{\ }$ indicates that we omit the letter (or the portion) to which $\widehat{\ }$ is attached. We note that the normal space of $X^{[j]}$ with respect our metric, denoted by $V_{\mathcal{A},j}$, is the subspace spanned by the $\{\nabla_{\mathcal{A}} F_1, \ldots, \widehat{\nabla_{\mathcal{A}} F_j}, \ldots, \nabla_{\mathcal{A}} F_p\}$. From the corollary 13, the vectors $\nabla_{\mathcal{A}} F_k(x,t)$, are linearly independent. Then by lemma 1.4 in [5], we obtain that the projection π_j, with respect to our metric, onto the tangent space $T_{\mathcal{A},x,t} X^{[j]} = V^{\perp}_{\mathcal{A},j}$ is expressed by the following form.

$$\pi_j(v) = \sum_{i=1}^{n} \frac{\langle dF^{[j]} \wedge dx_i, dF^{[j]} \wedge v \rangle}{\|dF^{[j]}\|^2_{\mathcal{A}}} \rho_i^2 \frac{\partial}{\partial x_i} + \frac{\langle dF^{[j]} \wedge dt, dF^{[j]} \wedge v \rangle}{\|dF^{[j]}\|^2_{\mathcal{A}}} \frac{\partial}{\partial t} \tag{3.6}$$

Since $N(F_j, (x,t)) = \pi_j(\nabla_{\mathcal{A}} F_j)$, we can easily see that

$$\|N(F_j, (x,t))\|^2_{\mathcal{A}} = \langle \pi_j(\nabla_{\mathcal{A}} F_j), \nabla_A F_j \rangle = \frac{\|dF\|^2_{\mathcal{A}}}{\|dF^{[j]}\|^2_{\mathcal{A}}}. \tag{3.7}$$

By the Cauchy-Schwartz inequality, we have

$$\left| \frac{\partial F}{\partial(x_{i_1}, \ldots, x_{i_{p-1}}, t)} \right|^2 \le \sum_{j=1}^{p} \left| \frac{\partial F^{[j]}}{\partial(x_{i_1}, \ldots, x_{i_{p-1}})} \right|^2 \left| \frac{\partial F_j}{\partial t} \right|^2,$$

so that

$$\left(\rho_{i_1} \cdots \rho_{i_{p-1}} \left| \frac{\partial F}{\partial(x_{i_1}, \ldots, x_{i_{p-1}}, t)} \right| \right)^2 \le \sum_{j=1}^{p} \left(\rho_{i_1} \cdots \rho_{i_{p-1}} \left| \frac{\partial F^{[j]}}{\partial(x_{i_1}, \ldots, x_{i_{p-1}})} \right| \right)^2 \left| \frac{\partial F_j}{\partial t} \right|^2,$$

then

$$\|d^x F\|^2_{\mathcal{A}} \le \sum_{j=1}^{p} \|dF^{[j]}\|^2_{\mathcal{A}} \left| \frac{\partial F_j}{\partial t} \right|^2. \tag{3.8}$$

Since $\left| \frac{\partial F_j}{\partial t} \right| = |f_j - g_j| \lesssim \rho^d$. This is clear because f and g are d-equivalent. It follows from Lemma 12 that

$$\left| \frac{\partial F_j}{\partial t} \right| \ll \rho^{d-\delta} \lesssim \|N(F_j, (x,t))\|_{\mathcal{A}}$$

and, using (3.8), we obtain

$$\|d^x F\|^2_{\mathcal{A}} \ll \sum_{j=1}^{p} \|dF^{[j]}\|^2_{\mathcal{A}} \, \|N(F_j, (x,t))\|^2_{\mathcal{A}}. \tag{3.9}$$

It now follows from (3.7) that (3.5) holds. This completes the proof of Theorem 6.

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