Bifurcation Model from Initial Nonlinear Term of Nonlinear Partial Differential Equations

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Abstract

The bifurcation model from the initial nonlinear term of partial differential equation is introduced. We show how these models work on the domain like rectangle and square etc. We introduce the non-degeneracy condition which ensures the (m, k)-bifurcation model describes the bifurcation of partial differential equation. We observe a perturbation of rectangle to a square creates new bifurcation, which is not a limit of the bifurcations on rectangles.

Contents

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1 Introduction

In this paper, let X,Y be Banach spaces, and let $\lambda\in\mathbb{R}$, we investigate the bifurcation of the nonlinear partial differential equation

$$\Phi(\lambda,u)=Lu+h(\lambda,u)-\lambda u=0,\qquad u\in X, \tag{1.1}$$

where $\lambda\in\mathbb{R}$, L is a linear self-adjoint operator, and $h(\lambda,u)\in C^1(\mathbb{R}\times X,X)$, $h(\lambda,0)=0$, $\frac{d}{du}h(\lambda,0)=0$.

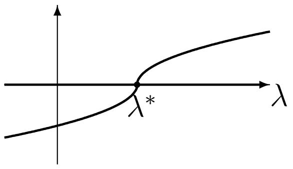
We first recall the notation of a bifurcation point [1, §1]. Suppose that $\Phi:\mathbb{R}\times X\to Y$ is a continuous map. Let $S_\lambda=\{x\in X\mid\Phi(\lambda,x)=0\}$ be the solution set of the equation $\Phi(\lambda,x)=0$, where λ is a parameter. Assume that $\Phi(\lambda,0)=0$, we call $(\lambda,0)$ a bifurcation point, if for any neighborhood U of $(\lambda,0)$, there exists $(\lambda,x)\in U$ with $x\in S_\lambda\setminus\{0\}.\lambda$ is a branching point if the solution set S_λ contains a connected set S such that $(\lambda,0)\in S$ and $S\ \{(\lambda,0)\}\neq\emptyset$.

The main purpose of this paper is to establish the (m,k)-bifurcation model with the initial higher order term (Definition 3.1) for the equation (1.1) at the bifurcation point $(\lambda^*,0)$, where λ^* is an eigenvalue with multiplicity m of L, $a_k(\lambda)u^k$ is the first nonzero term of the Taylor expansion of $h(\lambda,u)$. If the region Ω is k-non-degenerate (Definition 3.2), we show that the bifurcation equations of Lyapunov-Schmidt reduction are equivalent to the (m,k)-bifurcation model (Theorem 3.4).

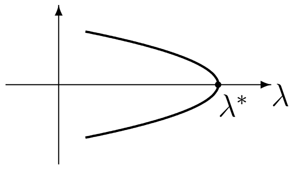
For a simple eigenvalue λ^*, the bifurcation model (Remark 3.3) is described by

$$(\lambda^*-\lambda)x+ax^k=0,\ a\neq 0 \tag{1.2}$$

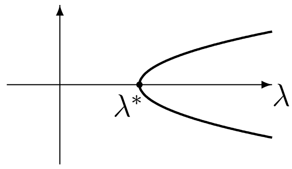
and the bifurcation of solutions is decided by k and a. See the following figures.



Transcritical bifurcation (k is even).



Subcritical bifurcation (k is odd, $a < 0$).



Supercritical bifurcation (k is odd, $a > 0$).

A motivation of this paper is to find a generalization of this phenomenon to the multiple eigenvalues case. The (m, k)-Bifurcation model is a weighted homogeneous system (Remark 3.3) and it provides a way to investigate the bifurcation of nonlinear partial differential equations on k-non-degenerate region Ω. As a series of the results of our methods, for the partial differential equation

$$-\Delta u = \lambda u - a_k(\lambda)u^k + o(u^k) \quad \text{on } \Omega$$

with Dirichlet or Neumann boundary value conditions, the $(2, 3)$ and $(2, 5)$-bifurcation models show the type of bifurcations for all the eigenvalues with multiplicity 2 on the region $\Omega = [0, \pi]^2$ (Dirichlet problem: Theorem 5.13 and 5.15, Neumann problem: Theorem 5.17 and 5.18). The $(3, 3)$-bifurcation model also shows the type of bifurcation for all the eigenvalues with multiplicity 3 on the region $\Omega = [0, \pi]^3$ (Dirichlet problem: Theorem 5.20, Neumann problem: Theorem 5.21). We observe that all the bifurcations of these eigenvalues are plurisupercritical (or plurisubcritical) bifurcation (Remark 3.3).

The methods presented here are mainly based on the nonlinear functional analysis and singularity theory.

The paper is organized as follows. In section 2, we recall the inverse function theorem, implicit function theorem, Lyapunov-Schmidt reduction and Schauder bases. In section 3, the (m, k)-Bifurcation model from the initial nonlinear term of partial differential equations is defined. In section 4, the equivalent conditions of k-non-degeneracy and the main theorem are proved. In section 5, we show how our method works on the domain like rectangle and square, etc. In section 6, when the rectangle degenerate to square, we observe that there is a new bifurcation on square, which is not a limit of bifurcations on rectangles.

2 Preliminary

Let us recall the following theorems first.

Inverse Function Theorem and Implicit Function Theorem. Let X, Y be Banach spaces, $u \in X$ and let F be a map $X \to Y$. In the particular case that $Y = \mathbb{R}$, F is called a function. We say that F is differentiable at $u \in X$ along the direction $v \in X$ if there exists

$$L_u[v] := \lim_{t \to 0} \frac{F(u + tv) - F(u)}{t}.$$

We say that F is (Fréchet) differentiable at $u \in X$ if there exists a linear continuous map $L_u : X \to Y$ such that

$$F(u+v) - F(u) = L_u[v] + o(\|v\|), \quad as \ \ \|v\| \to 0.$$

When F is Fréchet differentiable at $u \in X$, the map L_u is uniquely determined by F and u and is denoted by $dF(u)$ or $F'(u)$. It is easy to see that if F is Fréchet differentiable, then it is also differentiable along any direction. Conversely, if F is differentiable along any directions, $L_u \in L(X,Y)$ and the map $u \mapsto L_u$ is a continuous map from X to $L(X,Y)$. then F is Fréchet differentiable. The Fréchet derivative has the same properties as the usual differential in Euclidean spaces [2, 3].

Lemma 2.1 (Inverse Function Theorem (Theorem 3.1.1 in [3], [5]))**.** *Let $P : U \to V$, be a smooth map between Banach spaces, U, V are open sets of X, Y respectively. Suppose that for some $f_0 \in U$ the derivative $dP(f_0) : X \to Y$ is an invertible linear map. Then we can find neighborhoods $\tilde{U}$ of f_0 and $\tilde{V}$ of $g_0 = P(f_0)$ such that the map P gives a one-to-one map of $\tilde{U}$ onto $\tilde{V}$, and the inverse map $P^{-1} : \tilde{V} \subseteq Y \to \tilde{U} \subseteq X$ is smooth.*

Lemma 2.2 (Implicit Function Theorem (Theorem 3.2.1 in [3], [2, 5]))**.** *Let X, Y be Banach spaces and fix $(\lambda_0, u_0) \in \mathbb{R} \times X$. Assume that F is a C^1 map from a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$ into Y such that $F(\lambda_0, u_0) = 0$ and suppose that $d_u F(\lambda_0, u_0)$ is invertible. Then there exist a neighborhood Λ of λ_0 and a neighborhood U of u_0 such that the equation $F(\lambda, u) = 0$ has a unique solution $u = u(\lambda) \in U$ for all $\lambda \in \Lambda$. The function $u(\lambda)$ is of class C^1, and the following holds*

$$u'(\lambda_0) = -[d_u F(\lambda_0, u_0)]^{-1} d_\lambda F(\lambda_0, u_0).$$

Lyapunov-Schmidt reduction. Consider the case that

$$\Phi(\lambda, u) = Lu + h(u) - \lambda u = 0, \quad \lambda \in \mathbb{R}, \ u \in X,$$

where L is a linear operator, and $h \in C^1(X,Y)$, $h(\lambda, 0) = 0$, $\frac{d}{du} h(\lambda, 0) = 0$. Thus,

$$d_u \Phi(\lambda^*, 0)[v] = Lv + h'(0)[v] - \lambda^* v = Lv - \lambda^* v.$$

If $Lv - \lambda^* v \neq 0$, by the inverse function theorem, then $(\lambda^*, 0)$ is the unique solution of $\Phi(\lambda, u) = Lu + h(u) - \lambda u = 0$. That is, $(\lambda^*, 0)$ cannot be a bifurcation point. If $(\lambda^*, 0)$ is a bifurcation point, then λ^* is an eigenvalue of L.

Let $(\lambda^*, 0)$ is a bifurcation point of the equation $F(\lambda, u) = 0$. Set $V := \mathrm{Ker}(\, L - \lambda^* I)$, $R := \mathrm{Range}(L - \lambda^* I)$. Assume that R is closed, V has a topological complement W in X, R has a topological complement Z in Y, then

$$X = V \oplus W, \qquad Y = Z \oplus R,$$

where $\dim Z = n$. For every $u \in X$, u can be written in the form

$$u = v + w, \quad v \in V, \quad w \in W,$$

uniquely. Let P be the projection onto Z, then $I-P$ be the conjugate projection onto R. The equation $F(\lambda,u)=0$ is equivalent to

$$P\Phi(\lambda, v+w)=0, \tag{2.1}$$

$$(I-P)\Phi(\lambda, v+w)=0. \tag{2.2}$$

The equation (2.1) is called the *bifurcation equation* and the equation (2.2) the *auxiliary equation*. By implicit function theorem, (2.2) can be uniquely solved with respect to w locally. Denote that $w=r(\lambda,v)$ be the solution of (2.2), substitute $w=r(\lambda,v)$ in (2.1), one gets the bifurcation equation $PF(\lambda, v+r(\lambda,v))=0$ which determines the bifurcation of solutions to $\Phi(\lambda,u)=0$.

Schauder bases. A sequence $\{x_n\}$ of elements of a Banach space X is said to be a *Schauder bases* for X if for every x of X there is a unique sequence of numbers $\{a_n\}$ such that $x=\sum_{i=1}^{\infty} a_i x_i$ in the sense that $\lim_{n\to\infty}\|x-\sum_{i=1}^{n} a_i x_i\|=0$ (see [7, 9]).

- Every orthonormal bases in a separable Hilbert spaces is a Schauder bases (see [8, Example on the Page 134], [6, Theorem 1]).
- Let E, F be two Banach spaces with Schauder bases $\{x_n\}$, $\{y_n\}$, respectively. Then the system of all products $x_i\otimes y_j$ is a Schauder bases of $E\otimes F$ (see [9, Theorem 18.1]).

3 Bifurcation Model from the Initial Nonlinear Term

In this section, we are going to establish a bifurcation model for the equation (1.1).

Assume that $(\lambda^*,0)$ is a bifurcation point of $\Phi(\lambda,u)=0$, that is, λ^* is an eigenvalue with finite multiplicity m of L. Let v_i be the bases of $V=\mathrm{Ker}(\,L-\lambda^* I)$ and w_j be the bases of $V=\mathrm{Ker}(\,L-\lambda_j I)$ with eigenvalues $\lambda_j\neq\lambda^*$, where $1\le i\le m, 1\le j\le\infty$.

By Taylor expansion,

$$h(\lambda,u)=a_0(\lambda)+a_1(\lambda)u+a_2(\lambda)u^2+\cdots+a_k(\lambda)u^k+\cdots,$$

then $a_0(\lambda)=a_1(\lambda)=0$ for $h(\lambda,0)=0$, $\frac{d}{du}h(\lambda,0)=0$. Let m be the multiplicity of λ^*. Let k be the order of the least number so that $a_k(\lambda)\neq 0$.

Assume that there exists $\phi: X\to\mathbb{R}$, such that $v^*x=\phi(vx)$, $v^*\in V^*$, $x\in X$. The k-form $a_k(\lambda^*)v_p^*\sum_{i=1}^{m}(x_iv_i)^k$ is the partial derivative of the following $(k+1)$-form H by x_p, where

$$H=\frac{a_k(\lambda^*)}{k+1}\phi(x_1v_1+\cdots+x_mv_m)^{k+1}.$$

Considering the PDE:

$$Lu+h(x,\lambda)-\lambda u=0 \ \text{ in } \Omega, \quad h(\lambda,u)=a_k(\lambda)u^k+o(u^k),\ k\ge 2,$$

with one of the following boundary condition:

- Dirichlet boundary condition: $u|_{\partial\Omega} = 0$.
- Neumann boundary condition: $\partial_n u|_{\partial\Omega} = 0$.

where n denotes the (typically exterior) normal to the boundary $\partial\Omega$, ∂_n is the partial differential along the direction n.

Let $\widehat{Z}$ denote the set defined by the bifurcation equation $\hat{F} = 0$ in $\mathbb{R} \times \mathbb{R}^m$.

Definition 3.1 (Bifurcation model)**.** *Set* $F_i = (\lambda^* - \lambda)x_i + H_{x_i}$ $(i = 1, \ldots, m)$ *where*

$$H = \frac{a_k(\lambda^*)}{k+1}\phi(x_1 v_1 + \cdots + x_m v_m)^{k+1}.$$

We say the set Z *defined by* $F_i = 0$ $(i = 1, \ldots, m)$ *is* (m, k)**-bifurcation model**.

Definition 3.2 (Non-degenerate region)**.** *We say that the region* Ω *is* k**-non-degenerate** *if the restriction of* H *to* S *is a Morse function, and* 0 *is not a critical value of the restriction of* H *to* S*. Here* S *is the sphere defined by* $\sum_{i=1}^{m} x_i^2 = k + 1$.

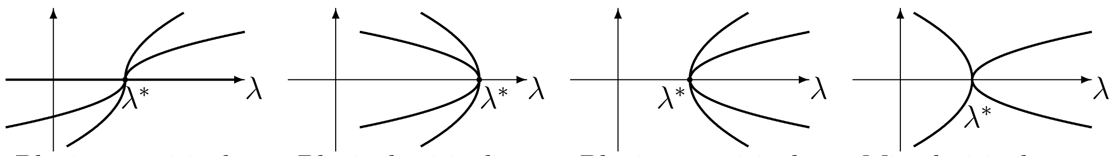
Remark 3.3 (Type of bifurcation portrait)**.** *When* $m = 1$ *and* k *is finite, the* $(1, k)$*-bifurcation model is defined by* (1.2)*.*

The (m, k)*-bifurcation model is a weighted homogeneous system with weight* $(k - 1, 1, \ldots, 1; k, \ldots, k)$*. There are* k^m *complex branches of the* (m, k)*-Bifurcation model. The solution curves of* (m, k)*-bifurcation model* (*Definition 3.1*) *are expressed in the following form:*

$$t \mapsto (\lambda, x_1, x_2, \ldots, x_m) = (a_0 t^{k-1}, a_1 t, a_2 t, \ldots, a_m t).$$

We call the image of the interval $t \geq 0$ (*or* $t \leq 0$) *a* **real semi-branch** *of the bifurcation model.*

(i) *If* k *is even, then all real branches go through from the region* $\lambda < \lambda^*$ *to the region* $\lambda > \lambda^*$*. Several transcritical bifurcations take place at the bifurcation point* $(\lambda^*, 0)$*. We say such a bifurcation* **pluritranscritical bifurcation** (*or multi-transcritical bifurcation*)*. See the left figure below.*



Pluritranscritical bifurcation (k is even). Plurisubcritical bifurcation (k is odd). Plurisupercritical bifurcation (k is odd). Mixedcritical bifurcation (k is odd).

(ii) *If* k *is odd, then the real branches of each solution stay in the region* $\lambda \leq \lambda^*$ *or* $\lambda \geq \lambda^*$*. Then one of the following types is possible as the bifurcation at* $(\lambda^*, 0)$*. See the right three figures above. We call them* **plurisupercritical bifurcation** (*or multi-supercritical bifurcation*)*,* **plurisubcritical bifurcation** (*or multi-subcritical bifurcation*)*,* **mixed critical bifurcation***, respectively.*

Theorem 3.4. *If the region Ω is k-non-degenerate, then the bifurcation equations $\hat{F}_i = 0$ $(i = 1, \dots, m)$ are equivalent to the (m, k)-bifurcation model $F_i = 0$ $(i = 1, \dots, m)$, that is, there is a homeomorphism germ*

$$\Xi : (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)) \to (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)),$$

preserving the hyperplane defined by $\lambda = \lambda^*$, with $\Xi(Z) = \hat{Z}$.

Remark 3.5. *The use of the function H has already appeared in [1, Theorem 1],[3, Page 66]. They showed $(\lambda^*, 0)$ is a branching point under non-degeneracy conditions. Since we use singularity theory, we are able to conclude the bifurcation model and the type of bifurcation portrait, which give more precise information for bifurcation.*

4 The Proof of the Main Theorem

Lyapunov-Schmidt Process. Suppose that $X = V \oplus W$, where $V = \mathrm{Ker}(L - \lambda^* I) = \mathrm{span}\{v_1, v_2, \dots, v_m\}$, W is the closure of $\mathrm{span}\{w_1, w_2, \dots\}$. We assume that the sequence $\{v_1, \dots, v_m, w_1, w_2, \dots\}$ is a Schauder bases of X. For any $u \in X$, u can be expressed as the following

$$u = \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j,$$

where $(x_1, x_2, \dots x_m) \in \mathbb{R}^m$, $(y_1, y_2, \dots) \in U \subset \mathbb{R}^\infty$, U is an open neighborhood of 0. Then the equation (1.1) is written as

$$\begin{aligned} \Phi(\lambda, u) &= L(\sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} y_j w_j) - \lambda(\sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} y_j w_j) + h(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} y_j w_j) \\ &= \sum_{i=1}^{m} (\lambda^* - \lambda) x_i v_i + \sum_{j}^{\infty} (\lambda_j - \lambda) y_j w_j + h(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} y_j w_j). \end{aligned}$$

We choose $v_i^* \in V^*$ and $w_j^* \in W^*$ such that $v_i^* v_s = \delta_{is}$, $w_j^* w_t = \delta_{jt}$, $v_i^* w_j = w_j^* v_i = 0$ where

$$\delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}, \quad 1 \le j, s \le m, \ 1 \le j, t \le \infty.$$

Let p_X denote the projection

$$p_X : X \to \mathbb{R}^m \times U, \quad u \mapsto (v_i^* u, w_j^* u),$$

and ι_X denote the injection

$$\iota_X : p_X(X) \to X, \quad (x_i, y_j) \mapsto \sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} y_j w_j,$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$. We have $p_X \circ \iota_X$ and $\iota_X \circ p_X$ are the identities. Then we define F by $F = p_Y \circ \Phi \circ \iota_X$, and have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{R} \times & X & \xrightarrow{\Phi} & Y & \\ & \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} & \\ \mathbb{R} \times & \mathbb{R}^m \times U & \xrightarrow[F]{} & \mathbb{R}^m \times \mathbb{R}^\infty & . \end{array}$$

The function

$$F(\lambda, x_1, \ldots, x_m, y_1, \ldots) = \tag{4.1}$$
$$((\lambda^* - \lambda)x_1 + h_{p1}, \ldots, (\lambda^* - \lambda)x_m + h_{pm},\ (\lambda_1 - \lambda)y_1 + h_{q1}, (\lambda_2 - \lambda)y_2 + h_{q2}, \ldots)$$

has the same bifurcations at $(\lambda^*, 0)$ as those of the $\Phi(\lambda, u) = Lu + h(\lambda, u) - \lambda u = 0$, where

$$\begin{aligned} h_{pi} =& v_i^* h(\lambda, \sum_{i=1}^m x_i v_i + \sum_j^\infty y_j w_j), && 1 \le i \le m, \\ h_{qj} =& w_j^* h(\lambda, \sum_{i=1}^m x_i v_i + \sum_j^\infty y_j w_j), && j = 1, 2, \ldots. \end{aligned}$$

By calculation, one can find the following derivatives directly,

$$\begin{aligned} \frac{\partial F}{\partial \lambda} &= (x_p, y_q), \\ \frac{\partial F}{\partial x_i} &= (\delta_{pi}(\lambda^* - \lambda) + \frac{\partial h_{pi}}{\partial x_i},\ \frac{\partial h_{qj}}{\partial x_i}), \\ \frac{\partial F}{\partial y_j} &= (\frac{\partial h_{pi}}{\partial y_j},\ \delta_{qj}(\lambda_j - \lambda) + \frac{\partial h_{qj}}{\partial y_j}), \end{aligned}$$

where $1 \le p \le m$, $1 \le q$. Since that λ^* is an eigenvalue of L, $\lambda_j \ne \lambda^*$, the component $\lambda_j - \lambda$ of (4.1) is non-zero at $(\lambda^*, 0)$. By implicit function theorem, and $F_{y_j}(\lambda^*, 0)$ is invertible, there exists a unique map

$$\varphi_j : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^\infty,$$

such that $y_j = \varphi_j(\lambda, x_1, x_2, \ldots, x_m)$, with

$$F(\lambda,\ x_1, \ldots, x_m,\ \varphi_1(\lambda, x_1, x_2, \ldots, x_m), \ldots) = 0.$$

Moreover, we have

$$(\varphi_j)_\lambda(\lambda, 0) = 0, \quad (\varphi_j)_{x_i}(\lambda, 0) = 0.$$

Hence $\varphi_j(\lambda, x) = o(\lambda - \lambda^*, x)$, where $x = (x_1, x_2, \ldots, x_m)$.

Let $\hat{F} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a map defined by

$$\hat{F}(\lambda, x_1, \ldots, x_m) = ((\lambda^* - \lambda)x_1 + h_{p1}(\lambda, \sum_{i=1}^m x_i v_i + \sum_j^\infty \varphi_j(\lambda, x_1, x_2, \ldots, x_m) w_j),\ \ldots,$$

$$(\lambda^* - \lambda)x_m + h_{pm}(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j}^{\infty} \varphi_j(\lambda, x_1, x_2, \dots, x_m) w_j).$$

By Lyapunov-Schmidt reduction, $\hat{F}(\lambda, x_1, \dots, x_m) = 0$ is the bifurcation equation of $F(\lambda, x_1, \dots, x_m, y_1, \dots) = 0$.

4.1 A Characterization of k-non-degeneracy

The definition of k-non-degeneracy can be characterized by the following singularity conditions.

Lemma 4.1. *The region Ω is k-non-degenerate if and only if the following conditions* (i) *and* (ii) *hold.*

(i) *Any irreducible component of $F_i = 0$ $(i = 1, \dots, n)$ is not in the hyperplane defined by $\lambda = \lambda^*$, that is, $\{\lambda = \lambda^*, H_{x_1} = \cdots = H_{x_m} = 0\} = \{0\}$.*

(ii) *$F_i = 0$ $(i = 1, \dots, n)$ defines curves with an isolated singularity at $(\lambda^*, 0)$, that is, $\mathrm{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x_i x_j}) = m$ if $F_i = 0$ $(i = 1, \dots, n)$ except $(\lambda^*, 0)$.*

Proof. First we remark that the conditions $F_i = 0$ $(i = 1, \dots m)$ is equivalent that $k(\lambda - \lambda^*)$ is an eigenvalue of $(H_{x_i x_j})_{i,j=1,\dots,m}$ with an eigenvector x, since $H_{x_i} = \frac{1}{k}\sum_{j=1}^{m} x_j H_{x_i x_j}$. So, the condition (i) is equivalent that 0 is not an eigenvalue of $(H_{x_i x_j})$ with eigenvector x.

Next we observe that (ii) is equivalent to the following condition (ii′).

(ii′) $k(\lambda - \lambda^*)$ is an eigenvalue of $(H_{x_i x_j})$ with an eigenvector x, and $\lambda - \lambda^*$ is not an eigenvalue of $(H_{x_i x_j})$.

In fact, if the condition (ii) does not hold and $F_i = 0$ $(i = 1, \dots, m)$, then $\lambda - \lambda^*$ is an eigenvalue of $(H_{x_i x_j})$. Conversely, if $\lambda - \lambda^*$ is a non-zero eigenvalue of $(H_{x_i x_j})$, then the corresponding eigenvector $y = (y_1, \dots, y_m)$ is perpendicular to x, and

$$(y_1, \dots, y_m)(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x_i x_j}) = 0.$$

This implies that $\mathrm{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x_i x_j}) < m$ and the condition (i) does not hold.

Suppose that the region Ω is k-non-degenerate. The critical points set of the restriction of H to the sphere S defined by $\sum_{i=1}^{m} x_i^2 = k + 1$ is $Z \cap S$, and $\lambda - \lambda^*$ is the value of H there, since $(k + 1)H = \sum_{i=1}^{m} x_i H_{x_i} = (\lambda - \lambda^*) \sum_{i=1}^{m} x_i^2$ on Z. We have

$$\begin{vmatrix} 0 & x_j \\ x_i & (\lambda^* - \lambda)\delta_{ij} + H_{x_i x_j} \end{vmatrix} \neq 0 \text{ on } Z \cap S,$$

and the conditions (i) and (ii) hold.

Suppose that the conditions (i) and (ii) hold. If the restriction of H to S is not a Morse function, then $\operatorname{rank}(x_i, (\lambda^* - \lambda)\delta_{ij} + H_{x_ix_j}) < m$. Thus the following equation

$$\begin{pmatrix} 0 & x_j \\ x_i & (\lambda^* - \lambda)\delta_{ij} + H_{x_ix_j} \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_m \end{pmatrix} = 0,$$

has a nonzero solution $(y_0, \dots, y_m)$ and $x_1y_1 + \cdots + x_my_m = 0$. Let $v_1 = {}^t(x_1, \dots, x_m)$, v_2, $\dots$, v_m are the eigenvectors of $(H_{x_ix_j})$, which are perpendicular each other, and set $y = {}^t(y_1, \dots, y_m) = b_1v_1 + \cdots + b_mv_m$. We have $b_1 = 0$, and

$$\begin{aligned} 0 &= y_0v_1 + [(\lambda^* - \lambda)\delta_{ij} + H_{x_ix_j}]y \\ &= y_0v_1 + [(\lambda^* - \lambda)\delta_{ij} + H_{x_ix_j}]\sum_{j=1}^{m} b_jv_j \\ &= y_0v_1 + \sum_{j=1}^{m} b_j(\lambda^* - \lambda + \lambda_j)v_j. \end{aligned}$$

Thus $y_0 = 0$ and $b_j(\lambda^* - \lambda + \lambda_j) = 0$, $j = 2, \dots, m$. Since y is not zero, there exists j such that $\lambda^* - \lambda + \lambda_j = 0$, then $\lambda - \lambda^*$ is an eigenvalue of $(H_{x_ix_j})$, which contradict to (ii′). □

4.2 The Proof of Theorem 3.4

Here we present the proof of Theorem 3.4 by singularity theory.

Replacing $\lambda - \lambda^*$ by λ, it is enough to show the theorem assuming $\lambda^* = 0$. Set $\rho = (\lambda^2 + x_1^{2(k-1)} + \cdots + x_m^{2(k-1)})^{\frac{1}{2(k-1)}}$. Let M denote the minimum of

$$\rho^2 \det((F_j)_{x_1}, \dots, (F_j)_{x_m})^2 + \lambda^2 \sum_{i=1}^{m} \det((F_j)_\lambda, (F_j)_{x_1}, \dots, \widehat{(F_j)_{x_i}} \dots, (F_j)_{x_m})^2$$

on $\rho^{-1}(1)$. By the conditions (i) and (ii), we have $M > 0$.

Let us consider a singular metric $\langle\ ,\ \rangle$ defined by

$$\langle \lambda\partial_\lambda, \lambda\partial_\lambda \rangle = 1,\ \langle \lambda\partial_\lambda, \rho\partial_{x_i} \rangle = 0,\ \langle \rho\partial_{x_i}, \rho\partial_{x_j} \rangle = \delta_{ij},\ i, j = 1, \dots, m. \tag{4.2}$$

We remark that the gradient of $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $(\lambda, x) \mapsto f(\lambda, x)$, is given by

$$\nabla f = \lambda^2 f_\lambda \partial_\lambda + \rho^2 \sum_{i=1}^{m} f_{x_i}\partial_{x_i}.$$

Then we have $\det(\langle \nabla F_i, \nabla F_j \rangle) + |F|^{2m} \geq M$ on $\rho^{-1}(1)$, since

$$\det(\langle \nabla F_i, \nabla F_j \rangle) = \rho^{2m} \det((F_j)_{x_1}, \dots, (F_j)_{x_m})^2$$

$$+\lambda^2\rho^{2(m-1)}\sum_{i=1}^{m}\det((F_j)_\lambda,(F_j)_{x_1},\ldots,\widehat{(F_j)_{x_i}},\ldots,(F_j)_{x_m})^2.$$

We thus have the following inequality on $\rho^{-1}(1)$ and therefore on $\mathbb{R}\times\mathbb{R}^m$,

$$\det(\langle\nabla F_i,\nabla F_j\rangle)+|F|^{2m}\geq M\rho^{2mk},$$

because of weighted homogeneity of both sides.

Define $K_i(\lambda,x)$ by $\hat{F}_i=F_i+K_i$. There is a positive constant C_i and δ so that

$$|K_i|\leq C_i\rho^{k+\delta}\quad\text{near } 0. \tag{4.3}$$

Set $\tilde{F}_j(\lambda,x,t)=\lambda x_j+H_{x_j}+tK_j$ which are functions on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$. We set $\tilde{\nabla}\tilde{F}_j=\nabla\tilde{F}_j+(F_j)_t\partial_t$, $\nabla\tilde{F}_j=\lambda^2(\tilde{F}_j)_\lambda\partial_\lambda+\rho^2\sum_{i=1}^{m}(\tilde{F}_j)_{x_i}\partial_{x_i}$. There is a positive constant C_i' so that

$$|\nabla\tilde{F}_i|\leq C_i'\rho^k\quad\text{near } 0. \tag{4.4}$$

Set $A(\lambda,x,t)=\det\langle\nabla\tilde{F}_i,\nabla\tilde{F}_j\rangle+|\tilde{F}|^{2m}$ and $A_0(\lambda,x)=\det\langle\nabla F_i,\nabla F_j\rangle+|F|^{2m}$. Then there is a function $A_1(\lambda,x,t)$ with $A(\lambda,x,t)=A_0(\lambda,x)+tA_1(\lambda,x,t)$. By (4.3) and (4.4), $0\leq|A_1(x,t)|\leq A_0(x)/2$ near $(\lambda,x)=(0,0)$.

$$A_0(x)-tA_0(x)/2\leq A_0(x)+tA_1(x,t)\text{ near }(\lambda,x)=(0,0)\text{ for }t\geq 0,$$

and thus

$$\tfrac{1}{2}A_0(x)\leq(1-\tfrac{t}{2})A_0(x)\leq A(x,t)\quad\text{near }(\lambda,x)=(0,0)\text{ for any }t\in[0,1].$$

Therefore we have

$$\det\langle\nabla\tilde{F}_i,\nabla\tilde{F}_j\rangle+|\tilde{F}|^{2m}\geq C_0\rho^{2km}\text{ near } 0. \tag{4.5}$$

Set

$$\xi=\frac{1}{\det\langle\nabla\tilde{F}_i,\nabla\tilde{F}_j\rangle+|\tilde{F}|^{2m}}\begin{vmatrix}\langle\tilde{\nabla}\tilde{F}_i,\tilde{\nabla}\tilde{F}_j\rangle & \nabla\tilde{F}_i\\ \langle\partial_t,\tilde{\nabla}\tilde{F}_j\rangle & 0\end{vmatrix}+\partial_t.$$

We show that $\xi\tilde{F}_i=0$ if $F_i(x)=0$ except $(\lambda,x)=(0,0)$. To see this, we consider the orthogonal projection to the tangent space of $\tilde{F}_j=0$, which is defined at its regular point, with respect to the singular metric induced by (4.2) and the Euclidean metric of t-axis. This is expressed by

$$v\mapsto\pi(v)=\frac{1}{\det\langle\tilde{\nabla}\tilde{F}_i,\tilde{\nabla}\tilde{F}_j\rangle}\begin{vmatrix}\langle\tilde{\nabla}\tilde{F}_i,\tilde{\nabla}\tilde{F}_j\rangle & \tilde{\nabla}\tilde{F}_i\\ \langle v,\tilde{\nabla}\tilde{F}_j\rangle & v\end{vmatrix}.$$

Then we have

$$\langle\pi(\partial_t),\partial_\lambda\rangle=\frac{1}{\det\langle\tilde{\nabla}\tilde{F}_i,\tilde{\nabla}\tilde{F}_j\rangle}\begin{vmatrix}\langle\tilde{\nabla}\tilde{F}_i,\tilde{\nabla}\tilde{F}_j\rangle & \langle\tilde{\nabla}\tilde{F}_i,\partial_\lambda\rangle\\ \langle\partial_t,\tilde{\nabla}\tilde{F}_j\rangle & 0\end{vmatrix},$$

$$\langle \pi(\partial_t), \partial_{x_i}\rangle = \frac{1}{\det\langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle} \begin{vmatrix} \langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle & \langle \tilde\nabla \tilde F_i, \partial_{x_i}\rangle \\ \langle \partial_t, \tilde\nabla \tilde F_j\rangle & 0 \end{vmatrix},$$

$$\langle \pi(\partial_t), \partial_t\rangle = \frac{1}{\det\langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle} \begin{vmatrix} \langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle & \langle \tilde\nabla \tilde F_i, \partial_t\rangle \\ \langle \partial_t, \tilde\nabla \tilde F_j\rangle & \langle \partial_t, \partial_t\rangle \end{vmatrix} = \frac{\det\langle \nabla F_i, \nabla F_j\rangle}{\det\langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle},$$

and conclude that $\xi = \frac{\det\langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle}{\det\langle \nabla F_i, \nabla F_j\rangle}\pi(\partial_t)$ if $\tilde F_i = 0$ $(i = 1, \dots, m)$. This shows $\xi \tilde F_i = 0$ whenever $\tilde F_i = 0$ and ξ is defined. Now we define $\tilde\xi$ by $\tilde\xi = \xi$ if $(\lambda, x) \neq (0,0)$; $\tilde\xi = \partial_t$ if $(\lambda, x) = (0,0)$. Let $\tilde\xi = \xi_0 \partial_\lambda + \sum_{i=1}^{m} \xi_i \partial_{x_i} + \partial_t$. By (4.3), (4.4) and (4.5), there is a positive constant C so that

$$|\xi_0| \leq \frac{1}{|\det\langle \nabla \tilde F_i, \nabla \tilde F_j\rangle + |\tilde F|^{2m}|} \left\| \begin{matrix} \langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle & \tilde F_\lambda \lambda^2 \\ \langle \partial_t, \tilde\nabla \tilde F_j\rangle & 0 \end{matrix} \right\| \leq \frac{C\rho^{2km+\delta}|\lambda|}{\rho^{2km}} = C\rho^\delta |\lambda|,$$

$$|\xi_i| \leq \frac{1}{|\det\langle \nabla \tilde F_i, \nabla \tilde F_j\rangle + |\tilde F|^{2m}|} \left\| \begin{matrix} \langle \tilde\nabla \tilde F_i, \tilde\nabla \tilde F_j\rangle & \tilde F_{x_i} \rho^2 \\ \langle \partial_t, \tilde\nabla \tilde F_j\rangle & 0 \end{matrix} \right\| \leq \frac{C\rho^{2km+\delta}\rho}{\rho^{2km}} = C\rho^{1+\delta},$$

near $(\lambda, x) = (0,0)$. These inequalities imply the uniqueness of the flow of $\tilde\xi$. (See [4, §2.2-4]) Thus the flow of $\tilde\xi$ yield a desired homeomorphism.

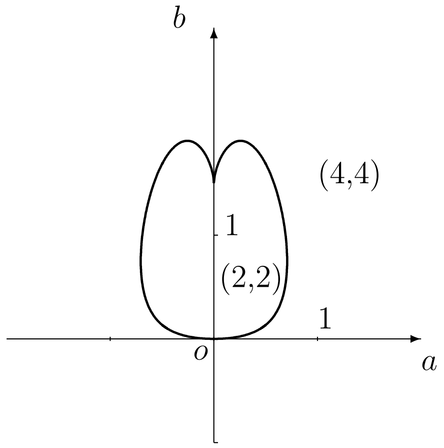
4.3 Examples of H and the Numbers of Real Semi-branches

Let b_- (resp. b_+) denote the number of real semi-branches of Z (in Definition 3.1) in the region $\lambda^* - \varepsilon < \lambda \leq \lambda^*$ (resp. $\lambda^* \leq \lambda < \lambda^* + \varepsilon$) where ε is a small positive number.

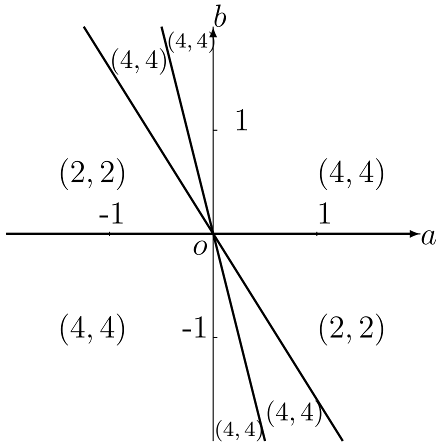
Example 4.2. *When* $H(x,y) = y((y-ax)^2 + bx^2)$, *we have* $(m,k) = (2,2)$. *and the bifurcation model is defined by*

$$\begin{cases} (\lambda^* - \lambda)x + y(2bx - 2a(-ax+y)) = 0 \\ (\lambda^* - \lambda)y + bx^2 + 2y(-ax+y) + (-ax+y)^2 = 0 \end{cases}. \tag{4.6}$$

The bifurcation of solutions are decided by a *and* b. *See the left figure below. The boundary of inner part is defined by* $f(a,b) = 0$, *where* $f(a,b) = 9a^2 + 26a^4 + 25a^6 + 8a^8 - 27b - 48a^2b + 14a^4b + 32a^6b + 54b^2 - 47a^2b^2 + 48a^4b^2 - 36b^3 + 32a^2b^3 + 8b^4$.



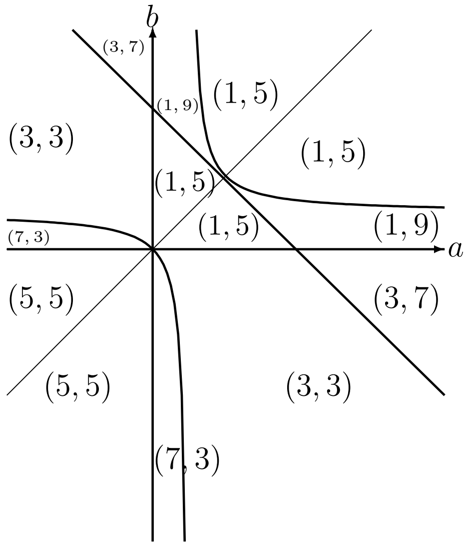
Example 4.2, (b_-, b_+) in (a,b)-plane.



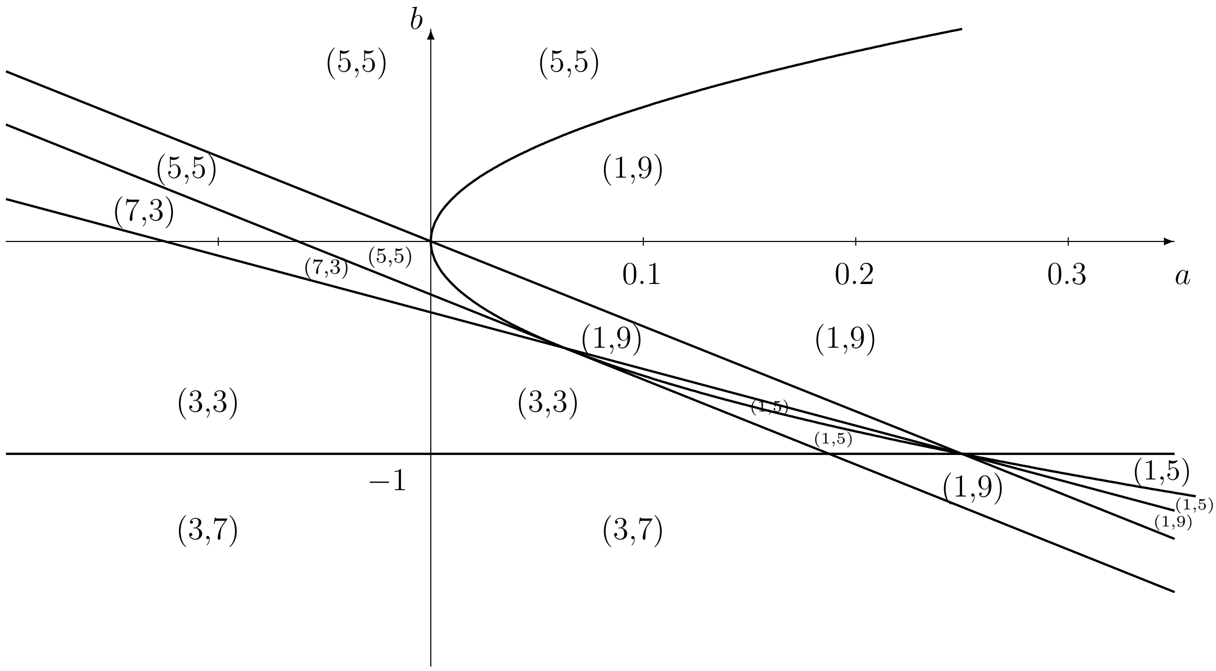
Example 4.3, (b_-, b_+) in (a,b)-plane.

Example 4.3. *When $H(x,y) = a(x+y)^3 + b(x+y)xy$, we have $(m,k) = (2,2)$, and the number of real semi-branches of bifurcation model is decided by a and b. See the right figure above. The lines in the figure are $b = -4a$, and $b = -\frac{12}{5}a$.*

Example 4.4. *When $H(x,y) = (ax^2 + y^2)(bx^2 + y^2)$, we have $(m,k) = (2,3)$ and the number of real semi-branches of bifurcation model is decided by a and b. See the figure below. The curves in the figure are $b = a$, $b = 2 - a$ and $b = \frac{a}{2a-1}$.*



Example 4.4, (b_-, b_+) in (a,b)-plane.



Example 4.5, (b_-, b_+) in (a,b)-plane.

Example 4.5. *When $H_1(x,y) = a'(x+y)^4 + b'(x+y)^2xy + c'x^2y^2$, we have $(m,k) = (2,3)$. If $c' = 0$, then the function H is degenerate.*

Assume that $c' \neq 0$*, let* $H(x,y) = a(x+y)^4 + b(x+y)^2xy + x^2y^2$*, where* $a = \frac{a'}{c'}$*,* $b = \frac{b'}{c'}$*. The number of real semi-branches of bifurcation model is decided by* a *and* b*. See the figure above. The curves in the figure are* $b^2 = 4a$*,* $3b + 8a + 1 = 0$*,* $b = -4a$*,* $b = -1$*, and* $4b + 16a + 1 = 0$*.*

5 The (m,k)-Bifurcation Model on $[0, l_1\pi] \times \cdots \times [0, l_n\pi]$

In this section, we show closed formulas of H in Definition 3.1 for the following differential equation on $\Omega = [0, l_1\pi] \times \cdots \times [0, l_n\pi]$,

$$-\Delta u = \lambda u - a_k(\lambda)u^k + o(u^k), \tag{5.1}$$

with one of the following boundary value condition:

- Dirichlet boundary value condition: $u|_{\partial\Omega} = 0$,
- Neumann boundary value condition: $\partial_n u|_{\partial\Omega} = 0$.

We first present some integral calculus, which we need later. For $\boldsymbol{p} = (p_1, \ldots, p_m)$, $\boldsymbol{a} = (a_1, \ldots, a_m)$, set

$$\begin{aligned} I(\boldsymbol{p};\boldsymbol{a}) &= \int_0^\pi \sin^{p_1} a_1 t \cdots \sin^{p_m} a_m t \, dt, \text{ and} \\ J(\boldsymbol{p};\boldsymbol{a}) &= \int_0^\pi \cos^{p_1} a_1 t \cdots \cos^{p_m} a_m t \, dt. \end{aligned}$$

Lemma 5.1. *Setting* $|\boldsymbol{j}| = j_1 + \cdots + j_m$ *for* $\boldsymbol{j} = (j_1, \ldots, j_m)$*, we have*

$$I(\boldsymbol{p};\boldsymbol{a}) = \begin{cases} \dfrac{(-1)^{\frac{|\boldsymbol{p}|}{2}}\pi}{2^{|\boldsymbol{p}|}} \displaystyle\sum_{\boldsymbol{j} \in S(\boldsymbol{p};\boldsymbol{a})} (-1)^{|\boldsymbol{j}|} \binom{p_1}{j_1} \cdots \binom{p_m}{j_m} & (|\boldsymbol{p}| \text{ is even}), \\ \dfrac{(-1)^{\frac{|\boldsymbol{p}|-1}{2}}}{2^{|\boldsymbol{p}|-1}} \displaystyle\sum_{\boldsymbol{j} \in S'(\boldsymbol{p};\boldsymbol{a})} (-1)^{|\boldsymbol{j}|} \binom{p_1}{j_1} \cdots \binom{p_m}{j_m} \frac{1}{\langle \boldsymbol{p} - 2\boldsymbol{j}, \boldsymbol{a} \rangle} & (|\boldsymbol{p}| \text{ is odd}), \end{cases}$$

$$J(\boldsymbol{p};\boldsymbol{a}) = \frac{\pi}{2^{|\boldsymbol{p}|}} \sum_{\boldsymbol{j} \in S(\boldsymbol{p};\boldsymbol{a})} \binom{p_1}{j_1} \cdots \binom{p_m}{j_m},$$

where

$$\begin{aligned} S(\boldsymbol{p};\boldsymbol{a}) &= \{\boldsymbol{j} \in \mathbb{Z}^m : 0 \leq j_i \leq p_i, \langle \boldsymbol{p} - 2\boldsymbol{j}, \boldsymbol{a} \rangle = 0\}, \text{ and} \\ S'(\boldsymbol{p};\boldsymbol{a}) &= \{\boldsymbol{j} \in \mathbb{Z}^m : 0 \leq j_i \leq p_i, \langle \boldsymbol{p} - 2\boldsymbol{j}, \boldsymbol{a} \rangle \neq 0\}. \end{aligned}$$

Remark that $S(\boldsymbol{p};\boldsymbol{a})$ is in the hyperplane containing $\boldsymbol{p}/2$ with normal vector $\boldsymbol{a}$.

Proof. Direct calculation.

$$I(\boldsymbol{p};\boldsymbol{a}) = \int_0^\pi \Big(\frac{e^{\sqrt{-1}a_1 t} - e^{-\sqrt{-1}a_1 t}}{2\sqrt{-1}}\Big)^{p_1} \cdots \Big(\frac{e^{\sqrt{-1}a_m t} - e^{-\sqrt{-1}a_m t}}{2\sqrt{-1}}\Big)^{p_m} dt$$

$$
\begin{aligned}
&=\sum_{j_1=0}^{p_1}\cdots\sum_{j_m=0}^{p_m}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}\int_0^{\pi}\frac{(-1)^{|\boldsymbol{j}|}e^{\sqrt{-1}\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t}}{(2\sqrt{-1})^{|\boldsymbol{p}|}}dt\\
&=\sum_{j_1=0}^{p_1}\cdots\sum_{j_m=0}^{p_m}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}\int_0^{\pi}(-1)^{|\boldsymbol{j}|}\frac{\cos\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t+\sqrt{-1}\sin\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t}{(2\sqrt{-1})^{|\boldsymbol{p}|}}dt
\end{aligned}
$$

When $|\boldsymbol{p}|$ is even, the real part of the integral is determined by the terms with cosine and

$$
I(\boldsymbol{p};\boldsymbol{a}) = \frac{\pi}{2^{|\boldsymbol{p}|}}\sum_{\boldsymbol{j}\in S(\boldsymbol{p};\boldsymbol{a})}(-1)^{|\boldsymbol{j}|+\frac{|\boldsymbol{p}|}{2}}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}.
$$

When $|\boldsymbol{p}|$ is odd, the real part of the integral is determined by the terms with sine and

$$
I(\boldsymbol{p};\boldsymbol{a}) = \sum_{\boldsymbol{j}\in S'(\boldsymbol{p};\boldsymbol{a})}\frac{(-1)^{|\boldsymbol{j}|+\frac{|\boldsymbol{p}|-1}{2}}}{2^{|\boldsymbol{p}|-1}}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}\frac{1}{\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle}.
$$

We also have

$$
\begin{aligned}
J(\boldsymbol{p};\boldsymbol{a}) =& \int_0^{\pi}\Big(\frac{e^{\sqrt{-1}a_1t}+e^{-\sqrt{-1}a_1t}}{2}\Big)^{p_1}\cdots\Big(\frac{e^{\sqrt{-1}a_mt}+e^{-\sqrt{-1}a_mt}}{2}\Big)^{p_m}dt\\
=&\sum_{j_1=0}^{p_1}\cdots\sum_{j_m=0}^{p_m}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}\int_0^{\pi}\frac{e^{\sqrt{-1}\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t}}{2^{|\boldsymbol{p}|}}dt\\
=&\sum_{j_1=0}^{p_1}\cdots\sum_{j_m=0}^{p_m}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}\int_0^{\pi}\frac{\cos\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t+\sqrt{-1}\sin\langle \boldsymbol{p}-2\boldsymbol{j},\boldsymbol{a}\rangle t}{2^{|\boldsymbol{p}|}}dt\\
=&\frac{\pi}{2^{|\boldsymbol{p}|}}\sum_{\boldsymbol{j}\in S(\boldsymbol{p};\boldsymbol{a})}\binom{p_1}{j_1}\cdots\binom{p_m}{j_m}.
\end{aligned}
$$

The proof is completed. □

Lemma 5.2. *The followings hold.*

- *If* $\sum_{a_i:\text{even}} p_i \equiv 1 \bmod 2$, *then* $I(\boldsymbol{p};\boldsymbol{a}) = 0$.
- *If* $\sum p_i \equiv 1 \bmod 2$, *then* $J(\boldsymbol{p};\boldsymbol{a}) = 0$.
- *If each* a_j *is even. then* $I(\boldsymbol{p};\boldsymbol{a}) = I(\boldsymbol{p};\frac{\boldsymbol{a}}{2})$, *and* $J(\boldsymbol{p};\boldsymbol{a}) = J(\boldsymbol{p};\frac{\boldsymbol{a}}{2})$.

Proof. Since $\sin a(t+\frac{\pi}{2}) = \sin at\cos\frac{a\pi}{2}+\cos at\sin\frac{a\pi}{2}$,

$$
\begin{aligned}
I(\boldsymbol{p};\boldsymbol{a}) &= \int_{-\pi/2}^{\pi/2}\sin^{p_1}a_1(t+\frac{\pi}{2})\cdots\sin^{p_m}a_m(t+\frac{\pi}{2})\,dt\\
&=\int_{-\pi/2}^{\pi/2}\prod_{a_i:\text{even}}(-1)^{\frac{a_ip_i}{2}}\sin^{p_i}a_it\prod_{a_i:\text{odd}}(-1)^{\frac{a_i-1}{2}}\cos^{p_i}a_it\,dt.
\end{aligned}
$$

This is zero when $\sum_{a_i:\text{even}} p_i$ is odd. If each a_i is even,

$$I(\boldsymbol{p};\boldsymbol{a}) = 2\int_0^{\pi/2} \sin^{p_1} a_1 t \cdots \sin^{p_m} a_m t\, dt = I(\boldsymbol{p};\frac{\boldsymbol{a}}{2}).$$

Since $\cos a(t+\frac{\pi}{2}) = \cos at \cos\frac{a\pi}{2} - \sin at \sin\frac{a\pi}{2}$

$$\begin{aligned} J(\boldsymbol{p};\boldsymbol{a}) &= \int_{-\pi/2}^{\pi/2} \cos^{p_1} a_1(t+\frac{\pi}{2}) \cdots \cos^{p_m} a_m(t+\frac{\pi}{2})\, dt \\ &= \int_{-\pi/2}^{\pi/2} \prod_{a_i:\text{even}} (-1)^{\frac{a_i p_i}{2}} \cos^{p_i} a_i t \prod_{a_i:\text{odd}} (-1)^{\frac{(a_i-1)p_i}{2}} \sin^{p_i} a_i t\, dt. \end{aligned}$$

This is zero when $\sum_{a_i:\text{odd}} p_i$ is odd. If each a_i is even,

$$J(\boldsymbol{p};\boldsymbol{a}) = 2\int_0^{\pi/2} \cos^{p_1} a_1 t \cdots \cos^{p_m} a_m t\, dt = J(\boldsymbol{p};\frac{\boldsymbol{a}}{2}).$$

The lemma is proved. □

5.1 Closed Formulas for Bifurcation Model

To consider the bifurcation of the equation (5.1) on $\Omega = [0, l_1\pi] \times \cdots \times [0, l_n\pi]$, there are infinite eigenvalues of the Laplacian on Ω,

$$\lambda_1 < \lambda_2 < \cdots < \lambda^* < \ldots.$$

Let λ^* be an eigenvalue with multiplicity m, that is, there are $a_j^{(i)} \in \mathbb{R}$ ($i = 1,\ldots,n$, $j = 1,\ldots,m$) with

$$\lambda^* = (a_j^{(1)}/l_1)^2 + \cdots + (a_j^{(n)}/l_n)^2.$$

- With Dirichlet boundary value conditions: $a_j^{(i)}$ are positive integer.
- With Neumann boundary value conditions: $a_j^{(i)}$ are non-negative integer.

Let $v_1, v_2, \ldots, v_m$ denote all the eigenfunctions of λ^*, and $w_1, w_2, \ldots$ be all the eigenfunctions of $\lambda_j \neq \lambda^*$, $j = 1, 2, \ldots$. With Dirichlet and Neumann boundary value conditions, $\{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots\}$ can be chosen a trigonometric system. The trigonometric system is an orthonormal bases in L^2 space. Then $\{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots\}$ is a Schauder bases of $L^2(\Omega)$. Setting $V = \text{Ker}(L - \lambda^* I) = \text{span}\{v_1, v_2, \ldots, v_m\}$, and W the closure of $\text{span}\{w_1, w_2, \ldots\}$, we have $L^2(\Omega) = V \oplus W$.

The following two closed formulas are useful.

Lemma 5.3. *We have the following closed formula for H in Definition 3.1, with Dirichlet boundary condition: $u|_{\partial\Omega} = 0$.*

$$H = \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} l_1 \cdots l_n I(\boldsymbol{p};\boldsymbol{a}^{(1)}) \cdots I(\boldsymbol{p};\boldsymbol{a}^{(n)}).$$

Proof. With Dirichlet boundary conditions, the eigenfunctions $v_j = \prod_{i=1}^{n} \sin \frac{a_j^{(i)} t_i}{l_i}$ are the bases of $V = \mathrm{Ker}(-\Delta - \lambda^* I)$. Setting $u = \sum_{j=1}^{m} x_j v_j$, we have

$$\begin{aligned} H =& \frac{1}{k+1} \int_0^{l_1 \pi} dt_1 \cdots \int_0^{l_n \pi} u^{k+1} dt_n \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} \prod_{i=1}^{n} \int_0^{l_i \pi} \sin^{p_1} \frac{a_1^{(i)} t_i}{l_i} \cdots \sin^{p_m} \frac{a_m^{(i)} t_i}{l_i} dt_i \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} \prod_{i=1}^{n} l_i \int_0^{l_i \pi} \sin^{p_1} a_1^{(i)} t_i \cdots \sin^{p_m} a_m^{(i)} t_i \, dt_i \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} l_1 \cdots l_n I(\boldsymbol{p}; \boldsymbol{a}^{(1)}) \cdots I(\boldsymbol{p}; \boldsymbol{a}^{(n)}). \end{aligned}$$

The lemma is proved. □

Lemma 5.4. *We have the following closed formula for H in Definition 3.1 with Neumann boundary condition: $\partial_n u|_{\partial\Omega} = 0$.*

$$H = \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} l_1 \cdots l_n J(\boldsymbol{p}; \boldsymbol{a}^{(1)}) \cdots J(\boldsymbol{p}; \boldsymbol{a}^{(n)}).$$

Proof. With Neumann boundary conditions, the eigenfunctions $v_j = \prod_{i=1}^{n} \cos \frac{a_j^{(i)} t_i}{l_i}$ are the bases of $V = \mathrm{Ker}(-\Delta - \lambda^* I)$. Setting $u = \sum_{j=1}^{m} x_j v_j$, we have

$$\begin{aligned} H =& \frac{1}{k+1} \int_0^{l_1 \pi} dt_1 \cdots \int_0^{l_n \pi} u^{k+1} dt_n \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} \prod_{i=1}^{n} \int_0^{l_i \pi} \cos^{p_1} \frac{a_1^{(i)} t_i}{l_i} \cdots \cos^{p_m} \frac{a_m^{(i)} t_i}{l_i} dt_i \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} \prod_{i=1}^{n} l_i \int_0^{l_i \pi} \cos^{p_1} a_1^{(i)} t_i \cdots \cos^{p_m} a_m^{(i)} t_i \, dt_i \\ =& \sum_{|\boldsymbol{p}|=k+1} \frac{x_1^{p_1} \cdots x_m^{p_m}}{k+1} \binom{k+1}{p_1 \ldots p_m} l_1 \cdots l_n J(\boldsymbol{p}; \boldsymbol{a}^{(1)}) \cdots J(\boldsymbol{p}; \boldsymbol{a}^{(n)}). \end{aligned}$$

The proof is completed. □

Lemma 5.5. *Assume that $\boldsymbol{a}^{(i)} = 2^{q_i} \boldsymbol{b}^{(i)}$, $\boldsymbol{b}^{(i)} \in \mathbb{Z}^m$. If, for any $\boldsymbol{p}$ with $|\boldsymbol{p}| = k+1$, there exist i with $\langle \boldsymbol{p}, \boldsymbol{b}^{(i)} \rangle$ is odd, then $H = 0$.*

Proof. We conclude $H = 0$, if, for any $\boldsymbol{p}$ with $|\boldsymbol{p}| = k+1$, there exists i with $S(\boldsymbol{p}; \boldsymbol{a}^{(i)}) = \emptyset$. If $\boldsymbol{j}^{(i)} \in S(\boldsymbol{p}; \boldsymbol{a}^{(i)})$, $i = 1, \dots, n$, and then

$$\langle \boldsymbol{p} - 2\boldsymbol{j}^{(i)}, \boldsymbol{b}^{(i)} \rangle = 0$$

and $\langle \boldsymbol{p}, \boldsymbol{b}^{(i)} \rangle$ are even. $\square$

We remark that the Neumann problem on the region $[0, (l_1/2^{q_1})\pi] \times \cdots \times [0, (l_n/2^{q_n})\pi]$ has the same eigenvalue λ^* with multiplicity m, since

$$\Big(\frac{b_j^{(1)}}{l_1/2^{q_1}}\Big)^2 + \cdots + \Big(\frac{b_j^{(n)}}{l_n/2^{q_n}}\Big)^2 = \lambda^*.$$

5.2 Parity test

Assume that $l_1 = \cdots = l_n$ (i.e., the region is hypercube). Then $\{\boldsymbol{a}^{(i)}\}$ has a natural $\mathfrak{S}_n$-action where $\mathfrak{S}_n$ is the symmetric group of order n. That is, for any $\sigma \in \mathfrak{S}_n$ and j, there is j' with $(a_j^{(\sigma(1))}, \dots, a_j^{(\sigma(n))}) = (a_{j'}^{(1)}, \dots, a_{j'}^{(n)})$. Without loss of generality, we may assume that at least one of $a_1^{(1)}, \dots, a_m^{(1)}$ is odd. If $S(\boldsymbol{p}; \boldsymbol{a}^{(i)})$ $(i = 1, \dots, n)$ are not empty, then there is $\boldsymbol{j}^{(i)} = (j_1^{(i)}, \dots, j_m^{(i)})$ $(0 \le j_i \le p_i)$ with

$$(p_1 - 2j_1^{(i)})a_1^{(i)} + \cdots + (p_m - 2j_m^{(i)})a_m^{(i)} = 0. \tag{5.2}$$

From this, we often conclude some restriction on parity of $\boldsymbol{p}$, which we call the parity test. Set $a_j = a_j^{(1)}$ $(j = 1, \dots, m)$ for simplicity.

Theorem 5.6. *If $n = 2$, and k is even, then $H = 0$ for Neumann problem.*

Let us show this theorem case by case for the multiplicity m. If $m = 1$, by (5.2), as we assumed that $\boldsymbol{a} = a_1$ is odd, we have $a_1 p_1 \equiv 0 \bmod 2$. that is $||\boldsymbol{p}| = p_1$ is even, then $H = 0$ when k is even.

Lemma 5.7. *In the case $(n, m) = (2, 2)$, if k is even, then $H = 0$ for Neumann problem.*

Proof. Let $(n, m) = (2, 2)$, $a_2^{(1)} = a_2$, $a_2^{(2)} = a_1$. By (5.2), we obtain that

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \equiv 0 \bmod 2.$$

- If $(a_1, a_2) \equiv (1, 1) \bmod 2$, then $p_1 + p_2 \equiv 0 \bmod 2$.
- If $(a_1, a_2) \equiv (1, 0) \bmod 2$, then $p_1 \equiv p_2 \equiv 0 \bmod 2$.

In each case, $|\boldsymbol{p}|$ is even. In particular, H is zero when k is even. $\square$

Lemma 5.8. *In the case $(n, m) = (2, 3)$, if k is even, then $H = 0$ for Neumann problem.*

Proof. Let $(n,m) = (2,3)$, $a_1^2 + a_2^2 = 2a_3^2$, $a_1 \neq a_2$. By (5.2), we obtain that

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \equiv 0 \bmod 2.$$

- If $(a_1, a_2, a_3) \equiv (1,1,1) \bmod 2$, then $p_1 + p_2 + p_3 \equiv 0 \bmod 2$.

In this case, $|\boldsymbol{p}|$ is even. The cases $(a_1, a_2, a_3) \equiv (1,0,1)$, $(1,1,0)$, and $(1,0,0) \bmod 2$ are not possible, because $a_1^2 + a_2^2 = 2a_3^2$. In particular, H is zero when k is even. □

Lemma 5.9. *In the case $(n,m) = (2,4)$, if k is even, then $H = 0$ for Neumann problem.*

Proof. Let $(n,m) = (2,4)$, $a_1^2 + a_2^2 = a_3^2 + a_4^2$, $a_1 \neq a_2$, $a_3 \neq a_4$, $\{a_1, a_2\} \cap \{a_3, a_4\} = \emptyset$. By (5.2), we obtain that

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_4 \end{pmatrix} \equiv 0 \bmod 2.$$

- If $(a_1, a_2, a_3, a_4) \equiv (1,1,1,1) \bmod 2$, then $p_1 + p_2 + p_3 + p_4 \equiv 0 \bmod 2$.
- If $(a_1, a_2, a_3, a_4) \equiv (1,0,1,0) \bmod 2$, then $p_1 + p_3 \equiv p_2 + p_4 \equiv 0 \bmod 2$.

In each case, $|\boldsymbol{p}|$ is even. The cases $(a_1, a_2, a_3, a_4) \equiv (1,1,1,0)$, $(1,1,0,0)$ and $(1,0,0,0) \bmod 2$ are not possible, because $a_1^2 + a_2^2 = a_3^2 + a_4^2$. In particular, H is zero when k is even. □

Proof of Theorem 5.6. As the proofs of the previous lemmas, for each m, a similar discussion shows that H is zero when $n = 2$ and k is even. □

We proceed the case $n = 3$ (the region is a cube), and show several conclusions of the parity test.

Case $(n,m) = (3,3)$**,** $a_1 \neq a_2 = a_3$**.** By (5.2), we obtain that

$$\begin{pmatrix} a_1 & a_2 & a_2 \\ a_2 & a_1 & a_2 \\ a_2 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \equiv 0 \bmod 2.$$

- If $(a_1, a_2) \equiv (1,1) \bmod 2$, then $p_1 + p_2 + p_3 \equiv 0 \bmod 2$.
- If $(a_1, a_2) \equiv (1,0) \bmod 2$, then $p_1 \equiv p_2 \equiv p_3 \equiv 0 \bmod 2$.
- ◦ If $(a_1, a_2) \equiv (0,1) \bmod 2$, then $p_1 + p_2 \equiv p_2 + p_3 \equiv p_1 + p_3 \equiv 0 \bmod 2$, that is, $p_1 \equiv p_2 \equiv p_3 \bmod 2$.

In cases with •, we have $|\boldsymbol{p}|$ is even. In the case with ◦, $|\boldsymbol{p}|$ may not be even.

Case $(n,m)=(3,6)$**,** a_1**,** a_2**,** a_3 **are distinct.** By (5.2), we obtain that

$$\begin{pmatrix} a_1 & a_1 & a_2 & a_2 & a_3 & a_3 \\ a_2 & a_3 & a_1 & a_3 & a_1 & a_2 \\ a_3 & a_2 & a_3 & a_1 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_6 \end{pmatrix} \equiv 0 \bmod 2.$$

- If $(a_1,a_2,a_3)\equiv(1,1,1) \bmod 2$, then $p_1+p_2+p_3+p_4+p_5+p_6\equiv 0 \bmod 2$.
- ◦ If $(a_1,a_2,a_3)\equiv(1,1,0) \bmod 2$, then $p_1+p_3\equiv p_2+p_4\equiv p_5+p_6 \bmod 2$.
- If $(a_1,a_2,a_3)\equiv(1,0,0) \bmod 2$, then $p_1+p_2\equiv p_3+p_5\equiv p_4+p_6\equiv 0 \bmod 2$.

Example 5.10. *The Neumann problem on the cube* $[0,\pi]^3$ *has eigenvalue* $2(=1^2+1^2+0^2)$ *with multiplicity* 3. *When* $k=2$, *we obtain*

$$H=\frac{1}{3}\int_0^\pi dt_1\int_0^\pi dt_2\int_0^\pi u^3 dt_3=\frac{\pi^3}{4}xyz,$$

where $u=x\cos t_1\cos t_2+y\cos t_1\cos t_3+z\cos t_2\cos t_3$.

Example 5.11. *The Neumann problem on the rectangle with* $(l_1,l_2)=(1,\sqrt{3})$ *has eigenvalues* $\frac{4}{3}=1^2+\frac{1^2}{3}=0^2+\frac{2^2}{3}$, *(resp.* $4=2^2+\frac{0^2}{3}=1^2+\frac{3^2}{3}$*) of multiplicities* 2. *When* $k=2$, *we have*

$$H=\frac{1}{3}\int_0^\pi dt_1\int_0^{\sqrt{3}\pi} u^3 dt_2=\frac{\sqrt{3}\pi^2}{8}x^2y$$

where $u=x\cos t_1\cos\frac{t_2}{\sqrt{3}}+y\cos\frac{2t_2}{\sqrt{3}}$ *(resp.* $x\cos 2t_1+y\cos t_1\cos\frac{3t_2}{\sqrt{3}}$*).*

5.3 Two dimensional case

The description of our bifurcation model becomes more explicit when we consider rectangles $\Omega=[0,l_1\pi]\times[0,l_2\pi]$.

Lemma 5.12. *With Dirichlet, Neumann boundary value conditions, all the eigenvalue of* $L=-\Delta$ *on the rectangle domain* $\Omega=[0,l_1\pi]\times[0,l_2\pi]$ *is simple eigenvalue, if and only if* $\frac{l_2^2}{l_1^2}$ *is an irrational number.*

Proof. Suppose that there exist non-negative integers $a\neq a'$, $b\neq b'$, such that $\lambda^*=\lambda_{a,b}=\lambda_{a',b'}$. Thus, we have

$$\Big(\frac{a}{l_1}\Big)^2+\Big(\frac{b}{l_2}\Big)^2=\Big(\frac{a'}{l_1}\Big)^2+\Big(\frac{b'}{l_2}\Big)^2,\quad \text{and}\quad \frac{l_2^2}{l_1^2}=\frac{(b'-b)^2}{(a-a')^2}.$$

This is a contradiction to l_2^2/l_1^2 is an irrational number. □

For the simple eigenvalue case, it is easy to analyze the bifurcation model (1.2). In the rest of this section, we discuss the case that $l_1=l_2=1$. In this case, most of the eigenvalues are multiple eigenvalue. We concentrate on analyzing the bifurcation model for $m\geq 2$.

5.3.1 Dirichlet boundary value problem

With Dirichlet boundary value conditions, the eigenvalues $\lambda_{a,b}$ can be listed out by their multiplicity m. See the following table:

Eigenvalues $\lambda_{a,b}$	Eigenfunctions	Multiplicity	Examples
$2 = 1^2 + 1^2$	$\sin(x)\sin(y)$	$m = 1$	The first eigenvalue
$5 = 1^2 + 2^2$	$\sin(x)\sin(2y)$ $\sin(2x)\sin(y)$	$m = 2$	The second eigenvalue
$\lambda_{a,b} = 2a^2$	$\sin(ax)\sin(ay)$	$m = 1$	$2 = 2 \times 1^2$
$\lambda_{a,b} = a^2 + b^2$	$\sin(ax)\sin(by)$ $\sin(bx)\sin(ay)$	$m = 2$	$5 = 1^2 + 2^2$ $25 = 3^2 + 4^2$
$\lambda_{a,b} = a^2 + b^2$ $= 2a_1^2$	$\sin(ax)\sin(by)$ $\sin(bx)\sin(ay)$ $\sin(a_1x)\sin(a_1y)$	$m = 3$	$50 = 1^2 + 7^2$ $= 2 \times 5^2$
$\lambda_{a,b} = a^2 + b^2$ $= a_1^2 + b_1^2$	$\sin(ax)\sin(by)$ $\sin(bx)\sin(ay)$ $\sin(a_1x)\sin(b_1y)$ $\sin(b_1x)\sin(a_1y)$	$m = 4$	$65 = 1^2 + 8^2$ $= 4^2 + 7^2$

Here a, a_i, b, b_i are positive integers with $a \neq a_i \neq a_j$, $b \neq b_i \neq b_j$ $(i \neq j)$.

Theorem 5.13. *For $k = 3$, and for all the eigenvalues λ^* with multiplicity $m = 2$, $(\lambda^*, 0)$ is a bifurcation point. The bifurcation model is non-degenerate and does not depend on the choice of λ^*.*

Explicitly, for the eigenvalue $\lambda^* = a^2+b^2$ with multiplicity $m = 2$, the $(2, 3)$-bifurcation models have the uniform H, where

$$H = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4).$$

If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$). The bifurcation at the point $(\lambda^*, 0) = (a^2 + b^2, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.

Proof. For the eigenvalue $\lambda^* = a^2 + b^2$ with multiplicity $m = 2$ where a and b are positive integers with $a \neq b$, we see

$$\begin{aligned} H =& \frac{1}{4} a_3(\lambda^*) \int_0^\pi \int_0^\pi [x_1 \sin(ax)\sin(by) + x_2 \sin(bx)\sin(ay)]^4 \, dx \, dy \\ =& \frac{1}{4} a_3(\lambda^*) \sum_{r=0}^{4} \binom{4}{r} x_1^r x_2^{4-r} \int_0^\pi \sin^r ax \sin^{4-r} bx \, dx \int_0^\pi \sin^r by \sin^{4-r} ay \, dy \\ =& \frac{1}{4} a_3(\lambda^*) \sum_{r=0}^{4} \binom{4}{r} x_1^r x_2^{4-r} A_{1r} A_{2r}, \end{aligned}$$

where $A_{1r} = \int_0^\pi \sin^r ax \sin^{4-r} bx\, dx,\ A_{2r} = \int_0^\pi \sin^r by \sin^{4-r} ay\, dy$.

Suppose that $b \neq 3a$ and $a \neq 3b$, a direct calculation shows that

$$\begin{aligned}
A_{14} &= \left[\frac{3x}{8} - \frac{\sin(2ax)}{4a} + \frac{\sin(4ax)}{32a}\right]_0^\pi = \frac{3\pi}{8},\\
A_{13} &= \frac{1}{8}\left[\frac{3\sin[(a-b)x]}{a-b} - \frac{\sin[(3a-b)x]}{3a-b} - \frac{3\sin[(a+b)x]}{a+b} + \frac{\sin[(3a+b)x]}{3a+b}\right]_0^\pi = 0,\\
A_{12} &= \frac{1}{16}\left[\frac{4x-(2\sin[2ax])}{a} + \frac{\sin[2(a-b)x]}{a-b} - \frac{2\sin[2bx]}{b} + \frac{\sin[2(a+b)x]}{a+b}\right]_0^\pi = \frac{\pi}{4},\\
A_{11} &= \frac{1}{8}\left[-\frac{\sin[(a-3b)x]}{a-3b} + \frac{3\sin[(a-b)x]}{a-b} - \frac{3\sin[(a+b)x]}{a+b} + \frac{\sin[(a+3b)x]}{a+3b}\right]_0^\pi = 0,\\
A_{10} &= \left[\frac{3x}{8} - \frac{\sin(2bx)}{4b} + \frac{\sin(4bx)}{32b}\right]_0^\pi = \frac{3\pi}{8},
\end{aligned}$$

and $A_{24} = \frac{3\pi}{8},\ A_{23} = 0,\ A_{22} = \frac{\pi}{4},\ A_{21} = 0,\ A_{20} = \frac{3\pi}{8}$. Thus

$$H = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4).$$

If $b = 3a$, then a similar calculation shows that

$$A_{14} = \frac{3\pi}{8},\ A_{13} = -\frac{\pi}{8},\ A_{12} = \frac{\pi}{4},\ A_{11} = 0,\ A_{10} = \frac{3\pi}{8}$$

$$A_{24} = \frac{3\pi}{8},\ A_{23} = 0,\ A_{22} = \frac{\pi}{4},\ A_{21} = -\frac{\pi}{8},\ A_{20} = \frac{3\pi}{8},$$

and we show the result. When $a = 3b$, we can prove the result similarly. □

Remark 5.14. *We may find the number of real semi-branches as consequences of Examples 4.2—4.5, when $m = 2$. For example, if $k = 3$, the H has the uniform form (Theorem 5.13)*

$$H(x_1, x_2) = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4) = \frac{9\pi^2}{256} a_3(\lambda^*)\left(\frac{4+\sqrt{7}}{3}x_1^2 + x_2^2\right)\left(\frac{4-\sqrt{7}}{3}x_1^2 + x_2^2\right).$$

This implies that $(a, b) = (\frac{4+\sqrt{7}}{3}, \frac{4-\sqrt{7}}{3})$. By Example 4.4, if $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$). The bifurcation on the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubacritical) bifurcation.

Theorem 5.15. *For $k = 5$ and for all the eigenvalues λ^* with multiplicity $m = 2$, $(\lambda^*, 0)$ is a bifurcation point. The bifurcation model is non-degenerate. If $a_5(\lambda^*) > 0$ (resp. $a_5(\lambda^*) < 0$), then $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$). The bifurcation on the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.*

Proof. The eigenvalue λ^* with multiplicity $m = 2$ be $\lambda^* = a_1^2 + a_2^2$, see the table above. Here a_1, a_2 are positive integers, $a_1 \neq a_2$.

If $a_2 = 2a_1$ or $a_1 = 2a_2$, then the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{25}{3072} a_5(\lambda^*)\pi^2(x_1^2 + x_2^2)(2x_1^4 + 7x_1^2x_2^2 + 2x_2^4).$$

If $a_2 = 3a_1$ or $a_1 = 3a_2$, then the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{1536} a_5(\lambda^*)\pi^2(5x_1^6 + 27x_1^4x_2^2 + 9x_1^3x_2^3 + 27x_1^2x_2^4 + 5x_2^6).$$

In the other case, the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{1536} a_5(\lambda^*)\pi^2(x_1^2 + x_2^2)(5x_1^4 + 22x_1^2x_2^2 + 5x_2^4).$$

The theorem is proved, by solving the bifurcation models in the three cases above. □

By Theorem 5.13 and Theorem 5.15, the second eigenvalue $\lambda^* = 5 = 1^2 + 2^2$ is a bifurcation point when $k = 3$ or $k = 5$ with Dirichlet boundary value conditions.

Theorem 5.16. *Suppose that $\lambda_{a,b} = a^2 + b^2$ is an eigenvalue with multiplicity $m = 2$. If k is even and $a \cdot b$ is even, then $H = 0$ and the square domain is k-degenerate.*

Proof. Let $a = a_1$, $b = 2b_1$, $k = 2z$, $z = 1, 2, \ldots$, then $a \cdot b \equiv 0 \mod 2$. Suppose that $\lambda_{a,b} = a^2 + b^2$ is an eigenvalue with multiplicity $m = 2$, the corresponding eigenfunctions are

$$v_{a,b} = \sin(a_1x)\sin(2b_1y), \;\; v_{b,a} = \sin(2b_1x)\sin(a_1y).$$

$$\begin{aligned}
H =& \frac{1}{2z+1} a_{2z}(\lambda_{a,b}) \int_0^\pi \int_0^\pi (x_1v_{a,b} + x_2v_{b,a})^{2z+1}\, dx\, dy \\
=& \frac{a_{2z}(\lambda_{a,b})}{2z+1} \int_0^\pi \int_0^\pi \sum_{r=0}^{2z+1} \binom{2z+1}{r} x_1^r x_2^{2z+1-r} \cdot \\
& \qquad (\sin(a_1x)\sin(2b_1y))^{2z+1-r}(\sin(2b_1x)\sin(a_1y))^r\, dx\, dy \\
=& \frac{a_{2z}(\lambda_{a,b})}{2z+1} \sum_{r=0}^{2z+1} \binom{2z+1}{r} x_1^r x_2^{2z+1-r} A_1A_2,
\end{aligned}$$

where $A_1 = \int_0^\pi (\sin(a_1x))^{2z+1-r}(\sin(2b_1x))^r\, dx$, $A_2 = \int_0^\pi (\sin(2b_1y))^{2z+1-r}(\sin(a_1y))^r\, dy$.

In fact, if r is odd, then $A_1 = 0$. If r is even, then $A_2 = 0$.

Let r is odd, set $t = x - \frac{\pi}{2}$, then $2z + 1 - r$ is even,

$$\begin{aligned}
A_1 =& \int_0^\pi (\sin(a_1x))^{2z+1-r}(\sin(2b_1x))^r\, dx \\
=& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin(a_1t + \frac{a_1}{2}\pi))^{2z+1-r}(\sin(2b_1t + b_1\pi))^r\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin(a_1 t)\cos\frac{a_1}{2}\pi) + \cos(a_1 t)\sin\frac{a_1}{2}\pi))^{2z+1-r}((-1)^{b_1}\sin(2b_1 t))^r \, dt \\
&= (-1)^{b_1 r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin(a_1 t)\cos\frac{a_1}{2}\pi) + \cos(a_1 t)\sin\frac{a_1}{2}\pi))^{2z+1-r}(\sin(2b_1 t))^r \, dt = 0.
\end{aligned}
$$

Here $(\sin(a_1t)\cos\frac{a_1}{2}\pi + \cos(a_1t)\sin\frac{a_1}{2}\pi)^{2z+1-r}$ is an even function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $(\sin(2b_1t))^r$ is an odd function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

If r is even, then $2z+1-r$ is odd. A similar discussion shows that $A_2 = 0$.

That is $A_1 A_2 = 0$. Thus

$$
H = \frac{a_{2z}(\lambda_{a,b})}{2z+1} \sum_{r=0}^{2z+1} \binom{2z+1}{r} x_1^r x_2^{2z+1-r} A_1 A_2 = 0.
$$

□

Let us mention the polynomials H explicitly for first several cases which we do not mention above. We remark the number (b_-, b_+) of semi-branches for $a_k(\lambda^*) > 0$, when the bifurcation model is non-degenerate. Set $m = 2$ and $\lambda^* = \lambda_{a,b} = a^2 + b^2$.

k	(a,b)	$H/a_k(\lambda_{a,b})$	(b_-, b_+)
2	$(1,3)$	$16(x_1+x_2)(175x_1^2 - 418x_1x_2 + 175x_2^2)/14175$	$(4,4)$
2	$(1,5)$	$16(x_1+x_2)(77x_1^2 - 102x_1x_2 + 77x_2^2)/10395$	$(4,4)$
2	$(3,5)$	$16(x_1+x_2)(1001x_1^2 + 5074x_1x_2 + 1001x_2^2)/405405$	$(4,4)$
4	$(1,3)$	$\frac{256(x_1+x_2)}{70945875}[21021(x_1^4+x_2^4) - 119436x_1x_2(x_1^2+x_2^2) + 46766x_1^2x_2^2]$	$(4,4)$
4	$(1,5)$	$\frac{256(x_1+x_2)}{1489508645625}[264801537(x_1^4+x_2^4) - 40348412x_1x_2(x_1^2+x_2^2) - 178745338x_1^2x_2^2]$	$(4,4)$
4	$(3,5)$	$\frac{256(x_1+x_2)}{44799836956875}[2654805153(x_1^4+x_2^4) + 26562147972x_1x_2(x_1^2+x_2^2) - 4336241722x_1^2x_2^2]$	$(4,4)$

We observe that all the bifurcation above is pluritranscritical bifurcation with $(b_-, b_+) = (4,4)$.

The smallest eigenvalue with multiplicity $m = 3$ is $\lambda^* = 50 = 1^2 + 7^2 = 2 \times 5^2$. We mention the data for the bifurcation models in the following table.

k	$H/a_k(\lambda^*)$	(b_-, b_+)
2	$\frac{1}{3}\left(\begin{array}{c}\frac{16}{63}(x_1^3+x_2^3) - \frac{112}{2925}x_1x_2(x_1+x_2) - \frac{112}{4275}(x_1^2+x_2^2)x_3 \\ +\frac{39200}{61347}x_1x_2x_3) + \frac{10000}{11781}(x_1+x_2)x_3^2 + \frac{16}{225}x_3^3\end{array}\right)$	$(8,8)$
3	$\frac{3\pi^2}{256}[3(x_1^4+x_2^4+x_3^4) + 8(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)]$	$(1,27)$
4	$\begin{array}{c}\frac{256}{7875}(x_1^5+x_2^5) - \frac{333312}{26558675}x_1^2x_2^2(x_1+x_2) + \frac{87808}{25193025}x_1x_2(x_1^3+x_2^3) \\ +\frac{87808}{9734175}(x_1^4+x_2^4)x_3 + \frac{1036763648}{417532482675}x_1^2x_2^2x_3 + \frac{4128029696}{10559352375}x_1x_2x_3(x_1^2+x_2^2) \\ -\frac{768298496}{3585421125}(x_1+x_2)x_1x_2x_3^2 + \frac{8000000}{28121247}(x_1^3+x_2^3)x_3^2 + \frac{10000000000}{42934740849}x_1x_2x_3^3 \\ -\frac{146944}{9132825}(x_1^2+x_2^2)x_3^3 + \frac{100000000}{549972423}(x_1+x_2)x_3^4 + \frac{256}{28125}x_3^5\end{array}$	$(8,8)$
5	$\frac{5\pi^2}{1536}\left(\begin{array}{c}5(x_1^6+x_2^6+x_3^6) + 72x_1^2x_2^2x_3^2 - 9x_1x_2x_3^3(x_1+x_2) \\ +27(x_1^4x_2^2 + x_1^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^4x_3^2 + x_2^2x_3^4)\end{array}\right)$	$(1,27)$

5.3.2 Neumann boundary value problem

With Neumann boundary value conditions, the eigenvalues $\lambda_{a,b}$ can be listed out by their multiplicity m. See the following table:

Eigenvalues $\lambda_{a,b}$	Eigenfunctions	Multiplicity	Examples
$0 = 0^2 + 0^2$	1	$m = 1$	The first eigenvalue
$1 = 0^2 + 1^2$	$\cos(x)$ $\cos(y)$	$m = 2$	The second eigenvalue
$\lambda_{a,b} = 2a^2$	$\cos(ax)\cos(ay)$	$m = 1$	$2 = 2 \times 1^2$
$\lambda_{a,b} = a^2 + b^2$	$\cos(ax)\cos(by)$ $\cos(bx)\cos(ay)$	$m = 2$	$5 = 1^2 + 2^2$
$\lambda_{a,b} = a_1^2 + b_1^2 = 2a_2^2$	$\cos(a_1x)\cos(b_1y)$ $\cos(b_1x)\cos(a_1y)$ $\cos(a_2x)\cos(a_2y)$	$m = 3$	$50 = 1^2 + 7^2 = 2 \times 5^2$

Here a, a_i, b, b_i are positive integers, $a \neq a_i \neq a_j$, $b \neq b_i \neq b_j$ $i \neq j$.

Theorem 5.17. *For $k = 3$ and for all the eigenvalues λ^* with multiplicity $m = 2$, $(\lambda^*, 0)$ is a bifurcation point. The bifurcation model is non-degenerate. If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$). The bifurcation on the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.*

Proof. Explicitly, the eigenvalue λ^* with multiplicity $m = 2$ be $\lambda^* = a_1^2 + a_2^2$, see the table above. Here a_1, a_2 are nonnegative integers, $a_1 \neq a_2$.

If $a_2 = 0$ or $a_1 = 0$, then the $(2, 3)$-bifurcation models have the uniform H, where

$$H = \frac{3}{32} a_3(\lambda^*)\pi^2(x_1^4 + 4x_1^2x_2^2 + x_2^4).$$

In the other case, the $(2, 3)$-bifurcation models have the uniform H, where

$$H = \frac{3}{256} a_3(\lambda^*)\pi^2(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4).$$

The theorem is proved, by solving the bifurcation models in both cases above. □

Theorem 5.18. *For $k = 5$ and for all the eigenvalues λ^* with multiplicity $m = 2$, $(\lambda^*, 0)$ is a bifurcation point. The bifurcation model is non-degenerate. If $a_5(\lambda^*) > 0$ (resp. $a_5(\lambda^*) < 0$), then $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$). The bifurcation on the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.*

Proof. The eigenvalue λ^* with multiplicity $m = 2$ be $\lambda^* = a_1^2 + a_2^2$, see the table above. Here a_1, a_2 are nonnegative integers, $a_1 \neq a_2$.

If $a_2 = 0$ or $a_1 = 0$, then the $(2, 5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{96} a_5(\lambda^*)\pi^2(x_1^2 + x_2^2)(x_1^4 + 8x_1^2x_2^2 + x_2^4).$$

If $a_2 = 2a_1$ or $a_1 = 2a_2$, then the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{3072} a_5(\lambda^*)\pi^2(x_1^2 + x_2^2)(10x_1^4 + 53x_1^2x_2^2 + 10x_2^4).$$

If $a_2 = 3a_1$ or $a_1 = 3a_2$, then the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{1536} a_5(\lambda^*)\pi^2(5x_1^6 + 27x_1^4x_2^2 + 9x_1^3x_2^3 + 27x_1^2x_2^4 + 5x_2^6).$$

In the other case, the $(2,5)$-bifurcation models have the uniform H, where

$$H = \frac{5}{1536} a_5(\lambda^*)\pi^2(x_1^2 + x_2^2)(5x_1^4 + 22x_1^2x_2^2 + 5x_2^4).$$

The theorem is proved, by solving the bifurcation models in the three cases above. □

By Theorem 5.17 and Theorem 5.18, the second eigenvalue $\lambda^* = 1 = 1^2 + 0^2$ is a bifurcation point when $k = 3$ or $k = 5$ with Neumann boundary value conditions.

Theorem 5.19. *When k is even and $m = 2$, then $H = 0$ and the square domain is $2n$-degenerate for the corresponding eigenvalue.*

Proof. It is clear from Lemma 5.2. □

We remark the polynomial H to describe the bifurcation model in first several cases which we do not mention above. We remark the number (b_-, b_+) of semi-branches for $a_k(\lambda^*) > 0$, when the bifurcation model is non-degenerate.

The first eigenvalue with multiplicity $m = 3$ is $\lambda^* = 50 = 1^2 + 7^2 = 2 \times 5^2$. We show the polynomial H in the following table.

k	$H/a_k(\lambda^*)$	(b_-, b_+)
2	0	
3	$\frac{3\pi^2}{256}[3(x_1^4 + x_2^4 + x_3^4) + 8(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)]$	$(1, 27)$
4	0	
5	$\frac{5\pi^2}{1536}\begin{pmatrix} 5(x_1^6 + x_2^6 + x_3^6) + 72x_1^2x_2^2x_3^2 + 9x_1x_2(x_1 + x_2)x_3^3 \\ +27(x_1^4x_2^2 + x_1^2x_2^4 + x_1^4x_3^2 + x_1^2x_3^4 + x_2^4x_3^2 + x_2^2x_3^4) \end{pmatrix}$	$(1, 27)$

5.4 Three dimensional case

When the region $\Omega = [0,\pi] \times [0,\pi] \times [0,\pi]$ is three dimensional cube, we show the type of bifurcation for the first few eigenvalues.

5.4.1 Dirichlet boundary value problem

The eigenvalues $\lambda_{a,b,c}$ can be listed out when we fix the multiplicity m. Set $v_{a,b,c} = \sin(ax)\sin(by)\sin(cz)$.

Eigenvalues $\lambda_{a,b,c}$	Eigenfunctions	Multiplicity	Example
$\lambda_{a,b,c} = 3a_1^2$	v_{a_1,a_1,a_1}	$m = 1$	$3 = 3 \times 1^2$
$\lambda_{a,b,c} = 2a_1^2 + a_2^2$	$v_{a_1,a_1,a_2},\ v_{a_1,a_2,a_1},\ v_{a_2,a_1,a_1}$	$m = 3$	$6 = 2 \times 1^2 + 2^2$
$\lambda_{a,b,c} = 2a_1^2 + a_2^2$ $= 3a_3^2$	$v_{a_1,a_1,a_2},\ v_{a_1,a_2,a_1},\ v_{a_2,a_1,a_1},$ v_{a_3,a_3,a_3}	$m = 4$	$27 = 2 \times 1^2 + 5^2$ $= 3 \times 3^2$
$\lambda_{a,b,c} = a_1^2 + a_2^2 + a_3^2$	$v_{a_1,a_2,a_3},\ v_{a_1,a_3,a_2},\ v_{a_2,a_1,a_3},$ $v_{a_2,a_3,a_1},\ v_{a_3,a_1,a_2},\ v_{a_3,a_2,a_1}$	$m = 6$	$14 = 1^2 + 2^2 + 3^2$
$\lambda_{a,b,c} = 2a_1^2 + a_2^2$ $= a_1^2 + 2a_3^2$	$v_{a_1,a_1,a_2},\ v_{a_1,a_2,a_1},\ v_{a_2,a_1,z1},$ $v_{a_1,a_3,a_3},\ v_{a_3,a_1,a_3},\ v_{a_3,a_3,a_1}$	$m = 6$	$51 = 2 \times 1^2 + 7^2$ $= 1^2 + 2 \times 5^2$
$\lambda_{a,b,c} = a_1^2 + a_2^2 + a_3^2$ $= a_1^2 + 2a_4^2$	$v_{a_1,a_2,a_3},\ v_{a_1,a_3,a_2},\ v_{a_2,a_1,a_3},$ $v_{a_2,a_3,a_1},\ v_{a_3,a_1,a_2},\ v_{a_3,a_2,a_1},$ $v_{a_1,a_4,a_4},\ v_{a_4,a_1,a_4}.\ v_{a_4,a_4,a_1}$	$m = 9$	$38 = 2^2 + 3^2 + 5^2$ $= 6^2 + 2 \times 1^2$

Here a_i are positive integers, $a_i \neq a_j$, $i \neq j$.

Theorem 5.20. *For $k = 3$ and for all the eigenvalues λ^* with multiplicity $m = 3$, $(\lambda^*, 0)$ is a bifurcation point. The bifurcation model is non-degenerate. If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-, b_+) = (1, 27)$ (resp. $(b_-, b_+) = (27, 1)$). The bifurcation on the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.*

Proof. On the three dimensional region, the eigenvalue λ^* with multiplicity $m = 3$ be $\lambda^* = 2a_1^2 + a_2^2$, see the table above. Here a_1, a_2 are positive integers, $a_1 \neq a_2$.

If $a_2 \neq 3a_1$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{9\pi^3}{2048} a_3(\lambda^*)[3(x_1^4 + x_2^4 + x_3^4) + 8(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)].$$

If $a_2 = 3a_1$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{3\pi^3}{2048} a_3(\lambda^*)[9(x_1^4 + x_2^4 + x_3^4) + 8x_1x_2x_3(x_1 + x_2 + x_3) + 24(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)].$$

The theorem is proved, by solving the bifurcation models in both cases. □

We show the data for our bifurcation model for first few cases with $m = 3$.

k	(a,b,c)	$H/a_k(\lambda_{a,b,c})$	(b_-,b_+)
2	$(1,1,2)$	0	
2	$(1,2,2)$	$\frac{8192}{3375}x_1x_2x_3$	$(5,5)$
2	$(1,1,3)$	$\begin{array}{c}\frac{64}{243}(x_1^3+x_2^3+x_3^3)-\frac{128}{3375}x_1x_2x_3\\ -\frac{64}{175}(x_1x_2^2+x_1x_3^2+x_1^2x_2+x_2x_3^2+x_1^2x_3+x_2^2x_3)\end{array}$	$(8,8)$
4	$(1,1,2)$	0	
4	$(1,2,2)$	$(16777216/10418625)x_1x_2x_3(x_1^2+x_2^2+x_3^2)$	$(5,5)$
4	$(1,1,3)$	$\begin{array}{c}\frac{4096}{50625}(x_1^5+x_2^5+x_3^5)-\frac{704512}{4244625}x_1x_2x_3(x_1^2+x_2^2+x_3^2)\\ -\frac{1384448}{1157625}x_1x_2x_3(x_1x_3+x_1x_2+x_2x_3)\\ -\frac{4579328}{16372125}(x_2^2x_3^3+x_1^3x_2^2+x_1^3x_3^2+x_2^3x_3^2+x_1^2x_3^3+x_1^2x_2^3)\\ -\frac{331776}{875875}(x_2x_3^4+x_1^4x_2+x_2^4x_3+x_1x_2^4+x_1x_3^4+x_1^4x_3)\end{array}$	$(8,8)$
5	$(1,1,2)$	$\frac{125}{98304}\pi^3\left(\begin{array}{c}4(x_1^6+x_2^6+x_3^6)+45x_1^2x_2^2x_3^2\\ +18(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_1^2x_3^4+x_2^4x_3^2+x_2^2x_3^4)\end{array}\right)$	$(1,27)$
5	$(1,2,2)$	$\frac{5}{49152}\pi^3\left(\begin{array}{c}50(x_1^6+x_2^6+x_3^6)+972x_1^2x_2^2x_3^2\\ +225(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_1^2x_3^4+x_2^4x_3^2+x_2^2x_3^4)\end{array}\right)$	$(1,27)$
5	$(1,1,3)$	$\frac{5\pi^3}{49152}\left(\begin{array}{c}50(x_1^6+x_2^6+x_3^6)+90(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3)\\ +972x_1^2x_2^2x_3^2+225x_1x_2x_3(x_1^3+x_2^3+x_3^3)\\ +270(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_2^4x_3^2+x_1^2x_3^4+x_2^2x_3^4\\ +x_1^3x_2^2x_3+x_1^3x_2x_3^2+x_1^2x_2^3x_3+x_1^2x_2x_3^3+x_1x_2^2x_3^3+x_1x_2^3x_3^2)\end{array}\right)$	$(1,27)$

The smallest eigenvalue with multiplicity $m=6$ is $\lambda^*=14=1^2+2^2+3^2$, and the data for the bifurcation model are shown as follows:

k	$H/a_k(\lambda^*)$	(b_-,b_+)
2	0	
3	$\frac{3\pi^3}{2048}\left(\begin{array}{c}9(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4+x_6^4)+16(x_1x_2x_5x_6+x_1x_3x_4x_6+x_2x_3x_4x_5)\\ +24(x_1^2x_2^2+x_1^2x_3^2+x_1^2x_6^2+x_2^2x_4^2+x_2^2x_5^2+x_3^2x_4^2+x_3^2x_5^2+x_4^2x_6^2+x_5^2x_6^2)\\ +16(x_1^2x_4^2+x_1^2x_5^2+x_2^2x_3^2+x_2^2x_6^2+x_3^2x_6^2+x_4^2x_5^2)\\ -8(x_1x_2^2x_6+x_1^2x_2x_5+x_1x_4^2x_6+x_2x_3^2x_5+x_3x_4x_5^2+x_3x_4x_6^2)\end{array}\right)$	$(1,345)$

The second eigenvalue with multiplicity $m=6$ is $\lambda^*=21=1^2+2^2+4^2$, and the data for the bifurcation model are shown as follows:

k	$H/a_k(\lambda^*)$	(b_-,b_+)
2	$\begin{array}{c}-\frac{65536}{99225}(x_1x_3x_4+x_1x_3x_6+x_2x_3x_4+x_2x_5x_6+x_2x_4x_5+x_1x_5x_6)\\ -\frac{65536}{1157265}(x_1x_4x_5+x_2x_3x_6)\end{array}$	$(25,25)$
3	$\frac{3\pi^3}{2048}\left(\begin{array}{c}9(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4+x_6^4)\\ +16(x_1x_2x_3x_5+x_1x_2x_4x_6+x_3x_4x_5x_6)\\ +24(x_1^2x_2^2+x_1^2x_3^2+x_1^2x_6^2+x_2^2x_4^2\\ +x_2^2x_5^2+x_3^2x_4^2+x_3^2x_5^2+x_4^2x_6^2+x_5^2x_6^2)\\ +16(x_4^2x_5^2+x_1^2x_4^2+x_1^2x_5^2+x_2^2x_3^2+x_2^2x_6^2+x_3^2x_6^2)\end{array}\right)$	$(1,249)$

5.4.2 Neumann boundary value problem

The eigenvalues $\lambda_{a,b,c}$ can be listed out when we fix the multiplicity m. Set $v_{a,b,c} = \cos(ax)\cos(by)\cos(cz)$.

Eigenvalues $\lambda_{a,b,c}$	Eigenfunctions	Multiplicity	Example
0	1	$m=1$	$0 = 3\times 0^2$
$\lambda_{a,b,c} = a_1^2 + 2\times 0^2$	$v_{a_1,0,0},\ v_{0,a_1,0},\ v_{0,0,a_1}$	$m=3$	$1 = 1^2 + 2\times 0^2$ $4 = 2^2 + 2\times 0^2$ $16 = 4^2 + 2\times 0^2$
$\lambda_{a,b,c} = 2\times a_1^2 + 0^2$	$v_{a_1,a_1,0},\ v_{a_1,0,a_1},\ v_{0,a_1,a_1}$	$m=3$	$2 = 2\times 1^2 + 0^2$ $8 = 2\times 2^2 + 0^2$
$\lambda_{a,b,c} = 2a_1^2 + a_2^2$	$v_{a_1,a_1,a_2},\ v_{a_1,a_2,a_1}\ v_{a_2,a_1,a_1}$	$m=3$	$6 = 2\times 1^2 + 2^2$ $11 = 2\times 1^2 + 3^2$

Theorem 5.21. *For $k=3$ and for all the eigenvalues λ^* with multiplicity $m=3$, $(\lambda^*,0)$ is a bifurcation point. The bifurcation model is non-degenerate. If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-,b_+) = (1,27)$ (resp. $(b_-,b_+) = (27,1)$). The bifurcation on the point $(\lambda^*,0)$ is plurisupercritical (resp. plurisubcritical) bifurcation.*

Proof. On the three dimensional region, the eigenvalue λ^* with multiplicity $m=3$ be $\lambda^* = 2a_1^2 + a_2^2$, see the table above. Here a_1, a_2 are non-negative integers, $a_1 \neq a_2$.

If $a_1 = 0$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{3\pi^3}{32} a_3(\lambda^*)(x_1^4 + 4x_1^2x_2^2 + 4x_1^2x_3^2 + x_2^4 + 4x_2^2x_3^2 + x_3^4).$$

If $a_2 = 0$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{9\pi^3}{256} a_3(\lambda^*)(x_1^4 + 4x_1^2x_2^2 + 4x_1^2x_3^2 + x_2^4 + 4x_2^2x_3^2 + x_3^4).$$

If $a_2 \neq 3a_1 \neq 0$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{9\pi^3}{2048} a_3(\lambda^*)[3(x_1^4 + x_2^4 + x_3^4) + 8(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)].$$

If $a_2 = 3a_1 \neq 0$, then the $(3,3)$-bifurcation models have the uniform H, where

$$H = \frac{3\pi^3}{2048} a_3(\lambda^*)[9(x_1^4 + x_2^4 + x_3^4) + 8x_1x_2x_3(x_1 + x_2 + x_3) + 24(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)].$$

The theorem is proved, by solving the bifurcation models in the four cases above. □

We show the data for our bifurcation model for first few cases with $m=3$.

k	(a,b,c)	$H/a_k(\lambda_{a,b,c})$	(b_-,b_+)
2	$(1,0,0)$	0	
2	$(1,1,0)$	$\frac{1}{4}\pi^3 x_1x_2x_3$	$(5,5)$
2	$(1,1,2)$	$\frac{1}{32}\pi^3 x_1x_2x_3$	$(5,5)$
4	$(1,0,0)$	0	
4	$(1,1,0)$	$\frac{9}{32}\pi^3 x_1x_2x_3(x_1^2+x_2^2+x_3^2)$	$(5,5)$
4	$(1,1,2)$	$\frac{3}{64}\pi^3 x_1x_2x_3(x_1^2+x_2^2+x_3^2)$	$(5,5)$
5	$(a,0,0)$	$\frac{5}{96}\pi^3\begin{pmatrix} x_1^6+x_2^6+x_3^6+36x_1^2x_2^2x_3^2 \\ +9(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_1^2x_3^4+x_2^4x_3^2+x_2^2x_3^4)\end{pmatrix}$	$(1,27)$
5	$(a,a,0)$	$\frac{5}{1536}\pi^3\begin{pmatrix} 5(x_1^6+x_2^6+x_3^6)+243x_1^2x_2^2x_3^2 \\ +45(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_1^2x_3^4+x_2^4x_3^2+x_2^2x_3^4)\end{pmatrix}$	$(1,27)$
5	$(1,1,2)$	$\frac{5}{98304}\pi^3\begin{pmatrix} 100(x_1^6+x_2^6+x_3^6)+3087x_1^2x_2^2x_3^2 \\ +630(x_1^4x_2^2+x_1^4x_3^2+x_1^2x_2^4+x_1^2x_3^4+x_2^4x_3^2+x_2^2x_3^4)\end{pmatrix}$	$(1,27)$

We observe that $H/a_k(\lambda_{a,b,c})$ do not depend on a, when $(a,b,c)=(a,0,0)$ or $(a,a,0)$ $(a\neq 0)$, and $k=5$.

The smallest eigenvalue with multiplicity $m=6$ is $\lambda^*=5=1^2+2^2+0^2$:

k	$H/a_k(\lambda^*)$	(b_-,b_+)
2	0	
3	$\frac{3\pi^3}{256}\begin{pmatrix} 3(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4+x_6^4)+16(x_1x_2x_3x_5+x_1x_2x_4x_6+x_3x_4x_5x_6) \\ +8(x_1^2x_3^2+x_1^2x_4^2+x_1^2x_5^2+x_2^2x_3^2+x_2^2x_4^2+x_2^2x_6^2+x_3^2x_6^2+x_4^2x_5^2+x_5^2x_6^2) \\ +12(x_1^2x_2^2+x_2^2x_5^2+x_3^2x_4^2+x_3^2x_5^2+x_1^2x_6^2+x_4^2x_6^2)\end{pmatrix}$	$(1,297)$

The second eigenvalue with multiplicity $m=6$ is $\lambda^*=9=1^2+2\times 2^2=3^2+2\times 0^2$:

k	$H/a_k(\lambda^*)$	(b_-,b_+)
2	0	
3	$\frac{3\pi^3}{2048}\begin{pmatrix} 9(x_1^4+x_2^4+x_3^4)+64(x_4^4+x_5^4+x_6^4)+24(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2) \\ +64(x_1^2x_4^2+x_1^2x_5^2+x_1^2x_6^2+x_2^2x_4^2+x_2^2x_5^2+x_2^2x_6^2+x_3^2x_4^2+x_3^2x_5^2+x_3^2x_6^2) \\ +128(x_1x_2x_4x_5+x_1x_3x_4x_6+x_2x_3x_5x_6)+256(x_4^2x_5^2+x_4^2x_6^2+x_5^2x_6^2)\end{pmatrix}$	$(1,53)$

6 Symmetry creates new bifurcation

Here, we consider the following Dirichlet problem.

$$\begin{cases} -\Delta u=\lambda u-a_3(\lambda)u^3+o(u^3), & \text{in } \Omega_t=[0,\pi]\times[0,t\pi], \text{ where } t>1, \\ u=0, & \text{on } \partial\Omega_t. \end{cases} \tag{6.1}$$

If $t=1+\varepsilon$, then the rectangle domain Ω_ε converge to the square domain $\Omega=[0,\pi]\times[0,\pi]$, the eigenvalues $\lambda_{1,2}=1^2+(\frac{2}{1+\varepsilon})^2$, $\lambda_{2,1}=2^2+(\frac{1}{1+\varepsilon})^2$ on Ω_ε converge to the eigenvalue

$\lambda^* = 5$ on Ω, as $\varepsilon \to 0$, where $\varepsilon > 0$ is small. For the bifurcation portrait, see the left of figure 6, where $v_{1,2} = \sin(x)\sin(\frac{2}{1+\varepsilon}y)$ is orthogonal to the vector $v_{2,1} = \sin(2x)\sin(\frac{1}{1+\varepsilon}y)$.

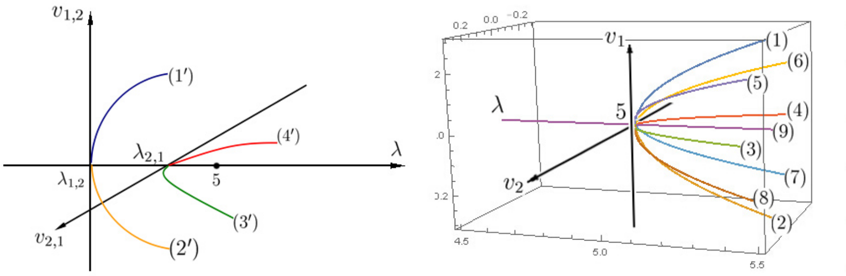
The eigenvalue $\lambda^* = 5$ is a multiple eigenvalue with multiplicity $m = 2$. By Theorem 5.13, the function $H = \frac{3\pi^2}{64}a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4)$ decides a $(2,3)$-bifurcation model. Solving this model, the solution curves are

$$(1)\begin{cases} x_1 = \frac{4}{3\pi}\sqrt{\frac{\lambda-5}{a_3(\lambda^*)}} \\ x_2 = 0 \end{cases}, \quad (2)\begin{cases} x_1 = -\frac{4}{3\pi}\sqrt{\frac{\lambda-5}{a_3(\lambda^*)}} \\ x_2 = 0 \end{cases}, \quad (3)\begin{cases} x_1 = 0 \\ x_2 = \frac{4}{3\pi}\sqrt{\frac{\lambda-5}{a_3(\lambda^*)}} \end{cases},$$

$$(4)\begin{cases} x_1 = 0 \\ x_2 = -\frac{4}{3\pi}\sqrt{\frac{\lambda-5}{a_3(\lambda^*)}} \end{cases}, \quad (5)\begin{cases} x_1 = \frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \\ x_2 = \frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \end{cases}, \quad (6)\begin{cases} x_1 = -\frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \\ x_2 = \frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \end{cases},$$

$$(7)\begin{cases} x_1 = -\frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \\ x_2 = -\frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \end{cases}, \quad (8)\begin{cases} x_1 = \frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}} \\ x_2 = -\frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}}6cm \end{cases}, \quad (9)\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}.$$

The bifurcation portrait see the following figure.



(a) Bifurcation portrait for rectangle domain

(b) Bifurcation portrait for square domain

Comparing the bifurcation portrait for the rectangle domain with that of the square domain, at $\lambda = 5$, as $\varepsilon \to 0$, there are 4 new semi-branches (5), (6), (7), (8) which are not come from the semi-branches of the rectangle domain.

Remark 6.1. *If t is a positive integer, the eigenvalue $\lambda^* = 5(= 1^2 + (\frac{2t}{t})^2 = 2^2 + (\frac{t}{t})^2)$ is the $(3t-1)$-th eigenvalue with multiplicity $m = 2$ of $L = -\Delta$ on Ω_t. By Theorem 5.13, the H of $(2,3)$-bifurcation model at the bifurcation point $(5,0)$ on each Ω_t is*

$$H = \frac{3t\pi^2}{256}a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4).$$

The bifurcation is exactly the same described as above. The details are left to the readers.

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