CUSPIDAL EDGES AND GENERALIZED CUSPIDAL EDGES IN THE LORENTZ-MINKOWSKI 3-SPACE

T. FUKUI, R. KINOSHITA, D. PEI, M. UMEHARA, AND H. YU

Abstract. It is well-known that every cuspidal edge in the Euclidean space $\mathbb{E}^3$ cannot have a bounded mean curvature function. On the other hand, in the Lorentz-Minkowski space $\mathbb{L}^3$, zero mean curvature surfaces admit cuspidal edges. One natural question is to ask when a cuspidal edge has bounded mean curvature in $\mathbb{L}^3$. We show that such a phenomenon occurs only when the image of the singular set is a light-like curve in $\mathbb{L}^3$. Moreover, we also investigate the behavior of principal curvatures in this case as well as other possible cases. In this paper, almost all calculations are given for generalized cuspidal edges as well as for cuspidal edges. We define the "order" at each generalized cuspidal edge singular point is introduced. As nice classes of zero-mean curvature surfaces in $\mathbb{L}^3$, "maxfaces" and "minfaces" are known, and generalized cuspidal edge singular points on maxfaces and minfaces are of order 4. One of the important results is that the generalized cuspidal edges of order 4 exhibit a quite similar behaviors as those on maxfaces and minfaces.

Contents

Introduction

We denote the Euclidean 3-space by $(\mathbb{E}^3; x, y, z)$ and the Lorentz-Minkowski 3-space by $(\mathbb{L}^3; x, y, z)$, where the signature of $\mathbb{L}^3$ is taken as $(++-)$. Let $f : U \to \mathbb{L}^3$ be a regular surface (i.e. an immersion). A point on U is said to be *space-like* (resp. *time-like*) if the pull-back of the canonical Lorentzian metric of $\mathbb{L}^3$ (i.e. the first fundamental form of f) induces a Riemannian metric (resp. a Lorentzian metric) at the point. In this paper, $\mathbb{R}^3$ denotes $\mathbb{L}^3$ or $\mathbb{E}^3$ ignoring the canonical metrics. Fix $\delta > 0$ and an open interval I on $\mathbb{R}$.

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Definition 0.1. Let U be a domain of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^3$ a C^∞-map. A point $p \in U$ is called a *cuspidal edge singular point* of f if there exist local diffeomorphisms φ and Ψ on $\mathbb{R}^2$ and $\mathbb{R}^3$ such that $\varphi(p) = (0, 0)$, $\Psi \circ f(p) = (0, 0, 0)$ and $\Psi \circ f \circ \varphi^{-1}(s, t) = (s, t^2, t^3)$.

We fix a regular curve $\Gamma : I \ni s \mapsto f(s, 0) \in \mathbb{R}^3$.

Definition 0.2. A C^∞-map $f : I \times (-\delta, \delta) \to \mathbb{R}^3$ is called a *cuspidal edge* along Γ if $(s, 0)$ is a cuspidal edge singular point of f for each $s \in I$.

We next define "generalized cuspidal edges":

Definition 0.3. Fix $\delta > 0$ and an open interval I on $\mathbb{R}$. A C^∞-map

$$f : I \times (-\delta, \delta) \ni (s, t) \mapsto f(s, t) \in \mathbb{R}^3$$

is called a *generalized cuspidal edge* along the curve $\Gamma : I \ni s \mapsto f(s, 0) \in \mathbb{R}^3$ if $f(s, t)$ satisfies the following properties:

(a) $f_t(s, 0)$ vanishes identically for each $s \in I$, and
(b) the vectors $f_{tt}(s, 0)$ and $\Gamma'(s)(= f_s(s, 0))$ are linearly independent for each $s \in I$.

Moreover, each point $(s, 0)$ is called a *generalized cuspidal edge singular point*.

Cuspidal edges are typical examples of generalized cuspidal edges (cf. Proposition 1.6). For a given generalized cuspidal edge f, we set

$$\mathcal{U}_f := I \times (-\delta, \delta), \qquad \Sigma_f := \{(s, 0) \in U \,;\, s \in I\}, \tag{0.1}$$

which are the domain of definition and the singular set of f, respectively. Then, the set $\mathcal{U}_f \setminus \Sigma_f$ of regular points of f has two connected components. The purpose of this paper is to investigate the behavior of the Gaussian curvature, the mean curvature, the principal curvatures and umbilical points of f. We are particularly interested in the behavior of mean curvature and umbilical points:

- Although the mean curvature functions H^E of cuspidal edges in $\mathbb{E}^3$ diverge along their singular sets, there are cuspidal edges in $\mathbb{L}^3$ whose mean curvature functions are bounded. Moreover, it is also known that cuspidal edge singular points appear as one of the most general singular points on zero-mean curvature (i.e. ZMC) surfaces in $\mathbb{L}^3$ ([1, 4, 21]). Compared to the case in $\mathbb{E}^3$, it would be interesting to investigate when the mean curvature function H^L around an arbitrarily given cuspidal edge singular point in $\mathbb{L}^3$ becomes bounded.
- A regular point $p \in \mathcal{U}_f$ of f is called an *umbilical point* (resp. *a quasi-umbilical point*) if it is a space-like or time-like point of f at which the two principal curvatures coincide and the Weingarten matrix (cf. (1.15)) is diagonal (resp. is non-diagonal). On a given surface, the umbilical points with $\mathbb{R}^3 = \mathbb{L}^3$ appear at a different location than the umbilical points with $\mathbb{R}^3 = \mathbb{E}^3$, in general. Moreover, quasi-umbilical points never appear on regular surfaces in $\mathbb{E}^3$ and space-like regular surfaces in $\mathbb{L}^3$, but often appear on time-like regular surfaces in $\mathbb{L}^3$.

  In $\mathbb{E}^3$, umbilical points never accumulate at any cuspidal edge singular point (cf. [5, 10, 20]). Also in $\mathbb{L}^3$, the same conclusion holds as long as a cuspidal edge singular point p is time-like or space-like (see Definition 1.8 and Proposition C.1 in the appendix). On the other hand, a light-like cuspidal edge singular point (cf. Definition 1.1) could be an accumulation point of umbilical points, and it is worth investigating this possibility.

Therefore, the goal of this paper is to give answers to the above questions as much as possible. Honda, Izumiya, Saji and Teramoto [7] gave normal forms of germs of cuspidal edges in $\mathbb{L}^3$ and investigated cuspidal edges at light-like points from a different point of view than ours.

Fix a generalized cuspidal edge $f : \mathcal{U}_f \to \mathbb{R}^3$ along Γ. At each singular point $p := (s, 0) \in \Sigma_f$, we consider the sign

$$\sigma^C(p) := \operatorname{sgn}(d^C(s)) \in \{-1, 0, 1\} \tag{0.2}$$

of the value

$$d^C(s) := \det(\Gamma'(s), f_{tt}(s, 0), \Gamma''(s)). \tag{0.3}$$

By definition, the sign $\sigma^C(p)$ at the singular point p is common in $\mathbb{E}^3$ and $\mathbb{L}^3$. If $\sigma^C(p)$ is positive (or negative, zero), we say that the germ of the generalized cuspidal edge f at p is *right-handed* (or *left-handed*, *neutral*). The following three assertions hold (see Propositions 1.15 and 1.23):

(1) $\sigma^C(p)$ vanishes if and only if the osculating plane of Γ at p coincides with the limiting tangent plane (which is the plane in $\mathbb{R}^3$ passing through $f(p)$ obtained as the limit of the tangent planes at its regular points) of f at p.
(2) The sign of $\sigma^C(p)$ coincides with that of the Euclidean limiting normal curvature $\kappa_\nu^E(s)$ if we think $\mathbb{R}^3$ as $\mathbb{E}^3$.
(3) If the normal vector of f at p points in a space-like or time-like direction, then the sign of $\sigma^C(p)$ also coincides with that of the Lorentzian limiting normal curvature $\kappa_\nu^L(s)$ if we think $\mathbb{R}^3$ as $\mathbb{L}^3$.

Regarding the above facts, we give the following:

Definition 0.4. A generalized cuspidal edge singular point p of $f : \mathcal{U}_f \to \mathbb{R}^3$ is said to be *generic* if $\sigma^C(p)$ does not vanish.

We remark that the invariant σ^C is generalized for an invariant of cuspidal edges in a Riemannian 3-manifold (M^3, g) (cf. [17], where σ^C is denoted by σ_g^C).

A regular curve $\Gamma : I \to \mathbb{L}^3$ defined on an interval I is called of *type* S (resp. *type* T) if the velocity vector field $\Gamma'(s)$ points in a space-like (resp. time-like) direction for each $s \in I$. If Γ is neither type S nor type T, then there exists $s \in I$ such that $\Gamma'(s)$ points in a light-like direction. We are interested in the special case that $\Gamma'(s)$ is a light-like direction for any $s \in I$. Such a Γ is called a *regular curve of type* L.

We next define the *order* of a singular point $p\,(= (s, 0) \in \pm_f)$ of a given generalized cuspidal edge $f : \mathcal{U}_f \to \mathbb{L}^3$, denoted by i_p (cf. Definition 1.25): We denote by $\Delta^L(s, t)$ the determinant of the symmetric matrix associated with the first fundamental form of f (cf. (1.11)). Roughly speaking, the order i_p is the minimal number k such that $\partial^k \Delta^L(s, 0)/\partial t^k$ does not vanish (cf. Definition 1.25). The order is a concept that only makes sense for f lying in $\mathbb{L}^3$ (when f lying in $\mathbb{E}^3$, i_p can be also defined in the same manner but is always equal to 2). If the order i_p $(p \in \Sigma_f)$ is an even constant, then all regular points on a sufficiently small neighborhood V_p of p have the same causal type, in particular, they are space-like or time-like on V_p. On the other hand, if i_p is an odd constant, then any regular points on a sufficiently small neighborhood of p is space-like on one side of the singular curve Σ_f and time-like on the opposite side. As a special case, $i_p = \infty$ can occur (see (1.27)). The following result describes the generic behavior of generalized cuspidal edges in $\mathbb{L}^3$ (as a consequence, most cuspidal edges in $\mathbb{L}^3$ have unbounded mean curvature functions):

Theorem A. *Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge along a regular curve Γ in $\mathbb{L}^3$. If Γ is of type S or of type T, then, for each singular point $p := (s, 0) \in \Sigma_f$, there exists a sufficiently small neighborhood $V_p(\subset \mathcal{U}_f)$ of p such that the following assertions hold* (*the first three assertions are about the causality of f which is defined in Definition 1.1*)*:*

(1) *f is space-like on $V_p \setminus \Sigma_f$ if Γ is of type S and the $\mathbb{L}^3$-cuspidal direction vector $\mathbf{D}^L_f(s)$ is space-like (cf. Definition 1.19).*
(2) *f is time-like on $V_p \setminus \Sigma_f$ if*
   - *Γ is of type T, or*
   - *Γ is of type S and $\mathbf{D}^L_f(s)$ is a time-like vector.*
(3) *If Γ is of type S and $\mathbf{D}^L_f(s)$ is a light-like vector, then the causal type of f is different between both sides of $V_p \setminus \Sigma_f$.*
(4) *If $\mathbf{D}^L_f(s)$ is not a light-like vector, then the limiting normal curvature $\kappa^L_\nu(s)$ is defined (cf. (1.34)), and the sign of $\kappa^L_\nu(s)$ coincides with $\sigma^C(p)$.*
(5) *The order i_p is equal to 2 if $\mathbf{D}^L_f(s)$ is not a light-like vector at p. On the other hand, if $\mathbf{D}^L_f(s)$ points in a light-like direction, then $i_p \geq 3$ holds. Moreover, the equality holds if and only if p is a cuspidal edge singular point.*
(6) *If p is a cuspidal edge singular point, then the mean curvature H^L of f is unbounded on V_p. In particular, if p is an accumulation point of the set of cuspidal edge singular points, then H^L is also unbounded at p.*
(7) *If p is a generic (i.e. $\sigma_C(p) \neq 0$) cuspidal edge singular point satisfying $i_p = 2$ (resp. $i_p = 3$), then the Gaussian curvature K^L of f is unbounded and takes different signs (resp. the same sign) on each side (resp. both sides) of Σ_f in V_p.*
(8) *If p is a cuspidal edge singular point then the two principal curvatures of f are both real-valued on V_p. Moreover, if p is generic (i.e. $\sigma_C(p) \neq 0$), then one of them is unbounded on V_p. In this setting, the other one is bounded on V_p unless $i_p = 3$.*
(9) *The umbilical or quasi-umbilical points cannot accumulate at any cuspidal edge singular point.*

The corresponding assertions for cuspidal edges in $\mathbb{E}^3$ are known (cf. [10, 5, 20]), which are summarized in Fact 2.5. In particular, for an arbitrarily given cuspidal edge in $\mathbb{E}^3$, its mean curvature function H^E is always unbounded, and its Gaussian curvature function K^E is bounded if and only if its limiting normal curvature κ^E_ν vanishes identically on Σ_f. So, Theorem A contains an analogue of these facts in $\mathbb{L}^3$. The strategy of the proof of Theorem A is an improvement of the corresponding result in $\mathbb{E}^3$ given by the first author [5]. In fact, we need more case separations and computations of higher-order derivatives. As a consequence of Theorem A, we have the following:

Corollary B. *Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a cuspidal edge along a regular curve Γ in $\mathbb{L}^3$. If the mean curvature H^L of f is bounded on $\mathcal{U}_f \setminus \Sigma_f$, then Γ is of type L.*

By this assertion, as long as Σ_f consists of cuspidal edge singular points, the mean curvature H^L of f can be bounded only when Γ is of type L. So we consider the case that Γ is of type L:

Definition 0.5. A C^∞-map $f : \mathcal{U}_f \to \mathbb{L}^3$ is called a *light-like generalized cuspidal edge of general type* along Γ (see Definition 4.6) if it is a generalized cuspidal edge such that

- Γ is of type L,
- the order i_p of f at each point $p \in \Sigma_f$ is equal to 2, and
- $\Gamma''(s)$ never vanishes for each $(s, 0) \in \Sigma_f$.

If Γ is of type L, the possibility of the normal direction of f is either space-like or light-like. More precisely, the following assertion holds:

Proposition C. *Let* $\Gamma : I \to \mathbb{L}^3$ *be a regular curve such that* $\Gamma'(s_0)$ $(s_0 \in I)$ *is a light-like vector. If* f *is a generalized cuspidal edge along* Γ *in* $\mathbb{L}^3$*. Then the following two assertions are equivalent:*

(1) *The normal direction of* f *at* $p := (s_0, 0)$ *points in a space-like direction* (*resp. a light-like direction*),
(2) *the order* i_p *of* $p := (s_0, 0)$ *is equal to* 2 (*resp. greater than or equal to* 4).

In particular, $i_p > 2$ *implies* $i_p \geq 4$*. Moreover, if* Γ *is of type L satisfying* $\Gamma''(s_0) \neq \mathbf{0}$*, then* (1) *and* (2) *hold if and only if* $\sigma^C(p) \neq 0$ (*resp.* $\sigma^C(p) = 0$).

In particular, the normal vectors of light-like generalized cuspidal edges of general type always point in space-like directions. Regarding this, we show the following:

Theorem D. *Let* f *be a light-like generalized cuspidal edge of general type along* Γ *in* $\mathbb{L}^3$*. Then, for each singular point* $p := (s, 0) \in \Sigma_f$*, there exists a sufficiently small neighborhood* $V_p(\subset \mathcal{U}_f)$ *of* p *such that the following assertions hold:*

(1) *The sign* $\sigma^C(p)$ *does not vanish for each* $p \in \Sigma_f$.
(2) f *is time-like on* $V_p \setminus \Sigma_f$.
(3) *The mean curvature* H^L *is bounded on* $V_p \setminus \Sigma_f$.
(4) *If* p *is a cuspidal edge singular point, then* K^L *is unbounded and takes different signs on each side of* Σ_f *on* V_p.
(5) *If* p *is a cuspidal edge singular point, then the two principal curvatures of* f *are both unbounded on* V_p*, amd they are real-valued on one side of* Σ_f *and are not real-valued on the other side on* V_p (*that is, the singular set* Σ_f *behaves rather like the locus of quasi-umbilical points*).
(6) *Umbilical points and quasi-umbilical points of* f *never accumulate at any cuspidal edge singular points of* f.

In the setting of Theorem D, the authors do not know of any light-like generalized cuspidal edge of general type along Γ in $\mathbb{L}^3$ for which H^L vanishes identically (if it happens, by Proposition 5.7, f cannot be a minface).

The case that $\Gamma'(s)$ points in a light-like direction only at a point $s = s_0$ is also discussed in Section 4. We note that (9) of Theorem A and (6) of Theorem D are special cases of the following statement:

Proposition E. *Let* f *be a cuspidal edge along a regular curve* Γ *in* $\mathbb{L}^3$*. If a given cuspidal edge singular point* $p := (s, 0) \in \Sigma_f$ *is of order less than* 4 (*i.e.* $i_p \leq 3$), *then umbilical points of* f *never accumulate at* p*. Moreover, quasi-umbilical points of* f *also cannot accumulate at* p *unless* p *is a light-like point* (*if* p *is an isolated light-like point, quasi-umbilical points can accumulate at* p*, see* (3) *of Proposition 4.5*).

For a light-like cuspidal edge singular point p, the authors know of no example in which umbilical points accumulate at p.

Definition 0.6. Let f be a generalized cuspidal edge along a regular curve Γ of type L. If $i_p = 4$ holds for each $p \in \Sigma_f$, we call f a *generalized cuspidal edge of order four*.

If f be a generalized cuspidal edge along a regular curve Γ of type L in $\mathbb{L}^3$, then the Lorentzian singular curvature function κ_s^L along Γ cannot defined. In this case, we use the Euclidean singular curvature function κ_s^E along Γ, instead. We focus on the property that the sign of κ_s^E is an identifier of whether the cuspidal edge f looks convex or concave in human's eyes (cf. [16, Section 5]):

Definition 0.7. Let f be a generalized cuspidal edge of order four along a regular curve Γ of type L. Then f is said to be of *convex type* (resp. *concave type*) at a singular point $p = (s, 0) \in \Sigma_f$ if $\kappa_s^E(s)$ is positive (resp. negative).

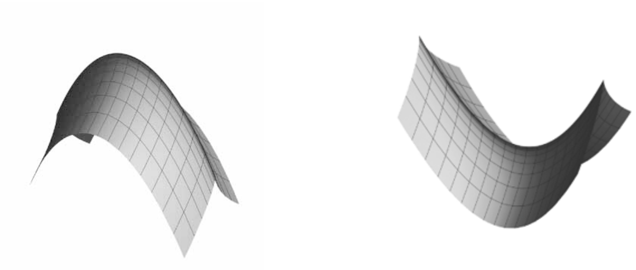

FIGURE 1. Cuspidal edges in $\mathbb{E}^3$ with positive (left) and negative (right) singular curvatures

If f is a generalized cuspidal edge of order four which is of convex (resp. concave) type, then, for any time-like vector $\mathbf{v}$, the image of the orthogonal projection of f into the plane $\Pi^L_{\mathbf{v}}$ which is perpendicular to $\mathbf{v}$ is locally convex (resp. locally concave), see [6, Proposition 2.19 and also Remark 2.20] for details.

Let

$$\pi : \mathbb{L}^3 \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2 \tag{0.4}$$

be the canonical projection, and consider the curve $\gamma := \pi \circ \Gamma$ in the space-like xy-plane. We denote by P_s the plane in $\mathbb{L}^3$ passing through $\gamma(s)$, which is spanned by $\mathbf{v}_3 := (0, 0, 1)$ and $\mathbf{n}(s)$, where $\mathbf{n}(s)$ is the normal vector of the curve γ at $\gamma(s)$. Then P_s passes through $\Gamma(s)$ as well as $\gamma(s)$ (cf. Figure 2). In this setting, we may assume that s is the arc-length parameter of γ. Let $\kappa(s)$ be the curvature of γ at $s \in I$, and $\mathbf{c}_s$ the section of the image of f by the plane P_s. We denote by $\mu(s, t)$ the function (cf. Definition A.6) associated with the cuspidal curvature of $\mathbf{c}_s$, which can be written as

$$\mu(s, t) = \mu_0(s) + \mu_1(s)t + \varphi(s, t)t^2, \tag{0.5}$$

where $\varphi(s, t)$ is a certain C^∞-function. Then, we prove the following:

Proposition F. *Let f be a cuspidal edge in $\mathbb{L}^3$ and p its singular point satisfying $i_p \geq 4$. Then $i_p > 4$ holds if and only if f is of concave type at p and*

$$|\kappa| = \mu_0^2 \tag{0.6}$$

holds at the point p.

As nice classes of zero-mean curvature surfaces in $\mathbb{L}^3$, "maxfaces" and "minfaces" are known, and if they give generalized cuspidal edges (in fact, cuspidal edges and cuspidal cross caps are typical examples of generalized cuspidal edges), then they are always of order four (cf. Proposition 5.7). We remark that "maxfaces" and "minfaces" themselves may admit singular points which are not appeared on generalized cuspidal edges, like as swallowtails and cone-like singular points. The following theorem summarizes the properties of the generalized cuspidal edges of order four, which is the deepest result in this paper:

Theorem G. *Let f be a generalized cuspidal edge of order four along a regular curve Γ of type L. Suppose that the regular curve $\gamma(s) := \pi \circ \Gamma(s)$ $(s \in I)$ in the xy-plane is parametrized by the arc-length and $\kappa(s)$ denotes the curvature function of $\gamma(s)$ as a plane curve. Then, for each singular point $p := (s_0, 0) \in \Sigma_f$, there exists a sufficiently small neighborhood $V_p(\subset \mathcal{U}_f)$ of p such that the following assertions hold:*

(a) *The causal type of f is the same on both sides of the singular set Σ_f on V_p.*

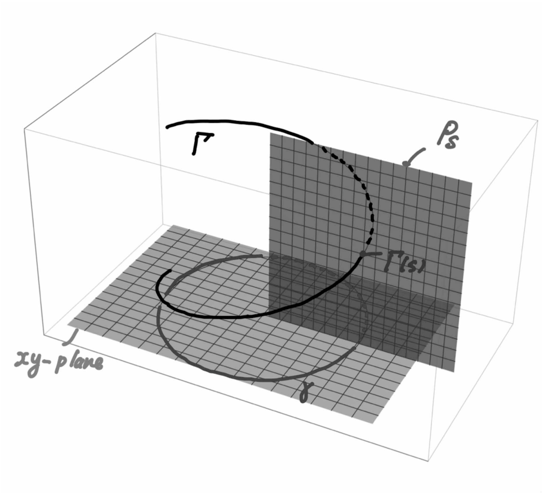

FIGURE 2. The plane P_s and the curves Γ (a helix) and γ (a circle)

(b) *The mean curvature function H^L of f is bounded on V_p if and only if (where the definition of the sign σ is given in* (5.1))

$$3\kappa\mu_1 = 8\mu_0\mu_0' + 3\sigma\kappa' \tag{0.7}$$

holds on $(I \times \{0\}) \cap V_p$, where μ_i $(i = 1, 2)$ are the coefficients of the function $\mu(s,t)$ as in (0.5).

(c) *The generalized cuspdial edge singular point p cannot be an accumulation point of umbilics of f.*

(d) *The point p is a cuspidal edge singular point if and only if $\mu_0(s_0) \neq 0$.*

Moreover, if $\Gamma''(s_0)$ does not vanish (i.e. $\kappa(s_0) \neq 0$), then the following assertions hold:

(1) *If f is space-like on $V_p \setminus \Sigma_f$, then f is of concave type and K^L is positive at each point on $V_p \setminus \Sigma_f$ (cf. Figure 3, right).*

(2) *If f is time-like on $V_p \setminus \Sigma_f$, then the Gaussian curvature K^L is positive (resp. negative) if and only if f is of convex type (resp. concave type) at p, see Definition 0.7 and Figure 3. As a special case, if p is not a cuspidal edge, then the Gaussian curvature K^L is positive and f is of convex type (cf. Corollary 5.2).*

(3) *The Gaussian curvature K^L diverges and takes the same sign on both sides of Σ_f, and the two principal curvatures are both unbounded. In this situation, if K^L is positive (resp. negative), then the two principal curvatures both take values in $\mathbb{R}$ (resp. $\mathbb{C} \setminus \mathbb{R}$) on V_p.*

(4) *The quasi-umbilical points of f cannot accumulate at p.*

Theorem G is not vacuous. In fact, there is a representation formula (cf. (5.2)) producing all generalized cuspidal edges of order four in $\mathbb{L}^3$, by which we can construct examples satisfying the assumptions of Theorem G. At first glance, the conditions (0.6) and (0.7) seem to depend on the choice of an orthogonal coordinate system in $\mathbb{L}^3$. However, these conditions are invariant under Lorentzian motions belonging to the identity component of the isometry group of $\mathbb{L}^3$ (cf. Proposition 4.4 and Corollary 5.1). The assertion (4) of Theorem G cannot be expected when $\Gamma''(s) = \mathbf{0}$. In fact, quasi-umbilical points may accumulate at the singular point $(s, 0)$ of f satisfying $\Gamma''(s) = \mathbf{0}$. More precisely, such an example is constructed in Akamine [1, Figure 4] (see Remark 5.4). As stated in Theorems D and G, cuspidal

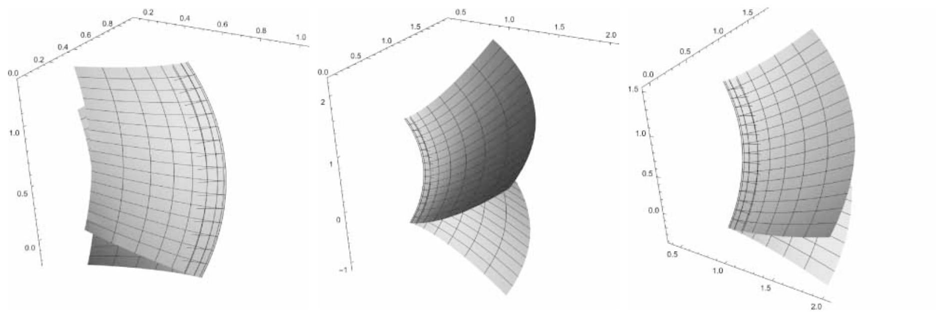

FIGURE 3. Cuspidal edges of order four along a light-like helix, which is time-like of convex type (left), and time-like of concave type (center) and space-like (right), respectively. These are explained in Example 5.5.

edges of type L whose order are at most four cannot change their causal types. However, there exist cuspidal edges of type L with order five such that they are space-like on one side of the singular set, and time-like on the other side (see Example 5.6). This means that cuspidal edges of type L also can change their causal type as well as in the case (3) in Theorem A.

In $\mathbb{E}^3$, minimal surfaces which are constructed using the Weierstrass formula do not admit cuspidal edges but only admit branch points as isolated singular points. As mentioned before Theorem G, there are two canonical classes of zero-mean curvature ("ZMC" in short) surfaces in $\mathbb{L}^3$ without branch points, called "maxfaces" and "minfaces". Roughly speaking, maxfaces (resp. minfaces) are ZMC surfaces which can be represented as a holomorphic (resp. para-holomorphic) data using the Weierstrass-type representation formulas, respectively.

Akamine [1] investigated the sign of the Gaussian curvature K^L near the singular points on a given time-like ZMC-surface (cf. [1, Theorem A]. In particular, at cuspidal edge points of a time-like surface f, he showed that the sign of K^L depends on the convexity or concavity of f (cf. [1, Theorem B]), and also investigated umbilical points and quasi-umbilical points near the singular points. As a consequence, Akamine [1] proved (a), (d), (2), (3) and (4) of Theorem G for "minfaces" ((1) for maxface was shown in [21] except for the concavity), and so, Theorem G can be considered as generalizations of Akamine's results (as we have already mentioned, generalized cuspidal edge singular points appeared on maxfaces and minfaces are all of order four).

Through Akamine's work, we can observe how the cuspidal edges are special amongst the non-degenerate singular points on zero-mean curvature surfaces. Moreover, by Theorem G, we are able to recognize that these interesting properties of generalized cuspidal edges having vanishing mean curvature functions attribute to the property that their order i_p are equal to four along the singular curve. Also, we note that the case where $\Gamma'(s)$ is a light-like vector only at a value $s = s_0$ is also investigated in Section 4.

We organize the paper as follows: In Section 1, we introduce the fundamental properties of generalized cuspidal edges in $\mathbb{L}^3$ and define "singular curvature" and "limit normal curvature" along their singular sets. In Section 2 (resp. Section 3), we investigate the behaviors of H^L and K^L for generalized cuspidal edges when Γ is time-like (resp. space-like), and prove Theorem A and Corollary B in Sections 2 and 3. In Section 4, we consider the case that Γ is light-like and prove Theorem D, Propositions C and E. In Section 5, we consider generalized cuspidal edges of order four, and prove Proposition F and Theorem G. We have three appendices: In the first appendix, a representation formula for cusps in Lorentz-Minkowski plane $\mathbb{L}^2$ is given. In the second appendix, the existence of a certain parametrization of the generalized cuspidal edge is shown. In the third appendix, we discuss umbilical points on wave fronts in $\mathbb{L}^3$.

1. Preliminaries

In this section, we prepare fundamentals of the geometry of generalized cuspidal edges in $\mathbb{L}^3$ comparing to the Euclidean case.

1.1. **Regular surfaces in $\mathbb{E}^3$.** Let U be a domain in the uv-plane $(\mathbb{R}^2; u, v)$. A C^∞-map $f : U \to \mathbb{R}^3$ is called a *regular surface* if it is an immersion on U. We denote by "$\cdot$" the canonical positive definite inner product on $\mathbb{E}^3$. When the image of f lies in $\mathbb{E}^3$, the functions on U

$$E := f_u \cdot f_u, \quad F := f_u \cdot f_v, \quad G := f_v \cdot f_v$$

are called the *coefficients of the first fundamental form* $ds^2 := Edu^2 + 2Fdudv + Gdv^2$. We set $\tilde{\nu}^E := f_u \times f_v$, which gives the normal vector field of f on U, where "$\times$"denotes the canonical vector product on $\mathbb{E}^3$. By definition,

$$(\Delta_E :=)\tilde{\nu}^E \cdot \tilde{\nu}^E = EG - F^2 \tag{1.1}$$

holds. We then set

$$\tilde{L} := f_{uu} \cdot \tilde{\nu}^E = \det(f_u, f_v, f_{uu}), \tag{1.2}$$

$$\tilde{M} := f_{uv} \cdot \tilde{\nu}^E = \det(f_u, f_v, f_{uv}), \tag{1.3}$$

$$\tilde{N} := f_{vv} \cdot \tilde{\nu}^E = \det(f_u, f_v, f_{vv}), \tag{1.4}$$

which are not the usual coefficients of the second fundamental form, because $\tilde{\nu}^E$ may not be a unit vector field. By setting

$$L := \frac{\tilde{L}}{\sqrt{\Delta_E}}, \quad M := \frac{\tilde{M}}{\sqrt{\Delta_E}}, \quad N := \frac{\tilde{N}}{\sqrt{'\Delta_E}},$$

$Ldu^2 + 2Mdudv + Ndv^2$ gives the second fundamental form of f with respect to the unit normal vector field $\nu^E := \tilde{\nu}^E/|\tilde{\nu}^E|_E$, where

$$|\mathbf{a}|_E := \sqrt{\mathbf{a} \cdot \mathbf{a}} \ (\mathbf{a} \in \mathbb{E}^3).$$

We set

$$\tilde{W}^E := \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} \tag{1.5}$$

and

$$W^E := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{\Delta_E^{3/2}} \tilde{W}^E. \tag{1.6}$$

The matrix W^E is called the *Weingarten matrix* or *the shape operator*. Using this,

$$K^E := \det(W^E) = \frac{\det(\tilde{W}^E)}{\Delta_E^3} = \frac{\tilde{L}\tilde{N} - \tilde{M}^2}{\Delta_E^2}, \tag{1.7}$$

$$H^E := \text{trace}(W^E) = \frac{\text{trace}(\tilde{W}^E)}{2\Delta_E^{3/2}} \tag{1.8}$$

give the Gaussian curvature function and the mean curvature function of f, respectively.

1.2. **Regular surfaces in $\mathbb{L}^3$.** We next consider the case that f is a regular surface (i.e. an immersion) into $\mathbb{L}^3$. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{L}^3$ as column vectors, the Lorentzian inner product and the vector product are defined by

$$(1.9)\qquad \langle \mathbf{a}, \mathbf{b} \rangle_L := \mathbf{a}^T E_3 \mathbf{a}, \quad \mathbf{a} \times_L \mathbf{b} := E_3(\mathbf{a} \times \mathbf{b}), \qquad E_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where the symbol ^{T} denotes matrix transposition. Then

$$(1.10)\qquad \tilde{\nu}_L := f_u \times_L f_v (= E_3 \tilde{\nu}_E)$$

is a normal vector field of f in $\mathbb{L}^3$.

Definition 1.1. Let $f : U \to \mathbb{R}^3$ be a regular surface. A point $p \in U$ is said to be *space-like, time-like, light-like* if the normal vector $\tilde{\nu}^L(p)$ is time-like, space-like, light-like in $\mathbb{L}^3$, respectively. Moreover, f is said to be *space-like* (resp. *time-like*) if all points of U are space-like (resp. time-like).

Here, we consider the case that f is a space-like or time-like surface on U. We set

$$E^L := \langle f_u, f_u \rangle_L, \quad F^L := \langle f_u, f_v \rangle_L, \quad G^L := \langle f_v, f_v \rangle_L,$$

and call the function

$$(1.11)\qquad \Delta_L := E^L G^L - (F^L)^2$$

the *identifier of the causality* of f which depends on the choice of coordinate system (u, v) and differs only by the multiplication of a positive function under a coordinate change. Since $\tilde{\nu}^L = E_3 \tilde{\nu}^E$, we have

$$(1.12)\qquad \begin{aligned} \left\langle \tilde{\nu}^L, \tilde{\nu}^L \right\rangle_L &= \left\langle \tilde{\nu}^E, \tilde{\nu}^E \right\rangle_L = (f_u \times f_v)^T E_3 (f_u \times f_v) \\ &= \det(E_3)(f_u \times f_v) \cdot (E_3 f_u \times E_3 f_v) \\ &= -\Big((f_u \cdot E_3 f_u)(f_v \cdot E_3 f_v) - (f_u \cdot E_3 f_v)(f_v \cdot E_3 f_u) \Big) \\ &= - \langle f_u, f_u \rangle_L \langle f_v, f_v \rangle_L + \langle f_u, f_v \rangle_L^2 = -\Delta_L. \end{aligned}$$

Moreover, since $\left\langle \mathbf{v}, \tilde{\nu}^L \right\rangle_L = \mathbf{v} \cdot \tilde{\nu}^E$ holds for any vector $\mathbf{v} \in \mathbb{R}^3$, we have (cf. (1.10))

$$\left\langle f_{uu}, \tilde{\nu}^L \right\rangle_L = \tilde{L}, \qquad \left\langle f_{uv}, \tilde{\nu}^L \right\rangle_L = \tilde{M}, \qquad \left\langle f_{vv}, \tilde{\nu}^L \right\rangle_L = \tilde{N}.$$

Regarding

$$\left\langle \tilde{\nu}^L, \tilde{\nu}^L \right\rangle_L = \varepsilon |\Delta_L| \qquad \varepsilon := \operatorname{sgn}\left(\left\langle \tilde{\nu}^L, \tilde{\nu}^L \right\rangle_L \right),$$

we set

$$L^L := \frac{\tilde{L}}{\sqrt{|\Delta_L|}}, \quad M^L := \frac{\tilde{M}}{\sqrt{|\Delta_L|}}, \quad N^L := \frac{\tilde{N}}{\sqrt{|\Delta_L|}}.$$

Then the Gaussian curvature (i.e. the sectional curvature with respect to the first fundamental form) K^L and the mean curvature H^L of f are defined by (cf. [13, page 107 and page 101])

$$(1.13)\qquad K^L := \varepsilon \frac{L^L N^L - (M^L)^2}{\Delta_L} \qquad H^L := \frac{E^L N^L - 2F^L M^L + G^L L^L}{2\Delta_L}.$$

Since

$$L^L N^L - (M^L)^2 = \frac{\tilde{L}\tilde{N} - \tilde{M}^2}{|\Delta_L|} = -\varepsilon \frac{\tilde{L}\tilde{N} - \tilde{M}^2}{\Delta_L},$$

we have that

$$(1.14)\qquad K^L = \frac{-(\tilde{L}\tilde{N} - \tilde{M}^2)}{\Delta_L^2}, \qquad H^L = \frac{E^L \tilde{N} - 2F^L \tilde{M} + G^L \tilde{L}}{2\Delta_L \sqrt{|\Delta_L|}}.$$

As an advantage of these expressions, we can use $\tilde{L}$, $\tilde{M}$ and $\tilde{N}$ given in (1.2), (1.3), (1.4) to compute K^L and H^L. About H^L, there might exist other definitions with different sign. However, our results are related only to the absolute value of H^L, and so this does not affect the latter discussions. We set

$$\tilde{W}^L := \begin{pmatrix} G^L & -F^L \\ -F^L & E^L \end{pmatrix} \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix}, \tag{1.15}$$

$$W^L := \begin{pmatrix} E^L & F^L \\ F^L & G^L \end{pmatrix}^{-1} \begin{pmatrix} L^L & M^L \\ M^L & N^L \end{pmatrix} = \frac{1}{\Delta_L \sqrt{|\Delta_L|}} \tilde{W}^L.$$

Then W_L is called the *Weingarten matrix* or *the shape operator*. Using these two matrices, we have the following expressions:

$$K^L = \varepsilon \det(W^L) = -\frac{\det(\tilde{W}^L)}{\Delta_L^3}, \qquad H^L = \operatorname{trace}(W^L) = \frac{\operatorname{trace}(\tilde{W}^L)}{2\Delta_L\sqrt{|\Delta_L|}}. \tag{1.16}$$

In particular, the determinant of W^L is $-K^L$ (resp. K^L) and the trace of W^L is $2H^L$ when f is space-like (resp. time-like). Thus, the eigenvalues of W^L are invariants of f, which are called the *principal curvatures* of f. Unlike the Euclidean case, the principal curvatures of f may not be real-valued if $\Delta_L < 0$.

As mentioned in the introduction, the results in $\mathbb{L}^3$ sometimes exhibit different phenomena from those in $\mathbb{E}^3$, which is the motivation for this research. In this paper, we will investigate the behavior of the principal curvatures around singularities of surfaces in $\mathbb{L}^3$ as well as K^L and H^L, keeping in mind the results obtained in the case of Euclidean geometry.

Definition 1.2. The point p on the domain of definition of a regular surface f is called an *umbilical point* (resp. a *quasi-umbilical point*) if the two principal curvatures coincide and W^L is a diagonal (resp. not a diagonal) matrix at p.

Remark 1.3. If p is a space-like regular point (i.e. $\Delta_L(p) > 0$), then there exists a regular matrix P such that $P^{-1}\tilde{W}^L P$ is a symmetric matrix at p, and so quasi-umbilical points never appear like as the case of regular surfaces in $\mathbb{E}^3$.

By (1.7) and (1.16), the sign of K^L is opposite of that of K^E, that is, we have (cf. [1, Remark 4.6])

$$\operatorname{sgn}(K^L) = -\operatorname{sgn}(K^E). \tag{1.17}$$

1.3. **Generalized cuspidal edges in $\mathbb{R}^3$ and $\mathbb{E}^3$.** In the introduction, we have defined generalized cuspidal edges. In this subsection, we show several important properties of them:

Definition 1.4. Let $f(s,t)$ be a smooth map satisfying $f_s(s,0) \neq \mathbf{0}$ and $f_t(s,0) = \mathbf{0}$ for each s. For such f, we consider a new local coordinate system (u,v) satisfying

$$t(u,0) = 0, \quad s_v(u,0) = 0, \quad s_u(u,0) > 0, \quad t_v(u,0) > 0.$$

In particular, $f_v(u,0) = \mathbf{0}$ holds for each u. We call such a local coordinate (u,v) an *admissible local coordinate system* associated with f, as well as the original coordinate system (s,t).

If $f(s,t)$ is a generalized cuspidal edge, then $f_{tt}(s,0) \neq \mathbf{0}$ holds, which can be reproved from the next lemma (we omit the proof):

Lemma 1.5. *Let (s,t) be an admissible coordinate system of a generalized cuspidal edge f. Let $\varphi(s,t)$ be a smooth $\mathbb{R}^l$ $(l \geq 1)$ valued function satisfying $\varphi_s(s,0) \neq \mathbf{0}$ and $\varphi_t(s,0) = \mathbf{0}$ for each s. Then, for a C^∞-function $\varphi(s,t)$, the first non-vanishing order k of the derivative $\partial^k\varphi(s,0)/\partial t^k$ is independent of the choice of an admissible* *coordinate system of f. Moreover, if k is even, then the sign of $\partial^k\varphi(s,0)/\partial t^k$ is also independent of the choice of an admissible coordinate system.*

We introduce here typical examples of generalized cuspidal edges:

Proposition 1.6. *A cuspidal edge along a regular curve Γ in $\mathbb{R}^3$ as in Definition 0.2 is a generalized cuspidal edge.*

Proof. Let $f(s,t)$ be a cuspidal edge along Γ as in Definition 0.2. Then, it has a unit normal vector field ν on $\mathcal{U}_f$, and if we set $\lambda := \det(f_s, f_t, \nu)$, then the exterior derivative $d\lambda$ at $(s,0)$ does not vanish for each $s \in I$ ([16, Chapter 2]). Since $\lambda_s(s,0) = 0$ and $f_t(s,0) = \mathbf{0}$, we have

$$0 \neq \lambda_t(s,0) = \det(f_s(s,t), f_t(s,t), \nu(s,t))_t|_{t=0} = \det(f_s(s,0), f_{tt}(s,0), \nu(s,0)),$$

which implies that $f_s(s,0)$ and $f_{tt}(s,0)$ are linearly independent, proving the assertion. □

Example 1.7. The C^∞-maps defined by

$$f_1(s,t) = (s, t^2, 0), \quad f_2(s,t) = (s, t^2, st^3),$$

give generalized cuspidal edges. The origin $(0,0)$ is a singular point of them, which is called the *standard fold singular point* and the *standard cuspidal cross cap singular point*, respectively.

Definition 1.8. Fix a domain U of $\mathbb{R}^2$. A C^∞-map $f : U \to \mathbb{E}^3$ is called a *frontal* if there exists a nowhere vanishing vector field $\tilde{\nu} : U \to \mathbb{E}^3$ which is perpendicular to f_u and f_v at each point of U. Let $P(\mathbb{R}^3)$ be the projective space associated with the vector space $\mathbb{R}^3$ and $\mathbb{R}^3 \ni \mathbf{v} \to [\mathbf{v}] \in P(\mathbb{R}^3)$ the canonical projection. Then f is called a *wave front* if $U \ni p \mapsto (f(p), [\tilde{\nu}(p)]) \in \mathbb{R}^3 \times P(\mathbb{R}^3)$ is an immersion.

The maps f_1 and f_2 given in Example 1.7 are frontals, but not are wave fronts. Let $f := f(s,t)$ be a generalized cuspidal edge along Γ, and consider the vector field defined by $\mathbf{v}(s,t) := f_t(s,t)/t$ ($|t| < \delta$), which is C^∞-differentiable at $t = 0$ such that $f_{tt}(s,0) = \mathbf{v}(s,0) (\neq \mathbf{0})$. By condition (c) of Definition 0.3,

$$\hat{\nu}^E := f_s(s,t) \times \mathbf{v}(s,t) \tag{1.18}$$

gives a normal vector field of f, which implies that f is a frontal, that is, $\hat{\nu}^E$ can be smoothly extended to the singular set. Then, we have

$$\hat{\nu}^E(s,0) = f_s(s,0) \times \mathbf{v}(s,0) = \Gamma'(s) \times f_{tt}(s,0). \tag{1.19}$$

In this setting,

$$\nu^E := \frac{\hat{\nu}^E}{|\hat{\nu}^E|_E} \tag{1.20}$$

gives a unit normal vector field of f on $\mathcal{U}_f$ which can be smoothly extended across the singular set Σ_f of f. As the converse of Proposition 1.6, the following is a well-known criterion for cuspidal edges (cf. [16, Theorem 2.6.3]):

Fact 1.9. *A point $(s,0)$ of a generalized cuspidal edge is a cuspidal edge singular point if f is a wave front at $(s,0)$, that is, the map $(s,t) \mapsto \big(f(s,t), \nu^E(s,t)\big)$ is an immersion at $(s,0)$.*

Using this, we prepare the following:

Proposition 1.10. *Let $f : \mathcal{U}_f \to \mathbb{R}^3$ be a generalized cuspidal edge along Γ. Then f is a frontal, that is, its normal vector field (as $\mathbb{R}^3 = \mathbb{E}^3$ or $\mathbb{R}^3 = \mathbb{L}^3$ see Remark 1.17) is defined at each singular point of f. Moreover, let P be a plane passing through $f(s_0,0)$ ($s_0 \in I$) which is transversal to the curve Γ at $\Gamma(s_0) = f(s_0,0)$. Then the section $\mathbf{c}$ of f by P is a generalized cusp in P (cf. Definition A.1), and the point*

$p := (s_0, 0)$ *is a cuspidal edge singular point if and only if* p *is a cusp of* $\mathbf{c}$ *in the plane* P.

Proof. This statement is essentially the same statement as in [9, Lemma 3.2]. However, our proof is new and is different from that in [9]: Take a basis spanning the plane of P at $f(s_0, 0)$, and extend it as a frame field $\{\mathbf{a}_1, \mathbf{a}_2\}$ along Γ. By Proposition B.1 in the appendix, f can be written in the form

$$f(s,t) = \Gamma(s) + x(s,t)\mathbf{a}_1(s) + y(s,t)\mathbf{a}_2(s),$$

where $x(s,t)$ and $y(s,t)$ are smooth functions. Since the map

$$(s,x,y) \mapsto \Gamma(s) + x\mathbf{a}_1(s) + y\mathbf{a}_2(s)$$

gives a tubular neighborhood of Γ in $\mathbb{R}^3$, $f_0(s,t) := (x(s,t), y(s,t), s)$ is a generalized cuspidal edge in $\mathbb{R}^3$ defined on a neighborhood of $(s_0, 0) \in U$. Without loss of generality, we can write

$$x(s,t) = t^2\hat{x}(s,t), \qquad y(s,t) = t^3\hat{y}(s,t),$$

where $\hat{x}(s,t)$ and $\hat{y}(s,t)$ are smooth functions satisfying $\hat{x}(s_0, 0) \neq 0$. Since the section $\mathbf{c}$ of f by P is a generalized cusp in P, which can be identified with the curve $\mathbf{c} : t \mapsto (x(s_0,t), y(s_0,t))$ in $\mathbb{R}^2$, and the point $(s_0, 0)$ corresponds to the singular point of $\mathbf{c}$. Moreover, $(s_0, 0)$ is a cusp point of $\mathbf{c}$ if and only if $\hat{y}(s_0, 0) \neq 0$. Then, it can be easily checked that

$$N := \Big(-t\,(t\hat{y}_t + 3\hat{y})\,, t\hat{x}_t + 2\hat{x}, \big((t\hat{y}_t + 3\hat{y})\,\hat{x}_s - (t\hat{x}_t + 2\hat{x})\,\hat{y}_s\big)t^3\Big) \tag{1.21}$$

is a normal vector field of f_0 in the Euclidean 3-space $\mathbb{E}^3$. So f_0 (and also f) is a frontal. The unit normal vector field $\nu^E(s,t) := N(s,t)/|N(s,t)|_E$ of f_0 in $\mathbb{E}^3$ has the following asymptotic expansion

$$\nu^E(s,t) = (0,1,0) + t\frac{3\hat{y}(s,0)}{2\hat{x}(s,0)}\mathbf{e}_2 + O(t^2) \qquad (\mathbf{e}_2 := (0,1,0)),$$

where $O(t^2)$ is the term such that $O(t^2)/|t|^2$ is a vector-valued bounded function of (s,t) around $(s_0, 0)$. So, the matrix

$$\begin{pmatrix} (f_0)_s(s_0,0) & (f_0)_t(s_0,0) \\ \nu_s(s_0,0) & \nu_t(s_0,0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_3 & 0 \\ * & \frac{3\hat{y}(s_0,0)}{2\hat{x}(s_0,0)}\mathbf{e}_2 \end{pmatrix} \qquad (\mathbf{e}_3 := (0,0,1))$$

is of rank two if and only if $\hat{y}(s_0, 0) \neq 0$. So f_0 (and also f) is a wave front on a sufficiently small neighborhood of $(s_0, 0)$ if and only if $\hat{y}(s_0, 0) \neq 0$. Since $(s_0, 0)$ is a cusp point of $\mathbf{c}$ iff $\hat{y}(s_0, 0) \neq 0$, we obtain the last assertion by Fact 1.9. □

Definition 1.11. The vector field

$$\tilde{\mathbf{D}}^E_f(s) := f_{tt}(s,0) - \frac{f_{tt}(s,0) \cdot \Gamma'(s)}{\Gamma'(s) \cdot \Gamma'(s)}\Gamma'(s) \tag{1.22}$$

is called an $\mathbb{E}^3$-*cuspidal direction vector* of f at $(s, 0)$. Then it is the projection of the vector $f_{tt}(s, 0)$ to the normal plane of Γ in $\mathbb{E}^3$. By (c) of Definition 0.3, we can set

$$\mathbf{D}^E_f(s) := \frac{\tilde{\mathbf{D}}^E_f(s)}{|\tilde{\mathbf{D}}^E_f(s)|_E} \tag{1.23}$$

and call it the *unit* $\mathbb{E}^3$-*cuspidal direction vector* of f at s.

By definition, $\tilde{\mathbf{D}}^E_f(s)$ is determined up to a positive scalar multiplication if we take other admissible parametrizations of f (cf. Definition 1.4). Thus, $\mathbf{D}^E_f(s)$ is uniquely determined from f.

Proposition 1.12. $\Gamma'(s) \times \mathbf{D}^E_f(s)$ *is a positive scalar multiplication of* $\nu^E(s, 0)$.

Proof. In fact, it holds that (cf. (1.19))

$$\Gamma'(s) \times \mathbf{D}_f^E(s) = \frac{\Gamma'(s) \times f_{tt}(s,0)}{|\tilde{\mathbf{D}}_f^E(s)|_E} = \frac{\hat{\nu}^E(s)}{|\tilde{\mathbf{D}}_f^E(s)|_E},$$

where ν^E is given in (1.20) and $\hat{\nu}^E = \nu^E(s,0)$. □

If we are thinking of $f : \mathcal{U}_f \to \mathbb{R}^3$ as a generalized cuspidal edge in $\mathbb{E}^3$, the singular curvature function along Γ is defined by (cf. [5])

$$\kappa_s^E(s) := \frac{\Gamma''(s) \cdot \mathbf{D}_f^E(s)}{\Gamma'(s) \cdot \Gamma'(s)}. \tag{1.24}$$

Remark 1.13. The original definition of the singular curvature κ_s^E (cf. [15, 16]) is

$$\kappa_s^E := \operatorname{sgn}(\lambda_t) \frac{\det(\Gamma', \Gamma'', \hat{\nu}^E)}{|\Gamma'|_E^3}. \tag{1.25}$$

We may assume $|\Gamma'|_E = 1$. By setting $\mathbf{n} := \hat{\nu}^E \times \Gamma'$, we have

$$\lambda_t = \det(\Gamma', f_{tt}, \hat{\nu}^E) = \mathbf{n} \cdot f_{tt} = \mathbf{n} \cdot D_f^E.$$

Since $\mathbf{n}$ is a unit vector field, we have $\mathbf{n} = \varepsilon \mathbf{D}_f^E$, where $\varepsilon := \operatorname{sgn}(\lambda_t)$. Thus

$$\kappa_s^E = \varepsilon \det(\Gamma', \Gamma'', \hat{\nu}^E) = \varepsilon(\hat{\nu}^E \times \Gamma') \cdot \Gamma'' = \varepsilon \mathbf{n} \cdot \Gamma'' = \mathbf{D}_f^E \cdot \Gamma''$$

holds, and (1.24) is obtained.

This definition of κ_s^E does not depend on the choice of the parameter (s,t) of f and the sign ambiguity of the normal vector field $\nu^E(s,t)$. When $\kappa_s^E > 0$ (resp. $\kappa_s^E < 0$) at $p := (s,0)$, then the orthogonal projection of the image of f to the limiting tangent plane looks convex (resp. concave), see [16, Fig. 5.2]. Regrading this, we give the following:

Definition 1.14. A generalized cuspidal edge singular point $p := (s,0)$ of f is called of *convex type* (resp *concave type*) if the sign of κ_s^E is positive (resp. negative), see Figure 1.

The limiting normal curvature function along the s-axis is defined by

$$\kappa_\nu^E(s) := \frac{\Gamma''(s) \cdot \nu^E(s,0)}{\Gamma'(s) \cdot \Gamma'(s)}. \tag{1.26}$$

We denote by $\kappa^E(s)$ the curvature function of Γ in $\mathbb{E}^3$. Then, it holds that

$$(\kappa^E)^2 = (\kappa_s^E)^2 + (\kappa_\nu^E)^2. \tag{1.27}$$

Proposition 1.15. *For a singular point $p := (s,0)$ of the generalized cuspidal edge f, the sign of $\kappa_\nu^E(s)$ coincides with the sign $\sigma^C(s)$ (cf. (0.2) and (0.3)), that is,*

$$\operatorname{sgn}(\kappa_\nu^E(s)) = \sigma^C(p) \tag{1.28}$$

holds. Moreover, $\kappa_\nu^E(s) = 0$ (i.e. $\sigma^C(p) = 0$) holds if and only if $\mathbf{D}_f^E(s)$ lies in the osculating plane of Γ at the point $\Gamma(s)$.

Proof. Since $\nu^E(s,0)$ is a positive scalar multiplication of $f_s(s,0) \times f_{tt}(s,0)$ by (1.19), the numerator $\Gamma'(s) \cdot \nu^E(s,0)$ of (1.26) is a non-zero scalar multiple of

$$\Gamma''(s) \cdot (f_s(s,0) \times f_{tt}(s,0)) = \det(f_{ss}(s,0), f_s(s,0), f_{tt}(s,0)) = d^C(p),$$

proving the assertion. By the definition, $\mathbf{D}_f^E(s)$ lies in the osculating plane of Γ if and only if $\det(f_{ss}(s,0), f_s(s,0), f_{tt}(s,0))$ vanishes. So the second assertion is obtained. □

1.4. **Generalized cuspidal edges in $\mathbb{L}^3$.** Fix a generalized cuspidal edge $f(s,t)$ along $\Gamma(s)$ in the Lorentz-Minkowski 3-space $(\mathbb{L}^3; x, y, z)$ of signature $(++-)$. We define the causal types of the regular curve Γ in $\mathbb{L}^3$:

Definition 1.16. A regular curve $\Gamma : I \to \mathbb{L}^3$ is said to be of *type S* (resp. *type T*) if $\Gamma'(s)$ is a space-like (resp. time-like) vector for each $s \in I$. Moreover, $\Gamma : I \to \mathbb{L}^3$ is said to be of type L if $\Gamma'(s)$ is a light-like vector for each $s \in I$.

We consider the vector field defined by

$$\mathbf{v}(s,t) := \frac{f_t(s,t)}{t} \qquad (|t| < \delta), \tag{1.29}$$

which is C^∞-differentiable at $t = 0$ such that $f_{tt}(s,0) = \mathbf{v}(s,0) (\neq \mathbf{0})$. By (c) of Definition 0.3,

$$\hat{\nu}^L(s,t) := f_s(s,t) \times_L \mathbf{v}(s,t) \tag{1.30}$$

gives a normal vector field of f in $\mathbb{L}^3$ for sufficiently small choice of δ, which implies that f is a frontal and

$$\hat{\nu}^L(s,0) = f_s(s,0) \times_L \mathbf{v}(s,0) = \Gamma'(s) \times_L f_{tt}(s,0) \tag{1.31}$$

holds. Unless $\hat{\nu}^L(s,0)$ is not a light-like vector, we can set (for sufficiently small $|t|$)

$$\nu^L(s,t) := \frac{\hat{\nu}^L(s,t)}{|\hat{\nu}^L(s,t)|_L},$$

which gives a unit normal vector field of f, where

$$|\mathbf{a}|_L := \sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle_L|} \ (\mathbf{a} \in \mathbb{L}^3).$$

Remark 1.17. Since $f_s \times_L \mathbf{v}$ coincides with $E_3(f_s \times_E \mathbf{v})$ (see (1.9) for the definition of E_3), a smooth map f defined on a domain in $\mathbb{R}^2$ into $\mathbb{L}^3$ admits a Lorentzian normal vector field if and only if f admits a Euclidean normal vector field.

Regarding this, a smooth map $f : \mathcal{U}_f \to \mathbb{L}^3$ is called a *frontal* if and only if it is a frontal as a map into $\mathbb{E}^3$. Then, by Proposition 1.10, a generalized cuspidal edge is a frontal.

Definition 1.18. Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge. A point $p \in \mathcal{U}_f$ is said to be *space-like, time-like, light-like* if the normal vector $\hat{\nu}^L(p)$ is time-like, space-like, light-like in $\mathbb{L}^3$, respectively. Moreover, f is said to be *space-like* (resp. *time-like*) if all points of $\mathcal{U}_f$ are space-like (resp. time-like).

By definition, $p \in \mathcal{U}_f$ is *space-like, time-like, light-like* if so is the limiting tangent plane. Similar to (1.22), we give the following definition, under the assumption that Γ is of type T or of type S.

Definition 1.19. The vector

$$\tilde{\mathbf{D}}^L_f(s) := f_{tt}(s,0) - \frac{\langle f_{tt}(s,0), \Gamma'(s) \rangle_L}{\langle \Gamma'(s), \Gamma'(s) \rangle_L} \Gamma'(s) \tag{1.32}$$

is called the $\mathbb{L}^3$-*cuspidal direction vector* of f at $(s,0)$ if Γ is of type S (resp. type T) on $\mathcal{U}_f$.

By (c) of Definition 0.3, $\tilde{\mathbf{D}}^L_f(s)$ never vanishes, and can be considered as the projection of the vector $f_{tt}(s,0)$ to the normal plane of Γ at $\Gamma(s)$ in $\mathbb{L}^3$. Modifying the proof of Proposition 1.12, we obtain the following assertion:

Proposition 1.20. *$\Gamma'(s) \times \tilde{\mathbf{D}}^L_f(s)$ is a positive scalar multiple of $\nu^L(s,0)$.*

In particular, the following assertion holds:

Corollary 1.21. *Let f be a generalized cuspidal edge. Then, for each $s \in I$,*

(1) *$\hat{\nu}^L(s,0)$ is a space-like vector if $\Gamma'(s)$ is time-like or $\tilde{\mathbf{D}}^L_f(s)$ is time-like,*

(2) *$\hat{\nu}^L(s,0)$ is a time-like vector if $\Gamma'(s)$ and $\tilde{\mathbf{D}}^L_f(s)$ are both space-like, and*

(3) *$\hat{\nu}^L(s,0)$ is light-like if $\Gamma'(s)$ is space-like and $\tilde{\mathbf{D}}^L_f(s)$ is light-like.*

When s is the arclength parameter of Γ, the *curvature function* of Γ in $\mathbb{L}^3$ is defined by $\kappa^L(s) := |\Gamma''(s)|_L$. The following assertion is obvious (cf. Definition 1.18):

Proposition 1.22. *If the limiting normal vector $\hat{\nu}^L(s,0)$ is space-like (resp. time-like), then f is time-like (resp. space-like) on its regular set $\mathcal{U}_f \setminus \Sigma_f$ (cf. (0.1)).*

From now on, we consider the case that $\nu^L(s,t) := \hat{\nu}^L/|\hat{\nu}^L|_L$ is well-defined (that is, $(s,0)$ is a space-like or a time-like point), and assume that $\Gamma'(s)$ and $\tilde{\mathbf{D}}^L_f(s)$ are not light-like vectors. Then

$$\mathbf{D}^L_f(s) := \frac{\tilde{\mathbf{D}}^L_f(s)}{|\tilde{\mathbf{D}}^L_f(s)|_L} \tag{1.33}$$

is well-defined, which is called the *unit* $\mathbb{L}^3$*-cuspidal direction vector* of f in $\mathbb{L}^3$. We define the *singular curvature* $\kappa^L_s(s)$ and the *limiting normal curvature* $\kappa^L_\nu(s)$ so that

$$\Gamma''(s) = \kappa^L_s(s)\mathbf{D}^L_f(s) + \kappa^L_\nu(s)\nu^L(s,0), \tag{1.34}$$

where s is the arc-length parameter of Γ. Then the following assertion holds:

Proposition 1.23. *Suppose that $\Gamma''(s)$ does not vanish. Then $\kappa^L_\nu(s)$ is equal to zero if and only if so is $\kappa^E_\nu(s)$.*

Proof. We may assume that s is the arc-length parameter of Γ. By the definition of $\nu^L(s)$, we have $\langle f_{tt}(s,0), \nu^L(s)\rangle_L = \langle \Gamma'(s), \nu^L(s)\rangle_L = 0$. Since $\nu^E(s,0)$ points in the same direction as $E_3\nu^L(s,0)$, we can write $\nu^E(s,0) = a(s)E_3\nu^L(s,0)$, where $a(s)$ is a non-zero smooth function. By (1.34), we have

$$\begin{aligned}\operatorname{sgn}(\kappa^E_\nu(s)) &= \operatorname{sgn}(\Gamma''(s)\cdot\nu^E) = \operatorname{sgn}(a(s))\operatorname{sgn}\Big(\langle\Gamma''(s), \nu^L(s,0)\rangle_L\Big)\\ &= \operatorname{sgn}\Big(a(s)\langle\nu^{\rm L}(\mathrm{s},0), \nu^{\rm L}(\mathrm{s},0)\rangle_{\rm L}\Big)\operatorname{sgn}(\kappa^L_\nu(s)).\end{aligned}$$

Since $a(s) \neq 0$, the function $\langle\nu^L(s,0), \nu^L(s,0)\rangle_L$ does not vanish, proving the assertion. ☐

By imitating the proof of Proposition 1.15 the following assertion is obtained:

Proposition 1.24. *The section of the image of f by the normal plane P_s of Γ at $\Gamma(s)$ in $\mathbb{L}^3$ is a generalized cusp (cf. Definition A.1) whose cuspidal direction vector in the plane P_s is $\tilde{\mathbf{D}}^L_f(s)$. Moreover, $\tilde{\mathbf{D}}^L_f(s)$ lies in the osculating plane of Γ at $\Gamma(s)$ if and only if $\kappa^L_\nu(s) = 0$.*

In the introduction, we mentioned the order i_p at each singular point p. We now define this concept as follows: Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge. Since $f_t(s,0) = \mathbf{0}$, the identifier of causality can be expressed as

$$\Delta_L(s,t)(:= E^L(s,t)G^L(s,t) - F^L(s,t)^2) = t^r\varphi_s(t),$$

where $r(\geq 2)$ is a positive integer and $\varphi_s(t)$ is a certain smooth function with respect to t depending on s.

Definition 1.25. In the above setting, if $\varphi_s(0) \neq 0$ ($p := (s,0) \in \Sigma_f$), we set

$$i_p := r \tag{1.35}$$

and call it the *order* of the generalized cuspidal edge f at the singular point p. If $\Delta_L(s,t)$ is a flat function with respect to t (that is, $\partial^k\Delta_L(s,0)/\partial t^k$ vanish for all k), then we set $i_p := \infty$.

It can be easily checked that the *order* i_p is an invariant of f for each $p \in \Sigma_f$ (if the image of f lies in $\mathbb{E}^3$, the same concept can be defined but it is always equal to 2). In $\mathbb{L}^3$, the order i_p is generically 2, and might be greater than 2 in special cases. If i_p is an odd number for each p, then f changes its causal type along the the curve $t \mapsto f(s,t)$.

Proposition 1.26. *The order i_p ($p \in \Sigma_f$) is independent of the choice of the admissible coordinate system (cf. Definition 1.4) of f.*

Proof. Let (u,v) be an admissible coordinate change of $f(s,t)$, and let $J(s,t)$ be its Jacobian. Then the identifier of the causality of f with respect to (u,v) is $J^2\Delta_L$. Since $J(s,0) \neq 0$, the assertion is obtained by applying Lemma 1.5 with $\varphi := \Delta_L$. □

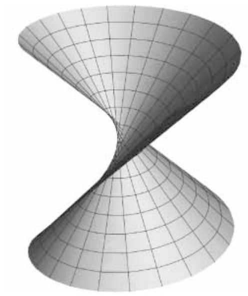

FIGURE 4. The null surface in $\mathbb{L}^3$ associated with a logarithmic spiral

We give here examples of cuspidal edges with $i_p = \infty$:

Example 1.27 (cf. [2])**.** We let $\gamma(s)$ be a locally convex regular curve in the xy-plane in $\mathbb{L}^3$ defined on an interval I whose curvature function κ and derivative $\kappa' = d\kappa/ds$ are both positive. We assume that s is the arc-length parameter of γ. We then denote by $\mathbf{n}(s)$ the left-ward unit normal vector field along γ, and consider a smooth map $f : I \times \mathbb{R} \to \mathbb{L}^3$ defined by

$$f(s,t) := (\gamma(s), 0) + t(\mathbf{n}(s), 1).$$

As shown in [2, Proposition 4.5], f gives a null wave front in $\mathbb{L}^3$, and the singular set Σ_f of f is given by

$$\Sigma_f = \left\{(s, \frac{1}{\kappa(s)})\,;\, s \in I\right\}(\subset I \times \mathbb{R}),$$

which consists only of cuspidal edge singular points. All points in the domain of definition of f are light-like points (that is, $\Delta_L(s,t)$ vanishes identically), and so $i_p = \infty$ for each point $p \in \Sigma_f$. Let q be a regular point which is not space-like nor time-like (in the domain of definition of f). If the matrix $\tilde{W}^L$ vanishes at q, then such a point q is called a *light-like umbilical point* (cf. [19]) which can be considered as "fake umbilical points". By the above construction, all regular points of f are fake umbilical points. For example, if γ is a logarithmic spiral, then the image of f is given as in Figure 4.

If a cuspidal edge singular point is not light-like, then it cannot be an accumulation point of umbilical points (cf. Appendix C). On the other hand, in the above example, we showed that light-like umbilical points (i.e. fake umbilics) can accumulate at a cuspidal edge singular point. Since our definition of umbilical points does not contain light-like points, the question of whether umbilical points can accumulate at a light-like cuspidal edge singular point naturally remains. In fact, as mentioned in the introduction, the authors know of any such examples.

2. General calculations

2.1. **General setting.** From now on, we will compute the Weingarten matrix of a generalized cuspidal edge f in $\mathbb{E}^3$ or $\mathbb{L}^3$. Let $\Gamma : I \to \mathbb{R}^3$ be a regular curve such that the origin $0 \in \mathbb{R}$ belongs to the interval I. Let $f : \mathcal{U}_f \to \mathbb{R}^3$ be a generalized cuspidal edge defined on $\mathcal{U}_f := I \times (-\delta, \delta)$, where $\delta > 0$. We will consider the following cases:

- (E) f lies in $\mathbb{E}^3$.
- (T) f lies in $\mathbb{L}^3$ and Γ is time-like.
- (S_s) f lies in $\mathbb{L}^3$, Γ is space-like and $\mathbf{D}_f^L$ is a space-like vector field along Γ.
- (S_t) f lies in $\mathbb{L}^3$, Γ is space-like, and $\mathbf{D}_f^L$ is a time-like vector field along Γ.
- (S_l) f lies in $\mathbb{L}^3$, Γ is space-like, and $\mathbf{D}_f^L(0)$ points in a light-like vector.
- (L_2) f lies in $\mathbb{L}^3$, $\Gamma'(0)$ is light-like, $\Gamma''(0) \neq \mathbf{0}$ and $i_{(0,0)}$ is equal to 2.
- (L_4) f lies in $\mathbb{L}^3$, Γ is light-like, and the order $i_{(s,0)}$ is equal to 4 for each $s \in I$.

We first give a framework which will be useful to compute the matrix $\tilde{W}^L$ except for the cases (L_2) and (L_4). We fix a symmetric bilinear form (,) on $\mathbb{R}^3$ and let $\{\mathbf{a}_0(s), \mathbf{a}_1(s), \mathbf{a}_2(s)\}$ be a frame field of $\mathbb{R}^3$ along $\Gamma(s)$ satisfying

$$(\mathbf{a}_i, \mathbf{a}_j) = \varepsilon_i \delta_{i,j} \qquad (i, j = 0, 1, 2), \tag{2.1}$$

where $\delta_{i,j}$ is Kronecker's delta, and $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. The cases (E) and (T) are discussed in this section, and the cases (S_s), (S_t) and (S_l) are discussed in the next section. Finally, the two remaining cases (L_2) and (L_4) are discussed in Sections 4 and 5.

In the following calculation, we assume that $\Gamma(s)$ is parametrized by arc-length with respect to the bilinear form (,), that is,

$$(\Gamma'(s), \Gamma'(s)) = \pm 1 \qquad (s \in I)$$

is assumed. Moreover, we set

$$\mathbf{a}_0(s) := \Gamma'(s). \tag{2.2}$$

There exists a 3×3 matrix $\mathcal{K}$ satisfying

$$(\mathbf{a}_0'(s),\ \mathbf{a}_1'(s),\ \mathbf{a}_2'(s)) = (\mathbf{a}_0(s),\ \mathbf{a}_1(s),\ \mathbf{a}_2(s))\mathcal{K}(s), \tag{2.3}$$

where $\Gamma'(s) = \mathbf{a}_0(s)$ for each $s \in I$ and $\mathbf{a}_i' := d\mathbf{a}_i/ds$ $(i = 0, 1, 2)$. By (2.1), $\mathcal{K}$ can be written in the following form

$$\mathcal{K} := \begin{pmatrix} 0 & -\varepsilon_0\varepsilon_1\kappa_1 & -\varepsilon_0\varepsilon_2\kappa_2 \\ \kappa_1 & 0 & -\varepsilon_1\varepsilon_2\Omega \\ \kappa_2 & \Omega & 0 \end{pmatrix}, \tag{2.4}$$

where κ_1, κ_2 and Ω are C^∞-functions on the interval I. In fact, if we write $K = (k_{ij})_{i,j=0,1,2}$, then

$$k_{ii} = (a_i', a_i) = \frac{(a_i, a_i)'}{2} = \frac{\varepsilon_i'}{2} = 0$$

holds for $i = 1, 2, 3$, and we have

$$\varepsilon_j k_{ji} = (a_i', a_j) = (a_i, a_j)' - (a_i, a_j') = -(a_i, a_j') = -\varepsilon_i k_{ij}$$

for $i \neq j$. In particular, we have $-\varepsilon_i \varepsilon_j k_{ij} = k_{ji}$, where $k_{10} := \kappa_1$, $k_{20} := \kappa_2$ and $k_{21} := \Omega$.

Since f is a generalized cuspidal edge along Γ, we can write (cf. Proposition B.1 in the appendix)

$$f(s,t) = \Gamma(s) + X(s,t)\mathbf{a}_1(s) + Y(s,t)\mathbf{a}_2(s). \tag{2.5}$$

Then the plane curve defined by $\mathbf{c}_s(t) := (X(s,t), Y(s,t))$ gives a cusp at $t = 0$ for each $s \in I$. In particular, $X(s,t)$ and $Y(s,t)$ can be written in the following form

$$X = \cos\theta A + \sin\theta B, \qquad Y = -\sin\theta A + \cos\theta B, \tag{2.6}$$

$$A(s,t) = \sum_{i=0}^{m} \frac{\alpha_i(s)}{i!} t^i + O(t^{m+1}), \qquad B(s,t) = \sum_{i=0}^{m} \frac{\beta_i(s)}{i!} t^i + O(t^{m+1}), \tag{2.7}$$

where $O(t^m)$ is a function written as $t^m \varphi(s,t)$ using a C^∞-function φ defined on a neighborhood of $(s, 0)$. In this setting,

$$\begin{aligned} &\alpha_0(s) = \alpha_1(s) = 0, \ \alpha_2(0) \neq 0, \\ &\beta_0(s) = \beta_1(s) = \beta_2(s) = 0. \end{aligned} \tag{2.8}$$

We remark that the angular function θ is a C^∞-function of one variable which vanishes identically if f is of type E, T, S_s or S_t like as Example 2.3. We will use the following asymptotic expansion of a given smooth function $\mu(s,t)$ with respect to t

$$\mu(s,t) = \mu_0(s) + \mu_1(s)t + \frac{\mu_2(s)}{2!}t^2 + \frac{\mu_3(s)}{3!}t^3 + O(t^4) \qquad (s \in I), \tag{2.9}$$

where $\mu_i(s)$ $(s = 1, 2, 3)$ are smooth functions on I as follows: If we think of $\mathbf{c}_s$ as lying in the Euclidean plane $\mathbb{E}^2$, we can write (cf. Fact A.2)

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t u \begin{pmatrix} \cos\lambda(s,u) \\ \sin\lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u)du. \tag{2.10}$$

In this case, we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \frac{1}{3}\begin{pmatrix} 0 \\ \mu_0 \end{pmatrix} t^3 + \frac{1}{8}\begin{pmatrix} -\mu_0^2 \\ \mu_1 \end{pmatrix} t^4 + \frac{1}{30}\begin{pmatrix} -3\mu_0\mu_1 \\ -\mu_0^3 + \mu_2 \end{pmatrix} t^5 + O(t^6) \tag{2.11}$$

and

$$\begin{aligned} &\alpha_2 = 1, \quad \alpha_3 = 0, \quad \alpha_4 = -3\mu_0^2, \quad \alpha_5 = -12\mu_0\mu_1, \\ &\beta_3 = 2\mu_0, \quad \beta_4 = 3\mu_1, \quad \beta_5 = 4(-\mu_0^3 + \mu_2). \end{aligned} \tag{2.12}$$

On the other hand, if we think of $\mathbf{c}_s$ as lying in the Lorentz-Minkowski plane $\mathbb{L}^2$, we can write (cf. (A.5))

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t u \begin{pmatrix} \cosh\lambda(s,u) \\ \sinh\lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u)du. \tag{2.13}$$

In this case, we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \frac{1}{3}\begin{pmatrix} 0 \\ \mu_0 \end{pmatrix} t^3 + \frac{1}{8}\begin{pmatrix} \mu_0^2 \\ \mu_1 \end{pmatrix} t^4 + \frac{1}{30}\begin{pmatrix} 3\mu_0\mu_1 \\ \mu_0^3 + \mu_2 \end{pmatrix} t^5 + O(t^6), \tag{2.14}$$

which implies that

$$\begin{aligned} &\alpha_2 = 1, \quad \alpha_3 = 0, \quad \alpha_4 = 3\mu_0^2, \quad \alpha_5 = 12\mu_0\mu_1, \\ &\beta_3 = 2\mu_0, \quad \beta_4 = 3\mu_1, \quad \beta_5 = 4(\mu_0^3 + \mu_2). \end{aligned} \tag{2.15}$$

Thus, in both of the two cases (i.e. (2.12) and (2.15)), we may assume

$$\alpha_3(s) = 0 \qquad (s \in I), \tag{2.16}$$

which is useful for simplifying future calculations. We have divided the expressions of A and B into two cases as above, where the difference between formulas (2.12) and (2.15) is simply the sign of α_4 and β_5. However, the coefficients α_4, α_5 and β_5

are not important, since they are not appear in the terms of asymptotic expansion of the matrix $\tilde{W}^E$ or $\tilde{W}^L$ that we will use in future discussions. Therefore, in the following discussion, we need to pay little attention to the above case separation of the definitions of A and B.

Remark 2.1. Let $P(s)$ be the plane passing through $f(s,0)$ which is spanned by two vectors $\mathbf{a}_1(s), \mathbf{a}_2(s) \in T_{f(s,0)}\mathbb{R}^3$, where $T_{f(s,0)}\mathbb{R}^3$ is the tangent space of $\mathbb{R}^3$ at $f(s,0)$. Then the section $C_s(\subset \mathbb{R}^3)$ of the image of f by the plane $P(s)$ is a generalized cusp in the plane $P(s)$, by Proposition B.1, and the map $\mathbf{c}_s(t) : t \mapsto (X(s,t), Y(s,t))$ can be identified with a parametrization of C_s. In our following representation formulas of generalized cuspidal edges f of type (E), (T), (S_s) and (S_t), the frame field $\{\mathbf{a}_0(s), \mathbf{a}_1(s), \mathbf{a}_2(s)\}$ is uniquely constructed by the same method for all generalized cuspidal edge f of the same type, and the image of $\hat{\mathbf{c}}_s : t \mapsto (A(s,t), B(s,t))$ is congruent to C_s in the plane $P(s)$. Thus, $\mu_i(s)$ $(s = 1, 2, 3)$ can be considered as invariants of $\mathbf{c}_s(t)$, and so they can be also considered as invariants of generalized cuspidal edge f.

The following example illustrates our setting:

Example 2.2. Let $\Gamma(s)$ be a regular curve in $\mathbb{E}^3$ parametrized by arc-length. If the curvature function $\kappa(s) := |\Gamma''(s)|_E$ does not vanish, then the principal normal vector $\mathbf{n}(s)$ and the bi-normal vector $\mathbf{b}(s)$ are defined along Γ. Any generalized cuspidal edges $f(s,t)$ along Γ can be expressed as (here X, Y are given by (2.6))

$$f(s,t) = \Gamma(s) + (\mathbf{n}(s), \mathbf{b}(s)) \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix}, \tag{2.17}$$

where

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} := \int_0^t u \begin{pmatrix} \cos\lambda(s,u) \\ \sin\lambda(s,u) \end{pmatrix} du, \quad \lambda := \int_0^t \mu(s,u)du,$$

and $\theta(s), \mu(s,u)$ are arbitrarily given smooth functions (cf. (A.2) in the appendix). This is just the formula of the first author [5], which enables us to produce generalized cuspidal edges from two geometric data θ, μ (the function $\theta(s)$ is called the *cuspidal angle* which is the angle from $\mathbf{n}(s)$ to the cuspidal direction vector $\mathbf{D}_f^E(s)$). Each coefficient of Taylor expansion of this formula can be considered as geometric invariants of f, since s is the arc-length parameter of Γ and t is the normalized half-arc-length parameter of sectional cusps. We set

$$\mathbf{a}_0 := \Gamma', \quad \mathbf{a}_1 := \mathbf{n}, \quad \mathbf{a}_2 := \mathbf{b}, \quad \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1, \quad \kappa_1 := \kappa, \ \kappa_2 := 0, \ \Omega := \tau,$$

where κ is the curvature function and τ is the torsion of the curve Γ. Then, (2.3) is reduced to the classical Frenet equation for regular space curves, and the formula (2.17) coincides with (2.5).

The above formula (2.17) is quite useful to construct concrete examples without solving any ordinary differential equations. However, if the curvature function $\kappa(s)$ vanishes for some s, it cannot make sense, since $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are not defined along Γ. In this case, we can give another formula producing all of generalized cuspidal edges along Γ as follows:

Example 2.3. Let $\Gamma(s)$ be a regular curve in $\mathbb{E}^3$ parametrized by the arc-length. We let f be a generalized cuspidal edge along Γ, which is written as in (2.5). Then we may set (cf. (1.23)) $\mathbf{a}_1(s) := D_f^E(s,0)$, which is a unit vector and is perpendicular to $\mathbf{a}_0 := \Gamma'$. By setting

$$\mathbf{a}_2(s) := \mathbf{a}_0(s) \times \mathbf{a}_1(s) (= \nu^L(s,0)),$$

$\mathbf{a}_0$, $\mathbf{a}_1, \mathbf{a}_2$ give an orthonormal frame field along Γ. Moreover, since f lies in $\mathbb{E}^3$ and $\mathbf{a}_1$ is the cuspidal direction vector, by setting $\theta = 0$, we can write (in this setting $X = A$ and $Y = B$ hold by (2.6))

$$f(s,t) = \Gamma(s) + A(s,t)\mathbf{a}_1(s) + B(s,t)\mathbf{a}_2(s) \tag{2.18}$$

and

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t \begin{pmatrix} \cos\lambda(s,u) \\ \sin\lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u)du, \tag{2.19}$$

where the function $\mu(s,t)$ can be considered as a geometric invariant of f. We set

$$\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1, \quad \theta = 0, \quad \kappa_1 := \kappa_s^E = \Gamma'' \cdot \mathbf{a}_1, \quad \kappa_2 := \kappa_\nu^E = \Gamma'' \cdot \mathbf{a}_2, \quad \Omega := -\omega^E.$$

Then (2.3) can be written as

$$\mathcal{F}' = \mathcal{F}\mathcal{K}, \qquad \mathcal{K} = \begin{pmatrix} 0 & -\kappa_s^E & -\kappa_\nu^E \\ \kappa_s^E & 0 & \omega^E \\ \kappa_\nu^E & -\omega^E & 0 \end{pmatrix}, \qquad \mathcal{F} := (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2). \tag{2.20}$$

The function ω^E is called the *cusp-directional torsion* of f in $\mathbb{E}^3$ (cf. [11]), which is an invariant of a generalized cusp f (in fact, $\omega^E = \tau - \theta'$ holds, where τ is the torsion of Γ and θ is the cuspidal angle defined in Example 2.2).

Conversely, if one gives data consisting of four functions

$$\Big(\kappa_s^E(s), \kappa_\nu^E(s), \omega^E(s), \mu(s,t)\Big) = \Big(k_1(s), k_2(s), \omega(s), m(s,t)\Big)$$

defined on an open interval containing $s = 0$, then, we solve the ordinary differential equation (2.20) with $\mathcal{F}$ as an unknown matrix-valued function under the initial condition that $\mathcal{F}(0)$ is the 3×3 identity matrix. By setting $\mathcal{F}(s) = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, $\Gamma(s) := \int_0^s \mathbf{a}_0(u)du$ and A, B as in (2.19), then the map f given by (2.18) is a generalized cuspidal edge along the curve Γ whose singular curvature, limiting normal curvature, cusp-directional torsion and the μ-function are κ_s^E, κ_ν^E, ω^E and μ, respectively. An advantage of this formula is that we can produce all of generalized cuspidal edges along Γ even when the curvature function of Γ admits zeros.

2.2. **Computation in the general setting.** For the sake of simplicity, we write

$$S(s) := \sin\theta(s), \qquad C(s) := \cos\theta(s).$$

By (2.5), we set

(2.21)
$$\begin{aligned} f_s = \mathbf{a}_0 &+ \frac{\varepsilon_0(-\varepsilon_1\kappa_1 C + \varepsilon_2\kappa_2 S)\mathbf{a}_0 + (\varepsilon_1\varepsilon_2\Omega - \theta')S\mathbf{a}_1 + (\Omega - \theta')C\mathbf{a}_2}{2}t^2 \\ &- \frac{\varepsilon_0\beta_3\,(\varepsilon_1\kappa_1 S + \varepsilon_2\kappa_2 C)\,\mathbf{a}_0}{6}t^3 \\ &+ \frac{(\beta_3(-\varepsilon_1\varepsilon_2\Omega + \theta')C + \beta_3' S)\,\mathbf{a}_1 + (\beta_3(\Omega - \theta')S + \beta_3' C)\mathbf{a}_2}{6}t^3 + O(t^4), \end{aligned}$$

(2.22)
$$\begin{aligned} f_t = (\mathbf{a}_1 C - S\mathbf{a}_2)t &+ \frac{\beta_3(S\mathbf{a}_1 + C\mathbf{a}_2)}{2}t^2 \\ &+ \frac{(\alpha_4 C + \beta_4 S)\mathbf{a}_1 + (-\alpha_4 S + \beta_4 C)\mathbf{a}_2}{6}t^3 + O(t^4), \end{aligned}$$

which imply that

$$(f_s, f_s) = \varepsilon_0 - (\varepsilon_1\kappa_1 C - \varepsilon_2\kappa_2 S)t^2 - \frac{\beta_3\left(\varepsilon_1\kappa_1 S + \varepsilon_2\kappa_2 C\right)}{3}t^3 + O(t^4), \tag{2.23}$$

$$(f_s, f_t) = \frac{1}{2}CS\big((-\varepsilon_1 + \varepsilon_2)\theta'\big)t^3 + O\left(t^4\right), \tag{2.24}$$

$$(f_t, f_t) = \left(\varepsilon_1 C^2 + \varepsilon_2 S^2\right)t^2 + \left(\varepsilon_0 - \varepsilon_2\right)\beta_3 CSt^3 + O\left(t^4\right). \tag{2.25}$$

If we set

$$\Delta := (f_s, f_s)(f_t, f_t) - (f_s, f_t)^2, \tag{2.26}$$

then, we have

$$\Delta = \varepsilon_0\left(\varepsilon_1 C^2 + \varepsilon_2 S^2\right)t^2 + \varepsilon_0\left(\varepsilon_1 - \varepsilon_2\right)\beta_3 CSt^3 + O\left(t^4\right) \tag{2.27}$$

and

$$\tilde{\nu}_E = f_s \times_E f_v = (S\mathbf{a}_1 + C\mathbf{a}_2)t + \frac{\beta_3}{2}(-C\mathbf{a}_1 + S\mathbf{a}_2)t^2 + O(t^3). \tag{2.28}$$

We next compute the second derivatives of f. We have

$$\begin{aligned} f_{ss} &= \kappa_1\mathbf{a}_1 + \kappa_2\mathbf{a}_2 \\ &- \frac{\varepsilon_0}{2}\Big(\Big(\varepsilon_2\kappa_2\left(\Omega - 2\theta'\right) + \varepsilon_1\kappa_1'\Big)C + \Big(\kappa_1(\varepsilon_2\Omega - 2\varepsilon_1\theta') - \varepsilon_2\kappa_2'\Big)S\Big)\mathbf{a}_0 t^2 \\ &- \frac{\Big(\varepsilon_1\varepsilon_2\Omega\left(\Omega - 2\theta'\right) + \theta'^2 + \varepsilon_0\varepsilon_1\kappa_1^2\Big)C + \Big(\theta'' - \varepsilon_0\varepsilon_2\kappa_1\kappa_2 - \varepsilon_1\varepsilon_2\Omega'\Big)S}{2}\mathbf{a}_1 t^2 \\ &+ \frac{\Big(-\theta'' - \varepsilon_0\varepsilon_1\kappa_1\kappa_2 + \Omega'\Big)C + \Big(-2\Omega\theta' + \theta'^2 + \varepsilon_0\varepsilon_2\kappa_2^2 + \varepsilon_1\varepsilon_2\Omega^2\Big)S}{2}\mathbf{a}_2 t^2 \\ &+ O(t^3), \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} f_{st} = &\Big(\varepsilon_0\Big(-\varepsilon_1\kappa_1 C + \varepsilon_2\kappa_2 S\Big)\mathbf{a}_0 + \Big(\varepsilon_1\varepsilon_2\Omega - \theta'\Big)S\mathbf{a}_1 + \Big(\Omega - \theta'\Big)C\mathbf{a}_2\Big)t \\ &+ \frac{1}{2}\Big(-\varepsilon_0\beta_3\left(\varepsilon_2\kappa_2 C + \varepsilon_1\kappa_1 S\right)\mathbf{a}_0 + \Big(\beta_3\left(\theta' - \varepsilon_1\varepsilon_2\Omega\right)C + \beta_3' S\Big)\mathbf{a}_1 \\ &\quad + \Big(\beta_3' C + \beta_3(\Omega - \theta')S\Big)\mathbf{a}_2\Big)t^2 + O(t^3), \end{aligned} \tag{2.30}$$

$$\begin{aligned} f_{tt} = &C\mathbf{a}_1 - S\mathbf{a}_2 + \beta_3(S\mathbf{a}_1 + C\mathbf{a}_2)t \\ &+ \frac{1}{2}\Big((\alpha_4 C + \beta_4 S)\mathbf{a}_1 + (\beta_4 C - \alpha_4 S)\mathbf{a}_2\Big)t^2 + O(t^3). \end{aligned} \tag{2.31}$$

By (0.3), we have

$$\begin{aligned} d^C((s,0)) &= -\det(f_s, f_{ss}, f_{tt})\Big|_{t=0} \\ &= -\det(\mathbf{a}_0, \kappa_1\mathbf{a}_1 + \kappa_2\mathbf{a}_2, C\mathbf{a}_1 - S\mathbf{a}_2) = \kappa_1 S + \kappa_2 C. \end{aligned} \tag{2.32}$$

Using (2.28) and (2.29), we have

$$\tilde{L} = (S\kappa_1 + C\kappa_2)t + \frac{\beta_3}{2}\left(-\kappa_1 C + \kappa_2 S\right)t^2 \tag{2.33}$$
$$+ \frac{1}{6}\Big(\left(-\beta_4\kappa_1 + \alpha_4\kappa_2\right)C + \left(\alpha_4\kappa_1 + \beta_4\kappa_2\right)S\Big)t^3$$
$$+ \Big(\left(\varepsilon_1\varepsilon_2 - 1\right)\Omega\theta' - \varepsilon_0\varepsilon_1\kappa_1^2 + \varepsilon_0\varepsilon_2\kappa_2^2\Big)CSt^3$$
$$+ \frac{1}{2}\Big(-\theta'' - 2\varepsilon_0\varepsilon_1\kappa_1\kappa_2 + \Omega'\Big)C^2t^3$$
$$+ \frac{1}{2}\Big(-\theta'' + 2\varepsilon_0\varepsilon_2\kappa_1\kappa_2 + \varepsilon_1\varepsilon_2\Omega'\Big)S^2t^3.$$

Similarly, using (2.28), (2.30), (2.31) with the relation $C^2 + S^2 = 1$, we have

$$\tilde{M} = \Big((\Omega(C^2 + \varepsilon_1\varepsilon_2 S^2) - \theta'\Big)t^2 + \frac{1}{2}\Big(2(1 - \varepsilon_1\varepsilon_2)\beta_3\Omega CS + \beta_3'\Big)t^3, \tag{2.34}$$

$$\tilde{N} = \frac{\beta_3}{2}t^2 + \frac{\beta_4}{3}t^3 + O(t^4). \tag{2.35}$$

Let U be a domain of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^3$ a C^∞-map. A point $p \in U$ is called a *cuspidal cross cap singular point* of f if there exist local diffeomorphisms φ and Ψ on $\mathbb{R}^2$ and $\mathbb{R}^3$ satisfying

$$\varphi(p) = (0,0), \quad \Psi \circ f(p) = (0,0,0), \quad \Psi \circ f \circ \varphi^{-1}(s,t) = f_2,$$

where f_2 is the standard map of cuspidal cross caps as in Example 1.7. Cuspidal cross cap singular points are known as a typical singular points appeared on generalized cuspidal edges as well as cuspidal edge singular points. The following is an application of the previous calculations.

Lemma 2.4. *The point $(s_0, 0)$ $(s_0 \in I)$ is a cuspidal edge singular point (resp. a cuspidal cross cap singular point) of f if and only if $\mu_0(s_0) \neq 0$ (resp. $\mu_0(s_0) = 0$ and $\mu_0'(s_0) \neq 0$).*

Proof. Let P_0 be the plane passing through $\Gamma(s_0)$ spanned by $\mathbf{a}_1(s_0)$ and $\mathbf{a}_2(s_0)$. Then $\mu_0(s_0) \neq 0$ if and only if the section of the image of f by the plane P_0 is a cusp. So the conclusion follows from Proposition 1.10.

On the other hand $(s_0, 0)$ $(s_0 \in I)$ is a cuspidal cross cap singular point if and only if $\varphi(s_0) = 0$ and $\varphi'(s_0) \neq 0$ (cf. [4] or [16, Section 2]), where

$$\varphi(s) := \det(\Gamma'(s), \check{\nu}_E(s,0), (\check{\nu}_E)_t(s,0)) \qquad (s \in I)$$

and (cf. (2.28))

$$\check{\nu}_E := \frac{\hat{\nu}_E}{t} = (S\mathbf{a}_1 + C\mathbf{a}_2) + \frac{\beta_3}{2}(-C\mathbf{a}_1 + S\mathbf{a}_2)t + O(t^2) \tag{2.36}$$

gives a non-vanishing normal vector field along f. We set

$$\mathbf{w}_1 := S\mathbf{a}_1 + C\mathbf{a}_2, \qquad \mathbf{w}_2 = -C\mathbf{a}_1 + S\mathbf{a}_2.$$

Since $\Gamma' = \mathbf{a}_0$, three vectors $\Gamma'(s), \mathbf{w}_1(s)$ and $\mathbf{w}_2(s)$ are linearly independent for each $s \in I$. So we have

$$\varphi(s) = \frac{\beta_3(s)}{2}\det(\Gamma'(s), \mathbf{w}_1(s), \mathbf{w}_2(s)) = \mu_0(s)\psi(s),$$
$$\psi(s) = \det(\Gamma'(s), \mathbf{w}_1(s), \mathbf{w}_2(s)).$$

Since $\psi(s) > 0$ for $s \in I$, the condition $\varphi(s_0) = 0$ is equivalent to $\mu_0(s_0) = 0$. Moreover, if $\mu_0(s_0) = 0$, then we have

$$\varphi'(s_0) = \mu_0'(s_0)\psi(s_0),$$

which proves the assertion for cuspidal cross caps. □

2.3. **Generalized cuspidal edges of type E.** Let $\Gamma : I \to \mathbb{E}^3$ be a regular curve parametrized by the arc-length. We use the setting given in Example 2.3, that is,

$$\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1, \quad \theta = 0, \quad \kappa_1 = \kappa_s^E, \quad \kappa_2 = \kappa_\nu^E.$$

Then, we have $C = 1$, $S = 0$ and $(X, Y) = (A, B)$. Since (A, B) satisfies (2.12), the equations (2.23), (2.24), (2.25) and (2.27) imply

$$E = 1 - \kappa_s^E t^2 - \frac{\beta_3 \kappa_\nu^E}{3} t^3 + O(t^4), \quad F = O(t^4), \quad G = t^2 + O(t^4) \tag{2.37}$$

and

$$\Delta_E (= \Delta) = t^2 + O(t^4). \tag{2.38}$$

Moreover, (2.32) implies that (we set $p := (s, 0)$)

$$\operatorname{sgn}(d^C(p)) = \operatorname{sgn}(\kappa_\nu^E(s)). \tag{2.39}$$

On the other hand, (2.33), (2.34) and (2.35) are reduced to

$$\tilde{L} = \kappa_\nu^E t - \frac{\beta_3 \kappa_s^E}{2} t^2 + \frac{1}{6}(-\beta_4 \kappa_s^E + \alpha_4 \kappa_\nu^E - 6\kappa_s^E \kappa_\nu^E - 3(\omega^E)')t^3 + O(t^4), \tag{2.40}$$

$$\tilde{M} = -\omega^E t^2 + \frac{\beta_3'}{2} t^3 + O\left(t^4\right), \quad \tilde{N} = \frac{1}{2} t^2 \beta_3 + \frac{1}{3} t^3 \beta_4 + O\left(t^4\right).$$

In this situation, we may assume α and β can be written as (2.12). As pointed out in [8] (see also Appendix B), $\mu_0(s), \mu_1(s)$ and $\mu_2(s)$ are geometric invariants of the generalized cuspidal edges in $\mathbb{E}^3$. By (2.37) and (2.40), we have

$$\begin{aligned}
(\tilde{W}^E :=) &\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{L} \end{pmatrix} \\
&= \begin{pmatrix} \kappa_\nu^E t^3 - \frac{\beta_3 \kappa_s^E}{2} t^4 + O\left(t^5\right) & -\omega^E t^4 + O\left(t^5\right) \\ -\omega^E t^2 + O\left(t^3\right) & \frac{1}{2} t^2 \beta_3 + \frac{1}{3} t^3 \beta_4 + O\left(t^4\right) \end{pmatrix} \\
&= \begin{pmatrix} \kappa_\nu^E t^3 - \kappa_s^E \mu_0 t^4 + O\left(t^5\right) & -\omega^E t^4 + O\left(t^5\right) \\ -\omega^E t^2 + O\left(t^3\right) & \mu_0 t^2 + \mu_1 t^3 + O\left(t^4\right) \end{pmatrix},
\end{aligned}$$

and then asymptotic behaviors of the Gaussian curvature and the mean curvature are given by

$$K^E = \frac{\kappa_\nu^E \mu_0}{t} - \kappa_s^E \mu_0^2 + \kappa_\nu^E \mu_1 - (\omega^E)^2 + O(t), \quad H^E = \frac{\mu_0}{2t} + \frac{\mu_1 + \kappa_\nu^E}{2} + O(t). \tag{2.41}$$

Moreover, since $\tilde{W}^E$ is a triangle matrix modulo $O(t^4)$-term, by (2.38), the diagonal components of $\tilde{W}^E$ divided by t^3 give the following asymptotic expansions of the principal curvatures of f:

$$\lambda_1 = \frac{\mu_0}{t} + \mu_1 + O(t), \qquad \lambda_2 = \kappa_\nu^E + O(t). \tag{2.42}$$

Summarizing these computations, the following facts are obtained, which have been already known (cf. [5], [10], [15] and [12]).

Fact 2.5. *Let $f : \mathcal{U}_f \to \mathbb{E}^3$ be a generalized cuspidal edge along a regular curve Γ. Then the following assertions hold:*

1. *The sign $\sigma^C(p)$ $(p := (s, 0))$ coincides with the sign of $\kappa_\nu^E(s)$.*
2. *The mean curvature H^E is unbounded near $p \in \Sigma_f$ if p is a cuspidal edge singular point.*
3. *If $p := (s, 0)$ is a cuspidal edge of non-vanishing limiting normal curvature, then the Gaussian curvature K^E is unbounded near p and takes different signs on each side of Σ_f.*

(4) *The two principal curvatures of f are real-valued on $\mathcal{U}_f \setminus \Sigma_f$, and one of them is bounded. Moreover, if p is a cuspidal edge singular point, the other principal curvature is unbounded.*
(5) *If $\mu_0(s) \neq 0$ (that is, p is a cuspidal edge singular point), the umbilical points of f never accumulate at p.*

Proof. (1) follows from (2.39). Suppose that $(s, 0)$ is a cuspidal edge singular point. Since $\mu_0(s) \neq 0$ (cf. Lemma 2.4), the second equation of (2.41) implies that H^E is unbounded, proving (2). On the other hand, (3) follows from the first equation of (2.41), since $\mu_0 \neq 0$ and $\kappa_\nu^L \neq 0$. Finally (4) and (5) follow from (2.42). □

2.4. **Generalized cuspidal edges of type T.** Consider a time-like regular curve $\Gamma : I \to \mathbb{L}^3$. Let f be a generalized cuspidal edge along Γ. Since Γ is time-like, it can be parametrized by arc-length, and can set (cf. (1.33))

$$\mathbf{a}_0(s) := \Gamma'(s), \qquad \mathbf{a}_1(s) := \mathbf{D}_f^L(s, 0), \tag{2.43}$$

which are both unit vector fields. By setting

$$\mathbf{a}_2(s) := \mathbf{a}_0(s) \times_L \mathbf{a}_1(s)(= \nu^L(s, 0)), \tag{2.44}$$

$(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ is an orthonormal frame field of $\mathbb{L}^3$ along Γ. By (2.5) and (2.6), we can write

$$f(s, t) = \Gamma(s) + A(s, t)\mathbf{a}_1(s) + B(s, t)\mathbf{a}_2(s). \tag{2.45}$$

Since $\mathbf{a}_1(s)$ points in the $\mathbb{L}^3$-cuspidal direction at $f(s, 0)$, we may set $\theta(s) = 0$. Since the normal plane of Γ is space-like, we can write

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t u \begin{pmatrix} \cos \lambda(s,u) \\ \sin \lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u) du,$$

and the function $\mu(s, t)$ can be considered as a geometric invariant of f (cf. Remark 2.1), that is, (2.12) holds. This corresponds to the case that

$$\varepsilon_0 = -1, \quad \varepsilon_1 = \varepsilon_2 = 1, \quad C = 1, \quad S = 0.$$

By (1.34), we have $\kappa_1 = \kappa_s^L$ and $\kappa_2 = \kappa_\nu^L$. The function $\omega^L := -\Omega$ should be called the *cusp-directional torsion* of f in $\mathbb{L}^3$. Then (2.3) can be written as

$$\mathcal{F}' = \mathcal{F}\mathcal{K}, \qquad \mathcal{K} = \begin{pmatrix} 0 & \kappa_s^L & \kappa_\nu^L \\ \kappa_s^L & 0 & \omega^L \\ \kappa_\nu^L & -\omega^L & 0 \end{pmatrix}, \quad \mathcal{F} := (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2). \tag{2.46}$$

Remark 2.6. If one gives data consisting of four functions

$$(\kappa_s^L, \kappa_\nu^L, \omega^L, \mu) := (k_1, k_2, \Omega, m)$$

defined on an open interval containing $s = 0$, then, like as in Example 2.3, a generalized cuspidal edge of type T is obtained by solving the ordinary differential equation (2.46) with $\mathcal{F}$ as an unknown matrix-valued function.

Here, (2.23), (2.24) and (2.25) are reduced to

$$E^L = -1 - \kappa_s^L t^2 - \frac{\beta_3 \kappa_\nu^L}{3} + O(t^4), \quad F^L = O(t^4), \quad G^L = t^2 + O(t^4) \tag{2.47}$$

and (2.27) and (2.32) imply

$$\Delta_L = -t^2 + O(t^4), \qquad d^C(p) = \kappa_\nu^L. \tag{2.48}$$

On the other hand, (2.33), (2.34) and (2.35) are reduced to

$$(2.49)\qquad \tilde{L} = \kappa_\nu^L t - \frac{\beta_3\kappa_s^L}{2}t^2 + \frac{1}{6}(-\beta_4\kappa_s^L + \alpha_4\kappa_\nu^L + 6\kappa_s^L\kappa_\nu^L - 3(\omega^L)')t^3 + O(t^4),$$
$$\tilde{M} = -\omega^L t^2 + \frac{\beta_3'}{2}t^3 + O\left(t^4\right), \quad \tilde{N} = \frac{1}{2}t^2\beta_3 + \frac{1}{3}t^3\beta_4 + O\left(t^4\right).$$

In this situation, we may assume α and β are written as (2.15). Here $\mu_0(s), \mu_1(s)$ and $\mu_2(s)$ are geometric invariants of generalized cuspidal edge f (cf. Remark 2.1). By (2.47) and (2.49), we have

$$\tilde{W}^L = \begin{pmatrix} \kappa_\nu^L t^3 - \frac{\beta_3\kappa_s^L}{2}t^4 + O\left(t^5\right) & -\omega^L t^4 + O\left(t^5\right) \\ \omega^L t^2 + O\left(t^3\right) & -\frac{1}{2}t^2\beta_3 - \frac{1}{3}t^3\beta_4 + O\left(t^4\right) \end{pmatrix}$$
$$= \begin{pmatrix} \kappa_\nu^L t^3 - \kappa_s^L\mu_0 t^4 + O\left(t^5\right) & -\omega^L t^4 + O\left(t^5\right) \\ \omega^L t^2 + O\left(t^3\right) & -\mu_0 t^2 - \mu_1 t^3 + O\left(t^4\right) \end{pmatrix},$$

and then the asymptotic behavior of the Gaussian curvature and the mean curvature are given by

$$(2.50)\qquad K^L = -\frac{\kappa_\nu^L\mu_0}{t} + \kappa_s^L\mu_0^2 - \kappa_\nu^L\mu_1 - (\omega^L)^2 + O(t),$$

$$(2.51)\qquad H^L = \frac{\mu_0}{2t} + \frac{\mu_1 - \kappa_\nu^L}{2} + O(t).$$

Moreover, since $\tilde{W}^L$ is a triangle matrix modulo $O(t^4)$-term, the diagonal components of $\tilde{W}^L$ divided by $-t^3$ give the following asymptotic expansions of the two principal curvatures of f:

$$(2.52)\qquad \lambda_1 = \frac{\mu_0}{t} + \mu_1 + O(t), \qquad \lambda_2 = -\kappa_\nu^L + O(t).$$

Summarizing these computations, we obtain the following:

Theorem 2.7. *Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge along a regular curve Γ of type T. Then, for each $p := (s, 0)$ $(s \in I)$, there exists a neighborhood V_p satisfying the following properties:*

1. *f is time-like at p.*
2. *The sign $\sigma^C(p)$ coincides with the sign of $\kappa_\nu^L(s)$, and the order i_p is 2.*
3. *If p is a cuspidal edge singular point with $\sigma^C(p) \neq 0$, then the mean curvature H^L is unbounded on V_p. Moreover, the Gaussian curvature K^L is unbounded and takes different signs on each side of Σ_f around p.*
4. *If p is a cuspidal edge singular point, then the two principal curvature functions of f are real-valued on V_p, and one of the principal curvature function is bounded.*
5. *Moreover, if p is a cuspidal edge singular point, then the other principal curvature function is unbounded. In particular, the umbilical points and quasi-umbilical points never accumulate at p.*

Proof. Since $\Gamma'(s_0)$ is a time-like vector in $\mathbb{L}^3$, the limiting tangent plane is time-like obviously, and (1) holds. (2) follows from the second equation of (2.48). We now assume that $p(= (s, 0))$ is a cuspidal edge singular point. Since $\mu_0(s) \neq 0$ holds, the second equation of (2.50) implies that H^L is unbounded on a neighborhood of p. By (2.50) and (2.51), we have

$$(H^L)^2 - K^L = \frac{1}{4t^2}\Big(\mu_0^2 + O(t)\Big).$$

This with the fact $\mu_0(s) \neq 0$ implies that the two principal curvatures are real-valued. Moreover, if $\sigma^C(p) \neq 0$, then $\kappa_\nu^L(s)$ does not vanish, which implies the coefficient $\kappa_\nu^L\mu_0$ of the first term of (2.50) does not vanish. So K^L is unbounded and takes different signs on each side of Σ_f around p. We have proved (3) and (4). Finally (5) follows from (2.52). □

3. Generalized cuspidal edges of type S

In this section, we fix a generalized cuspidal edge in $\mathbb{L}^3$ along a space-like regular curve $\Gamma : I \to \mathbb{L}^3$. Let f be a generalized cuspidal edge along Γ. Since Γ is space-like, Γ can be parametrized by the arc-length.

3.1. **Generalized cuspidal edge of types S_s and S_t.** We first consider the case that $\mathbf{D}_f^L(s)$ does not point in a light-like direction for any $s \in I$. We then set

$$\mathbf{a}_0(s) := \Gamma'(s), \qquad \mathbf{a}_1(s) := \mathbf{D}_f^L(s,0)$$

and

$$\mathbf{a}_2(s) := \mathbf{a}_0(s) \times_L \mathbf{a}_1(s) (= \nu^L(s,0)).$$

Then $\mathbf{a}_0$, $\mathbf{a}_1, \mathbf{a}_2$ give an orthonormal frame field in $\mathbb{L}^3$ along Γ.

Then (2.5) and (2.6) are reduced to

$$f(s,t) = \Gamma(s) + A(s,t)\mathbf{a}_1(s) + B(s,t)\mathbf{a}_2(s).$$

Since the image of f lies in $\mathbb{L}^3$ and $\mathbf{a}_1$ points in the cuspidal direction, we may set θ is identically zero. Since the plane perpendicular to Γ' is time-like, we can write

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t u \begin{pmatrix} \cosh \lambda(s,u) \\ \sinh \lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u)du.$$

In particular, (2.15) holds. The function $\mu(s,t)$ can be considered as an geometric invariant of f (cf. Remark 2.1). Since $\mathbf{D}_f^L(s,0)$ points in a space-like (resp. time-like) direction, we can set

$$\varepsilon_0 = 1,\ \varepsilon_1 = \varepsilon,\ \varepsilon_2 = -\varepsilon,\ C = 1,\ S = 0, \quad \kappa_1 = \kappa_s^L, \quad \kappa_2 = \kappa_\nu^L,$$

where

$$\varepsilon := \begin{cases} 1 & (\mathbf{D}_f^L \text{ is space-like}), \\ -1 & (\mathbf{D}_f \text{ is time-like}). \end{cases}$$

In particular, $\varepsilon = 1$ corresponds to the case (S_s) and $\varepsilon = -1$ corresponds to the case (S_t). Thus f is a generalized cuspidal edge of types S_s (resp. S_t) if $\varepsilon > 0$ (resp. $\varepsilon < 0$). We set

$$\omega^L := -\Omega$$

and call it the *torsional curvature* of f along Σ_f. Then (2.3) is reduced to

$$\mathcal{F}' = \mathcal{F}\mathcal{K}, \qquad \mathcal{K} = \begin{pmatrix} 0 & -\varepsilon\kappa_s^L & \varepsilon\kappa_\nu^L \\ \kappa_s^L & 0 & \omega^L \\ \kappa_\nu^E & -\omega^L & 0 \end{pmatrix}, \quad \mathcal{F} := (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2). \tag{3.1}$$

Remark 3.1. If one gives data consisting of four functions

$$(\kappa_s^E, \kappa_\nu^E, \omega^E, \mu) := (k_1, k_2, \Omega, m)$$

defined on an open interval containing $s = 0$, then, like as Example 2.3, solving the ordinary differential equation (3.1) with $\mathcal{F}$ as the unknown matrix-valued function, we can construct generalized cuspidal edges of type S_s or S_t.

Here, (2.23), (2.24) and (2.25) are reduced to

$$E^L = 1 - \varepsilon\kappa_s^L t^2 + \frac{\varepsilon\beta_3\kappa_\nu^L}{3} t^3 + O(t^4), \quad F^L = O(t^4), \quad G^L = \varepsilon t^2 + O(t^4) \tag{3.2}$$

and (2.27) implies

$$\Delta_L = \varepsilon t^2 + O(t^4). \tag{3.3}$$

Also, (2.32) implies

$$\operatorname{sgn}(d^C((s,0))) = \operatorname{sgn}(\kappa_\nu^L(s,0)). \tag{3.4}$$

On the other hand, (2.33), (2.34) and (2.35) yield

$$\tilde{L} = \kappa_\nu^L t - \frac{\beta_3 \kappa_s^L}{2} t^2 + \frac{1}{6}\Big(- \beta_4 \kappa_s^L + \alpha_4 \kappa_\nu^L - 6\varepsilon \kappa_s^L \kappa_\nu^L - 3(\omega^L)' \Big) t^3 + O(t^4), \tag{3.5}$$
$$\tilde{M} = -\omega^L t^2 + \frac{\beta_3'}{2} t^3 + O\left(t^4\right), \quad \tilde{N} = \frac{1}{2} t^2 \beta_3 + \frac{1}{3} t^3 \beta_4 + O\left(t^4\right).$$

In this situation, we may assume α and β can be written as (2.15). Then $\mu_0(s), \mu_1(s)$ and $\mu_2(s)$ are invariants of the generalized cuspidal edges in $\mathbb{L}^3$ (cf. Remark 2.1). By (3.2) and (3.5), we have

$$\begin{aligned} \tilde{W}^L &= \begin{pmatrix} \varepsilon \kappa_\nu^L t^3 - \varepsilon \frac{\beta_3 \kappa_s^L}{2} t^4 + O\left(t^5\right) & -\varepsilon \omega^L t^4 + O\left(t^5\right) \\ -\omega^L t^2 + O\left(t^3\right) & \frac{1}{2} t^2 \beta_3 + \frac{1}{3} t^3 \beta_4 + O\left(t^4\right) \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon \kappa_\nu^L t^3 - \varepsilon \kappa_s^L \mu_0 t^4 + O\left(t^5\right) & -\varepsilon \omega^L t^4 + O\left(t^5\right) \\ -\omega^L t^2 + O\left(t^3\right) & \mu_0 t^2 + \mu_1 t^3 + O\left(t^4\right) \end{pmatrix}. \end{aligned}$$

In particular, the Gaussian curvature and the mean curvature can be computed as

$$K^L = -\frac{\kappa_\nu^L \mu_0}{t} + \Big(\kappa_s^L \mu_0^2 - \kappa_\nu^L \mu_1 + (\omega^L)^2\Big) + O(t), \tag{3.6}$$

$$H^L = \frac{\varepsilon \mu_0}{2t} + \frac{1}{2}\left(\varepsilon \mu_1 + \kappa_\nu^L\right) + O(t). \tag{3.7}$$

Since $\tilde{W}^L$ is a triangle matrix modulo $O(t^4)$-term, the diagonal components of $\tilde{W}^L$ give the following asymptotic expansion of the two principal curvatures of f;

$$\lambda_1(s,t) = \frac{\varepsilon \mu_0(s)}{t} + \varepsilon \mu_1(s) + O(t), \qquad \lambda_2(s,t) = \kappa_\nu^L(s) + O(t). \tag{3.8}$$

By Lemma 2.4, the singular point $(s,0)$ is a cuspidal edge singular point of f if and only if $\mu_0(s) \neq 0$. Summarizing these computations, we obtain the following:

Theorem 3.2. *Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge along Γ of type S such that its $\mathbb{L}^3$-cuspidal direction $\mathbf{D}_f^L$ never points in light-like directions along Γ. Then, for each $p := (s,0)$ $(s \in I)$, there exists a neighborhood $V_p (\subset \mathcal{U}_f)$ satisfying the following properties:*

1. *The sign $\sigma^C(p)$ coincides with the sign of $\kappa_\nu^L(s)$, and the order i_p is 2.*
2. *f is space-like (resp. time-like) at p if and only if $\mathbf{D}_f^L(s)$ $(s \in I)$ points in a space-like (resp. time-like) direction.*
3. *If p is a cuspidal edge singular point, then the mean curvature H^L is unbounded on V_p. Moreover, if $\sigma^C(p) \neq 0$, then the Gaussian curvature K^L is unbounded and takes different signs on each side of Σ_f around p.*
4. *If p is a cuspidal edge singular point, then the two principal curvature functions of f are real-valued on V_p, and one of the principal curvature function is bounded.*
5. *Moreover, if p is a cuspidal edge singular point, then the other principal curvature function is unbounded. In particular, the umbilical points and quasi-umbilical points never accumulate at p.*

Proof. (1) follows from (3.4). By (2.44), it is obvious that $\mathbf{a}_2(s)$ points in the normal direction of f at p and is time-like (resp. space-like) if and only if $\mathbf{D}_f^L(s)$ points in a space-like (resp. time-like) direction. So (2) is obtained. The assertions (3), (4) and (5) follow from by the same reason in the proofs of (3), (4) and (5) of Theorem 2.7. □

3.2. **Generalized cuspidal edge of types** S_l**.** Here we consider the case that $\tilde{\mathbf{D}}^L_f(s_0)$ points in a light-like direction for some $s = s_0 \in I$. Since $\Gamma(s)$ is parametrized by the arc-length parameter,

$$\mathbf{a}_0(s) := \Gamma'(s) \qquad (s \in I)$$

is a unit space-like vector field along Γ. We choose a space-like vector field $\mathbf{a}_1$ and a time-like vector field $\mathbf{a}_2$ along Γ so that $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ forms an orthonormal frame field such that

$$1 = (\mathbf{a}_1, \mathbf{a}_1)_L = -(\mathbf{a}_2, \mathbf{a}_2)_L, \qquad \mathbf{a}_2 := \mathbf{a}_0 \times_L \mathbf{a}_1.$$

Without loss of generality, we assume that $0 \in I$ and $s_0 = 0$. Then the $\mathbb{L}^3$-cuspidal direction $\tilde{\mathbf{D}}^L_f(0)$ points in a light-like direction, which is the case that we have not discussed yet. Since $\tilde{\mathbf{D}}^L_f(0)$ is a light-like vector, it is parallel to one of the following two vectors

$$\mathbf{a}_1(0) + \mathbf{a}_2(0), \qquad \mathbf{a}_1(0) - \mathbf{a}_2(0).$$

If the latter case occurs, by replacing $\Gamma(s)$ by $\Gamma(-s)$, the three vector fields turn to be $-\Gamma'(-s)$, $\mathbf{a}_1(-s)$, $-\mathbf{a}_2(-s)$. So we may assume that $\tilde{\mathbf{D}}^L_f(0)$ is parallel to $\mathbf{a}_1(0) + \mathbf{a}_2(0)$. Moreover, since $\Gamma', \mathbf{a}_1, \mathbf{a}_2$ give the same orientation as $\Gamma', -\mathbf{a}_1, -\mathbf{a}_2$, we may also assume that

$$\tilde{\mathbf{D}}^L_f(0) = \frac{\mathbf{a}_1(0) + \mathbf{a}_2(0)}{\sqrt{2}}.$$

We then apply Proposition B.1 in the appendix for $(s,t) = (0,0)$, and can assume that f has the following expression

$$f(s,t) = \Gamma(s) + X(s,t)\mathbf{a}_1(t) + Y(s,t)\mathbf{a}_2(t), \tag{3.9}$$

where $X(s,t)$ and $Y(s,t)$ are smooth functions written as (2.6) and (2.7). We can write

$$\begin{pmatrix} A(s,t) \\ B(s,t) \end{pmatrix} = \int_0^t u \begin{pmatrix} \cos\lambda(s,u) \\ \sin\lambda(s,u) \end{pmatrix} du, \qquad \lambda(s,t) := \int_0^t \mu(s,u)du,$$

that is, (2.12) holds. Unlike the previous cases (E), (T), (S_s) and (S_t), the function $\mu(s,t)$ cannot be considered as a geometric invariant of f. One reason is that the two vector fields $\mathbf{a}_0$ and $\mathbf{a}_1$ are not uniquely determined from f, and the other reason is that we use the pair $(\cos\lambda, \sin\lambda)$ not $(\cosh\lambda, \sinh\lambda)$. We set $\mathbf{a}_0' = \kappa_1\mathbf{a}_1 + \kappa_2\mathbf{a}_2$. Since $(\mathbf{a}_1, \mathbf{a}_1) = 1$, we can write $\mathbf{a}_1' = -\kappa_1\mathbf{a}_1 + \Omega\mathbf{a}_2$, where Ω is a smooth function. In the setting of the previous section, this is the case that

$$\varepsilon_0 = \varepsilon_1 = 1,\ \varepsilon_2 = -1, \qquad C = \cos\theta,\ S = \sin\theta,$$

and

$$\theta(0) = \frac{\pi}{4}. \tag{3.10}$$

Then (2.3) can be written as

$$\mathcal{F}' = \mathcal{F}\mathcal{K}, \qquad \mathcal{K} = \begin{pmatrix} 0 & -\kappa_1 & \kappa_2 \\ \kappa_1 & 0 & \Omega \\ \kappa_2 & \Omega & 0 \end{pmatrix}, \qquad \mathcal{F} := (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2), \tag{3.11}$$

which can be considered as an ordinary differential equation with $\mathcal{F}$ as an unknown matrix-valued function, and we can produce all of cuspidal edges of type (S_l) along Γ like as the previous cases. If we set

$$\delta_1 := \frac{\kappa_1(0) + \kappa_2(0)}{\sqrt{2}}, \qquad \delta_2 := \frac{-\kappa_1(0) + \kappa_2(0)}{\sqrt{2}},$$

then

$$E^L(0,t) = 1 - \delta_1 t^2 + \frac{\beta_3(0)\delta_2}{3}t^3 + O(t^4), \quad F^L(0,t) = -\frac{\theta'(0)}{2}t^3 + O(t^4),$$
$$G^L(0,t) = \beta_3(0)t^3 + O(t^4)$$

and so

$$\Delta_L(0,t) = \beta_3(0)t^3 + O(t^4). \tag{3.12}$$

Moreover, we have (cf. (0.3))

$$d^C((0,0)) = \delta_1 \tag{3.13}$$

and

$$\begin{aligned}\tilde{L}(0,t) &= \delta_1 t + \frac{\beta_3(0)\delta_2}{2}t^2 \\ &\quad + \left(\frac{\beta_4(0)\delta_2 + \alpha_4(0)\delta_1}{6} - \delta_1^2 - \Omega(0)\theta'(0) - \frac{\theta''(0)}{2}\right)t^3 + O(t^4), \\ \tilde{M}(0,t) &= -\theta'(0)t^2 + \left(\Omega(0)\beta_3(0) + \frac{\beta_3'(0)}{2}\right)t^3 + O(t^4), \\ \tilde{N}(0,t) &= \frac{1}{2}\beta_3(0)t^2 + \frac{\beta_4(0)}{3}t^3 + O(t^4).\end{aligned}$$

By (2.15), we have

$$\begin{aligned}\tilde{W}^L(0,t) &= \begin{pmatrix}\beta_3(0)\delta_1 t^4 + O(t^5) & \frac{-3\beta_3(0)\theta'(0)}{4}t^5 + O(t^6) \\ -\theta'(0)t^2 + O(t^3) & \frac{\beta_3(0)}{2}t^2 + \frac{\beta_4(0)}{3}t^3 + O(t^4)\end{pmatrix} \\ &= \begin{pmatrix}2\mu_0(0)\delta_1 t^4 + O(t^5) & \frac{-3\mu_0(0)\theta'(0)}{2}t^5 + O(t^6) \\ -\theta'(0)t^2 + O(t^3) & \mu_0(0)t^2 + \mu_1(0)t^3 + O(t^4)\end{pmatrix}\end{aligned}$$

and

$$\det(\tilde{W}^L(0,t)) = 2\mu_0(0)^2\delta_1 t^6 + \frac{(4\mu_1(0)\delta_1 - 3\theta'(0)^2)\mu_0}{2}t^7 + O(t^8), \tag{3.14}$$

$$\operatorname{trace}(\tilde{W}^L(0,t)) = \mu_0(0)t^2 + \mu_1(0)t^3 + O(t^4). \tag{3.15}$$

Since $\tilde{W}^L(t,0)$ is a triangle matrix modulo $O(t^5)$-term, the diagonal components of $\tilde{W}^L(t,0)$ give the following asymptotic expansions of the two eigenvalues of $\tilde{W}^L$;

$$\tilde{\lambda}_1(0,t) := \mu_0(0)t^2 + \mu_1(0)t^3 + O(t^4), \quad \tilde{\lambda}_2(0,t) := 2\mu_0(0)\delta_1 t^4 + O(t^5). \tag{3.16}$$

We prove the following:

Theorem 3.3. *Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge along Γ of type S in $\mathbb{L}^3$. Suppose that $p := (s_0, 0)$ is a cuspidal edge singular point and the $\mathbb{L}^3$-cuspidal direction $\tilde{\mathbf{D}}^L_f(s_0,0)$ points in a light-like direction at p. Then the following assertions hold:*

1. *$i_p = 3$ holds, and f is light-like at p.*
2. *The sign $\sigma^C(p)$ vanishes if and only if $\Gamma''(s_0)$ points in a light-like direction which is different from the direction of $\tilde{\mathbf{D}}^L_f(s_0,0)$.*
3. *H^L is unbounded near a sufficiently small neighborhood of p.*
4. *f changes its causal type from space-like to time-like near p. Moreover, if $\sigma^C(p) \neq 0$ then K^L diverges near p and takes the different signs on each side of Σ_f in V_p.*
5. *The two principal curvatures of f are real-valued and one of them is unbounded near p. Moreover, other one is also unbounded if and only if $\sigma^C(p) \neq 0$.*
6. *The set of umbilical or quasi-umbilical points of f cannot accumulate at p.*

Proof. The fact that f is light-like at $p(=(s_0,0))$ follows from the fact that $\tilde{\mathbf{D}}^L_f(s_0,0)$ points in a light-like direction at p. We now assume p is a cuspidal edge singular point (i.e. $\mu_0(s_0)\neq 0$). Then $i_p=3$ follows from (3.12), proving (1). We now prove (2): By (3.11),

$$\Gamma''(s_0)=\mathbf{a}_0'(s_0)=\kappa_1(s_0)\mathbf{a}_1(s_0)+\kappa_2(s_0)\mathbf{a}_2(s_0).$$

By (3.13), $\sigma^C(p)=0$ if and only if $\kappa_1(s_0)=-\kappa_2(s_0)$, which is equivalent to the fact that $\Gamma''(s_0)=\kappa_1(s_0)(\mathbf{a}_1(s_0)-\mathbf{a}_2(s_0))$. Since $\mathbf{a}_1(s_0)-\mathbf{a}_2(s_0)$ is a light-like vector, we obtain the conclusion. On the other hand, (3) and (4) follow from (3.14) and (3.15) (cf. (1.16)). Since λ_i $(i=1,2)$ are the same order as $\tilde{\lambda}_i/|\Delta_L|^{3/2}$ with respect to t, (3.16) implies that λ_1 is unbounded (cf. (3.16)). Moreover, λ_2 is unbounded if and only if $\delta_1\neq 0$, which happens only when $\sigma^C(p)\neq 0$, proving (5). By (3.16), the two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ have different orders with respect to t, and (6) is obtained. □

Remark 3.4. In Theorem 3.3, we assumed that p is a cuspidal edge singular point. If we remove this assumption, that is, if p is a generalized cuspidal edge singular point such that $\tilde{\mathbf{D}}^L_f(s_0,0)$ points in a light-like direction at p, then (3.12) implies the order i_p is greater than or equal to 3. By (3.12), $i_p=3$ if and only if $\beta_3(0)(=2\mu_0)$ does not vanish. So $i_p=3$ holds only when p is a cuspidal edge singular point.

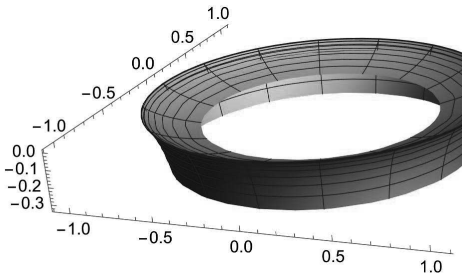

Figure 5. A cuspidal edge of order three along a circle

Example 3.5. We set $\Gamma(s):=(\cos s,\sin s,0)$ and $x(t):=t^2/2,\ y(t):=t^3/3$. Noticing that $\Gamma''=-\Gamma$ is the normal vector of the plane curve Γ, consider the cuspidal edge given by

$$f(s,t):=(1-x(t)-y(t))\Gamma(s)+(-x(t)+y(t))\mathbf{e}_3 \qquad (\mathbf{e}_3:=(0,0,1)).$$

Since the coefficients of the first fundamental form are computed as

$$E^L=\frac{1}{36}\left(2t^3+3t^2-6\right)^2,\quad F^L=0,\quad G^L=4t^3. \tag{3.17}$$

Since the coefficient of t^2 for E^L does not vanish, we have $\sigma^C\neq 0$. Moreover, we have

$$\Delta_L=\frac{1}{9}t^3\left(6-3t^2-2t^3\right)^2, \tag{3.18}$$

which is of order three at each point $(s,0)$ of Σ_f (see Figure 5). The functions $\tilde{L},\tilde{M},\tilde{N}$ are given by (cf. (1.2), (1.3) and (1.4))

$$\tilde{L}=\frac{t(1-t)\left(6-3t^2-2t^3\right)^2}{36},\quad \tilde{M}=0,\quad \tilde{N}=\frac{t^2}{3}\left(6-3t^2-2t^3\right). \tag{3.19}$$

By (1.14), (3.17), (3.18) and (3.19), the Gaussian curvature and the mean curvature do not depend on s and are computed as

$$K^L = \frac{-3(1-t)}{4t^3(6-3t^2-2t^3)}, \qquad \pm H^L = \frac{6+9t^2-14t^3}{8|t|^{5/2}(6-3t^2-2t^3)}.$$

In particular, the principal curvatures are both unbounded.

Proof of Theorem A and Corollary B. Summarizing the assertions of Theorems 2.7, 3.2 and 3.3, we obtain Theorem A. We remark that the second assertion of (5) follows from Remark 3.4.

We next prove Corollary B: If the mean curvature function of f is bounded, then (6) of Theorem A implies that Γ is a regular curve of type L, that is, Γ' always points in a light-like direction, proving Corollary B. □

4. The case that Γ' points in a light-like direction

In this section, we consider the case that $\Gamma'(s)$ points in a light-like direction for some s. The case (L_g) $(g = 2, 4)$ given in Subsection 2.1 is contained in this setting.

4.1. **Computations in the case that $\Gamma'(0)$ is a light-like vector.** Fix an open interval I containing $0 \in \mathbb{R}$. Let $\mathcal{E}^2$ be a space-like plane in $\mathbb{L}^3$. Without loss of generality, we may assume that $\mathcal{E}^2$ is the xy-plane. The following fact can be proved easily.

Fact 4.1. *Let $\Gamma : I \to \mathbb{L}^3$ be a regular curve of type L. If we set $\Gamma(s) = (x(s), y(s), z(s))$ and assume that s is the arc-length parameter of the curve $\gamma(s) := (x(s), y(s))$ in the xy-plane, then Γ has the expression $\Gamma(s) = (\gamma(s), \pm s)$ for $s \in I$.*

Without loss of generality, we may assume that $\Gamma(s)$ is future-pointing at $s = 0$. Regarding this, we set

$$\Gamma(s) := (\gamma(s), \varphi(s)) \qquad (s \in I), \tag{4.1}$$

where $\gamma(s)$ is a regular curve in $\mathcal{E}^2$ parametrized by the arc-length and $\varphi(s)$ is a smooth function satisfying

$$\varphi'(0) = 1, \tag{4.2}$$

which implies that $\Gamma'(0)$ points in a light-like direction. If Γ is of type L, then φ is the identity map, that is, φ satisfies

$$\varphi(s) = s \qquad (s \in I). \tag{4.3}$$

We set $\mathbf{e}(s) := (\gamma'(s), 0)$. We denote by $\mathbf{n}(s) \in \mathbb{L}^3$ the left-ward unit normal vector $\gamma'(s)$ in the xy-plane. For each $s \in I$, three vectors

$$\mathbf{e}(s), \quad \mathbf{n}(s), \quad \mathbf{v} := (0, 0, 1)$$

form an orthonormal basis of $\mathbb{L}^3$ for each $s \in I$, and $\mathbf{v}$ is a time-like vector. We fix our general setting as

$$\mathbf{a}_0 := \mathbf{e}(s), \quad \mathbf{a}_1 := \mathbf{n}(s), \quad \mathbf{a}_2 := \mathbf{v}, \quad \varepsilon_0 = \varepsilon_1 = 1, \quad \varepsilon_2 = -1.$$

Then,

$$\Gamma'(s) = \mathbf{e} + \varphi'(s)\mathbf{v}$$

holds. Let f be a generalized cuspidal edge along Γ. By applying Proposition B.1 in the appendix for $(s, t) = (0, 0)$, like as the general setting given in Section 2, we may assume that f has the following expression

$$f(s, t) = \Gamma(s) + X(s, t)\mathbf{n}(s) + Y(s, t)\mathbf{v}, \tag{4.4}$$

where $X(s,t)$ and $Y(s,t)$ are smooth functions of the form

$$(4.5)\qquad \begin{aligned} X(s,t) &= A(s,t)\cos\theta(s) + B(s,t)\sin\theta(s),\\ Y(s,t) &= B(s,t)\cos\theta(s) - A(s,t)\sin\theta(s) \end{aligned}$$

and

$$A(s,t) = \int_0^t u\cos\lambda(s,u)du, \qquad B(s,t) = \int_0^t u\sin\lambda(s,u)du,$$
$$\lambda(s,t) := \int_0^t \mu(s,u)du.$$

The vector field $\cos\theta(s)\mathbf{n}(s)+\sin\theta(s)\mathbf{v}$ gives the direction of the cusp for the section of the image of f, and the plane passing through $\Gamma(s)$ is spanned by $\mathbf{n}(s)$ and $\mathbf{v}_3$. Using the expansion of μ given in (2.9), we have (cf. (2.7))

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}t^2 + \frac{1}{3}\begin{pmatrix} 0 \\ \mu_0 \end{pmatrix}t^3 + \frac{1}{8}\begin{pmatrix} -\mu_0^2 \\ \mu_1 \end{pmatrix}t^4 + \frac{1}{30}\begin{pmatrix} -3\mu_0\mu_1 \\ -\mu_0^3+\mu_2 \end{pmatrix}t^5 + O(t^6)$$

and

$$(4.6)\qquad \begin{aligned} &\alpha_2 = 1, \quad \alpha_3 = 0, \quad \alpha_4 = -3\mu_0^2, \quad \alpha_5 = -12\mu_0\mu_1,\\ &\beta_3 = 2\mu_0, \quad \beta_4 = 3\mu_1, \quad \beta_5 = 4(-\mu_0^3+\mu_2). \end{aligned}$$

Let $\kappa(s)$ be the curvature function of the curve $\gamma(s)$ in the xy-plane (as $\mathcal{E}^2$). Then, we have (cf. (2.4))

$$(4.7)\qquad \mathcal{K} = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the sake of simplicity, we set

$$C(s) := \cos\theta(s), \quad S(s) := \sin\theta(s), \quad C_2(s) := \cos 2\theta(s), \quad S_2(s) := \sin 2\theta(s).$$

Since $\Gamma'(s)$ is different from $\mathbf{a}_0(s)$ (cf. (2.2)), we cannot apply the general computations given in Section 2, and must begin with the computations of f_s and f_t as follows:

$$(4.8)\qquad \begin{aligned} f_s &= \mathbf{e} + \varphi'\mathbf{v} - \frac{\kappa C\mathbf{e} + \theta' S\mathbf{n} + \theta' C\mathbf{v}}{2}t^2 \\ &\quad + \frac{-\kappa\beta_3 S\mathbf{e} + (\theta'\beta_3 C + \beta_3' S)\mathbf{n} + (-\theta'\beta_3 S + \beta_3' C)\mathbf{v}}{6}t^3 + O(t^4),\\ f_t &= (C\mathbf{n} - S\mathbf{v})t + \frac{\beta_3 S\mathbf{n} + \beta_3 C\mathbf{v}}{2}t^2 \\ &\quad + \frac{(\alpha_4 C + \beta_4 S)\mathbf{n} + (-\alpha_4 S + \beta_4 C)\mathbf{v}}{6}t^3 + O(t^4). \end{aligned}$$

Then, we have

$$E^L = 1 - \varphi'^2 + (-\kappa + \theta'\varphi')\,Ct^2 + O(t^3),$$
$$F^L = \varphi' St - \frac{\beta_3\varphi' C}{2}t^2 + O(t^3), \quad G^L = C_2t^2 + \beta_3 S_2 t^3 + O\left(t^4\right)$$

and

$$(4.9)\qquad \Delta_L = (C^2(1-\varphi'^2) - S^2)t^2 + \beta_3 CS(2-\varphi'^2)t^3 + O(t^4).$$

Moreover,

$$(4.10)\qquad \tilde{\nu}^E = (-\varphi' C\mathbf{e} + S\mathbf{n} + C\mathbf{v})t - \frac{\beta_3}{2}(\varphi' S\mathbf{e} + C\mathbf{n} - S\mathbf{v})t^2 + O(t^3)$$

and

$$f_{ss} = \kappa\mathbf{n} + \varphi''\mathbf{v} + \frac{(2\kappa\theta' S - \kappa' C)\mathbf{e} - ((\kappa^2+\theta'^2)C + \theta'' S)\mathbf{n} + (\theta'^2 S - \theta'' C)\mathbf{e}}{2}t^2 + O(t^3), \tag{4.11}$$

$$f_{st} = -\Big(\kappa C\mathbf{e} + \theta'(S\mathbf{n} + C\mathbf{v})\Big)t + \frac{-\kappa\beta_3 S\mathbf{e} + (\theta'\beta_3 C + \beta_3' S)\mathbf{n} + (-\theta'\beta_3 S + \beta_3' C)\mathbf{v}}{2}t^2 + O(t^3),$$

$$f_{tt} = C\mathbf{n} - S\mathbf{v} + (S\mathbf{n} + C\mathbf{v})t + \frac{(\alpha_4 C + \beta_4 S)\mathbf{n} + (-\alpha_4 S + \beta_4 C)\mathbf{v}}{2}t^2 + O(t^3).$$

In particular, we have

$$d^C((s,0)) := -\det(f_s, f_{ss}, f_{tt})|_{t=0} = S\kappa + C\varphi'' \tag{4.12}$$

and

$$\tilde{L} = (\kappa S + \varphi'' C)\,t - \frac{\beta_3(s)}{2}(\kappa C - \varphi'' S)t^2 + O\left(t^3\right),$$

$$\tilde{M} = \left(\kappa\varphi' C^2 - \theta'\right)t^2 + O\left(t^3\right), \quad \tilde{N} = \frac{\beta_3}{2}t^2 + O\left(t^3\right).$$

If we set

$$(\tilde{W}^L =)\begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} := \begin{pmatrix} G^L & -F^L \\ -F^L & E^L \end{pmatrix}\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix}, \tag{4.13}$$

then

$$w_{1,1} = \Big(S\theta'\varphi' + \kappa S(C_2 - C^2\varphi'^2) + \varphi'' C C_2\Big)t^3 + O(t^4), \tag{4.14}$$

$$w_{1,2} = -\frac{1}{2}\varphi'\beta_3 S t^3 + O\left(t^4\right),$$

$$w_{2,1} = \Big((\theta' - \kappa\varphi' C^2)(-1 + \varphi'^2) - \varphi' S(\kappa S + \varphi'' C)\Big)t^2 + O(t^3),$$

$$w_{2,2} = -\frac{\beta_3}{2}\left(\varphi'^2 - 1\right)t^2 + \frac{1}{3}\Big(3\varphi' S(\theta' - \varphi'\kappa C^2) + \beta_4(1 - \varphi'^2)\Big)t^3 + O(t^4)$$

hold. In particular, the square of the difference of the two eigenvalues of $\tilde{W}^L$ satisfies

$$(\operatorname{trace}\tilde{W}^L(s,t))^2 - 4\det(\tilde{W}^L(s,t)) = \frac{\beta_3^2\left(\varphi'^2 - 1\right)^2}{4}t^4 + O(t^5). \tag{4.15}$$

Substituting $\varphi'(0) = 1$, we have

$$\det(\tilde{W}^L(0,t)) = -\frac{\beta_3(0)S_0^2}{2}\Big(\kappa(0)S_0 + \varphi''(0)C_0\Big)t^5 + O\left(t^6\right), \tag{4.16}$$

$$\operatorname{trace}(\tilde{W}^L(0,t)) = \frac{1}{2}\Big(2S_0(-\kappa(0) + 2\theta'(0)) + (C_0 + C_{3,0})\varphi''(0)\Big)t^3 + O(t^4), \tag{4.17}$$

where

$$C_0 := \cos\theta(0), \quad S_0 := \sin\theta(0), \quad C_{3,0} = \cos 3\theta(0).$$

Using the above computations, we give the proof of Proposition C in the introduction:

Proof of Proposition C. Without loss of generality, we may assume that $s_0 = 0$ and $p = (0,0)$. By (4.10),

$$\tilde{\nu}^L(p) = -C_0\mathbf{e} + S_0\mathbf{n} - C_0\mathbf{v}$$

is a normal vector of f at p. Since $\Gamma'(0)$ points in a light-like direction, $\tilde{\nu}^L(p)$ is space-like or light-like. Since

$$1 = C_0^2 + S_0^2 \geq C_0^2,$$

the normal vector $\tilde{\nu}^L(p)$ is space-like (resp. light-like) if and only if $S_0 \neq 0$ (resp. $S_0 = 0$). By (4.18), this shows the equivalency of (1) and (2).

Since $\Gamma'(0)$ points in a light-like direction, we have $\varphi'(0) = 1$. So (4.9) implies

$$\Delta_L(0,t) = -S_0^2 t^2 + O(t^3). \tag{4.18}$$

So, if i_p is greater than 2, then $S_0 = 0$ holds, which implies $i_p \geq 4$. In particular, either $i_p = 2$ or $i_p \geq 4$ holds.

We now prove the last statement: So we assume that Γ is of type L satisfying $\Gamma''(0) \neq \mathbf{0}$. In this case $\varphi''(0) = 0$ and (4.12) yields that

$$d^C(p) := S_0 \kappa(0). \tag{4.19}$$

Since $\Gamma''(0) \neq \mathbf{0}$ and $\varphi''(0) = 0$, the first equation of (4.11) implies $\kappa(0) \neq 0$. Since the sign of $d^C(p)$ coincides with $\sigma^C(p)$, (4.19) implies the equivalency of (2) and the condition $\sigma^C(p) \neq 0$ (resp. $\sigma^C(p) = 0$). □

The following is an analogue of Lemma 2.4:

Lemma 4.2. *The point $(s_0, 0)$ $(s_0 \in I)$ is a cuspidal edge singular point (resp. a cuspidal cross cap singular point) of f if and only if $\mu_0(s_0) \neq 0$ (resp. $\mu_0(s_0) = 0$ and $\mu_0'(s_0) \neq 0$).*

Proof. The statement for cuspidal edge singular points can be proved by imitating the proof of Lemma 2.4. To prove the assertion for cuspidal cross caps, we set

$$\psi(s) := \det(\Gamma'(s), \check{\nu}_E(s,0), (\check{\nu}_E)_t(s,0))$$

and (cf. (4.10))

$$\check{\nu}_E := \frac{\hat{\nu}_E}{t} = (-\varphi' C\mathbf{e} + S\mathbf{n} + C\mathbf{v}) - \frac{\beta_3}{2}(\varphi' S\mathbf{e} + C\mathbf{n} - S\mathbf{v})t + O(t^2).$$

By (4.8), we have $\Gamma' = f_s = \mathbf{e} + \varphi' \mathbf{v}$. Using the relation $\beta_3 = 2\mu_0$, we have

$$\psi(s) = \mu_0(s)(1 + \varphi'(s)^2).$$

So $\psi(s_0) = 0$ if and only if $\mu_0(s_0) = 0$. If $\mu_0(s_0) = 0$, then $\psi'(s_0) = 0$ holds if and only if $\mu_0'(s_0) = 0$. So imitating the proof of Lemma 2.4, we obtain the assertion for cuspidal cross caps. □

We next prepare the following:

Lemma 4.3. *Let $\Gamma : I \to \mathbb{L}^3$ be a regular curve such that $\Gamma'(0)$ is a light-like vector, where we assume $0 \in I$. Let $f(s,t)$ be a generalized cuspidal edge along Γ and $p := (0,0)$ a cuspidal edge singular point of f. If the order i_p is equal to 2, then umbilical points of f never accumulate at p.*

Proof. Suppose $\{(s_k, t_k)\}_{k=1}^{\infty}$ is a sequence of umbilical points of f which converges to $p := (0,0)$. Then we have $w_{12}(s_k, t_k) = 0$. Since $\beta_3 = 2\mu_0 \neq 0$ and $\varphi'(0) = 1$, the second equation of (4.14) implies

$$0 = w_{12}(s_k, t_k) = -\mu_0(s_k) \sin\theta(s_k) t_k^3 + O(t_k^4).$$

However, since $\sin\theta(0) \neq 0$ and $\mu_0(0) \neq 0$, the right-hand side does not vanish when k is sufficiently large, a contradiction. □

We fixed a space-like plane $\mathcal{E}^2$ in $\mathbb{L}^3$ and is regarding it as the xy-plane. If we replace another space-like plane $\tilde{\mathcal{E}}^2$, then we have the two orthogonal projections:

$$\pi : \mathbb{L}^3 \to \mathcal{E}^2, \qquad \tilde{\pi} : \mathbb{L}^3 \to \tilde{\mathcal{E}}^2.$$

Proposition 4.4. *Even if we replace the orthogonal projection π by $\tilde{\pi}$, the condition that $\sin\theta(s) = 0$ does not change, that is, this condition is invariant under the choice of the orthogonal projection to a space-like plane in $\mathbb{L}^3$.*

Proof. As shown in Proposition 1.26, the order i_p is an invariant of generalized cuspidal edges, and $i_p \geq 3$ holds if and only if $\sin\theta = 0$ at the point p, as shown in the proof of Proposition C. So we obtain the conclusion. □

Using this, we can prove Proposition E in the introduction:

Proof of Proposition E. Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a generalized cuspidal edge along a regular curve $\Gamma : I \to \mathbb{L}^3$. Suppose that $p \in \mathcal{U}_f$ is a cuspidal edge singular point of f. Without loss of generality, we may assume that $0 \in I$ and $p := (0,0)$. If $\Gamma'(0)$ is not a light-like vector, p is not an accumulation point of the set of umbilics nor of the set of quasi-umbilics by (9) of Theorem A. So we may assume that $\Gamma'(0)$ is a light-like vector. In this case, umbilical points cannot accumulate at p by Lemma 4.3. □

We next prove the following:

Proposition 4.5. *Let $p := (0,0)$ $(0 \in I)$ be a cuspidal edge singular point of f of order 2 such that $\Gamma'(0)$ points in a light-like direction. Then there exists a neighborhood V_p of p such that the following assertions hold:*

1. *The normal vector of f at p is time-like.*
2. *If $\sigma^C(p) \neq 0$, then K^L is unbounded on V_p. Moreover, on one side of $\Sigma_f \cap V_p$, the principal curvatures of f are real-valued, and on the other side they are non-real-valued. In this situation, the two principal curvatures are both unbounded on V_p.*
3. *If $\sigma^C(p) \neq 0$ and also $\langle \Gamma'(0), \Gamma''(0)\rangle_L \neq 0$ (that is, $\varphi''(0) \neq 0$), then the set of quasi-umbilical points of f accumulates at p and lie in one side of $V_p \setminus \Sigma_f$.*

Proof. (1) follows from Proposition C. We assume $\sigma^C(p) \neq 0$. By (4.12), we have

$$0 \neq d^C(p) = S_0\kappa(0) + C_0\varphi''(0).$$

Since $\beta_3(0) = 2\mu_0(0)$, substituting $\varphi'(0) = 1$ to (4.16), we have

$$\det(\tilde{W}^L(0,t)) = \mu_0(0)S_0^2 d^C(p)t^5 + O(t^6), \tag{4.20}$$

which implies

$$K^L(0,t) = \frac{1}{S_0^4 t}\Big(\mu_0(0)d^C(p) + O(t)\Big). \tag{4.21}$$

Since p is a cuspidal edge singular point, we have $\mu_0(0) \neq 0$. Since $\sigma^C(p) \neq 0$, the unboundedness of K^L near the point p follows from (4.21). Finally, we prove (3): We set

$$F(s,t) := \frac{(\mathrm{trace}\tilde{W}^L(s,t))^2 - 4\det(\tilde{W}^L(s,t))}{t^4}.$$

By (4.15), F is a smooth function satisfying

$$F(s,t) = \frac{\beta_3^2\left(\varphi'^2 - 1\right)^2}{4} + O(t). \tag{4.22}$$

Since $(\mathrm{trace}(\tilde{W}^L(0,t))^2$ belongs to the class $O(t^6)$, (4.20) implies that

$$F(0,t) = -4\frac{\det(\tilde{W}^L(0,t))}{t^4} + O(t) = 4S_0^2\mu_0(0)d^C((0,0))t + O(t^2). \tag{4.23}$$

In particular, the two eigenvalues of $\tilde{W}^L(0,t)$ are

- real numbers if $\mu_0(0)d^C((0,0))t \geq 0$, and
- non-real numbers if $\mu_0(0)d^C((0,0))t < 0$.

By (4.16) and (4.17), the two principal curvatures λ_1, λ_2 have the same order with respect to t. Then, (4.21) implies λ_1 and λ_2 are both unbounded, so (2) is proved. By (4.23), we have

$$F_t(0,0) = 4\mu_0(0)S_0^2 d^C((0,0))(\neq 0).$$

By the implicit function theorem, there exists a C^∞-function of $\psi(s)$ near $s = 0$ such that $F(s, \psi(s)) = 0$ and $\psi(0) = 0$. Then we have

$$F_s(0,0) + F_t(0,0)\psi'(0) = 0.$$

In particular, if $(s, \psi(s))$ is not a singular point, it is a quasi-umbilical point or an umbilical point. By (4.22) with $\varphi'(0) = 1$, we have $F_s(0,0) = 0$, and so $\psi'(0) = 0$ holds. So, we have

$$F_{ss}(0,0) + F_t(0,0)\psi''(0) = 0. \tag{4.24}$$

Again, by (4.22) with $\varphi'(0) = 1$ and $\beta_3 = 2\mu_0 \neq 0$, we have

$$F_{ss}(0,0) = 8\mu_0(0)^2\varphi''(0)^2 \neq 0.$$

Since $F_t(0,0) \neq 0$, (4.24) implies $\psi''(0) \neq 0$. Thus, for each $s(\neq 0)$, $(s, \psi(s))$ is not a singular point, and so, it is a quasi-umbilical point or umbilical point. Since we have shown that (cf. Lemma 4.3) umbilical points of f never accumulate at $(s, 0)$, the point $(s, \psi(s))$ must be a quasi-umbilical point lying on the side of the s-axis where the two principal curvatures are not real. □

4.2. **Light-like cuspidal edges of general type.**

Definition 4.6. Let $f : I \times (-\delta, \delta) \to \mathbb{R}^3$ be a generalized cuspidal edge along a regular curve Γ of type L. Then f is called a *light-like cuspidal edge of general type* if $i_p = 2$ and $\Gamma''(s) \neq 0$ at each $p := (s, 0)$ $(s \in I)$.

We prove the following assertion:

Proposition 4.7. *If f is a light-like cuspidal edge of general type, then $\sigma^C(p) \neq 0$ for each $p \in \Sigma_f$ and the curvature function κ of $\gamma := \pi \circ \Gamma$ does not vanish everywhere, that is, γ has no inflection points in the xy-plane.*

Proof. Since Γ is of type L, we have $\varphi''(s) = 0$. Then, (4.12) is reduced to $d^C((s,0)) = S\kappa$. Since $i_p = 2$, we have $S \neq 0$. Since $f_{ss}(s,0) = \Gamma''(s) \neq \mathbf{0}$, the first equation of (4.11) implies $\kappa(s) \neq 0$. □

We now prove Theorem D in the introduction:

Proof of Theorem D. (1) follows from Proposition 4.7, and (2) follows from Propositions C. Since Γ is of type L, $\varphi''(0) = 0$ holds. Then, (4.17) implies

$$|H^L(0,t)| = \frac{1}{2S_0^2}\Big|\kappa(0) - 2\theta'(0)\Big| + O(|t|).$$

Since $\Gamma'(s)$ $(s \in I)$ points in a light-like direction, $0(\in I)$ can be replaced by any $s \in I$, we have

$$|H^L(s,t)| = \frac{1}{2S^2}\Big|\kappa(s) - 2\theta'(s)\Big| + O(|t|),$$

where $S = \sin\theta(s)$. So we obtain (3). Using the same method to obtain the expression of $H^L(s,t)$ from the expression of $H^L(0,t)$, the expression $K^L(0,t)$ given in (4.21) induces

$$K^L(s,t) = \frac{1}{t}\Big(\mu_0 S^{-4} d^C((s,0)) + O(t)\Big),$$

which proves (4). Since H_L is bounded and K^L is unbounded, the two principal curvatures λ_i $(i = 1, 2)$ are both unbounded. Since $\Gamma'(s)$ $(s \in I)$ always points in a light-like direction, we have (using $\beta_3 = 2\mu_0$)

$$\det(\tilde{W}^L) = -\mu_0 \kappa S^3 t^5 + O\left(t^6\right), \qquad \operatorname{trace}(\tilde{W}^L) = -(\kappa - 2\theta')St^3 + O(t^4).$$

Thus, the square of the difference of the two eigenvalues of $\tilde{W}^L$ satisfies

$$(\operatorname{trace} \tilde{W}^L)^2 - 4\det(\tilde{W}^L) = -4\det(\tilde{W}^L)t^5 + O(t^6) = 2\kappa\mu_0 S^3 t^5 + O\left(t^6\right).$$

Since f is a light-like cuspidal edge of general type, $\kappa\mu_0 S^3$ never vanishes on I, and so (5) and (6) are obtained. □

5. Proof of Proposition F and Theorem G

Let $\Gamma : I \to \mathbb{L}^3$ be a regular curve of type L, and set $\varphi(s) := s$ for each $s \in I$. Then (4.1) is reduced to $\Gamma(s) := (\gamma(s), s)$ $(s \in I)$, where $\gamma(s)$ is a regular curve in the xy-plane parametrized by the arc-length. We let $f : I \times (-\delta, \delta) \to \mathbb{R}^3$ be a generalized cuspidal edge satisfying $f(s, 0) = \Gamma(s)$ $(s \in I)$ whose singular points are of order four, that is, $\sin\theta$ is identically zero in the setting of the previous section. So, we set

$$\sigma := \cos\theta \in \{1, -1\} \tag{5.1}$$

and

$$f(s, t) = \Gamma(s) + \sigma\, x(s, t)\mathbf{n}(s) + \sigma\, y(s, t)\mathbf{v}, \tag{5.2}$$

where

$$x(s, t) = \int_0^t u \cos\lambda(s, u)du, \quad y(s, t) = \int_0^t u \sin\lambda(s, u)du, \quad \lambda(s, t) := \int_0^t \mu(s, u)du.$$

The formula (5.2) produces all generalized cuspidal edges of order greater than or equal to four. By Lemma 4.2, the singular point $(s, 0)$ is a cuspidal edge if and only if $\mu_0(s) \neq 0$. To obtain information on terms of higher order, we start with the computations of f_s and f_t;

$$f_s = (\mathbf{e} + \mathbf{v}) - \frac{\sigma\kappa\mathbf{e}}{2}t^2 + \frac{\sigma\beta_3'}{6}\mathbf{v}t^3 + \frac{-\sigma\kappa\alpha_4\mathbf{e} + \sigma\alpha_4'\mathbf{a}_1 + \sigma\beta_4'\mathbf{v}}{24}t^4 + O\left(t^5\right),$$

$$f_t = \sigma\mathbf{n}t + \frac{\sigma\beta_3}{2}\mathbf{v}t^2 + \frac{\sigma\alpha_4\mathbf{n} + \sigma\beta_4\mathbf{v}}{6}t^3 + \frac{\sigma}{24}t^4\left(\alpha_5\mathbf{n} + \beta_5(s, 0)\mathbf{v}\right) + O\left(t^5\right).$$

The first three terms of the above expansions of f_t and f_s coincide with those of (4.8) by substituting $\theta' = 0$, $C = 1$ and $S = 0$. We have

$$E^L = -\sigma\kappa t^2 - \frac{\sigma\beta_3'}{3}t^3 + O\left(t^4\right), \quad F^L = -\frac{\sigma\beta_3}{2}t^2 - \frac{\sigma\beta_4}{6}t^3 + O\left(t^4\right), \quad G^L = t^2 + O\left(t^4\right)$$

and

$$\Delta_L = -\frac{\beta_3^2 + 4\sigma\kappa}{4}t^4 + O\left(t^5\right) = -(\mu_0^2 + \sigma\kappa)t^4 + O\left(t^5\right). \tag{5.3}$$

Moreover, we have

$$\tilde{\nu}^E = \sigma(-\mathbf{e} + \mathbf{v})t - \frac{\sigma\beta_3}{2}t^2 + \frac{-\sigma\alpha_4\mathbf{e} + \sigma\beta_4\mathbf{n} + (-3\kappa + \sigma\alpha_4)\mathbf{v}}{6}t^3 + O(t^4)$$

and

$$f_{ss} = \kappa\mathbf{n} - \frac{\sigma}{2}\left(\kappa'\mathbf{e} + \kappa^2\mathbf{n}\right)t^2 + \frac{\sigma\beta_3''}{6}t^3\mathbf{v} + O\left(t^4\right),$$

$$f_{st} = -\sigma\kappa\mathbf{e}t + \frac{\sigma\beta_3'}{2}\mathbf{v}t^2 + \frac{\sigma}{6}\left(-\kappa\alpha_4\mathbf{e} + \alpha_4'\mathbf{n} + \beta_4'\mathbf{v}\right)t^3 + O(t^4),$$

$$f_{tt} = \sigma\mathbf{n} + \sigma\beta_3\mathbf{v}t + \frac{\sigma}{2}(\alpha_4\mathbf{n} + \beta_4\mathbf{v}) + \frac{\sigma}{6}(\alpha_5\mathbf{n} + \beta_5\mathbf{v})t^3 + O(t^4).$$

Then we have

$$\tilde{L} = -\frac{\sigma\kappa\beta_3}{2}t^2 + \frac{\sigma}{6}\left(3\sigma\kappa' - \kappa\beta_4\right)t^3 + O\left(t^4\right),$$
$$\tilde{M} = \kappa t^2 + \frac{\beta_3'}{2}t^3 + O\left(t^4\right), \quad \tilde{N} = \frac{\beta_3}{2}t^2 + \frac{\beta_4}{3}t^3 + O\left(t^4\right).$$

If we set $\tilde{W}^L = (w_{i,j})_{i,j=1,2}$, then

$$\begin{aligned}
w_{1,1} &= \frac{\sigma}{4}\left(\beta_3\beta_3' + 2\kappa'\right)t^5 + O\left(t^6\right),\\
w_{1,2} &= \frac{\sigma}{4}\left(\beta_3^2 + 4\sigma\kappa\right)t^4 + \frac{\sigma}{4}\left(2\sigma\beta_3' + \beta_3\beta_4\right)t^5 + O\left(t^6\right),\\
w_{2,1} &= \frac{-\kappa}{4}\left(\beta_3^2 + 4\sigma\kappa\right)t^4 + \frac{1}{12}\Big(3\sigma\beta_3\kappa' - 2\kappa\left(5\sigma\beta_3' + \beta_3\beta_4\right)\Big)t^5 + O\left(t^6\right),\\
w_{2,2} &= \frac{\sigma}{12}\left(\beta_3\beta_3' - 2\beta_4\kappa\right)t^5 + O\left(t^6\right)
\end{aligned}$$

hold. Using the fact that $\beta_3 = 2\mu_0$ and $\beta_4 = 3\mu_1$ (cf. (4.6)), we have

$$\det(\tilde{W}^L) = \sigma\kappa(\mu_0^2 + \sigma\kappa)^2 t^8 + O(t^9), \tag{5.4}$$

$$\operatorname{trace}(\tilde{W}^L) = \frac{\sigma}{6}\left(-3\kappa\mu_1 + 8\mu_0\mu_0' + 3\sigma\kappa'\right)t^5 + O(t^6). \tag{5.5}$$

In particular, we have

$$\begin{aligned}
(\operatorname{trace}\tilde{W}^L)^2 - 4\det(\tilde{W}^L) &= -4\det(\tilde{W}^L)t^8 + O(t^9)\\
&= -4\sigma\kappa\left(\mu_0^2 + \sigma\kappa\right)^2 t^8 + O(t^9).
\end{aligned}$$

Proof of Proposition F. Let $f : \mathcal{U}_f \to \mathbb{L}^3$ be a cuspidal edge along a curve Γ of type L. We fix a singular point $p = (s, 0)$ ($s \in I$) arbitrarily and assume $i_p > 2$. Then, by Lemma 4.3, we have $i_p \geq 4$. By (5.3), $i_p > 4$ if and only if $\beta_3(s)^2 + 4\sigma\kappa(s) = 0$. Without loss of generality, we may assume $\kappa(s) \geq 0$ (if necessary, we can reverse the orientation of the curve). Since p is a cuspidal edge singular point, $\beta_3(s) = 2\mu_0(s) \neq 0$ (cf. Lemma 4.2), and so $\sigma\kappa(s) < 0$. Then we have $\kappa(s) > 0$ and $\sigma < 0$ follows. Since $\kappa_s = \sigma\kappa$, the map f is of concave type at p and $|\kappa(s)| = \mu_0(s)^2$ holds. □

Proof of Theorem G. Since $(s, 0)$ is of order four, $\Delta(s,t)$ does not change sign at $t = 0$. So (a) holds. By (5.5), we have

$$|H^L| = \frac{1}{12|\mu_0^2 + \sigma\kappa|^{3/2}|t|}\left|-3\kappa\mu_1 + 8\mu_0\mu_0' + 3\sigma\kappa'\right| + O(1),$$

which implies (b). By Lemma 4.2, f has cuspidal edge singular point at $p := (s_0, 0)$ if and only if $\mu_0(s_0) \neq 0$, proving (d). We next consider the behavior of umbilical points of f: Since f is of order four, we have $\mu_0^2 + \sigma\kappa \neq 0$ and the facts

$$w_{1,2} = \sigma\left(\mu_0^2 + \sigma\kappa\right)t^4 + O(t^5) = -\sigma\Delta_L + O(t^5)$$

implies that matrix $\tilde{W}^L$ cannot be a diagonal matrix at p, and so p cannot be an accumulation point of a sequence of umbilical points $\{(s_k, t_k)\}_{k=1}^{\infty}$. If not, $t_k \neq 0$ and $\tilde{W}^L(s_k, t_k)$ is diagonal for each k, however it is impossible, because $\mu_0^2 + \sigma\kappa \neq 0$ at $s = 0$ by (5.3) and $w_{1,2}(0,0) \neq 0$, proving (c).

We now assume that $\Gamma'' \neq \mathbf{0}$ on I, which implies the curvature function κ of the plane curve γ does not vanish. Without loss of generality, we may assume that $\kappa > 0$. By (5.4), we have

$$K^L = \operatorname{sgn}(\mu_0^2 + \sigma\kappa)\frac{\sigma\kappa}{|\mu_0^2 + \sigma\kappa|t^4} + O(t^{-3}) = \frac{\sigma\kappa}{(\mu_0^2 + \sigma\kappa)t^4} + O(t^{-3}). \tag{5.6}$$

If f is space-like, then $\mu_0^2+\sigma\kappa$ is negative, so $\sigma\kappa<0$. In particular, K^L is positive, and f is of concave type, proving (1). On the other hand, if f is time-like, then $\mu_0^2+\sigma\kappa$ is positive, so the sign of K^L coincides with that of $\sigma\kappa$. Since $\mu_0^2+\sigma\kappa$ and κ are positive, it happens either $\sigma>0$ or "$\sigma<0$ and $\mu_0^2>\kappa$". So we obtain the first part of (2) (the second part of (2) is the case that $(s,0)$ is not a cuspidal edge singular point, which is discussed in the proof of Corollary 5.2 below). We prove (3): The fact K^L diverges is a consequence of (5.6) since $\mu_0^2+\sigma\kappa\neq 0$. Since $|H^L|$ is of order $1/|t|$, by (5.4) and (5.5), the absolute values $|\lambda_i|$ $(i=1,2)$ of two principal curvatures are of order $1/|t|$, and are unbounded. Finally, we prove (4): Since $\beta_3=2\mu_0$, (5.3) implies

$$\Delta_L=-(\mu_0^2+\sigma\kappa)t^4+O\left(t^5\right). \tag{5.7}$$

Since f is of order four, we have $\mu_0^2+\sigma\kappa\neq 0$ and so $\operatorname{trace}(\tilde{W}^L)^2-4\det(\tilde{W}^L)$ never vanishes near $t=0$, which implies (4). □

Corollary 5.1. *The sign of the function $\mu_0^2+\sigma\kappa$ and the zero set of the function $-3\kappa\mu_1+8\mu_0\mu_0'+3\sigma\kappa'$ are both independent of the choice of the projection of Γ to a space-like plane $\mathcal{E}^2$.*

Proof. In fact, the top-terms of the asymptotic expansions of $K^L(s,t)$ and $|H^L(s,t)|$ along the curve Γ with respect to the parameter t do not depend on the choice of an admissible coordinate change (cf. Definition 0.3 and Lemma 1.5), and replacement of the space-like plane is obtained only by such an admissible coordinate change. So, we obtain the conclusion. □

If $f(s,t)$ does not give cuspidal edge at $(s,t)=(s_0,0)$, the following assertion holds:

Corollary 5.2. *Let f be a generalized cuspidal edge of order four along a regular curve Γ of type L. If $p:=(s_0,0)$ $(s_0\in I)$ is not a cuspidal edge singular point and $\Gamma''(s_0)$ does not vanish (i.e. $\kappa(s_0)\neq 0$), then*

1. *f is space-like (resp. time-like) near p if and only if f is of concave type (resp. convex type, see Definition 0.7), that is, $\sigma\kappa(s_0)<0$ (resp. $\sigma\kappa(s_0)>0$) holds,*
2. *the Gaussian curvature K^L diverges to ∞ when (s,t) tends to p, and*
3. *umbilical points and quasi-umbilical points never accumulate at p.*

Proof. Since $p=(s_0,0)$ $(s_0\in I)$ is not a cuspidal edge singular point, $\mu_0(s_0)$ vanishes (cf. Lemma 4.2). So, (5.7) is reduced to

$$\Delta_L=-\sigma\kappa t^4+O\left(t^5\right).$$

This with (1) and (2) of Theorem G imply (1). On the other hand, since $\Gamma''(s_0)$ does not vanish, $\kappa(s_0)\neq 0$ and as long as s is close to s_0, (5.6) can be written as

$$K^L(s_0,t)=\operatorname{sgn}(\sigma\kappa(s_0))\frac{\sigma\kappa(s_0)}{|\sigma\kappa(s_0)|t^4}+O(t^{-3})=\frac{1}{t^4}+O(t^{-3}),$$

which implies (2). The last assertion (4) follows from (c) and (4) of Theorem G. □

Remark 5.3. This corollary is a generalization of the corresponding assertion in Akamine [1, (3) of Theorem A] for time-like zero mean curvature surfaces, where the assumption $\Gamma''(s_0)\neq 0$ is dropped. However, if f is a maxface or a minface, the order of f at a generalized cuspidal edge singular point is equal to four. Then $\kappa(s_0)\neq 0$, that is, $\Gamma''(s_0)\neq 0$ holds.

Remark 5.4. By Theorem G, umbilical points never accumulate at any cuspidal singular points of f. However, the quasi-umbilical points of f can accumulate, in general. A concrete example of a cuspidal edge of order four with zero mean curvature function whose quasi-umbilical points accumulate at a cuspidal edge singular point is given in Akamine [1, Figure 4].

We give a family of cuspidal edges with order four whose mean curvatures are bounded:

Example 5.5. We fix a real number a, and set

$$\begin{aligned} &\Gamma(s) := (\cos s, \sin s, s)\,, \\ &\mathbf{n}(s) := -(\cos s, \sin s, 0) \qquad (s \in \mathbb{R}), \\ &x(t) := \frac{\sigma t^2}{2}, \quad y(t) := \frac{\sigma a t^3}{3} \qquad (\sigma \in \{1, -1\},\ \mathbf{v} := (0,0,1)\,), \end{aligned}$$

where $|t|$ is a sufficiently small number. Then $f(s,t) := \Gamma(s) + x(t)\mathbf{n}(s) + y(t)\mathbf{v}$ gives a light-like cuspidal edge along Γ, which satisfies

$$E^L = \frac{t^2}{4}\left(t^2 - 4\sigma\right), \quad F^L = -a\sigma t^2, \quad G^L = t^2 - a^2 t^4 \tag{5.8}$$

and

$$\Delta_L = t^4\left(-a^2 - \sigma\right) + t^6\left(a^2\sigma + \frac{1}{4}\right) - \frac{a^2 t^8}{4}. \tag{5.9}$$

Moreover, we have (cf .(1.2), (1.3) and (1.4))

$$\tilde{L} = -\frac{a\sigma t^2\left(\sigma t^2 - 2\right)^2}{4}, \qquad \tilde{M} = t^2, \qquad \tilde{N} = -\frac{a t^2\left(\sigma t^2 - 2\right)}{2}. \tag{5.10}$$

In particular, f is of order four if $a^2 + \sigma \neq 0$. By (1.14), (5.8), (5.9) and (5.10), the mean curvature function H^L and the Gaussian curvature are given by

$$\begin{aligned} K^L(s,t) &= \frac{t^4}{8\Delta_L^2}\Big(8\left(a^2\sigma + 1\right) - 12a^2t^2 + 6a^2\sigma t^4 - a^2t^6\Big), \\ H^L(s,t) &= \pm\frac{a t^6}{8|\Delta_L|^{3/2}}\Big(\left(8a^2 + 14\sigma\right) - t^2\left(8a^2\sigma + 3\right) + 2a^2t^4\Big). \end{aligned}$$

Consequently, the Gaussian curvatrue is unbounded and the mean curvature function is bounded. Moreover, if $\sigma = 1$, then f is of convex type and time-like at $(s,0)$ satisfying $K^L > 0$ (cf. Figure 3, left). If $\sigma = -1$ and $a^2+\sigma$ is positive (cf. Figure 3, center), then f around $t = 0$ is of concave type and time-like satisfying $K^L < 0$. If $a^2 + \sigma$ is negative, f is space-like, and $\sigma = -1$, that is, f is of concave type (cf. Figure 3, right). In this case, the Gaussian curvature of f is positive.

By the above discussions, we can observe that cuspidal edges along a light-like curve cannot change their causal type if their order is less than or equal to four. However, if the order is greater than four, such a cuspidal edge exists:

Example 5.6. We fix a real number β, and set

$$\Gamma(s) := (\cos s, \sin s, s), \quad \mathbf{n}(s) := -(\cos s, \sin s, 0) \qquad (s \in \mathbb{R}).$$

Consider the cuspidal edge given by $f(s,t) := \Gamma(s) - x(t)\mathbf{n}(s) - y(t)\mathbf{v}$, where

$$x(t) := \frac{t^2}{2}, \qquad y(t) := \frac{\beta t^3}{6} - \frac{t^4}{8}, \qquad \mathbf{v} = (0,0,1)$$

for sufficiently small $|t|(> 0)$. The coefficients of the first fundamental form of f are computed as

$$E^L = \frac{1}{4}t^2\left(t^2+4\right), \quad F^L = \frac{1}{2}t^2(\beta - t), \qquad (5.11)$$
$$G^L = \frac{1}{4}t^2\left(4-\beta^2t^2-2\beta t^3+t^4\right).$$

Since

$$\Delta_L = (1-\frac{\beta^2}{4})t^4 + \frac{\beta}{2}t^5 + O(t^6),$$

by setting $\beta = 2$, f gives a cuspidal edge of order 5 at each point of Γ. This f changes its causal type along Γ. Moreover, we have

$$\tilde{L} = \frac{-t^2}{8}(t+2)(t^2+2)^2, \quad \tilde{M} = t^2, \quad \tilde{N} = \frac{-t^2}{2}(t+1)(t^2+2). \qquad (5.12)$$

Using(1.14), (5.11) and (5.12), we obtain

$$K^L = -\frac{16(-24+32t-36t^2+24t^3-18t^4+8t^5-3t^6+t^7)}{t^5(-16+16t-16t^2+8t^3-4t^4+t^5)^2},$$
$$H^L = \pm\frac{-16+24t+8t^2-44t^3+44t^4-32t^5+16t^6-6t^7+t^8}{2|t|^{5/2}|-16+16t-16t^2+8t^3-4t^4+t^5|^{3/2}}.$$

In particular, K^L and H^L are both unbounded near $(s,t) = (s,0)$.

In a joint work with Yamada, the fourth author [21] introduced the concept of "maxfaces" as a canonical class of space-like zero mean curvature surfaces with admissible singular points. Similarly, Takahashi [18] and Akamine [1] defined "minfaces" as a class of time-like zero mean curvature surfaces with admissible singular points. In addition, Akamine [1] showed essentially the same assertion as in Theorem G for cuspidal edges on maxfaces and minfaces. The following assertion implies Theorem G is a generalization of it.

Proposition 5.7. *Generalized cuspidal edges on maxfaces or on minfaces are light-like cuspidal edges of order four.*

Proof. Let f be a maxface (resp. a minface). As shown in the proof of [4, Theorem 2.4] (resp. the two line before the equation (8) in [1]), the (Euclidean) area density function λ^E of a maxface (resp. a minface) is a positive function multiple of

$$1-|g|^2 \qquad (\text{resp. } 1-g_1g_2), \qquad (5.13)$$

where g (resp. g_1 and g_2) is one of a pair of Weierstrass data of f, which corresponds (resp. correspond) to the Gauss map of f. We set $\Delta_L := E^LG^L - (F^L)^2$. If f is a maxface, its first fundamental form is given by (cf. [4, (2.1)]) $ds^2 = (1-|g|^2)^2|\omega|^2$. Thus Δ_L is a positive function multiple of $(1-|g|^2)^4$. On the other hand, if f is a minface, it holds that (cf. [1, Proof of Fact 4.2] where $f_u \times_E f_v$ is given)

$$f_u \times_E f_v = \frac{\hat{\omega}_1\hat{\omega}_2}{2}(1-g_1g_2)(g_1-g_2, -1-g_1g_2, -g_1-g_2),$$

where we remark that the signature of $\mathbb{L}^3$ in [1] is $(-++)$, which is different from the one in this paper. So, we have

$$f_u \times_L f_v = \frac{\hat{\omega}_1\hat{\omega}_2}{2}(1-g_1g_2)(g_1-g_2, -1-g_1g_2, g_1+g_2),$$

which implies

$$\begin{aligned}\Delta_L &= \langle f_u \times_L f_v, f_u \times_L f_v\rangle_L \\ &= \frac{\hat{\omega}_1^2\hat{\omega}_2^2}{4}(1-g_1g_2)^2\Big((g_1-g_2)^2+(1+g_1g_2)^2-(g_1+g_2)^2\Big) \\ &= \frac{\hat{\omega}_1^2\hat{\omega}_2^2}{4}(1-g_1g_2)^4.\end{aligned}$$

So Δ_L is a positive function multiple of $(1-g_1g_2)^4$. Regarding (5.13), in both of two cases, Δ_L is a positive function multiple of $(\lambda^E)^4$. On the other hand, if (s,t) is an admissible coordinate system of f, we have $(\lambda^E)_t(p) \neq 0$, since f is a cuspidal edge singular point. So we obtain the conclusion. □

Appendix A. A representation formula for cusps

Definition A.1. Let $\mathbf{c} : I \to \mathbb{R}^2$ be a C^∞-map defined on an open interval I in $\mathbb{R}$ containing the origin $0 \in \mathbb{R}$. We say that $\mathbf{c}(t)$ has a *generalized cusp* at $t=0$ if $\mathbf{c}$ satisfies $\mathbf{c}'(0)=\mathbf{0}$ and $\mathbf{c}''(0)\neq\mathbf{0}$. We remark that $\mathbf{c}''(0)$ is determined up to a positive constant if we change the parameter of $\mathbf{c}$. Moreover, $\mathbf{c}''(0)$ points in the *cuspidal direction*, that is, it points in the direction where the image of $\mathbf{c}$ exists. We say that $\mathbf{c}(t)$ has a *cusp* at $t=0$ if $\mathbf{c}''(0)$ and $\mathbf{c}'''(0)$ are linearly independent.

We set

$$\tau^E := \frac{\det(\mathbf{c}''(0), \mathbf{c}'''(0))}{|\mathbf{c}''(0)|_E^{5/2}},$$

which is called the *cuspidal curvature* of $\mathbf{c}$ at $t=0$, where

$$|\mathbf{a}|_E := \sqrt{\mathbf{a}\cdot\mathbf{a}} \qquad (\mathbf{a} \in \mathbb{E}^2).$$

The (Euclidean) curvature $\kappa^E(t)$ of $\mathbf{c}$ at $t \neq 0$ is given by

$$\kappa^E(t) = \frac{\det(\mathbf{c}'(0), \mathbf{c}''(0))}{|\mathbf{c}'(0)|_E^3}.$$

The function

$$s^E(t) := \int_0^t |\mathbf{c}'(0)|_E dt$$

gives the signed-arc-length of $\mathbf{c}(t)$ from $t=0$. As in [14], it holds that

$$\tau^E = 2\sqrt{2}\lim_{t\to 0}\kappa^E(t)\sqrt{|s^E(t)|}. \tag{A.1}$$

Moreover, the following formula is known:

Fact A.2 (Shiba-Umehara [14])**.** *Let $\mu(t)$ be a C^∞-function defined on an open interval I containing $0 \in \mathbb{R}$. If we set*

$$\mathbf{c}(t) := \int_0^t u(\cos\lambda(u), \sin\lambda(u))du, \qquad \lambda(t) := \int_0^t \mu(u)du, \tag{A.2}$$

then $\mathbf{c}$ is a generalized cusp at $t=0$ in $\mathbb{R}^2$. In particular, $t=0$ is a cusp if and only if $\mu(0)\neq 0$. Conversely, any map germs of a generalized cusp in $\mathbb{R}^2$ can be obtained by (A.2) up to motions in $\mathbb{E}^2$. Moreover, the coefficients $\{\mu_i\}_{i=0}^\infty$ of the Taylor expansion of μ satisfying

$$\mu(t) \approx \mu_0 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \frac{\mu_3}{3!}t^3 + \cdots$$

gives the series of invariants for generalized cusps in $\mathbb{E}^2$. In particular, $2\mu_0$ coincides with τ_E.

This formula is slightly different from the original one in [14] but is essentially the same, and the proof of the formula (A.2) is given in the appendix of [8] and in the book [16].

Remark A.3. Let $\kappa^E(t)$ be a C^∞-function defined on an open interval I containing $0 \in \mathbb{R}$. If we set

$$\text{(A.3)} \qquad \mathbf{c}(t) := \int_0^t (\cos\lambda(u), \sin\lambda(u))du, \qquad \theta(t) := \int_0^t \kappa^E(u)du,$$

then $\mathbf{c}$ is a regular curve with arc-length parameter at $t = 0$ in $\mathbb{E}^2$ whose curvature function is κ^E. Conversely, a regular curve with arc-length parameter can be obtained by this formula up to motions in $\mathbb{E}^2$. So Fact A.2 can be considered as an analogue of this classical formula in $\mathbb{E}^2$. The parameter t of $\mathbf{c}$ in the formula (A.3) is called a *normalized half-arc-length parameter* of $\mathbf{c}$ (see [16, Appendix B] for details).

In the plane $\mathbb{L}^2$, we can construct a similar representation formula as follows: We denote by $\langle\,,\,\rangle$ the Lorentzian inner product of $\mathbb{L}^2$.

Definition A.4. A generalized cusp of $\mathbf{c}(t)$ at $t = 0$ is said to be *space-like* (resp. *time-like*) if $\mathbf{c}''(0)$ is space-like, that is, $\langle \mathbf{c}''(0), \mathbf{c}''(0)\rangle_L > 0$ (resp. time-like, that is, $\langle \mathbf{c}''(0), \mathbf{c}''(0)\rangle_L < 0$).

If $\mathbf{c}$ is a space-like (resp. time-like) cusp, then $\mathbf{c}'(t)$ $(t \neq 0)$ is a space-like (resp. time-like) vector for each sufficiently small $|t|(> 0)$. By the transformation $(x, y) \mapsto (y, x)$, the generalized space-like cusps become generalized time-like cusps, so we only consider here generalized space-like cusps:

We set

$$\tau^L := \frac{\det(\mathbf{c}''(0), \mathbf{c}'''(0))}{|\mathbf{c}''(0)|_L^{5/2}},$$

which is called the *cuspidal curvature* of $\mathbf{c}$ at $t = 0$, where

$$|\mathbf{a}|_L := \sqrt{\langle \mathbf{a}, \mathbf{a}\rangle_L} \qquad (\mathbf{a} \in \mathbb{L}^2).$$

The (Lorentzian) curvature $\kappa^L(t)$ of $\mathbf{c}$ at $t \neq 0$ is given by

$$\kappa^L(t) = \frac{\det(\mathbf{c}'(0), \mathbf{c}''(0))}{|\mathbf{c}'(0)|_L^3}.$$

The function

$$s^L(t) := \int_0^t |\mathbf{c}'(0)|_L dt$$

gives the signed-arc-length of $\mathbf{c}(t)$ from $t = 0$ in $\mathbb{L}^2$. By imitating the argument in [14], it holds that

$$\text{(A.4)} \qquad \tau^L = 2\sqrt{2}\lim_{t\to 0} \kappa^L(t)\sqrt{|s^L(t)|}.$$

Proposition A.5. *Let $\mu(t)$ be a C^∞-function defined on an open interval I containing $0 \in \mathbb{R}$. If we set*

$$\text{(A.5)} \qquad \mathbf{c}(t) := \int_0^t u(\cosh\lambda(u), \sinh\lambda(u))du, \qquad \lambda(t) := \int_0^t \mu(u)du,$$

then $\mathbf{c}$ is a space-like generalized cusp at $t = 0$ in $\mathbb{L}^2$. In this situation, $t = 0$ is a cusp if and only if $\mu(0) \neq 0$. Conversely, any map germs of space-like generalized cusp can be obtained by this formula up to motions in $\mathbb{L}^2$. In particular, the coefficients $\{\mu_i\}_{i=0}^\infty$ of the Taylor expansion of μ satisfying

$$\mu(t) \approx \mu_0 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \frac{\mu_3}{3!}t^3 + \cdots$$

gives the series of geometric invariants of the generalized cusps in $\mathbb{L}^2$. In particular, $2\mu_0$ coincides with τ_L.

Proof. We denote by $\langle , \rangle_L$, which is the Lorentzian inner product on $\mathbb{L}^2$. For a vector $\mathbf{a} \in \mathbb{R}^2$, we set $|\mathbf{a}|_L := \sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle_L|}$. Let $\mathbf{c}(t)$ be a curve given in the formula of Proposition A.5. If we set

$$\mathbf{e}(t) := \begin{pmatrix} \cosh\theta(t) \\ \sinh\theta(t) \end{pmatrix},$$

then we have $\mathbf{c}'(t) = t\mathbf{e}(t)$. Thus

$$\mathbf{n}(t) := \begin{pmatrix} \sinh\theta(t) \\ \cosh\theta(t) \end{pmatrix}$$

gives a smooth unit normal vector field along $\mathbf{c}$. So $\mathbf{c}$ gives a generalized cusp at $t = 0$. The (Lorentzian) arc-length parameter of $\mathbf{c}$ satisfies

$$s(t) := \int_0^t |\mathbf{c}'(u)|_L du = \int_0^t |u| du = \mathrm{sgn}(t)\frac{t^2}{2}.$$

In particular, we have

$$t = \mathrm{sgn}(s)\sqrt{2|s|}. \tag{A.6}$$

Then, $\mathbf{c}''(t) = \mathbf{e}(t) + t\mu(t)\mathbf{n}(t)$ and the curvature function satisfies

$$\kappa^L(t) = \frac{\det(\mathbf{c}'(t), \mathbf{c}''(t))}{|\mathbf{c}'(t)|_L^3} = \frac{t^2\mu(t)\det(\mathbf{e}(t), \mathbf{n}(t))}{|t|^3} = \frac{\mu(t)}{|t|}. \tag{A.7}$$

By this with (A.6), we have

$$\mu(t) = \kappa^L(t)\sqrt{2|s(t)|} \qquad (\text{if } t \neq 0). \tag{A.8}$$

By (A.4), we have $\mu^L(0) := \tau^L/2$. Thus, $t = 0$ is a cusp if and only if $\mu^L(0) \neq 0$.

Conversely, let $\mathbf{c}(t)$ ($t \in I$) be a space-like generalized cusp at $t = 0$ in $\mathbb{L}^2$ defined on an open interval I containing $0 \in \mathbb{R}$. Since $t = 0$ is a cusp, the arc-length $s(t) := \int_0^t |\mathbf{c}'(t)|_L dt$ of the curve $\mathbf{c}$ is not smooth at $t = 0$. However (cf [16, Proposition B.2.1])

$$v(t) := \mathrm{sgn}(t)\sqrt{|2s(t)|} \tag{A.9}$$

is a C^∞-function on I satisfying $v'(t) > 0$. In fact, $\mathbf{w}(t) := (1/t)\mathbf{c}'(t)$ is a vector-valued C^∞-function of t even at $t = 0$. Since $\mathbf{w}(0) \neq \mathbf{0}$, the absolute value $\psi(t) := |\mathbf{w}(t)|_L$ is also C^∞-differentiable. By [16, (B.8) and Proposition A.4], we have $v(t) = t\sqrt{\Psi(t)}$, where Ψ is a C^∞-function of t given by $\Psi(t) := \int_0^1 u\psi(tu)du$. Since $\Psi(0) \neq 0$ (cf. [16, (A.2)]), $v(t)$ is C^∞-differentiable at $t = 0$. So we can use v as a new parameter of $\mathbf{c}$ around the cusp, which is called the *normalized half-arc-length parameter* of $\mathbf{c}$. By L'Hopital's law, we have

$$\begin{aligned} v'(0) &= \lim_{t\to 0} \mathrm{sgn}(t)\frac{\sqrt{2|s(t)|}}{t} = \lim_{t\to 0}\left|\frac{s(t)}{t}\right| \\ &= \left|\lim_{t\to 0}\frac{s(t)}{t^2}\right| = \left|\lim_{t\to 0}\frac{|\mathbf{c}'(t)|_L}{2t}\right| = \frac{|\mathbf{c}''(0)|_L}{2} > 0, \end{aligned}$$

and so v can be taken as a new parameter of $\mathbf{c}$. Since $\mathbf{c}(v)$ has a cusp at $v = 0$, there exists a smooth unit normal vector field $\mathbf{n}(t)$ along $\mathbf{c}$. We the take a smooth normal vector field $\mathbf{e}(v)$ along $\mathbf{c}$ such that $(\mathbf{e}(v), \mathbf{n}(v))$ gives an orthonormal frame field satisfying $\det(\mathbf{e}(v), \mathbf{n}(v)) = 1$ for each $v \in I$.

Since $\mathbf{c}(t)$ has a cusp at $t = 0$, the direction of the unit normal vector $\mathbf{n}(t)$ changes from the right side of the curve $\mathbf{c}(t)$ to the left side, or from the left side of $\mathbf{c}(t)$ to the right side, just at t=0. There exists a smooth unit tangent vector field

$\mathbf{e}(t)$ along $\mathbf{c}$, which change direction with respect to $\mathbf{c}'$ at $t = 0$. So, by changing $\mathbf{n}(v)$ by $-\mathbf{n}(v)$, we may assume that $\mathbf{e}$ satisfies

$$\mathbf{e}(s) = \begin{cases} \mathbf{c}_s(s) & \text{if } s > 0, \\ -\mathbf{c}_s(s) & \text{if } s < 0, \end{cases} \tag{A.10}$$

where $\mathbf{c}_s := d\mathbf{c}/ds$. Since $v^2/2 = \mathrm{sgn}(s)s$, we have

$$\mathbf{c}_v(v) = \mathbf{c}_s(v)s_v(v) = (\mathrm{sgn}(v)\mathbf{e}(v))(\mathrm{sgn}(v)(v^2/2)_v) = v\mathbf{e}(v).$$

By (A.10), $\mathbf{n}(v)$ is the left-ward unit normal vector field of $\mathbf{c}$ if $v > 0$. By the Frenet equation, we have

$$\mathbf{e}_v(v) = \mathbf{e}_s(v)s_v(v) = \Big(\mathrm{sgn}(v)\kappa(v)\mathbf{n}(v)\Big)\Big(\mathrm{sgn}(v)v^2/2\Big)_v = \kappa(v)v\mathbf{n}(v).$$

We set $\mu(v) := \kappa(v)v$, which can be extended as a smooth function of v. By definition, $\mathbf{e}_v = \mu\mathbf{n}$ holds, and $\mu(v)$ coincides with the function defined by (A.7) (by setting $t := v$). Since $\mathbf{n}_v$ is perpendicular to $\mathbf{n}$, we can write $\mathbf{n}_v(v) = a(v)\mathbf{e}(v)$, and

$$a = \mathbf{n}_v \cdot \mathbf{e} = (\mathbf{n} \cdot \mathbf{e})_v - \mathbf{n} \cdot \mathbf{e}_v = -\mathbf{n} \cdot \mathbf{e}_v = \mu.$$

Consequently, we obtain the formula

$$\big(\mathbf{c}(v), \mathbf{e}(v), \mathbf{n}(v)\big)_v = \big(\mathbf{c}(v), \mathbf{e}(v), \mathbf{n}(v)\big) \begin{pmatrix} 0 & 0 & 0 \\ v & 0 & \mu(v) \\ 0 & \mu(v) & 0 \end{pmatrix},$$

which can be considered as a linear ordinary differential equation when we think $\mu(v)$ is a known function and $\mathbf{c}(v), \mathbf{e}(v), \mathbf{n}(v)$ are unknown vector valued functions. By replacing v by t, the curve given in the formula of Proposition A.5 gives a solution of this equation. Then, the uniqueness of the solution with an initial value condition implies that any generalized cusp can be represented as the formula in Proposition A.5 up to a orientation preserving motion in $\mathbb{L}^2$. □

Definition A.6. In this paper, we call the function $\mu(t)$ appeared in the formulas (A.2) and (A.5) the *μ-function* associated with the germ of cusp $\mathbf{c}$.

Appendix B. Special coordinate systems for generalized cuspidal edges

In this section, we show the following:

Proposition B.1. *Let $f(s,t)$ ($s \in I$, $|t| < \varepsilon$) be a generalized cuspidal edge along a regular curve $\Gamma : I \to \mathbb{R}^3$, where I is an open interval. Suppose that there exist vector fields $\mathbf{a}_1(s)$ and $\mathbf{a}_2(s)$ along Γ such that $\Gamma'(s)$, $\mathbf{a}_1(s)$, $\mathbf{a}_2(s)$ are linearly independent in $\mathbb{R}^3$ for each $s \in I$. Then, for each $s_0 \in I$, there exist a local diffeomorphism germ $(u,v) \mapsto (s(u,v), t(u,v))$ defined on a sufficiently small neighborhood $(s_0, 0)$ such that*

1. *$s(u,0) = u$ and $t(u,0) = 0$ for each u,*
2. *f is written in the form*
$$f \circ \varphi(u,v) = \Gamma(u) + x(u,v)\mathbf{a}_1(u) + y(u,v)\mathbf{a}_2(u),$$
where $x(u,v)$ and $y(u,v)$ are C^∞-functions defined on a neighborhood of $(s_0, 0)$, and
3. *the map $v \mapsto (x(u,v), y(u,v))$ is a generalized cusp as defined in Appendix A.*

If $\{\mathbf{a}_1(s),\ \mathbf{a}_2(s)\}$ spans the plane orthogonal to $\Gamma'(s)$ in $\mathbb{E}^3$, almost the same result was proved in [8, Lemma 3.2]. The above statement can be seen as a generalization of this, but the proof is different.

Proof. Without loss of generality, we may assume that I contains 0 and set $s_0 := 0$. Consider the map defined by

$$\Phi : \mathbb{R}^2 \times I \ni (x, y, z) \mapsto \Gamma(z) + x\mathbf{a}_1(z) + y\mathbf{a}_2(z) \in \mathbb{R}^3.$$

Since $\Gamma'(0)$, $\mathbf{a}_1(0)$, $\mathbf{a}_2(0)$ are linearly independent, Φ gives a diffeomorphism defined on a neighborhood of the origin in $\mathbb{R}^3$. Since $\Gamma(s) = f(s, 0)$ holds, $g(s,t) := \Phi^{-1} \circ f(s,t)$ satisfies $g(s, 0) = (0, 0, s)$ for sufficiently small $|s|$. Then we can apply the argument given in [16, Page 105] (at which only the fact that f is a generalized cuspidal edge is needed until the final argument given in [16, Page 106]). We can take an admissible local coordinate system (see Definition 1.4) $(u, v) \mapsto (s(u, v), t(u, v))$ such that $s(0, 0) = t(0, 0) = 0$ and

$$g(u, v) = (x(u, v), y(u, v), u), \qquad x(u, v) = v^2\alpha(u, v), \quad y(u, v) = v^2\beta(u, v),$$

where α and β are C^∞-functions defined on a neighborhood of the origin in $\mathbb{R}^2$. So we have

$$f(u, v) = \Phi(v^2\alpha(u, v), v^2\beta(u, v), u) = \Gamma(u) + v^2\alpha(u, v)\mathbf{a}_1(u) + v^2\beta(u, v)\mathbf{a}_2(u).$$

Since f is a generalized cuspidal edge, we have

$$\mathbf{0} \neq f_{vv}(u, 0) = 2\alpha(u, 0)\mathbf{a}_1(u) + 2\beta(u, 0)\mathbf{a}_2(u),$$

which implies that $(x_{vv}(u, 0), y_{vv}(u, 0)) \neq \mathbf{0}$, and $v \mapsto (x(u, v), y(u, v))$ gives a generalized cusp for sufficiently small $|v|$. □

Appendix C. Umbilic points on a wave front in $\mathbb{L}^3$

Consider a wave front $f : U \to \mathbb{R}^3$, where U is a domain of $(\mathbb{R}^2; u, v)$. Since cuspidal edge singular points appear on wave fronts, this fits the setting of this paper. If we think $\mathbb{R}^3 = \mathbb{E}^3$, umbilical points never accumulate at cuspidal edge singular points of f (cf. [16, Corollary 5.3.3]). As an analogue of this fact, for $\mathbb{R}^3 = \mathbb{L}^3$, we show the following:

Proposition C.1. *If p is a space-like or time-like cuspidal edge singular point, then umbilical points of f never accumulate at p.*

As we have mentioned at the end of Section 1, we do not know whether umbilical points can accumulate at a light-like cuspidal edge singular point or not. To prove the proposition, we first consider an immersion $f : U \to \mathbb{L}^3$ and assume that U consists only of space-like points or only of time-like points. Then we can take a unit normal vector field $\nu^L : U \to \mathbb{L}^3$ of f. Define two 3×3 matrices by $P_1 := (f_u, f_v, \nu^L)$ and $P_2 := (\nu_u, \nu_v, \nu^L)$. By setting,

$$I := \begin{pmatrix} E^L & F^L \\ F^L & G^L \end{pmatrix}, \qquad II := \begin{pmatrix} L^L & M^L \\ M^L & N^L \end{pmatrix},$$

$W := I^{-1}II$ coincides with the matrix given in (1.15). Then, by (1.9), we have that

$$\begin{aligned} -(P_1^{-1})^T P_2 &= -(P_1^T E_3 P_1)^{-1}(P_1^T E_3 P_2) \\ &= \begin{pmatrix} I & 0 \\ 0 & \pm 1 \end{pmatrix}^{-1} \begin{pmatrix} II & 0 \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} I^{-1}II & 0 \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & \pm 1 \end{pmatrix}. \end{aligned}$$

So we obtain the following Lorentzian version of the Weingarten formula

$$(\nu_u^L, \nu_v^L) = -(f_u, f_v)W. \tag{C.1}$$

Lemma C.2. *Let $f : U \to \mathbb{L}^3$ be a wave front, and $p \in U$ its singular point at which f is space-like or time-like. Consider the family of parallel surfaces of f given by $f^t := f + t\nu^L$ ($t \in \mathbb{R}$), where ν^L is the unit normal vector field of f. Then for sufficiently small $t \neq 0$, p is a regular point of f^t.*

Proof. The proof is parallel to the case of $\mathbb{R}^3 = \mathbb{E}^3$, see [16, page 65]. □

Proposition C.3. *Let $f : U \to \mathbb{L}^3$ be a wave front and $p \in U$ a regular point which is space-like or time-like. Then the parallel surface f^t $(t \neq 0)$ has a singular point at p if and only if $1/t$ coincides with the principal curvature of f at p.*

Proof. Since ν^L is a unit vector field, $(\nu^L)_u$ and $(\nu^L)_v$ are perpendicular to ν^L. So ν^L is a common unit normal vector field of f^t $(t \in \mathbb{R})$. By (C.1), we have

$$(f^t)_u = f_u + t\nu^L_u = f_u(I + tW), \qquad (f^t)_v = f_v + t\nu^L_v = f_v(I + tW)$$

and

$$\Big((f^t)_u, (f^t)_v\Big) = (f_u, f_v)(I + tW). \tag{C.2}$$

Thus, p is a singular point of f^t if and only if $\det(I + tW) = 0$, that is, $1/t$ is an eigenvalue of the matrix W, proving the assertion. □

Corollary C.4. *If p is an umbilical point of f, then it is also an umbilical point of f^t unless $1/t$ coincides with the principal curvature of f.*

Proof of Proposition C.1. Suppose that there exists a sequence of umbilics $\{p_k\}_{k=1}^{\infty}$ of f on U converging to p. By Lemma C.2, we can choose $t \neq 0$ so that p is a regular point of f^t. Then there exists a positive integer N so that each p_k $(k \geq N)$ is an umbilical point of f^t. Since p is the limit of $\{p_k\}_{k=1}^{\infty}$, p itself is an umbilical point of f^t. Since $f = (f^t)^{-t}$, the value $-1/t$ must coincides with one of the two principal curvature of f^t at p. Moreover, by (C.2), $(f_u, f_v) = ((f^t)^{-t}_u, (f^t)^{-t}_v)$ vanishes at p, which contradicts the fact that the Jacobi matrix is of rank one at p. □

If f is a generalized cuspidal edge, then umbilical points may accumulate at a singular point:

Example C.5. We set $\Gamma(u) := (u, u^2, u^4)$, and consider a map $f(u, v) = \Gamma(u) + v\Gamma'(u)$ $(u, v \in \mathbb{R})$, which is the map given in [4, Example 1.13]. As pointed out in [4], $o := (0, 0)$ is a cuspidal cross cap singular point of f. In $\mathbb{L}^3$, the origin o is a space-like singular point, and the second fundamental form vanishes along the v-axis. Since $(0, v)$ $(v \neq 0)$ are regular points of f, they are umbilics of f which accumulate at the singular point o.

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(Toshizumi Fukui) Department of Mathematics, Saitama University, 255, Shimo-Okubo, Saitama, 338-8570, Japan
Email address: tfukui@rimath.saitama-u.ac.jp

(Ryosuke Kinosita) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1-W8-34, O-okayama, Meguro-ku, Tokyo 152-8552, Japan.
Email address: kinosita.r.math@gmail.com

(Donghe Pei) School of Mathematics and Statics, Northeast Normal University, Changchun 130024 Jilin China
Email address: peidh340@nenu.edu.cn

(Masaaki Umehara) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1-W8-34, O-okayama, Meguro-ku, Tokyo 152-8552, Japan.
Email address: umehara@is.titech.ac.jp

(Haiou Yu) School of Mathematics, Jilin University of Finance and Economics, Changchun 130117 Jilin, China
Email address: yuhaiou@jlufe.edu.cn