Versality of the folding families of regular surfaces

Toshizumi Fukui* and Atsuki Hiramatsu†

Department of Mathematics, Saitama University
255 Shimo-Okubo, Sakura-ku, Saitama 338-8570, Japan

E-mail addresses: tfukui@rimath.saitama-u.ac.jp, a.hiramatsu.2357@gmail.com

March 14, 2024

Abstract

We investigate $\mathcal{A}$-versality of the folding family introduced by Bruce and Wilkinson, which describes infinitesimal reflectional symmetry of a regular surface in Euclidean 3-space. We obtain several geometric conditions which ensure $\mathcal{A}$-versality of the folding family.

We consider the restriction of the folding map

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad (x, y, z) \longmapsto (x, y^2, z), \tag{0.1}$$

to the surface $\mathcal{M}$ defined by an embedding g whose 2-jet is given by

$$(x, y) \mapsto (x, y, a_{10}x + a_{01}y + a_{20}\tfrac{x^2}{2} + a_{11}xy + a_{02}\tfrac{y^2}{2}).$$

We easily see the following:

- the map $f|_{\mathcal{M}}$ is nonsingular at $(0, 0)$ if $\boldsymbol{v}$ is not tangent to $\mathcal{M}$, that is, $(a_{10}, a_{01}) \neq 0$,
- the map $f|_{\mathcal{M}}$ has a singularity $\mathcal{A}$-equivalent to cross-cap (S_0) at $(0, 0)$ if and only if $\boldsymbol{v}$ is tangent to $\mathcal{M}$ and does not generate a principal direction of $\mathcal{M}$ at 0, that is, $(a_{10}, a_{01}) = 0$ and $a_{11} \neq 0$,

where $\boldsymbol{v}$ denotes a unit vector which is perpendicular to the reflection plane $y = 0$.

So if we investigate more degenerate singularity of $f|_{\mathcal{M}}$, it is natural to assume that the embedding g is given by the following Monge form:

$$g(x, y) = (x,\, y,\, h(x, y)) \quad h(x, y) = \tfrac{k_1x^2 + k_2y^2}{2} + \sum_{i+j\geq 3}^{m} a_{ij}\tfrac{x^iy^j}{i!j!} + O(m + 1). \tag{0.2}$$

2020 Mathematics subject classification. 58K05, 58K60, 58K70

Key words and phrases. $\mathcal{A}$-versal unfolding, the folding family, focal set, symmetry set, ridge line, subparabolic line.

*The first author is supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (C) Grant Number 23K03106.

†Supported by RIMS symposia (open) "Extension of singularity theory", Kyoto University.

where m is an integer ≥ 3. Here $O(m+1)$ denotes a term whose absolute value is at most a positive constant multiple of $|(x,y)|^{m+1}$ near 0. When the origin is not umbilic (that is, $k_1 \neq k_2$), the vectors ∂_x and ∂_y generate principal directions at the origin. They can be extended to the principal vectors on the surface which we denote by v_1 and v_2, respectively.

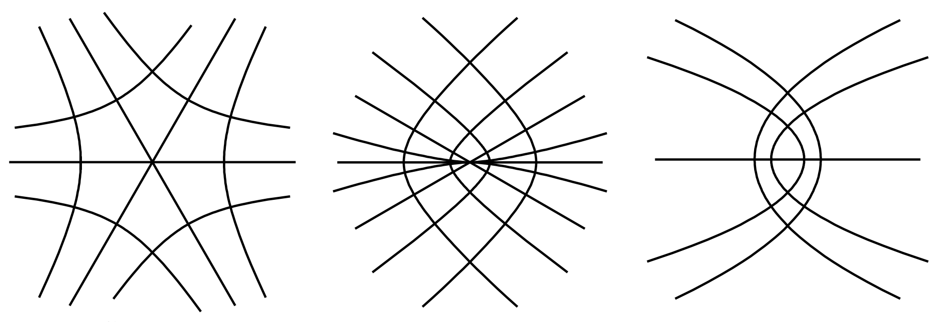
Bruce and Wilkinson showed the list of singularities of the folding map $f|_{\mathcal{M}}$ in a generic context, mentioning several geometric meaning ([2, Page 68]), as follows:

S_1	general smooth point
S_2	parabolic smooth point of focal set
S_3	cusp of gauss at smooth point of focal set
B_2	general cusp point of focal set
B_3	(cusp) point of focal set in closure of parabolic curve on symmetry set
C_3	intersection point of cuspidal edge and parabolic curve on focal set

Here we use the notations introduced by Mond ([10]).

Bruce and Wilkinson ([2]) also introduced the folding family, which is the restriction to $\mathcal{M}$ of the family of maps obtained by conjugating the map (0.1) by Euclidean motions. They showed that the folding family is $\mathcal{A}$-versal for a residual set of embeddings $\mathcal{M} \subset \mathbb{R}^3$. We recall these results as Theorem 1.2. Since Bruce and Wilkinson ([2]) did not show any explicit criteria for $\mathcal{A}$-versality in [2], it is an interesting problem to describe them. The folding map is motivated by describing infinitesimal reflectional symmetry of a regular surface, and the conditions being $\mathcal{A}$-versal should have several geometric meanings.

In this paper, we first give criteria of singularities of the folding map $f|_{\mathcal{M}}$ in terms of the double point locus of $f|_{\mathcal{M}}$ (Theorem 1.11). The main topic is to describe explicit criteria for $\mathcal{A}$-versality of the folding family and discuss their geometric meaning. Our main results are stated as Theorem 1.4 for non-umbilic points, and Theorems 4.19 for umbilic points. These are based on Lemma 1.9, which shows the necessary and sufficient conditions for $\mathcal{A}$-versality in terms of the coefficients of (0.2). We describe several consequences here. For non-umbilic points, the geometric criteria for $\mathcal{A}$-versality are stated using subparabolic lines and ridge lines. For example, if the folding map has a B_2 singularity, then the folding family is $\mathcal{A}$-versal if and only if the corresponding ridge line is nonsingular there (Theorem 1.4 (iv)). For umbilic points, we claim that the folding family is always $\mathcal{A}$-versal when the folding map has S_1, S_2, S_3 and B_2 singularity at Darbouxian umbilics (star, monstar and lemon) (see Theorem 4.31).



Configuration of curvature lines at Darbouxian umbilics

The paper is organized as follows. In §1, we recall the definition of the folding family, and state a main theorems at a non-umbilic point clarifying several geometric meaning of its $\mathcal{A}$-verality. We also discuss here the crirteria of singularutuies of folding map $f|_{\mathcal{M}}$ in terms of the double point locus of $f|_{\mathcal{M}}$. In §2, we recall the duality between focal/symmetry sets and the bifurcation sets of the folding families. In §3, we investigate the conditions appeared in our main theorem for non-umbilic points. To do this we describe derivatives of principal curvatures by principal vectors including higher orders. In §4, we recall classification Darbouxian umbilics and show our main theorems for umbilic points. In §5, we show Lemma 1.9, which is a key lemma in our calculation.

The authors are grateful to Farid Tari, since discussion with him motivates the paper.

Contents

1 The folding family F

1.1 Definition of the folding family F

Bruce and Wilkinson ([2]) defined the folding family F as follows:

Let $\mathcal{M}$ be a nonsingular surface in $\mathbb{R}^3$. Let $\mathcal{G}$ denote the group of motions of the Euclidean space $\mathbb{R}^3$. We define

$$\overline{F} : \mathcal{M} \times \mathcal{G} \longrightarrow \mathbb{R}^3 \quad \text{by} \quad \overline{F}(\boldsymbol{p}, A) = A^{-1} \circ f \circ A(\boldsymbol{p}). \tag{1.1}$$

Remark that this map is actually defined on $\mathbb{R}^3 \times \mathcal{G}$ and we are thinking its restriction to $\mathcal{M} \times \mathcal{G}$. Let Π_0 denote the plane defined by $y = 0$. If $\mathcal{H}$ denotes the subgroup of $\mathcal{G}$ preserving the region $y \geq 0$, then $\overline{F}$ gives rise to a family of foldings at the plane $\Pi = A^{-1}\Pi_0$. Remark that the quotient group $\mathcal{G}/\mathcal{H}$ parametrizes the planes in $\mathbb{R}^3$. Identifying the quotient group $\mathcal{G}/\mathcal{H}$ with the space $\mathcal{P}$ of all planes in $\mathbb{R}^3$, we define the **folding family**

$$F : \mathcal{M} \times \mathcal{P} \longrightarrow \mathbb{R}^3, \quad \text{by} \quad (\boldsymbol{p}, \Pi) \longmapsto \overline{F}(\boldsymbol{p}, A),$$

where A is a motion with $\Pi = A\Pi_0$. We also define $f^{\Pi} : \mathcal{M} \longrightarrow \mathbb{R}^3$ by $f^{\Pi}(\boldsymbol{p}) = F(\boldsymbol{p}, A)$.

Theorem 1.2 ([2, Proposition 2.2])**.** *For a residual set of embedding $\mathcal{M} \subset \mathbb{R}^3$ the folding maps $f|_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbb{R}^3$, have singularities $\mathcal{A}$-equivalent to one of the following types:*

type	normal form	$\mathcal{A}_e$-codimension	order of $\mathcal{A}$-determinacy	C
S_0	(x, y^2, xy)	0	2	1
$S_1^{\pm}$	$(x, y^2, y^3 \pm x^2y)$	1	3	2
S_2	$(x, y^2, y^3 + x^3y)$	2	4	3
$S_3^{\pm}$	$(x, y^2, y^3 \pm x^4y)$	3	5	4
$B_2^{\pm}$	$(x, y^2, x^2y \pm y^5)$	2	5	2
$B_3^{\pm}$	$(x, y^2, x^2y \pm y^7)$	3	7	3
$C_3^{\pm}$	$(x, y^2, xy^3 \pm x^3y)$	3	4	3

Here C is an invariant due to Mond, which bounds the number of cross cap appeared in stable deformations of each singularity. Moreover, these singularities are $\mathcal{A}$-versally unfolded by the family F.

We do not recall the theory on $\mathcal{A}$-versality in the paper. We just remark that the condition equivalent to the $\mathcal{A}$-versality of the folding family is stated as (5.3). The notion of $\mathcal{A}$-versality is important, since two $\mathcal{A}$-versal unfoldings of a map-germ are equivalent. See also [11, §3] more for $\mathcal{A}$-versality.

Since Bruce and Wilkinson ([2]) did not mention explicit conditons for $\mathcal{A}$-versality of the situation above, the theorem above becomes much useful after we clarify several geometric meanings of the criteria of $\mathcal{A}$-versality of F.

We recall the notions of ridge points and subparabolic points here.

Definition 1.3 ([7])**.** Let p be non umbilical point of a regular surface with principal vectors v_1, v_2, and the corresponding principal curvatures κ_1, κ_2, which are defined near p.

- We say that the point p is a v_i**-ridge** point, $i = 1, 2$, if $v_i\kappa_i(p) = 0$, where $v_i\kappa_i$ is the directional derivative of κ_i in v_i. Moreover, we say p is the **first order** v_i**-ridge** if $v_i^2\kappa_i(p) \neq 0$. The closure of the set of v_i-ridge points is called a v_i**-ridge line** if it is of one-dimensional.

- We say that the point p is a v_i**-subparabolic** point if $v_i\kappa_j(p) = 0$ $(i \neq j)$. The closure of the set of v_i-subparabolic points is called a v_i**-subparabolic line** if it is of one-dimensional.

We now state several geometric criteria of the singularity of the folding map and $\mathcal{A}$-versality of the folding families at non-umbilic points as follows.

Theorem 1.4. *Assume that we consider a point on the surfaces, which is not umbilic, and* $\boldsymbol{v}$ *generetes a principal direction there. We asuume that* v_2 *is the principal vector, which is an extension of* $\boldsymbol{v}$*.*

(i) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* $S_1^{\pm}$ *if and only if the point is neither* v_2*-ridge nor* v_2*-subparabolic. Moreover, the folding family* F *is automatically* $\mathcal{A}$*-versal there.*

(ii) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* S_2 *if and only if the point is* v_2*-subparabolic, but not* v_2*-ridge and the* v_2*-subparabolic line is not tangent to the reflection plane* Π_0 *there. Moreover, the folding family* F *is automatically* $\mathcal{A}$*-versal there.*

(iii) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* $S_3^{\pm}$ *if and only if the point is* v_2*-subparabolic, but not* v_2*-ridge and* $v_2^2\kappa_1(0) \neq 0$*. Moreover, the folding family* F *is* $\mathcal{A}$*-versal if and only if the* v_2*-subparabolic line is nonsingular. In this case, we automatically have that the* v_2*-subparabolic line has 2-point contact with the reflection plane* Π_0 *there.*

(iv) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* $B_2^{\pm}$ *if and only if the point is* v_2*-ridge, but not* v_2*-subparabolic and the double point locus* $D(f|_{\mathcal{M}})$ *has* $A_3^{\pm}$ *singularity with tangent property with respect to* $\boldsymbol{v}$ *(see Definition 1.10). Moreover, the folding family* F *is* $\mathcal{A}$*-versal if and only if the* v_2*-ridge line is nonsingular there.*

(v) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* $B_3^{\pm}$ *if and only if the point is* v_2*-ridge, but not* v_2*-subparabolic and the double point locus* $D(f|_{\mathcal{M}})$ *has* $A_5^{\pm}$ *singularity with tangent property with respect to* $\boldsymbol{v}$ *(see Definition 1.10). Moreover, the folding map* f^{Π_θ} *is* $\mathcal{A}$*-versally unfolded by the folding family* F *for a generic choice of the 6-jet of* (0.2)*. The condition for* $\mathcal{A}$*-versality is explicitly stated in Lemma 1.9.*

(vi) *The folding map* $f|_{\mathcal{M}}$ *has a singularity* $\mathcal{A}$*-equivalent to* $C_3^{\pm}$ *if and only if the point is* v_2*-subparabolic and* v_2*-ridge and the* v_2*-subparabolic line and the* v_2*-ridge line are nonsingular and intersect the reflection plane* Π_0 *transversely. Moreover, the folding family* F *is* $\mathcal{A}$*-versal if and only if the* v_2*-subparabolic line and* v_2*-ridge line intersect transversely there.*

Please refer to §1.3 for the definition (and several properties) of the double point locus $D(f|_{\mathcal{M}})$,

Remark 1.5. • The authors found that the item (iii), the condition for $\mathcal{A}$-versality for S_3 singularity, is already obtained by Wilkinson (see after Corollary 3.3 of [1]) and that the item (vi), the condition for $\mathcal{A}$-versality for C_3 singularity, is already obtained in [1, Theorem 4.6 (i)]. The authors show Theorem 1.4 without knowing [1]. The authors are not able to find litertures to state the items (i), (ii) and (iv).

- The geometric meaning of the condition ($\mathcal{B}_3 \neq 0$ in the notation of Lemma 1.9 below) of $\mathcal{A}$-versality for $B_3^\pm$ singularity is not clear for the authors.

Remark 1.6. In [5], we have discussed the conditions for $\mathcal{A}$-versality of the subunfolding of the folding family, obtained by restricting the motions to the rotations.

1.2 Criteria of singularities of $f|_{\mathcal{M}}$ and $\mathcal{A}$-versality of F

We start to describe a criteria of singularity of $f|_{\mathcal{M}}$ in terms of Monge form (0.2).

Lemma 1.7. *Let $f|_{\mathcal{M}}$ be the folding map of the regular surface $\mathcal{M}$. Then criteria of singularities of $f|_{\mathcal{M}}$ is given by the following table.*

type	condition
$S_1^\pm$	$\pm a_{21}a_{03} > 0.$
S_2	$a_{21} = 0,\ a_{03} \neq 0,\ a_{31} \neq 0.$
$S_3^\pm$	$a_{21} = 0,\ a_{31} = 0,\ \pm a_{03}a_{41} > 0.$
$B_2^\pm$	$a_{21} \neq 0,\ a_{03} = 0,\ \pm B_2 > 0.$
$B_3^\pm$	$a_{21} \neq 0,\ a_{03} = 0,\ B_2 = 0,\ \pm B_3 > 0.$
$C_3^\pm$	$a_{21} = 0,\ a_{03} = 0,\ \pm a_{31}a_{13} > 0.$

where $B_2 = \frac{a_{05}}{5} - \frac{a_{13}^2}{3a_{21}}$ *and* $B_3 = \frac{a_{07}}{7} - a_{15}\frac{a_{13}}{a_{21}} + \frac{5}{3}a_{23}(\frac{a_{13}}{a_{21}})^2 - \frac{5}{9}a_{31}(\frac{a_{13}}{a_{21}})^3$.

Proof. Routine calculation. See [4, Proposition 2.2] or [9, page 707] for some detailed computation. One can find the equivalent descriptions in other terminology at [1, page 254]. □

Remark 1.8. Bruce and Wilkinson ([2, page 64, lines 19–21]) stated that the key idea in this approach is that singularities of $f|_{\mathcal{M}}$ corresponds to infinitesimal reflectional symmetries of $\mathcal{M}$ in the plane $y = 0$. It is clear that $\mathcal{M}$ has reflectional symmetry in the plane $y = 0$ if and only if $h(x, y)$ is an odd function in y, that is, $h(x, y) = h(x, -y)$. So a naive condition for infinitesimal reflectional symmetry in the plane $y = 0$ is concerning the limit of $\frac{h(x,y)-h(x,-y)}{2y}$ tending $y \to 0$. For example, being $h_y(x, 0) = cx^k + o(x^k)$, $c \neq 0$, for some positive integer k is such a condition. But if we investigate singularities of $f|_{\mathcal{M}}$, we find several other infinitesimal reflectional symmetries in the plane $y = 0$.

Remark that the conditions appearing in Lemma 1.7 depend only on a_{ij}, where j is odd. This is a consequence of the fact that to investigate singularities of fold maps is descriptions of various infinitesimal reflectional symmetries of surfaces.

Lemma 1.9. *The folding family F is $\mathcal{A}$-versal if and only if the conditions shown in the following table hold.*

Singularity of $f\|_{\mathcal{M}}$	*Condition for* $\mathcal{A}$*-versality of* F
$S_1^{\pm}$	*always* $\mathcal{A}$*-versal.*
S_2	$k_1 \neq k_2$ *or* $a_{12} \neq 0$.
$S_3^{\pm}$	$(a_{22} - k_1 k_2^2)(k_1 - k_2) + a_{12}(2a_{12} - a_{30}) \neq 0$.
$B_2^{\pm}$	$a_{12} \neq \frac{a_{13}(k_1-k_2)}{3a_{21}}$ *or* $a_{04} - 3k_2^3 \neq \frac{a_{12}a_{13}}{a_{21}}$.
$B_3^{\pm}$	$\mathcal{B}_3 \neq 0$.
$C_3^{\pm}$	$\begin{vmatrix} k_2 - k_1 & -3a_{12} + \frac{a_{13}}{a_{31}}(a_{30} - 2a_{12}) \\ a_{12} & a_{04} - 3k_2^3 + \frac{a_{13}}{a_{31}}(a_{22} - k_1 k_2^2) \end{vmatrix} \neq 0.$

Here we define $\mathcal{B}_3$ *by* $\mathcal{B}_3 = \begin{vmatrix} a_{12}+\frac{a_{13}(k_2-k_1)}{3a_{21}} & p \\ a_{04}-3k_2^3-\frac{a_{12}a_{13}}{a_{21}} & q \end{vmatrix}$ *where*

$$p = \frac{a_{14}}{2} + \frac{a_{15}}{10a_{21}}(k_2 - k_1) + \frac{a_{13}}{3a_{21}}\left(a_{04} - 3a_{22} + \frac{a_{23}(k_1-k_2)}{a_{21}}\right) + \frac{a_{13}^2}{6a_{21}^2}\left(a_{30} - 2a_{12} + \frac{a_{31}}{a_{21}}(k_2 - k_1)\right),$$
$$q = \frac{3}{10}a_{06} - \frac{9}{2}a_{04}k_2^2 - \frac{3}{10}\frac{a_{12}a_{15}}{a_{21}} + \frac{a_{13}}{a_{21}}\left(-a_{14} + 6a_{12}k_2^2 + \frac{a_{12}a_{23}}{a_{21}}\right) + \frac{a_{13}^2}{2a_{21}^2}\left(a_{22} - k_1k_2^2 - \frac{a_{12}a_{31}}{a_{21}}\right).$$

The proof of Lemma 1.9 is long and we do not give it here, but in §5. Here, we simply note that the $\mathcal{A}$-versatility condition concerns the 3 (4, 6, respectively)-jet of h if $f|_{\mathcal{M}}$ has S_2 or S_3 (B_2 or C_3, B_3, respectively) singularity.

1.3 Double point locus of $f|_{\mathcal{M}}$

We consider the double point locus $D(f|_{\mathcal{M}})$ of the folding map $f|_{\mathcal{M}}$:

$$(x, y) \mapsto (x, y^2, h(x, y)), \quad h(x, y) = \frac{k_1x^2+k_2y^2}{2} + \sum_{i+j\geq 3}^{m} \frac{a_{ij}}{i!j!}x^i y^j + O(m + 1).$$

The double point locus $D(f|_{\mathcal{M}})$ is defined by $h'(x, y) = 0$ where

$$h'(x, y) = (h(x, y) - h(x, -y))/2y.$$

Remark that

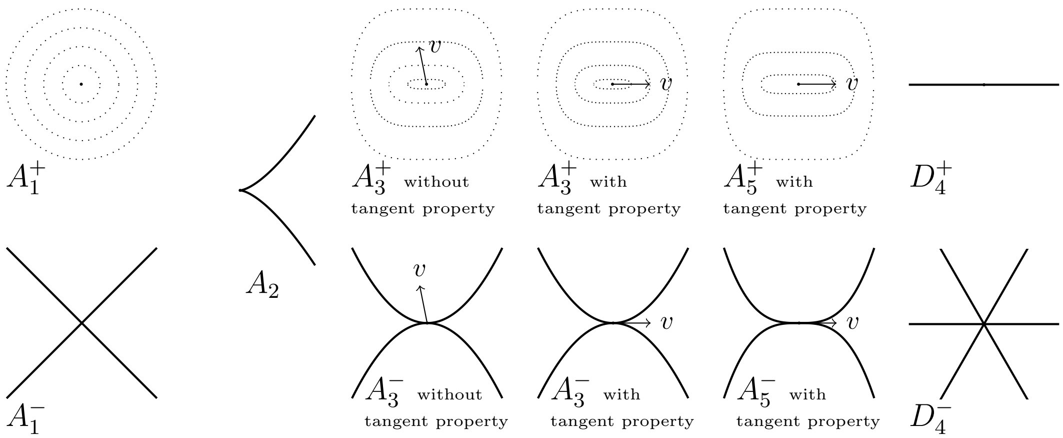
$$h' = \frac{a_{21}}{2}x^2 + \frac{a_{03}}{6}y^2 + \frac{a_{31}}{6}x^3 + \frac{a_{13}}{6}xy^2 + O(4).$$

Definition 1.10. We say that $D(f|_{\mathcal{M}})$ has **tangent property** with respect to the vecor $\boldsymbol{v}$ if $a_{21} \neq 0$ and $a_{03} = 0$ in the notation above. Geometrically this means that the vector $\boldsymbol{v}$ is in the limit of tangent lines of $g(D(f|_{\mathcal{M}}))$ at 0 when the zero of $h'(x, y)$ is not isolated at 0.

We now able to state criteria of singularities of the folding map $f|_{\mathcal{M}}$ in terms of the double point locus $D(f|_{\mathcal{M}})$.

Theorem 1.11. *There is a correspondence between singularities of the folding map* $f|_{\mathcal{M}}$ *and singularities of the double point locus* $D(f|_{\mathcal{M}})$ *as follows:*

Singularities of $f\|_{\mathcal{M}}$	Singularities of $D(f\|_{\mathcal{M}})$
$S_1^{\pm}$	$A_1^{\pm}$
S_2	A_2
$S_3^{\pm}$	$A_3^{\pm}$ without tangent property with respect to $\boldsymbol{v}$
$B_2^{\pm}$	$A_3^{\pm}$ with tangent property with respect to $\boldsymbol{v}$
$B_3^{\pm}$	$A_5^{\pm}$ with tangent property with respect to $\boldsymbol{v}$
$C_3^{\pm}$	$D_4^{\pm}$



Proof. The proof is given by comparing Lemma 1.7 with the following lemma. □

Lemma 1.12. *(a) If $\pm a_{21}a_{03} > 0$, then h' defines $A_1^\pm$ singularity.*

(b) When $a_{21} = 0$, $a_{03} \neq 0$, the kernel direction of the Hessian of h' is generated by ∂_x.

- *If $a_{31} \neq 0$, then h' defines A_2 singularity.*
- *If $a_{31} = 0$ and $\pm a_{03}a_{41} > 0$, then h' defines $A_3^\pm$ singularity.*

(c) When $a_{21} \neq 0$, $a_{03} = 0$, the kernel direction of the Hessian of h' is generated by ∂_y.

- *If $\pm B_2 > 0$, then h' defines $A_3^\pm$ singularity.*
- *If $B_2 = 0$ and $\pm B_3 > 0$, then h' defines $A_5^\pm$ singularity.*

(d) When $a_{21} = a_{03} = 0$, and $\pm a_{31}a_{13} > 0$, then h' defines $D_4^\pm$ singularity.

(e) If none of the conditions above hold, then h' does not define $A_1^\pm$, A_2, $A_3^\pm$, $A_5^\pm$, $D_4^\pm$ singularities.

Proof. The proof is routine and we show below its outline. For example, a detailed proof except for the case of A_5 can be found in [3, §4]. The A_5 case can be proved similarly.

(a): The assertion (a) is trivial.

(b): When $a_{21} = 0$, we have

$$h' = \tfrac{1}{6}(a_{03}y^2 + a_{31}x^3 + a_{13}x^2y) + O(4).$$

Thus if $a_{31}a_{03} \neq 0$, h' defines A_2 singularity. When $a_{21} = a_{31} = 0$, we have

$$h' = \tfrac{1}{6}(a_{03}y^2 + a_{13}x^2y) + \tfrac{a_{41}}{24}x^4 + \tfrac{a_{23}}{12}x^2y^2 + \tfrac{a_{05}}{120}y^4 + O(5).$$

Thus if $\pm a_{41}a_{03} > 0$, h' defines $A_3^\pm$-singularity.

(c): When $a_{03} = 0$ and $a_{21} \neq 0$, we have

$$h' = \tfrac{a_{21}}{6}(x + \tfrac{a_{13}}{6a_{21}}y^2)^2 + \tfrac{B_2}{24a_{21}}y^4 + \tfrac{a_{41}}{24}x^4 + \tfrac{a_{23}}{12}x^2y^2 + O(5),$$

and we obtain the first subcase. When $B_2 = 0$, replacing x by $x - \tfrac{a_{13}}{6a_{21}}y^2$, we obtain

$$h' = \tfrac{a_{21}}{6}x^2 + \tfrac{B_3}{6!a_{21}^3}y^6 + *x^4 + *x^2y^2 + *x^6 + *x^4y^2 + *x^2y^4 + O(7),$$

which implies the second subcase.

(d): When $a_{21} = a_{03} = 0$, the cubic part of h' defines three real lines (resp. one real line) if $a_{13} < 0$ (resp. > 0), and we are done.

(e): The assertion is trivial. □

1.4 Non-umbilical points

When the surface $\mathcal{M}$ is not umbilic at the origin (i.e., $k_1 \neq k_2$), we can define the principal curvatures κ_1, κ_2 and the principal vectors v_1, v_2 and we can state the conditions above in terms of κ_i and v_i.

Lemma 1.13. *If the origin is not an umbilic point of $\mathcal{M}$, the conditions in Lemmas 1.7 and 1.9 are rephrased as follows.*

type	*condition for singularities*	*condition for $\mathcal{A}$-versality*
$S_1^\pm$	$v_2\kappa_1(0) \neq 0$, $v_2\kappa_2(0) \neq 0$.	
S_2	$v_2\kappa_1(0) = 0$, $v_2\kappa_2(0) \neq 0$, $v_1v_2\kappa_1(0) \neq 0$.	
$S_3^\pm$	$v_2\kappa_1(0) = 0$, $v_2\kappa_2(0) \neq 0$, $v_1v_2\kappa_1(0) = 0$, $v_1^2v_2\kappa_1(0) \neq 0$.	$v_2^2\kappa_1(0) \neq 0$.
$B_2^\pm$	$v_2\kappa_1(0) \neq 0$, $v_2\kappa_2(0) = 0$, $v_2^3\kappa_2(0) \neq \frac{5}{3}\frac{v_1v_2\kappa_2(0)^2}{v_2\kappa_1(0)}$.	$(v_1v_2\kappa_2(0), v_2^2\kappa_2(0)) \neq 0$.
$C_3^\pm$	$v_2\kappa_1(0) = 0$, $v_2\kappa_2(0) = 0$, $v_1v_2\kappa_1(0) \neq 0$, $v_1v_2\kappa_2(0) \neq 0$.	$\begin{vmatrix} v_1v_2\kappa_1 & v_2v_2\kappa_1 \\ v_1v_2\kappa_2 & v_2v_2\kappa_2 \end{vmatrix}(0) \neq 0$.

We give a proof of Lemma 1.13 in §3.

2 Dual map and bifurcation sets

2.1 Dual map

For a regular surface X in $\mathbb{R}^3$, we consider the dual map δ defined by

$$\delta : X \longrightarrow \mathcal{P}, \quad \boldsymbol{p} \longmapsto T_{\boldsymbol{p}}X.$$

Lemma 2.1. *(i) The map δ is singular at $\boldsymbol{p}$ if and only if $\boldsymbol{p}$ is a parabolic point of X. Moreover, the rank of $d\delta_{\boldsymbol{p}}$ is 1 (resp. 0) if it is not umbilic (resp. umbilc).*

(ii) The map δ has a singularity $\mathcal{A}$-equivalent to cuspidal edge at $\boldsymbol{p}$ if and only if $\boldsymbol{p}$ is parabolic, neither umbilic, nor η-ridge where η is a principal vector corresponding to the zero principal curvature.

Proof. (i): For a surface given by

$$(u, v) \longmapsto \boldsymbol{p} = (u, v, f(u, v)),$$

the tangent plane $T_{\boldsymbol{p}}X$ is defined by $\boldsymbol{v} \cdot \boldsymbol{x} = c$, $|\boldsymbol{v}| = 1$, where

$$\boldsymbol{v} = \tfrac{1}{\sqrt{1+f_u^2+f_v^2}}(-f_u, -f_v, 1), \quad c = \tfrac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{vmatrix} 1 & 0 & u \\ 0 & 1 & v \\ f_u & f_v & f \end{vmatrix}.$$

We consider the map

$$(u, v) \mapsto (\boldsymbol{v}, c) = \tfrac{1}{\sqrt{1+f_u^2+f_v^2}}(-f_u, -f_v, 1, f - uf_u - vf_v).$$

Composing the inverse of the transformation

$$(x_1, x_2, c) \mapsto \Big(\tfrac{x_1}{\sqrt{1+x_1^2+x_2^2}}, \tfrac{x_2}{\sqrt{1+x_1^2+x_2^2}}, \tfrac{1}{\sqrt{1+x_1^2+x_2^2}}, c\Big),$$

it is enough to consider the map represented by

$$(u,v)\mapsto \big(-f_u, -f_v, \tfrac{f-uf_u-vf_v}{\sqrt{1+f_u^2+f_v^2}}\big). \tag{2.2}$$

We see that the jacobian of (2.2) is not of full rank if and only if $f_{uu}f_{vv}-f_{uv}^2=0$. When $f=k_1\frac{u^2}{2}+k_2\frac{v^2}{2}+\sum_{i+j\geq 3}a_{ij}\frac{u^iv^j}{i!j!}$, the 2-jet of (2.2) is

$$(-k_1u-a_{30}\tfrac{u^2}{2}-a_{21}uv-a_{12}\tfrac{v^2}{2}, -k_2v-a_{21}\tfrac{u^2}{2}-a_{12}uv-a_{03}\tfrac{v^2}{2}. -k_1\tfrac{u^2}{2}-k_2\tfrac{v^2}{2})$$

and 1-jet of the Jacobi's matrix of (2.2) is

$$\begin{pmatrix} -k_1-a_{30}u-a_{21}v & -a_{21}u-a_{12}v & -k_1u \\ -a_{21}u-a_{12}v & -k_2-a_{12}u-a_{03}v & -k_2v \end{pmatrix}. \tag{2.3}$$

If $k_1=0$ and $k_2\neq 0$, then the rank of (2.3) at 0 is 1. The null direction is generated by $\eta=-f_{uv}\partial_u+f_{uu}\partial_v$ there. This can be shown, checking by the identity:

$$\eta\big(-f_u,-f_v,\tfrac{f-uf_u-vf_v}{\sqrt{1+f_u^2+f_v^2}}\big)=(f_{uu}f_{vv}-f_{uv}^2)(0,1,\tfrac{v+ff_v-uf_uf_v+vf_u^2}{(1+f_u^2+f_v^2)^{3/2}}).$$

If $k_1=k_2=0$, then the rank of (2.3) at 0 is 0.

(ii): We assume that $k_1\neq 0$ and $k_2=0$. Since a unit normal of the map (2.2) is given by

$$\boldsymbol{\nu}=\frac{\tilde{\boldsymbol{\nu}}}{|\tilde{\boldsymbol{\nu}}|},\quad \tilde{\boldsymbol{\nu}}=(-u-uf_v^2-ff_u+vf_uf_v, -v-vf_u^2-ff_v+uf_uf_v, (1+f_u^2+f_v^2)^{3/2}),$$

its Taylor expansion is expressed as

$$\boldsymbol{\nu}=(-u+\tfrac{1+2k_1^2}{2}u^3+\tfrac{1}{2}uv^2, -v+\tfrac{1+k_1^2}{2}u^2v+\tfrac{1}{2}v^3, 1-\tfrac{u^2+v^2}{2})+O(4).$$

We now use Lemma A.1 and the notation there. We can take $\lambda=f_{uu}f_{vv}-f_{uv}^2$. Then we have $\eta\lambda(0)=k_1^2a_{03}$. Since $\psi(0)=k_1$, we have the result. □

Remark 2.4. Under the notation of the proof above, the map δ has singularity $\mathcal{A}$-equivalent to swallowtail at $\boldsymbol{p}$ if and only if $\boldsymbol{p}$ is parabolic, the first order v_2-ridge ($v_2\kappa_2(0)=0$, $v_2^2\kappa_2(0)\neq 0$), but not umbilic. For proof, we apply Lemma A.1, using $\eta^2\lambda(0)=k_1a_{04}+3(a_{21}a_{03}-a_{12}^2)$ and (3.5). Remark that $\eta^2\lambda(0)$ is non zero if and only if $v_2^2\kappa_2(0)\neq 0$ also.

Remark 2.5. We remark that the Gauss map of the surface X is represented by

$$(u,v)\mapsto(-f_u,-f_v). \tag{2.6}$$

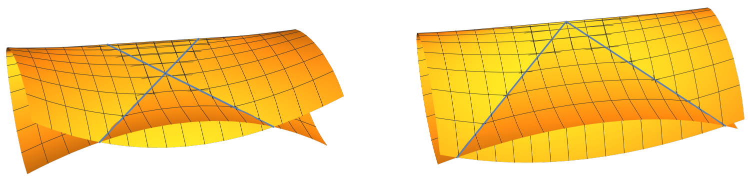
in the notation of the proof above. When we assume $k_1\neq 0$ and $k_2=0$, $-f_{uv}\partial_u+f_{uu}\partial_v$ represents the null direction at 0 along the singular locus, and the singular locus is defined by $\lambda=f_{uu}f_{vv}-f_{uv}^2$. Then the map (2.6) has a singularity $\mathcal{A}$-equivalent to

- a fold if $\boldsymbol{p}$ is not v_2-ridge, that is, $a_{03}\neq 0$,
- a cusp if $\boldsymbol{p}$ is the first order v_2-ridge, that is, $a_{03}=0$, and $k_1a_{04}+3(a_{21}a_{03}-a_{12}^2)\neq 0$.

2.2 Bifurcation sets of the folding family

The set of plane Π for which the folding map f^{Π} is not stable is the **bifurcation set** $\mathcal{B}(F)$ of the folding family F.

Remark that f^{Π} fails to be stable if f^{Π} has more degenerate singularity than a cross cap (S_0), or if f^{Π} has a self-tangent point, that is, two distinct points $\boldsymbol{p}$ and $\boldsymbol{p}'$ with $f^{\Pi}(\boldsymbol{p}) = f^{\Pi}(\boldsymbol{p}')$ and $\operatorname{Im} df^{\Pi}(\boldsymbol{p}) = \operatorname{Im} df^{\Pi}(\boldsymbol{p}')$.



A surface with a self-tangent point (left) as a deformation of B_2^- singularity (right).

The **focal set** $\mathcal{F}$ of a surface $\mathcal{M}$ in $\mathbb{R}^3$ is the locus of the centers of curvature of $\mathcal{M}$, and the **symmetry set** $\mathcal{S}$ of $\mathcal{M}$ is the closure of the locus of centers of spheres bi-tangent to $\mathcal{M}$. We denote $\mathcal{F}^{\circ}$ (resp. $\mathcal{S}^{\circ}$) the nonsingular locus of $\mathcal{F}$ (resp. $\mathcal{S}$).

Theorem 2.7. $\mathcal{B}(F) = \overline{\delta(\mathcal{F}^{\circ})} \cup \overline{\delta(\mathcal{S}^{\circ})}$.

Proof. See [2, Proposition 2.3]. □

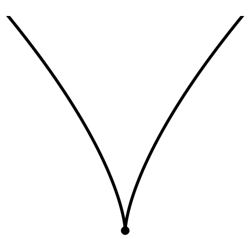
When the folding family F is $\mathcal{A}$-versal, one can deduce local models for the bifurcation sets $\mathcal{B}(F)$.

Example 2.8 ($S_1^{\pm}$)**.** An $\mathcal{A}$-versal unfolding of $S_1^{\pm}$ singularity defined by $(x, y) \mapsto (x, y^2, y^3 \pm x^2 y)$ is given by $f = (x, y^2, y^3 \pm x^2 y + ay)$. The S_1 locus in the parameter space is defined by $a = 0$ and there is no A_1^* locus.

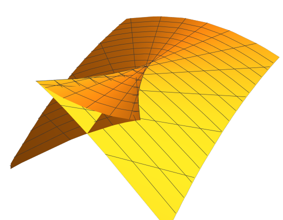
Example 2.9 (S_2)**.** An $\mathcal{A}$-versal unfolding of S_2 singularity defined by $(x, y) \mapsto (x, y^2, y^3 + x^3 y)$ is given by $f = (x, y^2, y^3 + x^3 y + ay + bxy)$. The S_1 locus in the parameter space is parametrized by

$$t \mapsto (a, b) = (-2t^3, 3t^2),$$

which corresponds to the mono-germ of f at $(t, 0)$ under (a, b) described above, and there is no A_1^* locus.



Bifurcation set for S_2 (Example 2.9)



Bifurcation set for $S_3^{\pm}$ (Example 2.10)

Example 2.10 ($S_3^{\pm}$)**.** An $\mathcal{A}$-versal unfolding of $S_3^{\pm}$ singularity defined by $(x, y) \mapsto (x, y^2, y^3 \pm x^4 y)$ is given by $f = (x, y^2, y^3 \pm x^4 y + ay + bxy + cx^2 y)$, $S_{\geq 1}$ locus is parametrized by

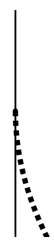
$$(t, c) \mapsto (a, b, c) = (-ct^2 \mp t^4 + 2t(ct \pm 2t^3), -2(ct \pm 2t^3), c)$$

which corresponds to the mono-germ of f at $(t, 0)$, and there is no A_1^* locus.

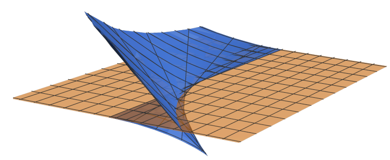
Example 2.11 ($B_2^{\pm}$)**.** An $\mathcal{A}$-versal unfolding of $B_2^{\pm}$ singularity defined by $(x, y) \mapsto (x, y^2, y^5 \pm x^2 y)$ is given by $f = (x, y^2, y^5 \pm x^2 y + ay + by^3)$. The S_1 locus is defined by $a = 0$, which corresponds to the mono-germs of f at the origin, while A_1^* locus is parametrized by

$$t \mapsto (a, b) = (t^4, -2t^2)$$

which corresponds to the bi-germ of f at $(0, t)$ and $(0, -t)$.



Bifurcation set for $B_2^{\pm}$ (Example 2.11)



Bifurcation set for $B_3^{\pm}$ (Example 2.12)

Example 2.12 ($B_3^{\pm}$)**.** An $\mathcal{A}$-versal unfolding of $B_3^{\pm}$ singularity defined by $(x, y) \mapsto (x, y^2, y^7 \pm x^2 y)$ is given by $f_{\pm} = (x, y^2, y^7 - x^2 y + ay + by^3 + cy^5)$. The S_1 locus is defined by $a = 0$, which corresponds to the mono-germs of f at the origin, while A_1^* locus is parametrized by

$$(t, c) \mapsto (a, b, c) = (t^4(c + 2t^2), -t^2(2c + 3t^2), c),$$

which corresponds to the bi-germ of $f_{\pm}$ at $(0, t)$ and $(0, -t)$.

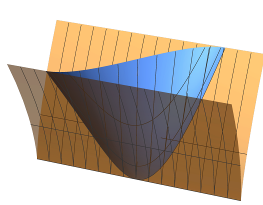
Example 2.13 ($C_3^{\pm}$)**.** An $\mathcal{A}$-versal unfolding of $C_3^{\pm}$ singularity defined by $(x, y) \mapsto (x, y^2, xy^3 \pm x^3 y)$ is given by $f = (x, y^2, xy^3 \pm x^3 y + ay + bxy + cy^3)$. The $S_{\geq 1}$ locus is parametrized by

$$(t, c) \mapsto (a, b, c) = (\mp 2t^3, \pm 3t^2, c),$$

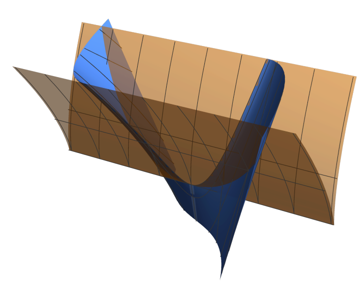
which corresponds to the mono-germ of f at $(t, 0)$, while A_1^* locus is parametrized by

$$(s, t) \mapsto (a, b, c) = (\pm 2s^3 + st^2, \mp 3s^2 - t^2, -s),$$

which corresponds to the bi-germ of f at (s, t) and $(s, -t)$.



Bifurcation set for C_3^+ (Example 2.13)



Bifurcation set for C_3^- (Example 2.13)

We remark that the figure right is missing in [2, Fig. 2, page 67].

3 Non-umbilic points: Proof of Lemma 1.13.

Let us describe the several computation of a regular surface defined by (0.2) at non-umbilical point. We thus assume that $k_1 \neq k_2$. The first observations are as follows.

$$(3.1)\qquad v_1\kappa_1(0) = a_{30}, \quad v_2\kappa_1(0) = a_{21}, \quad v_1\kappa_2(0) = a_{12}, \quad v_2\kappa_2(0) = a_{03},$$

$$(3.2)\qquad v_1^2\kappa_1(0) = a_{40} - 3k_1^3 + \tfrac{3a_{21}^2}{k_1-k_2}, \quad v_2v_1\kappa_1(0) = a_{31} + \tfrac{3a_{21}a_{12}}{k_1-k_2},$$

$$(3.3)\qquad v_1v_2\kappa_1(0) = a_{31} + \tfrac{a_{21}(2a_{12}-a_{30})}{k_1-k_2}, \quad v_2^2\kappa_1(0) = a_{22} - k_1k_2^2 + \tfrac{a_{12}(2a_{12}-a_{30})}{k_1-k_2},$$

$$(3.4)\qquad v_1^2\kappa_2(0) = a_{22} - k_1^2k_2 + \tfrac{a_{21}(2a_{21}-a_{03})}{k_2-k_1}, \quad v_2v_1\kappa_2(0) = a_{13} + \tfrac{a_{12}(2a_{21}-a_{03})}{k_2-k_1},$$

$$(3.5)\qquad v_1v_2\kappa_2(0) = a_{13} + \tfrac{3a_{21}a_{12}}{k_2-k_1}, \quad v_2^2\kappa_2(0) = a_{04} - 3k_2^3 + \tfrac{3a_{12}^2}{k_2-k_1}.$$

These are obtained by direct computations. See [3, 2.3] for some of the detail, for example. We also have the expressions of the principal curvatures as follows:

$$(3.6)\qquad \begin{aligned}\kappa_1 =& k_1 + a_{30}x + a_{21}y + \big(a_{40} - 3k_1^3 + \tfrac{2a_{21}^2}{k_1-k_2}\big)\tfrac{x^2}{2}\\ &+\big(a_{31} + \tfrac{2a_{21}a_{12}}{k_1-k_2}\big)xy + \big(a_{22} - k_1^2k_2 + \tfrac{2a_{12}^2}{k_2-k_1}\big)\tfrac{y^2}{2} + O(3),\end{aligned}$$

$$(3.7)\qquad \begin{aligned}\kappa_2 =& k_2 + a_{12}x + a_{03}y + \big(a_{22} - k_1^2k_2 + \tfrac{2a_{21}^2}{k_2-k_1}\big)\tfrac{x^2}{2}\\ &+\big(a_{13} + \tfrac{2a_{21}a_{12}}{k_2-k_1}\big)xy + \big(a_{04} - 3k_2^3 + \tfrac{2a_{12}^2}{k_2-k_1}\big)\tfrac{y^2}{2} + O(3).\end{aligned}$$

A principal vector v_2 is expressed by

$$\begin{aligned}v_2 =& \Big(\tfrac{a_{21}x+a_{12}y}{k_2-k_1} + \big(\tfrac{a_{31}}{k_2-k_1} + \tfrac{2a_{21}(a_{12}-a_{30})}{(k_2-k_1)^2}\big)\tfrac{x^2}{2} + \big(\tfrac{a_{22}-k_1k_2^2}{k_2-k_1} + \tfrac{a_{30}a_{12}-a_{12}^2-a_{21}^2+a_{21}a_{03}}{(k_2-k_1)^2}\big)xy\\ &+\big(\tfrac{a_{13}}{k_2-k_1} + \tfrac{2a_{12}(a_{21}-a_{03})}{(k_2-k_1)^2}\big)\tfrac{y^2}{2} + O(3)\Big)\partial_x\\ &+\Big(1 - \tfrac{a_{21}^2}{(k_2-k_1)^2}\tfrac{x^2}{2} + \tfrac{a_{21}a_{12}}{(k_2-k_1)^2}xy - \big(k_2^2 + \tfrac{a_{12}^2}{(k_2-k_1)^2}\big)\tfrac{y^2}{2} + O(3)\Big)\partial_y.\end{aligned}$$

We thus conclude that

$$(3.8)\qquad v_2\kappa_1 = a_{21} + \big(a_{31} + \tfrac{a_{21}(a_{30}-2a_{12})}{k_2-k_1}\big)x + \big(a_{22} - k_1k_2^2 + \tfrac{a_{12}(a_{30}-2a_{12})}{k_2-k_1}\big)y + O(2),$$

$$(3.9)\qquad v_2\kappa_2 = a_{03} + \big(a_{13} + \tfrac{3a_{21}a_{12}}{k_2-k_1}\big)x + \big(a_{04} - 3k_2^3 + \tfrac{3a_{12}^2}{k_2-k_1}\big)y + O(2).$$

Proof of Lemma 1.13. We first consider the condition for singularities of the folding map $f|_{\mathcal{M}}$.

The assertion for $S_1^{\pm}$ is clear by (3.1). By (3.3), we have $a_{31} = v_1v_2\kappa_1(0)$ when $a_{21} = 0$ and the assertion is clear. In a similar way to the computation above, we obtain that the coefficient of x^2 in the expression of $v_2\kappa_1$ is

$$(3.10)\qquad \begin{aligned}a_{41} - k_1a_{21}(5k_1+k_2) &+ \tfrac{2a_{21}(2a_{22}-a_{40}+k_1^3)+a_{31}(2a_{12}-a_{30})}{k_1-k_2}\\ &+\tfrac{a_{21}((a_{12}-a_{30})(2a_{12}-a_{30})a_{12}+2a_{03}a_{12}^2-7a_{12}^3)}{(k_1-k_2)^2}.\end{aligned}$$

The assertion for $S_3^{\pm}$ follows, since (3.10) is non-zero when $a_{21} = a_{31} = 0$ and $a_{41} \neq 0$. The assertion for $C_3^{\pm}$ follows by (3.1), (3.3) and (3.4).

Remark that, if $a_{03}=0$, we have

$$v_2^3\kappa_2(0) = a_{05} - 18a_{03}k_2^2 + \tfrac{10a_{12}a_{13}}{k_2-k_1} + \tfrac{3a_{12}^2(5a_{21}-3a_{03})}{(k_2-k_1)^2} \tag{3.11}$$

$$a_{13} = v_1v_2\kappa_2(0) - \tfrac{3v_2\kappa_1(0)v_1\kappa_2(0)}{k_2-k_1} \tag{3.12}$$

and we conclude that

$$B_2 = \tfrac{a_{05}}{5} - \tfrac{a_{13}^2}{3a_{21}} = \tfrac{v_2^3\kappa_2(0)}{5} - \tfrac{(v_1v_2\kappa_2(0))^2}{3v_1\kappa_2(0)}.$$

So the condition that $\pm(a_{05}-5a_{13}^2/2a_{21})>0$ for $B_2^\pm$ singularity is equivalent that

$$\pm(v_1\kappa_2(0)\cdot v_2^3\kappa_2(0) - \tfrac{5}{3}(v_1v_2\kappa_2(0))^2) > 0.$$

From now on, we consider $\mathcal{A}$-versality of the folding family.

The assertions for $S_1^\pm$ and S_2 are clear.

The assertion for $S_3^\pm$ follows, since $v_2v_2\kappa_1(0)\neq 0$ by (3.3).

For $B_2^\pm$ singularity the condition in Lemma 1.9 is equivalent that

$$(v_1v_2\kappa_2(0), v_2^2\kappa_2(0)) \neq 0$$

by (3.5), and thus shows the assertion.

For $C_3^\pm$ singularity, the condition in Lemma 1.9 is equivalent that

$$\begin{vmatrix} v_1v_2\kappa_1(0) & v_2^2\kappa_1(0) \\ v_1v_2\kappa_2(0) & v_2^2\kappa_2(0) \end{vmatrix} \neq 0$$

from (3.3) and (3.5). This shows the assertion. □

Remark 3.13. The origin is v_2-subparabolic (resp. v_2-ridge) if and only if the constant principal curvature line $\kappa_1=k_1$ (resp. $\kappa_2=k_2$) is perpendicular to the reflection plane $y=0$ there, whenever it is not v_1-ridge (resp. v_1-subparabolic), by (3.6) and (3.7).

Remark 3.14. We can conclude that the v_1-curvature line is parametrized by

$$t\mapsto(x,y) = (t, \tfrac{a_{21}}{k_1-k_2}\tfrac{t^2}{2} + (\tfrac{a_{31}}{k_1-k_2} + \tfrac{a_{21}(3a_{12}-2a_{30})}{(k_1-k_2)^2})\tfrac{t^3}{6} + a\tfrac{t^4}{24} + O(5))$$

where $a = \frac{a_{41}}{k_1-k_2} + \frac{3a_{21}(2a_{22}-a_{40}+k_1^3)+(4a_{12}-3a_{30})a_{31}}{(k_1-k_2)^2} + \frac{a_{21}(3a_{30}-4a_{12})(3a_{12}-2a_{30})-3a_{21}^2a_{03}+9a_{21}^3}{(k_1-k_2)^3}$, working the equation of curvature lines:

$$\begin{vmatrix} h_{xx} & 1+h_x^2 & dy^2 \\ h_{xy} & h_xh_y^2 & -dx\,dy \\ h_{yy} & 1+h_y^2 & dx^2 \end{vmatrix} = 0.$$

This shows that the folding map $f|_\mathcal{M}$ has a S_2 (resp. S_3, S_4) singularity at 0 with respect to the principal direction v_2 if and only if 0 is v_2-subparabplic but not v_2-ridge and the v_1-curvature line throgh 0 has 2 (resp. 3, 4)-point contact with the reflecftion plane $y=0$.

Remark 3.15. Let (u,v) denote a curvature coordinate of a surface $\boldsymbol{p}=\boldsymbol{p}(u,v)$. Let $\boldsymbol{\nu}$ denote its unit normal. When the principal curvature κ_2 is not zero, we can define a focal set $\boldsymbol{q}=\boldsymbol{p}+(1/\kappa_2)\boldsymbol{\nu}$, and its Gauss map is $\boldsymbol{g}=\boldsymbol{p}_v/|\boldsymbol{p}_v|$. Since

$$\boldsymbol{g}_u = \tfrac{(\kappa_1)_v}{\kappa_1-\kappa_2}\tfrac{\boldsymbol{p}_u}{|\boldsymbol{p}_v|}, \quad \text{and} \quad \boldsymbol{g}_v = \tfrac{\boldsymbol{p}_{vv}\cdot\boldsymbol{p}_u}{|\boldsymbol{p}_v\cdot\boldsymbol{p}_v||\boldsymbol{p}_u\cdot\boldsymbol{p}_u|^2}\boldsymbol{p}_u - \kappa_2\boldsymbol{\nu},$$

the Gauss map $\boldsymbol{g}$ is singular when $v_2\kappa_1=0$, where $v_2=\partial_v$, and $v_1=\partial_u$ generates the kernel field there. Then the Gauss map $\boldsymbol{g}$ has a singularity at 0 if and only if $v_2\kappa_1(0)=0$ (that is, $a_{21}=0$). Moreover, the Gauss map $\boldsymbol{g}$ has a singularity $\mathcal{A}$-equivalent to

- a fold at 0 if and only if $v_1 v_2 \kappa_1(0) \neq 0$ (that is, $a_{31} \neq 0$).
- a cusp at 0 if and only if $v_1 v_2 \kappa_1(0) = 0$ (that is, $a_{31} = 0$) and $v_1^2 v_2 \kappa_1(0) \neq 0$ ($a_{41} \neq 0$).

Remark 3.16. Since the Gauss curvature of the focal set $\boldsymbol{q} = \boldsymbol{p} + (1/\kappa_2)\boldsymbol{\nu}$ at $(u, v) = (0, 0)$ is given by

$$-\tfrac{v_2 \kappa_1 (\kappa_2)^4}{v_2 \kappa_2 (\kappa_1 - \kappa_2)^2}(0) = -\tfrac{a_{21} k_2^4}{a_{03}(k_1 - k_2)^2},$$

elliptic (resp. hyperbolic) points of the focal set correspond to S_1^- (resp. S_1^+) singularities of the folding maps. This fact mentioned in the third paragraph from the bottom in page 68 in [2] with changing the sign.

4 Umbilics

4.1 Classification of umbilics

We consider a nonsingular surface

$$(4.1) \quad \boldsymbol{p} : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}, \quad z \mapsto (z, h(z)), \quad \text{where } h(z) = \tfrac{k}{2} z\bar{z} + \sum_{k=3}^{m} H_k(z) + O(m+1),$$

and $H_k(z)$ is a real-valued homogeneous polynomial of degree k in variables z, $\bar{z}$. We remark that this surface has an umbilic point at the origin.

The first fundamental form is expressed as

$$I = d\boldsymbol{p} \cdot d\boldsymbol{p} = dz\, d\bar{z} + dh\, d\bar{h} = h_z^2\, dz^2 + (1 + 2|h_z|^2) dz\, d\bar{z} + h_{\bar{z}}^2\, d\bar{z}^2.$$

Since $\boldsymbol{p}_x \times \boldsymbol{p}_y = (-h_x - h_y\sqrt{-1}, 1) = (-2\sqrt{-1} h_{\bar{z}}, 1)$, a unit normal is expressed as

$$\boldsymbol{\nu} = \frac{1}{1 + |2h_z|^2}(-2\sqrt{-1} h_{\bar{z}}, 1).$$

The second fundamental form is thus expressed as

$$\mathrm{II} = d^2\boldsymbol{p} \cdot \boldsymbol{\nu} = \frac{1}{1 + |2h_z|^2}(h_{zz}\, dz^2 + 2h_{z\bar{z}}\, dz\, d\bar{z} + h_{\bar{z}\bar{z}}\, d\bar{z}^2).$$

Therefore the equation of curvature lines is

$$(4.2) \qquad \sqrt{-1} \begin{vmatrix} h_z^2 & h_{zz} & d\bar{z}^2 \\ 1+2|h_z|^2 & h_{z\bar{z}} & -dz\, d\bar{z} \\ h_{\bar{z}}^2 & h_{\bar{z}\bar{z}} & dz^2 \end{vmatrix} = \frac{1}{\sqrt{-1}}((H_3)_{zz}\, dz^2 - (H_3)_{\bar{z}\bar{z}}\, d\bar{z}^2) + \text{h.o.t.} = 0.$$

Set

$$(4.3) \qquad H_3(z) = \alpha z^3/6 + \beta z^2 \bar{z}/2 + \bar{\beta} z \bar{z}^2/2 + \bar{\alpha} \bar{z}^3/6.$$

We consider the resultant of $(H_3)_z$ and $(H_3)_{\bar{z}}$ as

$$(4.4) \qquad D_{H_3} = \begin{vmatrix} \alpha & 2\beta & \bar{\beta} & 0 \\ 0 & \alpha & 2\beta & \bar{\beta} \\ \beta & 2\bar{\beta} & \bar{\alpha} & 0 \\ 0 & \beta & 2\bar{\beta} & \bar{\alpha} \end{vmatrix} = |\alpha|^4 - 6|\alpha|^2|\beta|^2 - 3|\beta|^4 + 8\,\mathrm{Re}\,\alpha\bar{\beta}^3.$$

The cubic H_3 has three real roots (resp. one real root) if and only if the origin is **elliptic** (resp. **hyperbolic**) umbilic, that is, $D_{H_3} > 0$ (resp. < 0).

We also consider the **characteristic polynomial** H_3' for (4.2), which is defined by

$$H_3'(z) = \frac{1}{\sqrt{-1}}(z^2(H_3)_{zz}(z) - \bar{z}^2(H_3)_{\bar{z}\bar{z}}(z))). \tag{4.5}$$

Its zeros define the **characteristic directions** of the singularity of curvature lines at the origin. The characteristic polynomial H_3' has three real roots (resp. one real root) if and only if $D_{H_3'} > 0$ (resp. < 0) where

$$D_{H_3'}(z) = \begin{vmatrix} \frac{3\alpha}{\sqrt{-1}} & \frac{2\beta}{\sqrt{-1}} & \frac{-\bar{\beta}}{\sqrt{-1}} & 0 \\ 0 & \frac{3\alpha}{\sqrt{-1}} & \frac{2\beta}{\sqrt{-1}} & \frac{-\bar{\beta}}{\sqrt{-1}} \\ \frac{\beta}{\sqrt{-1}} & \frac{-2\bar{\beta}}{\sqrt{-1}} & \frac{-3\bar{\alpha}}{\sqrt{-1}} & 0 \\ 0 & \frac{\beta}{\sqrt{-1}} & \frac{-2\bar{\beta}}{\sqrt{-1}} & \frac{-3\bar{\alpha}}{\sqrt{-1}} \end{vmatrix} = 3(27|\alpha|^4 - 18|\alpha|^2|\beta|^2 - |\beta|^4 - 8\operatorname{Re}\alpha\bar{\beta}^3). \tag{4.6}$$

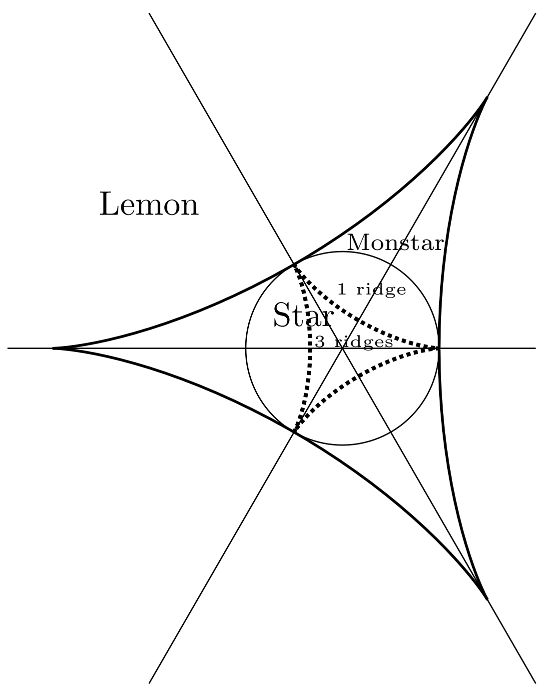
We say a characteristic direction is a **double characteristic direction** if it is generated by a double root of $H_3'(z)$.

An umbilic is said to be **right-angled** if there are two characteristic directions that are orthogonal to each other. It is well-known that this is equivalent that $|\alpha| = |\beta|$. This also implies $D_{H_3} \leq 0$ and $D_{H_3'} \geq 0$.

We are now able to state the classification result of Darbouxian umbilics.

- We say that the umbilic is **star** if $|\alpha| > |\beta|$.
  - ∗ If $D_{H_3} > 0$, then there are three directions which are limits of principal directions.
  - ∗ If $D_{H_3} < 0$, then there is one direction which is a limit of principal directions.
- We say that the umbilic is **monstar** if $|\alpha| < |\beta|$ and $D_{H_3'} > 0$.
- We say that the umbilic is **lemon** if $D_{H_3'} < 0$.

When $\alpha = 1$, the bifurcation of generic umbilics is shown in β-plane as follows:



$|\beta| = 1$ (circle), $D_{H_3} = 0$ (dotted), $D_{H_3'} = 0$ (thick)

Remark 4.7. The locus $D_{H_3} = 0$ is parametrized by

$$\mathbb{C} \times S^1 \to \mathbb{C}^2,\ (\alpha, \phi) \mapsto (\alpha, \beta), \quad \text{where} \quad \beta = (\bar{\alpha} e^{-2\phi\sqrt{-1}} - 2\alpha e^{\phi\sqrt{-1}})/3, \tag{4.8}$$

and its singular locus is defined by $\cos\left(\arg\alpha + \frac{3\phi}{2}\right) = 0$. Similarly, the locus $D_{H_3'} = 0$ is parametrized by

$$\mathbb{C} \times S^1 \to \mathbb{C}^2,\ (\alpha, \phi) \mapsto (\alpha, \beta), \quad \text{where} \quad \beta = -\bar{\alpha} e^{-2\phi\sqrt{-1}} - 2\alpha e^{\phi\sqrt{-1}}, \tag{4.9}$$

and its singular locus is defined by $\sin\left(\arg\alpha + \frac{3\phi}{2}\right) = 0$.

Remark 4.10. Replacing z by $e^{-\frac{\arg\alpha + 2n\pi}{3}\sqrt{-1}} z$ in $H_3(z)$, $n \in \mathbb{Z}$, in (4.3), we can reduce to the case $\alpha \in \mathbb{R}$. Then the argument of β becomes $\arg\beta - \frac{\arg\alpha + 2n\pi}{3}$.

Definition 4.11 (S_2-direction, B_2-direction and C_3-direction)**.** Set

$$w_\theta = e^{\theta\sqrt{-1}}, \quad \text{and} \quad v_\theta = \frac{e^{\theta\sqrt{-1}}}{\sqrt{-1}}.$$

Let Π_θ denote the plane generated by $(w_\theta, 0)$ and $(0, 1)$ in $\mathbb{C} \times \mathbb{R}$ passing through the origin. A normal vector to Π_θ is given by a vector $\boldsymbol{v}_\theta$, represented by the complex number v_θ.

We say that $\boldsymbol{v}_\theta$ generates a S_2**-direction** (resp. B_2**-direction**, C_3**-direction**) if

$$a_{21}(w_\theta) = 0 \text{ (resp. } \neq 0, = 0), \quad \text{and} \quad a_{03}(w_\theta) \neq 0 \text{ (resp. } = 0, = 0),$$

where the definition of $a_{ij}(w_\theta)$ $(i + j = 3)$ is given as follows:

$$H_3(wz) = a_{30}(w)\frac{x^3}{6} + a_{21}(w)\frac{x^2 y}{2} + a_{12}(w)\frac{xy^2}{2} + a_{03}(w)\frac{y^3}{6}.$$

In other words, $\boldsymbol{v}_\theta$ generates a S_2-direction (resp. B_2-direction, C_3-direction) if and only if

$$\begin{aligned} |\alpha|\sin(3\theta + \arg\alpha) + |\beta|\sin(\theta + \arg\beta) &= 0 \quad (\text{resp.} \neq 0,\ = 0), \text{ and} \\ |\alpha|\sin(3\theta + \arg\alpha) - 3|\beta|\sin(\theta + \arg\beta) &\neq 0 \quad (\text{resp.} = 0,\ = 0). \end{aligned}$$

Moreover, we say that $\boldsymbol{v}_\theta$ generates a **simple** (resp. **double**) S_2-direction (or B_2-direction), if w_θ is a simple (resp. double) root of the cubic $a_{21}(w)$ (or $a_{03}(w)$).

If $\boldsymbol{v}_\theta$ is not such a direction, f^{Π_θ} has a singularity $\mathcal{A}$-equivalent to $S_1^{\pm}$. Moreover, the singularity of f^{Π_θ} is $\mathcal{A}$-versally unfolded by the family F.

We remark that S_2-direction (or C_3-direction) is orthogonal to a characteristic direction (see the second formula of (4.24)).

Lemma 4.12. *We consider an umiblic defined by* (4.1)*. Then the numbers of S_2-directions, B_2-directions and C_3-directions are summarized as follows:*

D_{H_3}	$D_{H_3'}$	$\alpha\bar{\beta}^3 \neq \bar{\alpha}\beta^3$	$\alpha\bar{\beta}^3 = \bar{\alpha}\beta^3$
$+$	$+$	$3S_2 + 3B_2$	$2S_2 + 2B_2 + C_3$ $(\beta \neq 0)$, $3C_3$ $(\beta = 0)$
0	$+$	$3S_2 + 2B_2$	$2S_2 + B_2 + C_3$
$-$	$+$	$3S_2 + B_2$	$2S_2 + C_3$
0	0	$-$	$S_2 + C_3$
$-$	0	$2S_2 + B_2$	C_3
$-$	$-$	$S_2 + B_2$	C_3

- *Case:* $D_{H_3} \neq 0$, $D_{H_3'} = 0$

  * *If* $\alpha\overline{\beta}^3 \neq \bar{\alpha}\beta^3$, *then there are one simple* S_2*-direction and one double* S_2*-direction.*
  * *If* $\alpha\overline{\beta}^3 = \bar{\alpha}\beta^3$, *then there are one double* S_2*-direction and one* C_3*-direction.*

- *Case:* $D_{H_3} = 0$, $D_{H_3'} \neq 0$.

  * *If* $\alpha\overline{\beta}^3 \neq \bar{\alpha}\beta^3$, *then there are one simple* B_2*-direction and one double* B_2*-direction.*
  * *If* $\alpha\overline{\beta}^3 = \bar{\alpha}\beta^3$, *then there are one simple* B_2*-direction and one* C_3*-direction.*

- *When* $D_{H_3} = 0$, $D_{H_3'} = 0$, *we automatically have* $\alpha\overline{\beta}^3 = \bar{\alpha}\beta^3$ *and there are one double* S_2*-direction and one* C_3*-direction.*

Proof. Routine calculation. See the items (i)–(iv) in the proof of Proposition 4.20 below also.

Assume that $\alpha\overline{\beta}^3 = \bar{\alpha}\beta^3$. When $D_{H_3'} = 0$ and $D_{H_3} \neq 0$ (resp. $D_{H_3'} \neq 0$ and $D_{H_3} = 0$), the computation reduces to case $\alpha = 1$ and $\beta = -3$ (resp. -1), which is analyzed in Example 4.32 (resp. 4.37). When $D_{H_3'} = D_{H_3} = 0$, the computation reduces to case $\alpha = \beta = 1$, which is analyzed in Example 4.33. □

4.2 A criteria of S_2 and S_3 singularities

We here formulate a criterion that the folding map f^{Π_θ} has S_2 or S_3 singularities using curvature lines.

Theorem 4.13. *Let* L_θ *denote the section of the surface by the reflection plane* Π_θ*. If* $\boldsymbol{v}_\theta$ *generates an* S_2*-direction, then* f^{Π_θ} *has a singularity* $\mathcal{A}$*-equivalent to*

- S_2 *if a nonsingular curvature line approaching the umbilic in the direction generated by* w_θ *has 2-point contact with* L_θ*.*
- S_3 *if a nonsingular curvature line approaching the umbilic in the direction generated by* w_θ *has 3-point contact with* L_θ*.*

Before the proof of this theorem we introduce the notion of asymptotic curvature line. We say that a curve

$$\gamma : s \mapsto z = \gamma(s) = p_1 s + p_2 \tfrac{s^2}{2} + p_3 \tfrac{s^3}{6} + O(s^4), \quad p_1 \neq 0, \tag{4.14}$$

represents an **asymptotic curvature line** of order k if it satisfies the equation for curvature lines (4.2) up to order k, that is,

$$\sqrt{-1} \begin{vmatrix} h_z^2 & h_{zz} & d\bar{z}^2 \\ 1+2|h_z|^2 & h_{z\bar{z}} & -dz\, d\bar{z} \\ h_{\bar{z}}^2 & h_{\bar{z}\bar{z}} & dz^2 \end{vmatrix} (\gamma(s)) = O(s^{k+1}).$$

In order to show Theorem 4.13, it is enough to show the following.

Proposition 4.15. *Let* L_θ *denote the section of the surface by the reflection plane* Π_θ*. If* $\boldsymbol{v}_\theta$ *generates an* S_2*-direction, then* f^{Π_θ} *has a singularity* $\mathcal{A}$*-equivalent to*

- *S_2, if and only if $H_4'(w_\theta) \neq 0$, that is, an asymptotic curvature line of order 2 approaching the umbilic in the direction generated by w_θ has 2-point contact with L_θ.*
- *S_3, if and only if $H_4'(w_\theta) = 0$ and $H_5'(w_\theta) \neq 0$, that is, an asymptotic curvature line of order 3 approaching the umbilic in the direction generated by w_θ has 3-point contact with L_θ.*

Proof. The assertions are proved by evaluating (4.2) along a curve defined by (4.14). By this evaluation, the left hand side of (4.2) becomes

$$H_3'(p_1)s + [H_4'(p_1) + O(|p_2|)]s^2 + [H_5'(p_1) + \tfrac{k^2}{2}|p_1|^2 H_3'(p_1) + O(|p_2|, |p_3|))]s^3 + O(s^4), \tag{4.16}$$

where

$$H_k'(z) = \tfrac{1}{\sqrt{-1}}[z^2(H_k)_{zz}(z) - \bar{z}^2(H_k)_{\bar{z}\bar{z}}(z)], \quad \text{for } k = 3, 4, 5, \ldots. \tag{4.17}$$

If the curve (4.14) has at least 3-point contact with L_θ, we have $p_1 = w_\theta$ and $p_2 = 0$. If $H_4'(w_\theta) \neq 0$, (4.16) is not zero. This shows the first assertion.

If the curve (4.14) has at least 4-point contact with L_θ, we have $p_1 = w_\theta$ and $p_2 = p_3 = 0$. If $H_5'(w_\theta) \neq 0$, (4.16) is not zero. This shows the second assertion. □

Remark 4.18. Computation in the previous section has several interesting consequences at umbilic. Consider the surfaces defined by (0.2). When $k_1 \neq k_2$, the tangent direction of the locus $\kappa_1 = k_1$ (resp. $\kappa_2 = k_2$) is generated by $a_{21}\partial_x - a_{30}\partial_y$ (resp. $a_{03}\partial_x - a_{12}\partial_y$) by (3.1), whenever $(a_{21}, a_{30}) \neq 0$ (resp. $(a_{12}, a_{03}) \neq 0$). Tending $k_2 \to k_1$, we obtain that the limit of the tangent directions is generated by $a_{21}\partial_x - a_{30}\partial_y$ (resp. $a_{03}\partial_x - a_{12}\partial_y$). A similar argement using (3.2), (3.3), (3.4) and (3.5) shows that, tending $k_2 \to k_1$, the limit of tangent directions to the levels of $v_i\kappa_j$, $i, j = 1, 2$, at 0 is generated by $a_{12}\partial_x - a_{21}\partial_y$, whenever $(a_{21}, a_{12}) \neq 0$.

Setting $z = x + y\sqrt{-1}$ in (4.3) and (4.5), we have

$$\begin{aligned} H_3 &= \mathrm{Re}(\alpha + 3\beta)x^3/6 - \mathrm{Im}(\alpha + \beta)x^2y/2 - \mathrm{Re}(\alpha - \beta)xy^2/2 + \mathrm{Im}(\alpha - 3\beta)y^3/6, \\ H_3' &= \mathrm{Im}(\alpha + \beta)x^3 + \mathrm{Re}(3\alpha + \beta)x^2y - \mathrm{Im}(3\alpha - \beta)xy^2 - \mathrm{Re}(\alpha - \beta)y^3. \end{aligned}$$

If the origin is v_2-subparabolic (that is, $a_{21} = 0$), then $\mathrm{Im}(\alpha + \beta) = 0$, and the limit direction is generated by ∂_x. We remark that this direction is a characteristic direction since this is a root of H_3'.

If we have a C_3-direction with respect to v_2 (that is, $a_{21} = a_{03} = 0$), then, a discussion similar to the above shows that, tending $k_2 \to k_1$, the corresponding subparabolic line and the corresponding ridge line have the same limiting tangent direction (generated by ∂_x) at the umbilic whenever $a_{12} \neq 0$.

4.3 Criteria of $\mathcal{A}$-versality of the folding family

We consider criteria of $\mathcal{A}$-versality of the folding family at umbilics of surfaces. Since the case for S^1 singularity is always $\mathcal{A}$-versal (see Lemma 1.9), we state other singularities cases.

Theorem 4.19. *We use the notations prepared in §4.1.*

(1) Assume that $\boldsymbol{v}_\theta$ generates an S_2-direction,

- *If the folding map f^{Π_θ} has an S_2-singularity, then the folding map f^{Π_θ} is $\mathcal{A}$-versally unfolded by the folding family F, if and only if $\boldsymbol{v}_\theta$ does not generate a characteristic direction.*
- *If the folding map f^{Π_θ} has an S_3-singularity, then the folding map f^{Π_θ} is $\mathcal{A}$-versally unfolded by the folding family F, if and only if the both of following conditions hold.*
  - *$\boldsymbol{v}_\theta$ does not generate a characteristic direction, and*
  - *$\boldsymbol{v}_\theta$ generates a simple S_2-direction (equivalently, w_θ generates a simple characteristic direction).*

(2) Assume that $\boldsymbol{v}_\theta$ generates a B_2-direction and the folding map f^{Π_θ} has a B_2-singularity. The folding map f^{Π_θ} is $\mathcal{A}$-versally unfolded by the folding family F, if and only if one of the following conditions holds.

- *$\boldsymbol{v}_\theta$ generates a simple B_2-direction, or*
- *$\boldsymbol{v}_\theta$ generates a double B_2-direction and the circle C_θ has 4-point contact with the surface (i.e., $H_4(v_\theta) \neq k^3/8$), where C_θ is the section of the curvature sphere (or the tangent plane when $k = 0$) with the plane generated by the normal $(0,0,1)$ and $\boldsymbol{v}_\theta$.*

(3) Assume that $\boldsymbol{v}_\theta$ generates a C_3-direction and $f|_{\mathcal{M}}$ has a C_3 singularity. Then the folding map is $\mathcal{A}$-versally unfolded by the folding family if and only if one of the following conditions holds.

- *H_3' is a cube (that is, we have a triple characteristic direction), or*
- *H_3 is not a cube and the corresponding subparabolic line has 2-point contact with the corresponding ridge.*

This theorem is a consequence of the following proposition. The reason that the criterion for $\mathcal{A}$-versality for B_3 singularity is missing is that the authors are not aware the geometric meaning of the $\mathcal{A}$-versality condition (that is, $\mathcal{B}_3 \neq 0$) for B_3 singularity.

Proposition 4.20. *We consider an umbilic defined by (4.1). Then the conditions for singularities of the folding map $f|_{\mathcal{M}}$ and $\mathcal{A}$-versality of the folding family are summarized as follows:*

	Condition for singularity type	*Condition for $\mathcal{A}$-versality*
$S_1^\pm$	$\pm H_3'(w_\theta)H_3(v_\theta) > 0$	*always versal.*
S_2	$H_3'(w_\theta) = 0,\ H_3(v_\theta) \neq 0,\ H_4'(w_\theta) \neq 0$	$H_3'(v_\theta) \neq 0$
$S_3^\pm$	$H_3'(w_\theta) = 0, \quad H_4'(w_\theta) = 0$ $\pm H_3(v_\theta)H_5'(w_\theta) > 0$	$H_3'(v_\theta)(H_3'(v_\theta) + 3H_3(w_\theta)) \neq 0$
$B_2^\pm$	$H_3'(w_\theta) \neq 0,\ H_3(v_\theta) = 0,\ \pm B_2(w_\theta) > 0$	$H_3'(v_\theta) \neq 0$ *or* $H_4(v_\theta) \neq \frac{k^3}{8}$
$B_3^\pm$	$H_3'(w_\theta) \neq 0, \quad H_3(v_\theta) = 0,$ $B_2(w_\theta) = 0, \quad \pm B_3(w_\theta) > 0$	$\mathcal{B}_3(w_\theta) \neq 0$
$C_3^\pm$	$H_3'(w_\theta) = 0,\ H_3(v_\theta) = 0,$ $\mp H_4'(w_\theta)H_4'(v_\theta) > 0$	$H_3'(v_\theta)\begin{vmatrix} 3H_4'(w_\theta) & H_3'(v_\theta)+3H_3(w_\theta) \\ 2H_4'(v_\theta) & H_3'(w_\theta) \end{vmatrix} \neq 0$

Here $H'_k(z)$ is defined as (4.17)*. The definitions of $B_2(w_\theta)$, $B_3(w_\theta)$ and $\mathcal{B}_3(w_\theta)$ will be given later as* (4.26)*,* (4.27) *and* (4.28)*.*

Proof. By the rotation defined by $z \mapsto w_\theta z$, we can send Π_θ to Π_0 and $\boldsymbol{v}_\theta$ to ∂_y, and we can apply Lemmas 1.7 and 1.9, which are summarized the criteria of singularities and $\mathcal{A}$-versality as follows:

	Condition for singularity type	Condition for $\mathcal{A}$-versality
$S_1^\pm$	$\pm a_{21}a_{03} > 0$.	always versal.
S_2	$a_{21} = 0$, $a_{03} \neq 0$, $a_{31} \neq 0$.	$a_{12} \neq 0$.
$S_3^\pm$	$a_{21} = 0$, $a_{31} = 0$, , $\pm a_{03}a_{41} > 0$.	$a_{12}(2a_{12} - a_{30}) \neq 0$.
$B_2^\pm$	$a_{21} \neq 0$, $a_{03} = 0$, $\pm(\frac{a_{05}}{5} - \frac{1}{3}\frac{a_{13}^2}{a_{21}}) > 0$.	$a_{12} \neq 0$ or $a_{04} \neq 3k^3$.
$B_3^\pm$	$a_{21} \neq 0$, $a_{03} = 0$, $3a_{05} = 5a_{13}^2/a_{21}$, $\pm(\frac{a_{07}}{7} - a_{15}\frac{a_{13}}{a_{21}} + \frac{5}{3}a_{23}(\frac{a_{13}}{a_{21}})^2 - \frac{5}{9}a_{31}(\frac{a_{13}}{a_{21}})^3) > 0$.	$\begin{vmatrix} a_{12} & p \\ a_{04} - 3k^3 - \frac{a_{12}a_{13}}{a_{21}} & q \end{vmatrix} \neq 0$.
$C_3^\pm$	$a_{21} = 0$, $a_{03} = 0$, $\pm a_{31}a_{13} > 0$.	$a_{12}(3a_{31}a_{12} + a_{13}(2a_{12} - a_{30})) \neq 0$.

where

$$p = \frac{a_{14}}{2} + \frac{a_{13}}{3a_{21}}(a_{04} - 3a_{22}) + \frac{a_{13}^2}{6a_{21}^2}(a_{30} - 2a_{12}), \tag{4.21}$$

$$q = \frac{3}{10}a_{06} - \frac{9}{2}a_{04}k^2 - \frac{3}{10}\frac{a_{12}a_{15}^2}{a_{21}^2} + \frac{a_{13}}{a_{21}}(-a_{14} + 6a_{12}k^2 + \frac{a_{12}a_{23}}{a_{21}}) + \frac{a_{13}^2}{a_{21}^2}(a_{22} - k^3 - \frac{a_{12}a_{31}}{a_{21}}). \tag{4.22}$$

We define $a_{ij}(w)$ by

$$h(wz) = \frac{k}{2}z\bar{z} + \sum_{i+j\geq 3}^{m} a_{ij}(w)\frac{x^iy^j}{i!j!} + O(m+1). \tag{4.23}$$

Then, by direct computation, we have

(4.24)
$$a_{30}(w) = 6H_3(w), \quad a_{21}(w) = -H'_3(w), \quad a_{12}(w) = -H'_3(\tfrac{w}{\sqrt{-1}}), \quad a_{03}(w) = -6H_3(\tfrac{w}{\sqrt{-1}}),$$

and we also conclude

$$2a_{12}(w) - a_{30}(w) = -2(H'_3(\tfrac{w}{\sqrt{-1}}) + 3H_3(w)).$$

Using these relations, we can prove the following assertions, taking resultants of the corresponding cubics.

(i) There is a non-zero w with $a_{21}(w) = a_{12}(w) = 0$ if and only if $|\alpha| = |\beta|$.
(ii) There is a non-zero w with $a_{21}(w) = 2a_{12}(w) - a_{30}(w) = 0$ if and only if $D_{H'_3} = 0$.
(iii) There is a non-zero w with $a_{12}(w) = a_{03}(w) = 0$ if and only if $D_{H_3} = 0$.
(iv) There is a non-zero w with $a_{21}(w) = a_{03}(w) = 0$ if and only if $\alpha\bar{\beta}^3 = \bar{\alpha}\beta^3$.

In the same way as above, we can further show the following relations:

$$a_{31}(w) = -2H'_4(w), \qquad a_{13}(w) = 2H'_4(\tfrac{w}{\sqrt{-1}}),$$

and, we obtain that

$$3a_{31}a_{12}+a_{13}(2a_{12}-a_{30}) = 6H_4(w_\theta)H_3'(w_\theta)-4H_4'(v_\theta)(H_3'(v_\theta)+3H_3(w_\theta)).$$

Furthermore, we also have

$$\begin{array}{llllll} a_{22}(w)= & \frac{1}{4}(6H_4(w)-K_4'(w)), & a_{04}(w)= & 4!H_4(\frac{w}{\sqrt{-1}}), & a_{41}(w)= & -3!H_5'(w), \\ a_{23}(w)= & 3H_5'''(\frac{w}{\sqrt{-1}})-4H_5''(\frac{w}{\sqrt{-1}}), & a_{14}(w)= & -3!H_5'(\frac{w}{\sqrt{-1}}), & a_{05}(w)= & -5!H_5(\frac{w}{\sqrt{-1}}), \\ a_{15}(w)= & 4!H_6'(\frac{w}{\sqrt{-1}}), & a_{06}(w)= & 6!H_6(\frac{w}{\sqrt{-1}}), & a_{07}(w)= & -7!H_7(\frac{w}{\sqrt{-1}}), \end{array} \tag{4.25}$$

where

$$H_k''=\frac{1}{\sqrt{-1}}[z^3(H_k)_{zzz}-\bar{z}^3(H_k)_{\bar{z}\bar{z}\bar{z}}],\ H_k'''=\frac{1}{\sqrt{-1}}[z^4(H_k)_{zzzz}-\bar{z}^4(H_k)_{\bar{z}\bar{z}\bar{z}\bar{z}}],\ K_k'=z^2(H_k)_{zz}+\bar{z}^2(H_k)_{\bar{z}\bar{z}}.$$

Finally we obtain the corresponding expression for B_2, B_3 and $\mathcal{B}_3$ as follows:

$$B_2(w_\theta) = -24H_5(v_\theta)+\tfrac{4}{3}\tfrac{H_4'(v_\theta)^2}{H_3'(w_\theta)}, \tag{4.26}$$

$$\begin{aligned} B_3(w_\theta) = & 6!H_7(v_\theta)+48H_6(v_\theta)\tfrac{H_4'(v_\theta)}{H_3'(w_\theta)}+\tfrac{20}{3}(3H_5'''(v_\theta)-4H_5''(v_\theta))(\tfrac{H_4'(v_\theta)}{H_3'(w_\theta)})^2 \\ & -\tfrac{80}{9}H_4'(w_\theta)(\tfrac{H_4'(v_\theta)}{H_3'(w_\theta)})^3, \end{aligned} \tag{4.27}$$

$$\mathcal{B}_3(w_\theta) = \begin{vmatrix} H_3'(w_\theta) & p(w_\theta) \\ 24H_4(v_\theta)-3k_2^3-\frac{2H_3'(v_\theta)H_4'(v_\theta)}{H_3(v_\theta)H_3'(w_\theta)} & q(w_\theta) \end{vmatrix}, \tag{4.28}$$

where $p(w_\theta)$ (resp. $q(w_\theta)$) is defined by changing a_{ij} by $a_{ij}(w_\theta)$ in (4.21) (resp. (4.22)) and substituting using (4.25). We complete the proof. □

Example 4.29. When $\alpha > 0$ and $\beta = 0$, this is a star, and we have

$$H_3(w_\theta z) = |\alpha|(\cos 3\theta\,\tfrac{x^3-3xy^2}{6}-\sin 3\theta\,\tfrac{3x^2y-y^3}{6}).$$

We then conclude that the folding map f^{Π_θ} has a singularity $\mathcal{A}$-equivalent to

- $S_1^{\pm}$ singularity, if $3\theta \not\equiv 0 \mod \pi$.
- $C_3^{\pm}$ singularity, if $3\theta \equiv 0 \mod \pi$, and $H_4'(w_\theta) \neq 0$, $H_4'(v_\theta) \neq 0$. Moreover, the folding family F is $\mathcal{A}$-versal at f^{Π_θ} if $H_4'(w_\theta)-H_4'(v_\theta) \neq 0$.

Example 4.30. When $\alpha = 0$ and $\beta \neq 0$, this is a lemon, and we have

$$H_3(w_\theta z) = |\beta|(\cos(\theta+\arg\beta)\tfrac{x^3+3xy^2}{2}-\sin(\theta+\arg\beta)\tfrac{3x^2y+y^3}{2}).$$

We then conclude that the folding map f^{Π_θ} has a singularity $\mathcal{A}$-equivalent to

- $S_1^{\pm}$ singularity, if $\theta+\arg\beta \not\equiv 0 \mod \pi$.
- $C_3^{\pm}$ singularity, if $\theta+\arg\beta \equiv 0 \mod \pi$, and $H_4'(w_\theta) \neq 0$, $H_4'(v_\theta) \neq 0$. Moreover, the folding family F is $\mathcal{A}$-versal at f^{Π_θ} if $3H_4'(w_\theta)+H_4'(v_\theta) \neq 0$.

There are cases where $\mathcal{A}$-versality can be determined by 3-jet, which is worth stating as a theorem.

Theorem 4.31. *Assume that the umbilic is star, monstar or lemon. If the folding family f^{Π_θ} has S_2, S_3 or B_2 singularity, then f^{Π_θ} is $\mathcal{A}$-versally unfolded by the folding family F.*

Proof. A consequence of the table and the items (i) – (iv) in the proof of Proposition 4.20. □

Example 4.32. When $\alpha > 0$ and $\beta = \alpha$, the H_3 is a cube. This is the case that $D_{H_3} = D_{H_3'} = 0$, we have

$$a_{21}(w_\theta) = -8|\alpha|\cos^2\theta\sin\theta, \qquad a_{03}(w_\theta) = -8|\alpha|\sin^3\theta.$$

In this case we have one double S_2-direction (that is, $\theta = \pi/2$) and one C_3-direction (that is, $\theta = 0$). Since $a_{12}(w_\theta) = 8|\alpha|\cos\theta\sin^2\theta$, we obtain that the folding map f^{Π_θ} is not $\mathcal{A}$-versally unfolded by the folding family F, even though f^{Π_θ} may define S_2, S_3 or C_3 singularities.

Example 4.33. When $\alpha > 0$ and $\beta = -3\alpha$, we have H_3' is a cube. Then $D_{H_3'} = 0$ and

$$a_{21}(w_\theta) = 8|\alpha|\sin^3\theta, \qquad a_{03}(w_\theta) = 4|\alpha|(5 + \cos 2\theta)\sin\theta.$$

In this case, we have one C_3-directions (that is, $\theta = 0$). Since

$$a_{12}(w_\theta) = -8|\alpha|\cos^3\theta, \qquad 2a_{12}(w_\theta) - a_{30}(w_\theta) = 24|\alpha|\cos\theta\sin^2\theta.$$

If $\boldsymbol{v}_\theta$ generates a C_3-direction and f^{Π_θ} defines C_3 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever $H_4'(w_\theta) \neq 0$.

We first show the item (3) of Theorem 4.19.

Proof of Theorem 4.19 (3). We assume that $\alpha\bar{\beta}^3 = \bar{\alpha}\beta^3$. By Remark 4.10, we can assume that both α and β are non-zero real. Since

$$\begin{aligned} a_{21}(w_\theta) &= -2(\beta\sin\theta + \alpha\sin 3\theta), \\ a_{03}(w_\theta) &= -2(\beta\sin\theta - \alpha\sin 3\theta), \end{aligned}$$

we have $\sin\theta = \sin 3\theta = 0$. It is enough to consider the case $\theta = 0$. We assume that f^{Π_0} defines a C_3 singularity, which means $a_{31}(w_\theta)a_{13}(w_\theta) \neq 0$. Then

$$a_{12}(w_\theta) = 2(\beta\cos\theta - \alpha\cos 3\theta) = 2(\beta - \alpha), \tag{4.34}$$

$$2a_{12}(w_\theta) - a_{30}(w_\theta) = -2(\beta\cos\theta + 3\alpha\cos 3\theta) = -2(\beta + 3\alpha). \tag{4.35}$$

If H_3 is a cube, the folding family is not $\mathcal{A}$-versal, by Example 4.32. We assume that (4.34) is not zero. If (4.35) is zero (that is H_3' is a cube), then the folding family is $\mathcal{A}$-versal, since $a_{31}(w_\theta) \neq 0$. We then assume that (4.35) is not zero.

We consider the surface defined by (0.2). We remark that the coefficient of $x^2/2$ in (3.8) is

$$a_{41} - a_{21}k_1(5k_1 + k_2) + \tfrac{a_{21}(4a_{22} - 3a_{40} + 2k_1^3) + (2a_{12} - a_{30})a_{31}}{k_1 - k_2} + \tfrac{a_{21}(2a_{03} - 7a_{21}) + a_{12}(4a_{12} - 6a_{30}) + 2a_{30}^2}{(k_1 - k_2)^2}$$

and the coefficient of $x^2/2$ in (3.9) is

$$a_{23} + 3a_{21}k_2^2 + \tfrac{3(a_{12}a_{31}+2a_{21}(a_{22}-k_2^3))}{k_1-k_2} + \tfrac{6a_{21}(a_{21}^2-a_{12}^2+a_{12}a_{30})}{(k_1-k_2)^2}.$$

Assume that $a_{21} = a_{03} = 0$ and consider parametrizations of the zeros of (3.8) and (3.9). Tending $k_2 \to k_1$ we obtain the following: The limit of v_2-subparabolic lines is represented by

$$t \mapsto (x, y) = (t, \tfrac{a_{30}a_{31}}{a_{12}(a_{30}-2a_{12})}\tfrac{t^2}{2} + O(3)),$$

and the limit of v_2-ridge lines is represented by

$$t \mapsto (x, y) = (t, -\tfrac{2a_{12}a_{13}-2a_{30}a_{13}+3a_{12}a_{31}}{3a_{12}^2}\tfrac{t^2}{2} + O(3)).$$

We thus complete the proof, since

$$\tfrac{a_{30}a_{31}}{a_{12}(a_{30}-2a_{12})} + \tfrac{2a_{12}a_{13}-2a_{30}a_{13}+3a_{12}a_{31}}{3a_{12}^2} = \tfrac{2(a_{30}-a_{12})(-3a_{12}a_{31}+(a_{30}-2a_{12})a_{13})}{3a_{12}^2(2a_{12}-a_{30})}. \qquad \square$$

We also see several examples, as consequences of Proposition 4.20.

Example 4.36. When $\alpha > 0$ and $\beta = -(1/3)\alpha$, which is the case that $D_{H_3} = 0$, $D_{H_3'} \neq 0$ with C_3-direction, we have

$$a_{21}(w_\theta) = -\tfrac{4}{3}|\alpha|\sin\theta(1 + 3\cos 2\theta), \qquad a_{03}(w_\theta) = 8|\alpha|\cos^2\theta\sin\theta.$$

In this case, we have two simple S_2-directions (that is, $\theta = \pm\tan^{-1}\sqrt{2}$), one double B_2-direction (that is, $\theta = \pi/2$) and one C_3-direction (that is, $\theta = 0$). Since

$$a_{12}(w_\theta) = \tfrac{4}{3}|\alpha|\cos\theta(1 - 3\cos 2\theta), \qquad 2a_{12}(w_\theta) - a_{30}(w_\theta) = \tfrac{4}{3}|\alpha|\cos\theta(9\cos 2\theta - 5),$$

we obtain the following:

- If $\boldsymbol{v}_\theta$ generates a simple S_2-directions, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever f^{Π_θ} is S_2 or S_3 singularity.
- If $\boldsymbol{v}_\theta$ generates a double B_2-direction, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ} whenever $H_4(v_\theta) \neq k^3/8$.
- If $\boldsymbol{v}_\theta$ generates a C_3-direction and f^{Π_θ} defines a C_3 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever $3H_4'(w_\theta) - 2H_4'(v_\theta) \neq 0$.

Example 4.37. When $\alpha > 0$ and $\beta = -\alpha$, which is the right-angled umbilic with a C_3-direction, we have

$$a_{21}(w_\theta) = 8|\alpha|\sin\theta\sin\big(\tfrac{\pi}{4} + \theta\big)\sin\big(\tfrac{\pi}{4} - \theta\big), \qquad a_{03}(w_\theta) = 4|\alpha|\sin\theta(2 + \cos 2\theta).$$

In this case, we have two S_2-directions (that is, $\theta = \pm\pi/4$), which generate characteristic directions, and one C_3-direction (that is, $\theta = 0$). Since

$$a_{12}(w_\theta) = 4|\alpha|\cos\theta\sin\big(\tfrac{\pi}{4} + \theta\big)\sin\big(\tfrac{\pi}{4} - \theta\big), \quad 2a_{12}(w_\theta) - a_{30}(w_\theta) = 4|\alpha|\cos\theta(2 - 3\cos 2\theta),$$

we obtain the following:

- If $\boldsymbol{v}_\theta$ generates a simple S_2-directions, then the folding family F defines $\mathcal{A}$-versal at f^{Π_θ}, whenever f^{Π_θ} defines an S_2 or S_3 singularity.
- If $\boldsymbol{v}_\theta$ generates a C_3-direction and f^{Π_θ} defines C_3 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever $3H_4'(w_\theta) - 2H_4'(v_\theta) \neq 0$.

Proof of Theorem 4.19 (1), (2). The proof is already done when the umbilic is star, monstar and lemon. So we consider the case $|\alpha| = |\beta|$ or $D_{H_3'} = 0$ or $D_{H_3} = 0$. The following cases have been already analyzed.

- $D_{H_3'} = D_{H_3} = 0$ (Example 4.32).
- singular locus of $D_{H_3'} = 0$ (Example 4.33).
- $D_{H_3} = 0$, $\alpha\bar{\beta}^3 = \bar{\alpha}\beta^3$, $D_{H_3} \neq 0$ (Example 4.36).
- $|\alpha| = |\beta|$, $\alpha\bar{\beta}^3 = \bar{\alpha}\beta^3$, $D_{H_3} \neq 0$ (Example 4.37).

Without loss of generality, we can assume that $\alpha \geq 0$. We first consider the case that the umbilic is right-angled (that is, $|\alpha| = |\beta|$) with no C_3-direction (that is, $\alpha\bar{\beta}^3 \neq \bar{\alpha}\beta^3$). We can assume that $\alpha > 0$. We obtain that

$$
\begin{aligned}
a_{21}(w_\theta) =& -4|\alpha| \cos\big(\theta - \tfrac{\arg\beta}{2}\big) \sin\big(2\theta + \tfrac{\arg\beta}{2}\big), \\
a_{03}(w_\theta) =& 2|\alpha|(\sin 3\theta - 3\sin(\theta + \arg\beta)).
\end{aligned}
$$

Thus there are three simple S_2-direction (that is, $\theta = -\frac{\arg\beta}{4}, \frac{2\pi - \arg\beta}{4}, \frac{\pi + \arg\beta}{2}$) and one simple B_2-direction. Since

$$
a_{12}(w_\theta) = 4|\alpha| \sin\big(\theta - \tfrac{\arg\beta}{2}\big) \sin\big(2\theta + \tfrac{\arg\beta}{2}\big),
$$

we have the following:

- If $\boldsymbol{v}_\theta$ generates a simple S_2-directions with $\theta = \frac{\pi + \arg\beta}{2}$, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever f^{Π_θ} defines an S_2 or S_3 singularity.
- If $\boldsymbol{v}_\theta$ generates a simple S_2-directions with $\theta = \frac{\arg\beta}{4}, \frac{2\pi + \arg\beta}{4}$, then the folding family F is not $\mathcal{A}$-versal at f^{Π_θ}.
- If $\boldsymbol{v}_\theta$ generates a B_2-direction and f^{Π_θ} defines B_2 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever $H_4(v_\theta) \neq k^3/8$.

We next consider the case that $D_{H_3'} = 0$ with no C_3-direction (that is, $\alpha\bar{\beta}^3 \neq \bar{\alpha}\beta^3$). Using the notation of (4.9), we obtain that

$$
\begin{aligned}
a_{21}(w_\theta) =& 8\sin^2(\theta - \tfrac{\phi}{2}) \sin(\theta + \phi), \\
a_{03}(w_\theta) =& 2|\alpha|(\sin 3\theta + 3\sin(\theta - 2\phi) + 6\sin(\theta + \phi)).
\end{aligned}
$$

Thus there are one simple S_2-direction (that is, $\theta = -\phi$), one double S_2-direction (that is, $\theta = \phi/2$), and one simple B_2-direction. Since

$$
\begin{aligned}
a_{12}(w_\theta) =& 8\sin^2(\theta - \tfrac{\phi}{2}) \cos(\theta + \phi), \\
2a_{12}(w_\theta) - a_{30}(w_\theta) =& 2|\alpha| \sin\big(\theta - \tfrac{\phi}{2}\big)\big(3\sin\big(2\theta + \tfrac{\phi}{2}\big) + \sin\tfrac{3\phi}{2}\big),
\end{aligned}
$$

we have the following:

- If $\boldsymbol{v}_\theta$ generates a simple S_2-direction, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever f^{Π_θ} defines an S_2 or S_3 singularity.
- If $\boldsymbol{v}_\theta$ generates a double S_2-direction, then the folding family F is not $\mathcal{A}$-versal at f^{Π_θ}. (In [2, line 9, page 70], Bruce and Wilkinson mentioned that "S_2 is not versally unfolded by F", which should be read as pointing out this fact.)
- If $\boldsymbol{v}_\theta$ generates a B_2-direction and f^{Π_θ} defines B_2 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}.

Thirdly, we consider the case that $D_{H_3} = 0$ with no C_3-direction (that is, $\alpha\bar{\beta}^3 \neq \bar{\alpha}\beta^3$). Using the notation of (4.8), we obtain that

$$\begin{aligned} a_{21}(w_\theta) =& -\tfrac{4}{3}|\alpha|\sin\big(\theta-\tfrac{\phi}{2}\big)\big(3\cos\big(2\theta+\tfrac{\phi}{2}\big)+\cos\tfrac{3\phi}{2}\big),\\ a_{03}(w_\theta) =& 8|\alpha|\cos^2\big(\theta-\tfrac{\phi}{2}\big)\sin(\theta+\phi). \end{aligned}$$

Thus there are one simple B_2-direction, one double B_2-direction and three simple S_2-directions. Since

$$a_{12}(w_\theta) = -\tfrac{4}{3}|\alpha|\cos\big(\theta-\tfrac{\phi}{2}\big)\big(3\cos\big(2\theta+\tfrac{\phi}{2}\big)+\cos\tfrac{3\phi}{2}\big),$$

we have the following:

- If $\boldsymbol{v}_\theta$ generates a simple S_2-direction, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}, whenever f^{Π_θ} defines an S_2 or S_3 singularity.
- If $\boldsymbol{v}_\theta$ generates a simple B_2-direction, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ}.
- If $\boldsymbol{v}_\theta$ generates a double B_2-direction and f^{Π_θ} defines B_2 singularity, then the folding family F is $\mathcal{A}$-versal at f^{Π_θ} if and only if $H_4(v_\theta) \neq k^3/8$. □

Remark 4.38. When $\alpha = \beta = 0$, then any direction v_θ is a C_3-direction and the folding map can have C_3 singularity, but the folding family F is not $\mathcal{A}$-versal at f^{Π_θ}.

5 Proof of Lemma 1.9

We consider a motion $\boldsymbol{p} \longmapsto A(\boldsymbol{p}) = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3)\boldsymbol{p} + \boldsymbol{a}_0$ where

$$\boldsymbol{a}_0 = w\begin{pmatrix}\tau_1\\1\\\tau_3\end{pmatrix},\ \boldsymbol{a}_1 = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix}\sqrt{1-u^2-v^2}\\-u\\0\end{pmatrix},\ \boldsymbol{a}_2 = \begin{pmatrix}u\\\sqrt{1-u^2-v^2}\\v\end{pmatrix},\quad \boldsymbol{a}_3 = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix}-uv\\-v\sqrt{1-u^2-v^2}\\1-v^2\end{pmatrix}.$$

Here we remark that $(\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3)$ is an orthogonal matrix. We consider the motions

$$A(\boldsymbol{p}) = \begin{pmatrix}\frac{\sqrt{1-u^2-v^2}x-uvz}{\sqrt{1-v^2}}+uy+w\tau_1\\-\frac{ux+vz\sqrt{1-u^2-v^2}}{\sqrt{1-v^2}}+\sqrt{1-u^2-v^2}y+w\\vy+\sqrt{1-v^2}z+w\tau_3\end{pmatrix},\quad \boldsymbol{p} = \begin{pmatrix}x\\y\\z\end{pmatrix},$$

in $F = A^{-1}\circ f\circ A(\boldsymbol{p})$ (see (1.1)), and we obtain

$$(5.1)\qquad F_u|_{(u,v,w)=0} = \begin{pmatrix}y(1-y)\\x(1-2y)\\0\end{pmatrix},\ F_v|_{(u,v,w)=0} = \begin{pmatrix}0\\z(1-2y)\\y(1-y)\end{pmatrix},\ F_w|_{(u,v,w)=0} = \begin{pmatrix}0\\2y-1\\0\end{pmatrix}.$$

Setting $f = F|_{(u,v,w)=0}$, we have

$$f = \Big(x, y^2, \tfrac{k_1x^2+k_2y^2}{2} + \sum_{i+j\geq 3}^{m} a_{i,j}\tfrac{x^iy^j}{i!j!} + O(m+1)\Big). \tag{5.2}$$

We are looking for the condition that F is an $\mathcal{A}$-versal unfolding of f, that is,

$$\mathcal{E}_2^{\oplus 3} = T\mathcal{R}f + T\mathcal{L}f + V_F. \tag{5.3}$$

where $T\mathcal{R}f = \langle f_x, f_y\rangle_{\mathcal{E}_2}$, $T\mathcal{L}f = f^{-1}\mathcal{E}_3^{\oplus 3}$, $V_F = \langle \dot{F}_u, \dot{F}_v, \dot{F}_w\rangle_{\mathbb{R}}$. Here $\dot{F}_u = F_u|_{(u,v,w)=0}$, $\dot{F}_v = F_v|_{(u,v,w)=0}$, and $\dot{F}_w = F_w|_{(u,v,w)=0}$.

In the notation in [11, §3], this is $\mathcal{G}_e$-versality with $\mathcal{G} = \mathcal{A}$. See Versality Theorem 3.3 loc. cite. also.

If f is m-$\mathcal{A}$-determined, then we have

$$(\langle x, y\rangle_{\mathcal{E}_2}^{m+1})^{\oplus 3} \subset T\mathcal{R}f + T\mathcal{L}f.$$

Now we return to the case for the folding family. We assume that the map-germ $(x, y) \mapsto (x, y^2, f(x, y))$ is m-determined. We consider the condition that the matrix

$$\tilde{M} = \begin{pmatrix} \tilde{T}_1 & \tilde{W}\boldsymbol{e}_1 & O & O & \tilde{V}_1 \\ \tilde{T}_2 & O & \tilde{W}\boldsymbol{e}_2 & O & \tilde{V}_2 \\ \tilde{T}_3 & O & O & \tilde{W}\boldsymbol{e}_3 & \tilde{V}_3 \end{pmatrix}$$

is of full rank, where

$$\begin{aligned}
\tilde{T}_s &= ((\phi^j_{j_1,j_2})^*(x^{i_1}y^{i_2})^*(\phi^1 f_x + \phi^2 f_y)\boldsymbol{e}_s)_{i_1+i_2\leq m;\ j=1,2, j_1+j_2\leq m} \qquad (s = 1, 2, 3),\\
\tilde{W} &= \begin{pmatrix}\tilde{W}_0 & \tilde{W}_1 & \dots & \tilde{W}_{\lfloor m/2\rfloor}\end{pmatrix}, \quad \tilde{W}_k = ((x^{i_1}y^{i_2})^*(x^{j_1}y^{j_2}f^k))_{i_1+i_2\leq m,\ j_1+j_2\leq m-2k},\\
\tilde{V}_1 &= ((x^{i_1}y^{i_2})^*(y(1-y)\boldsymbol{e}_1\ 0\ 0))_{i_1+i_2\leq m},\\
\tilde{V}_2 &= ((x^{i_1}y^{i_2})^*(x(1-2y)\boldsymbol{e}_2\ f(1-2y)\boldsymbol{e}_2\ 2y\boldsymbol{e}_2))_{i_1+i_2\leq m},\\
\tilde{V}_3 &= ((x^{i_1}y^{i_2})^*(0\ y(1-y)\boldsymbol{e}_3\ 0))_{i_1+i_2\leq m}.
\end{aligned}$$

Here we define

$$(x^{j_1}y^{j_2})^*(x^{i_1}y^{i_2}) = \begin{cases} 1 & (i_1, i_2) = (j_1, j_2) \\ 0 & \text{otherwise} \end{cases}$$

$$(\phi^j_{j_1j_2})^*\phi^i_{i_1i_2} = \begin{cases} 1 & (i, i_1, i_2) = (j, j_1, j_2) \\ 0 & \text{otherwise} \end{cases}$$

where $\phi^i = \sum_{i_1,i_2} \phi^i_{i_1.i_2}x^{i_1}y^{i_2}$.

Because of the submatrices $\tilde{W}_0\boldsymbol{e}_s$ $(s = 1, 2, 3)$, we can remove

- the columns corresponding to $x^{i_1}y^{2i_2}\boldsymbol{e}_s$ $(i_1 + 2i_2 \leq m,\ s = 1, 2, 3)$, and
- the rows corresponding to $x^{j_1}y^{2j_2}\boldsymbol{e}_s$ $(j_1 + 2j_2 \leq m,\ s = 1, 2, 3)$

from the matrix $\tilde{M}$. The matrix obtained by this operation is denoted by

$$M = \begin{pmatrix} T_1 & W\boldsymbol{e}_1 & O & O & V_1 \\ T_2 & O & W\boldsymbol{e}_2 & O & V_2 \\ T_3 & O & O & W\boldsymbol{e}_3 & V_3 \end{pmatrix} \quad \text{where} \quad W = \begin{pmatrix} W_1 & W_2 & \dots & W_{\lfloor m/2\rfloor}\end{pmatrix}.$$

We set $T = \left(\begin{smallmatrix} T_1 \\ T_2 \\ T_3 \end{smallmatrix}\right)$.

5.1 S_1 singularity

We assume that f is $\mathcal{A}$-equivalent to S_1 singularity, that is, $a_{21} \neq 0$ and $a_{03} \neq 0$. Remark that S_1 singularity is 3-determined ($m = 3$). The matrix M is expressed as follows:

	ϕ^1_{00}	ϕ^1_{10}	ϕ^1_{01}	ϕ^1_{11}	ϕ^1_{21}	ϕ^1_{03}	ϕ^2_{00}	ϕ^2_{10}	ϕ^2_{01}	ϕ^2_{11}	ϕ^2_{03}	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\dot{F}_u$	$\dot{F}_v$
$y\boldsymbol{e}_1$			1												1	
$xy\boldsymbol{e}_1$				$\boxed{1}$												
$x^2y\boldsymbol{e}_1$					$\boxed{\frac{1}{2}}$							$\utilde{\frac{a_{21}}{2}}$				
$y^3\boldsymbol{e}_1$						$\boxed{\frac{1}{6}}$						$\utilde{\frac{a_{03}}{6}}$				
$y\boldsymbol{e}_2$							$\boxed{2}$									
$xy\boldsymbol{e}_2$								2							-2	
$x^2y\boldsymbol{e}_2$										$\underline{\underline{1}}$			$\utilde{\underline{\frac{a_{21}}{2}}}$			$\utilde{\utilde{\frac{a_{21}}{2} - k_1}}$
$y^3\boldsymbol{e}_2$											$\underline{\underline{1}}$		$\utilde{\underline{\frac{a_{03}}{6}}}$			$\utilde{\utilde{\frac{a_{03}}{6} - k_2}}$
$y\boldsymbol{e}_3$							$\utilde{k_2}$									$\underline{1}$
$xy\boldsymbol{e}_3$	$\underline{\underline{a_{21}}}$		$\utilde{k_1}$				$\utilde{a_{12}}$	$\utilde{k_2}$								
$x^2y\boldsymbol{e}_3$	$\utilde{\frac{a_{31}}{2}}$	$\boxed{a_{21}}$	$\utilde{\frac{a_{30}}{2}}$	$\utilde{k_1}$			$\utilde{\frac{a_{22}}{2}}$	$\utilde{a_{12}}$	$\utilde{\frac{a_{21}}{2}}$	$\utilde{\frac{k_2}{2}}$				$\utilde{\frac{a_{21}}{2}}$		
$y^3\boldsymbol{e}_3$	$\utilde{\utilde{\frac{a_{13}}{6}}}$		$\utilde{\utilde{\frac{a_{12}}{2}}}$				$\utilde{\frac{a_{04}}{6}}$		$\utilde{\utilde{\frac{a_{03}}{2}}}$		$\utilde{\utilde{\frac{k_2}{2}}}$			$\underline{\frac{a_{03}}{6}}$		

First, by Gauss's elimination method using boxed elements as pivots, we eliminate elements with wavy lines below. Next, by Gauss's elimination method using the underlined elements as pivots, we eliminate elements with double wavy lines below. Thirdly, by Gauss's elimination method using the double underlined elements as pivots, we eliminate elements with wavy lines below with underlines. Now it is easy to see that this matrix is always of full rank.

5.2 S_2 singularity

We assume that f is $\mathcal{A}$-equivalent to S_2 singularity, that is, $a_{21} = 0$, $a_{03} \neq 0$ and $a_{31} \neq 0$. Remark that S_2 singularity is 4-determined ($m = 4$). The non-zero entries of the matrix T is shown in the following tables.

	ϕ^1_{00}	ϕ^1_{10}	ϕ^1_{01}	ϕ^1_{20}	ϕ^1_{11}	ϕ^1_{02}	ϕ^1_{30}	ϕ^1_{21}	ϕ^1_{12}	ϕ^1_{03}	ϕ^1_{40}	ϕ^1_{31}	ϕ^1_{22}	ϕ^1_{13}	ϕ^1_{04}
$y\boldsymbol{e}_1$			1												
$xy\boldsymbol{e}_1$					1										
$x^2y\boldsymbol{e}_1$								$\frac{1}{2}$							
$y^3\boldsymbol{e}_1$										$\frac{1}{6}$					
$x^3y\boldsymbol{e}_1$												$\frac{1}{6}$			
$xy^3\boldsymbol{e}_1$														$\frac{1}{6}$	
$y\boldsymbol{e}_3$															
$xy\boldsymbol{e}_3$		k_1													
$x^2y\boldsymbol{e}_3$	$\frac{a_{31}}{2}$		$\frac{a_{30}}{2}$		k_1										
$y^3\boldsymbol{e}_3$	$\frac{a_{13}}{6}$		$\frac{a_{12}}{2}$												
$x^3y\boldsymbol{e}_3$	$\frac{a_{41}}{6}$	$\frac{a_{31}}{2}$	$\frac{a_{40}}{6}$		$\frac{a_{30}}{2}$			$\frac{k_1}{2}$							
$xy^3\boldsymbol{e}_3$	$\frac{a_{23}}{6}$	$\frac{a_{13}}{6}$	$\frac{a_{22}}{2}$		$\frac{a_{12}}{2}$					$\frac{k_1}{6}$					

	ϕ^2_{00}	ϕ^2_{10}	ϕ^2_{01}	ϕ^2_{20}	ϕ^2_{11}	ϕ^2_{02}	ϕ^2_{30}	ϕ^2_{21}	ϕ^2_{12}	ϕ^2_{03}
$y\boldsymbol{e}_2$	2									
$xy\boldsymbol{e}_2$		2								
$x^2y\boldsymbol{e}_2$				1						
$y^3\boldsymbol{e}_2$						1				
$x^3y\boldsymbol{e}_2$							$\frac{1}{3}$			
$xy^3\boldsymbol{e}_2$									1	
$y\boldsymbol{e}_3$	k_2									
$xy\boldsymbol{e}_3$	a_{12}	k_2								
$x^2y\boldsymbol{e}_3$	$\frac{a_{22}}{2}$	a_{12}		$\frac{k_2}{2}$						
$y^3\boldsymbol{e}_3$	$\frac{a_{04}}{6}$		$\frac{a_{03}}{2}$			$\frac{k_2}{2}$				
$x^3y\boldsymbol{e}_3$	$\frac{a_{32}}{6}$	$\frac{a_{22}}{2}$	$\frac{a_{31}}{6}$	$\frac{a_{12}}{2}$			$\frac{k_2}{2}$			
$xy^3\boldsymbol{e}_3$	$\frac{a_{14}}{6}$	$\frac{a_{04}}{6}$	$\frac{a_{13}}{2}$		$\frac{a_{03}}{2}$	$\frac{a_{12}}{2}$			$\frac{k_2}{2}$	

The non-zero elements of the matrix W are given as follows:

	$\boldsymbol{e}_1$	$x\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$x\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$x\boldsymbol{e}_3$	$\dot{F}_u$	$\dot{F}_v$	$\dot{F}_w$
$y\boldsymbol{e}_1$								-1	
$xy\boldsymbol{e}_1$									
$x^2y\boldsymbol{e}_1$									
$y^3\boldsymbol{e}_1$	$\frac{a_{03}}{6}$								
$x^3y\boldsymbol{e}_1$	$\frac{a_{31}}{6}$								
$xy^3\boldsymbol{e}_1$	$\frac{a_{13}}{6}$	$\frac{a_{03}}{6}$							
$y\boldsymbol{e}_2$									2
$xy\boldsymbol{e}_2$							-2		
$x^2y\boldsymbol{e}_2$								$-k_1$	
$y^3\boldsymbol{e}_2$			$\frac{a_{03}}{6}$						$\frac{a_{03}}{6} - k_2$
$x^3y\boldsymbol{e}_2$			$\frac{a_{31}}{6}$						$\frac{a_{31}}{6} - \frac{a_{30}}{3}$
$xy^3\boldsymbol{e}_2$			$\frac{a_{13}}{6}$	$\frac{a_{03}}{6}$					$\frac{a_{13}}{6} - a_{12}$
$y\boldsymbol{e}_3$								1	
$xy\boldsymbol{e}_3$									
$x^2y\boldsymbol{e}_3$									
$y^3\boldsymbol{e}_3$					$\frac{a_{03}}{6}$				
$x^3y\boldsymbol{e}_3$					$\frac{a_{31}}{6}$				
$xy^3\boldsymbol{e}_3$					$\frac{a_{13}}{6}$	$\frac{a_{03}}{6}$			

We thus looking for the condition so that the following matrix is of full rank.

$$
\begin{array}{ccccccccccccccccccccccccc}
0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0\\
0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\mathbf{\frac{1}{2}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&0&0&0&\frac{a_{03}}{6}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&0&0&\frac{a_{31}}{6}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&0&\frac{a_{13}}{6}&\frac{a_{03}}{6}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{2}\\
0&0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{-2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&-k_1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&\frac{a_{03}}{6}&0&0&0&0&a_{03}-k_2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{\frac{1}{3}}&0&0&0&\frac{a_{31}}{6}&0&0&0&0&\frac{a_{31}}{6}-\frac{a_{30}}{3}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&\frac{a_{13}}{6}&\frac{a_{03}}{6}&0&0&0&\frac{a_{13}}{6}-a_{12}&0\\
0&0&0&0&0&0&0&0&k_2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
0&0&\boldsymbol{k_1}&0&0&0&0&0&\boldsymbol{a_{12}}&\boldsymbol{k_2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\boldsymbol{\frac{a_{31}}{2}}&0&\frac{a_{30}}{2}&k_1&0&0&0&0&\frac{a_{22}}{2}&a_{12}&0&\frac{k_2}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\frac{a_{13}}{6}&0&\frac{a_{12}}{2}&0&0&0&0&0&\frac{a_{04}}{6}&0&\boldsymbol{\frac{a_{03}}{2}}&0&0&\frac{k_2}{2}&0&0&0&0&0&0&\frac{a_{03}}{6}&0&0&0&0\\
\frac{a_{41}}{6}&\boldsymbol{\frac{a_{31}}{2}}&\frac{a_{40}}{6}&\frac{a_{30}}{2}&\frac{k_1}{2}&0&0&0&\frac{a_{32}}{6}&\frac{a_{22}}{2}&\boldsymbol{\frac{a_{31}}{6}}&\frac{a_{12}}{2}&0&0&\frac{k_2}{6}&0&0&0&0&0&\frac{a_{31}}{6}&0&0&0&0\\
\frac{a_{23}}{6}&\frac{a_{13}}{6}&\frac{a_{22}}{2}&\frac{a_{12}}{2}&0&\frac{k_1}{6}&0&0&\frac{a_{14}}{6}&\frac{a_{04}}{6}&\frac{a_{13}}{2}&0&\boldsymbol{\frac{a_{03}}{2}}&\frac{a_{12}}{2}&0&\frac{k_2}{2}&0&0&0&0&\frac{a_{13}}{6}&\frac{a_{03}}{6}&0&0&0
\end{array}
$$

By applying the row-addition transformation to this matrix 12 times, we can remove the first 12 rows and columns 3-10, 12 and 14-16, yielding the following matrix.

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&1&-k_2\\
0&0&0&0&0&0&0&0&0&0&k_2-k_1&0&-a_{12}\\
\frac{a_{31}}{2}&0&0&0&0&0&0&0&0&0&*&*&*\\
\frac{a_{13}}{6}&0&\frac{a_{03}}{2}&0&0&0&*&0&*&0&*&*&*\\
\frac{a_{41}}{6}&\frac{a_{31}}{2}&\frac{a_{31}}{6}&0&0&0&*&0&*&0&*&*&*\\
\frac{a_{23}}{6}&*&*&\frac{a_{03}}{2}&*&0&*&*&*&*&*&*&*
\end{pmatrix}
$$

Since $a_{31} \neq 0$, $a_{03} \neq 0$, we conclude that this matrix is of full rank if and only if the upper-right 2×3 matrix is of full rank, that is, $k_1 \neq k_2$ or $a_{12} \neq 0$.

5.3 C_3 singularity

We assume that f is $\mathcal{A}$-equivalent to S_2 singularity, that is, $a_{21} = a_{03} = 0$, $a_{31} \neq 0$ and $a_{13} \neq 0$. Remark that C_3 singularity is 4-determined ($m = 4$). In a similar way to the discussion above, we obtain the following matrix for T.

$$
\begin{array}{ccccccccccccccccccccc}
0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0\\
0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\mathbf{\frac{1}{2}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&0&\frac{a_{31}}{6}&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{\frac{1}{6}}&0&0&0&0&0&0&0&\frac{a_{13}}{6}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&0&0&0&\mathbf{2}\\
0&0&0&0&0&0&0&0&0&\mathbf{2}&0&0&0&0&0&0&0&0&\mathbf{-2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&-k_1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&-k_2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{\frac{1}{3}}&0&0&\frac{a_{31}}{6}&0&0&\frac{a_{31}}{6}-\frac{a_{30}}{3}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&\frac{a_{13}}{6}&0&0&\frac{a_{13}}{6}-a_{12}&0\\
0&0&0&0&0&0&0&0&k_2&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
0&0&\boldsymbol{k_1}&0&0&0&0&0&\boldsymbol{a_{12}}&\boldsymbol{k_2}&0&0&0&0&0&0&0&0&0&0&0\\
\boldsymbol{\frac{a_{31}}{2}}&0&\frac{a_{30}}{2}&k_1&0&0&0&0&\frac{a_{22}}{2}&a_{12}&0&\frac{k_2}{2}&0&0&0&0&0&0&0&0&0\\
\frac{a_{13}}{6}&0&\frac{a_{12}}{2}&0&0&0&0&0&\frac{a_{04}}{6}&0&0&0&\frac{k_2}{2}&0&0&0&0&0&0&0&0\\
\boldsymbol{\frac{a_{41}}{6}}&\boldsymbol{\frac{a_{31}}{2}}&\frac{a_{40}}{6}&\frac{a_{30}}{2}&\frac{k_1}{2}&0&0&0&\frac{a_{32}}{6}&\frac{a_{22}}{2}&\boldsymbol{\frac{a_{31}}{6}}&\frac{a_{12}}{2}&0&\frac{k_2}{6}&0&0&0&\frac{a_{31}}{6}&0&0&0\\
\boldsymbol{\frac{a_{23}}{6}}&\boldsymbol{\frac{a_{13}}{6}}&\frac{a_{22}}{2}&\frac{a_{12}}{2}&0&\frac{k_1}{6}&0&0&\frac{a_{14}}{6}&\frac{a_{04}}{6}&\boldsymbol{\frac{a_{13}}{2}}&0&\frac{a_{12}}{2}&0&\frac{k_2}{2}&0&0&\frac{a_{13}}{6}&0&0&0
\end{array}
$$

By applying the row-addition transformation to this matrix 12 times, we can remove the first 12 rows and columns 3-10, 12-15, yielding the following matrix:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -k_2 \\
0 & 0 & 0 & 0 & 0 & 0 & k_2-k_1 & 0 & -a_{12} \\
\frac{a_{31}}{2} & 0 & 0 & 0 & 0 & 0 & a_{12}-\frac{a_{30}}{2} & \frac{k_1k_2}{2} & -\frac{a_{22}}{2} \\
\frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & -\frac{a_{12}}{2} & \frac{k_2^2}{2} & -\frac{a_{04}}{6} \\
\frac{a_{41}}{6} & \frac{a_{31}}{2} & \frac{a_{31}}{6} & 0 & * & * & * & * & * \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & \frac{a_{13}}{2} & 0 & * & * & * & * & *
\end{pmatrix}$$

Since $a_{31}a_{13} \neq 0$, we then conclude that this matrix is of full rank if and only if

$$\begin{vmatrix}
k_2-k_1 & -3a_{12}+\frac{a_{13}}{a_{31}}(a_{30}-2a_{12}) \\
a_{12} & a_{04}-3k_2^3+\frac{a_{13}}{a_{31}}(a_{22}-k_1k_2^2)
\end{vmatrix} \neq 0.$$

5.4 S_3 singularity

We assume that f is $\mathcal{A}$-equivalent to S_2 singularity, that is, $a_{21}=a_{31}=0$, $a_{03}\neq 0$ and $a_{41}\neq 0$. Remark that S_3 singularity is 5-determined ($m=5$). In a similar way to the discussion above, the matrix T, removing zero columns, is obtained as follows:

$$\begin{array}{cccccccccccccccccccccccccc}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\frac{1}{6}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{6}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{6}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{24}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{12}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{120}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{3}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{12}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{12}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&k_2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&k_1&0&0&0&0&0&0&0&0&0&0&a12&k_2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\frac{a_{30}}{2}&0&k_1&0&0&0&0&0&0&0&0&\frac{a_{22}}{2}&a_{12}&0&\frac{k_2}{2}&0&0&0&0&0&0&0&0&0\\
\frac{a_{13}}{6}&0&\frac{a_{12}}{2}&0&0&0&0&0&0&0&0&0&0&\frac{a_{04}}{6}&0&\frac{a_{03}}{2}&0&0&\frac{k_2}{2}&0&0&0&0&0&0&0\\
\frac{a_{41}}{6}&0&\frac{a_{40}}{6}&0&\frac{a_{30}}{2}&0&\frac{k_1}{2}&0&0&0&0&0&0&\frac{a_{32}}{6}&\frac{a_{22}}{2}&0&\frac{a_{12}}{2}&0&0&\frac{k_2}{6}&0&0&0&0&0&0\\
\frac{a_{23}}{6}&\frac{a_{13}}{6}&\frac{a_{22}}{2}&0&\frac{a_{12}}{2}&0&0&\frac{k_1}{6}&0&0&0&0&0&\frac{a_{14}}{6}&\frac{a_{04}}{6}&\frac{a_{13}}{2}&0&\frac{a_{03}}{2}&\frac{a_{12}}{2}&0&0&\frac{k_2}{2}&0&0&0&0\\
\frac{a_{51}}{24}&\frac{a_{41}}{6}&\frac{a_{50}}{24}&0&\frac{a_{40}}{6}&0&\frac{a_{30}}{4}&0&\frac{k_1}{6}&0&0&0&0&\frac{a_{42}}{24}&\frac{a_{32}}{6}&\frac{a_{41}}{24}&\frac{a_{22}}{4}&0&0&\frac{a_{12}}{6}&0&0&0&\frac{k_2}{24}&0&0\\
\frac{a_{33}}{12}&\frac{a_{23}}{6}&\frac{a_{32}}{4}&\frac{a_{13}}{12}&\frac{a_{22}}{2}&0&\frac{a_{12}}{4}&\frac{a_{30}}{12}&0&\frac{k_1}{6}&0&0&0&\frac{a_{24}}{12}&\frac{a_{14}}{6}&\frac{a_{23}}{4}&\frac{a_{04}}{12}&\frac{a_{13}}{2}&\frac{a_{22}}{4}&0&\frac{a_{03}}{4}&\frac{a_{12}}{2}&0&0&\frac{k_2}{4}&0\\
\frac{a_{15}}{120}&0&\frac{a_{14}}{24}&0&0&\frac{a_{13}}{12}&0&\frac{a_{12}}{12}&0&0&0&0&0&\frac{a_{06}}{120}&0&\frac{a_{05}}{24}&0&0&\frac{a_{04}}{12}&0&0&0&\frac{a_{03}}{12}&0&0&\frac{k_2}{24}
\end{array}$$

The non-zero elements of the matrix W are given as follows:

	f	xf	x^2f	y^2f	f^2
y	0	0	0	0	0
xy	0	0	0	0	0
x^2y	0	0	0	0	0
y^3	$\frac{a_{03}}{6}$	0	0	0	0
x^3y	0	0	0	0	0
xy^3	$\frac{a_{13}}{6}$	$\frac{a_{03}}{6}$	0	0	0
x^4y	$\frac{a_{41}}{24}$	0	0	0	0
x^2y^3	$\frac{a_{23}}{12}$	$\frac{a_{13}}{6}$	$\frac{a_{03}}{6}$	0	$\frac{k_1a_{03}}{6}$
y^5	$\frac{a_{05}}{120}$	0	0	$\frac{a_{03}}{6}$	$\frac{k_2a_{03}}{6}$

We are thus looking for the condition that the following matrix is of full rank:

$$\begin{array}{cccccccccccccccccccccccccccccccccccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & \frac{k_2a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & \frac{a_{03}}{6} & \frac{k_2a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{03}}{6}-k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_{30}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6}-a_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & 0 & 0 & 0 & \frac{k_1a_{03}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24}-\frac{a_{40}}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & \frac{k_2a_{03}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12}-\frac{a_{22}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120}-\frac{a_{04}}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a_{30}}{2} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{22}}{2} & a_{12} & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{13}}{6} & 0 & \frac{a_{12}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{04}}{6} & 0 & \frac{a_{03}}{2} & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{41}}{6} & 0 & \frac{a_{40}}{6} & 0 & \frac{a_{30}}{2} & 0 & \frac{k_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{32}}{6} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{2} & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{a_{14}}{6} & \frac{a_{04}}{6} & \frac{a_{13}}{2} & 0 & \frac{a_{03}}{2} & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & \frac{a_{50}}{24} & 0 & \frac{a_{40}}{6} & 0 & \frac{a_{30}}{4} & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & \frac{a_{42}}{24} & \frac{a_{32}}{6} & \frac{a_{41}}{24} & \frac{a_{22}}{4} & 0 & 0 & \frac{a_{12}}{6} & 0 & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{32}}{4} & \frac{a_{13}}{12} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{4} & \frac{a_{30}}{12} & 0 & \frac{k_1}{6} & 0 & 0 & 0 & \frac{a_{24}}{12} & \frac{a_{14}}{6} & \frac{a_{23}}{4} & \frac{a_{04}}{12} & \frac{a_{13}}{2} & \frac{a_{22}}{4} & 0 & \frac{a_{03}}{4} & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & \frac{a_{03}}{6} & 0 & \frac{k_1a_{03}}{6} & 0 & 0 & 0 \\
\frac{a_{15}}{120} & 0 & \frac{a_{14}}{24} & 0 & 0 & \frac{a_{13}}{12} & 0 & \frac{a_{12}}{12} & 0 & 0 & 0 & 0 & 0 & \frac{a_{06}}{120} & 0 & \frac{a_{05}}{24} & 0 & 0 & \frac{a_{04}}{12} & 0 & 0 & 0 & \frac{a_{03}}{12} & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & \frac{a_{03}}{6} & \frac{k_2a_{03}}{6} & 0 & 0 & 0
\end{array}$$

By applying the row-addition transformation to this matrix 18 times, we can remove the first 18 rows and columns 3, 5, 8, 10-17, 19, 21, 22, 24, 26-28, yielding the following matrix:

$$\begin{pmatrix}
 & & & & & & & & & & & & & & & & & & & & & 1 & -k_2 \\
 & & & & & & & & & & & & & & & & & & & & k_2-k_1 & & -a_{12} \\
 & & & & & & & & & & & & & & & & & & & & a_{12}-\frac{a_{30}}{2} & \frac{k_1k_2}{2} & -\frac{a_{22}}{2} \\
\frac{a_{13}}{6} & & & & \frac{a_{03}}{2} & & & & & & * & & & & & * & & & & & * & * & * \\
\frac{a_{41}}{6} & & & & & & & & & & & & & & & & & & & & * & * & * \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & & & \frac{a_{13}}{2} & \frac{a_{03}}{2} & & & * & & * & & & & & * & * & & & & * & * & * \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & & & \frac{a_{41}}{24} & & & & & & * & & & & & * & & & & & * & * & * \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{13}}{4} & & \frac{a_{23}}{4} & \frac{a_{13}}{2} & \frac{a_{03}}{4} & & * & * & * & * & * & & * & * & * & * & & * & * & * & * \\
\frac{a_{15}}{120} & & & \frac{a_{13}}{12} & \frac{a_{05}}{24} & & & \frac{a_{03}}{12} & * & & * & & & * & * & * & & & * & * & * & * & *
\end{pmatrix}$$

Since $a_{03}a_{41} \neq 0$, we then conclude that this matrix is of full rank if and only if the upper right 3×3 matrix is of full rank, that is,

$$(a_{22} - k_1k_2^2)(k_1 - k_2) + a_{12}(2a_{12} - a_{30}) \neq 0.$$

5.5 B_2 singularity

We assume that f is $\mathcal{A}$-equivalent to B_2 singularity, that is,

$$a_{21} \neq 0,\ a_{03} = 0, \text{and } 3a_{05} - 5a_{13}^2/a_{21} \neq 0.$$

Remark that B_2 singularity is 5-determined ($m = 5$). In a similar way to the discussion above, the matrix T, removing zero columns, obtained as follows:

$$
\begin{array}{cccccccccccccccccccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{31}}{2} & a_{21} & \frac{a_{30}}{2} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{22}}{2} & a_{12} & \frac{a_{21}}{2} & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{13}}{6} & 0 & \frac{a_{12}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{04}}{6} & 0 & 0 & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{41}}{6} & \frac{a_{31}}{2} & \frac{a_{40}}{6} & \frac{a_{21}}{2} & \frac{a_{30}}{2} & 0 & 0 & \frac{k_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{32}}{6} & \frac{a_{22}}{2} & \frac{a_{31}}{6} & \frac{a_{12}}{2} & \frac{a_{21}}{2} & 0 & \frac{k_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{2} & \frac{a_{21}}{2} & 0 & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{a_{14}}{6} & \frac{a_{04}}{6} & \frac{a_{13}}{2} & 0 & 0 & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & \frac{a_{50}}{24} & \frac{a_{31}}{4} & \frac{a_{40}}{6} & 0 & \frac{a_{21}}{6} & \frac{a_{30}}{4} & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & \frac{a_{42}}{24} & \frac{a_{32}}{6} & \frac{a_{41}}{24} & \frac{a_{22}}{4} & \frac{a_{31}}{6} & 0 & \frac{a_{12}}{6} & \frac{a_{21}}{4} & 0 & 0 & \frac{k_2}{24} & 0 & 0 \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{32}}{4} & \frac{a_{13}}{12} & \frac{a_{22}}{2} & \frac{a_{31}}{4} & 0 & \frac{a_{12}}{4} & \frac{a_{21}}{2} & \frac{a_{30}}{12} & 0 & \frac{k_1}{6} & 0 & 0 & 0 & \frac{a_{24}}{12} & \frac{a_{14}}{6} & \frac{a_{23}}{4} & \frac{a_{04}}{12} & \frac{a_{13}}{2} & \frac{a_{22}}{4} & 0 & 0 & \frac{a_{12}}{2} & \frac{a_{21}}{12} & 0 & \frac{k_2}{4} & 0 \\
\frac{a_{15}}{120} & 0 & \frac{a_{14}}{24} & 0 & 0 & \frac{a_{13}}{12} & 0 & 0 & 0 & \frac{a_{12}}{12} & 0 & 0 & 0 & 0 & 0 & \frac{a_{06}}{120} & 0 & \frac{a_{05}}{24} & 0 & 0 & \frac{a_{04}}{12} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_2}{24}
\end{array}
$$

The non-zero elements of the matrix W are given as follows:

	f	xf	x^2f	y^2f	f^2
y	0	0	0	0	0
xy	0	0	0	0	0
x^2y	$\frac{a_{21}}{2}$	0	0	0	0
y^3	0	0	0	0	0
x^3y	$\frac{a_{31}}{6}$	$\frac{a_{21}}{2}$	0	0	0
xy^3	$\frac{a_{13}}{6}$	0	0	0	0
x^4y	$\frac{a_{41}}{24}$	$\frac{a_{31}}{6}$	$\frac{a_{21}}{2}$	0	$\frac{k_1a_{21}}{2}$
x^2y^3	$\frac{a_{23}}{12}$	$\frac{a_{13}}{6}$	0	$\frac{a_{21}}{2}$	$\frac{k_2a_{21}}{2}$
y^5	$\frac{a_{05}}{120}$	0	0	0	0

Thus we look for the condition that the following matrix is of full rank.

$$
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccc}
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{24}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & \frac{k_1a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & 0 & \frac{a_{21}}{2} & \frac{k_2a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{120}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{21}}{2}-k_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{31}}{6}-\frac{a_{30}}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6}-a_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & \frac{k_1a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24}-\frac{a_{40}}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & 0 & \frac{a_{21}}{2} & \frac{k_2a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12}-\frac{a_{22}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{1}{12}} & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120}-\frac{a_{04}}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\
\mathbf{a_{21}} & 0 & \mathbf{k_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a_{12}} & \mathbf{k_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{0} \\
\frac{a_{31}}{2} & \mathbf{a_{21}} & \frac{a_{30}}{2} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{22}}{2} & a_{12} & \frac{a_{21}}{2} & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{\frac{a_{13}}{6}} & 0 & \mathbf{\frac{a_{12}}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{\frac{a_{04}}{6}} & 0 & 0 & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{0} \\
\frac{a_{41}}{6} & \frac{a_{31}}{2} & \frac{a_{40}}{6} & \mathbf{\frac{a_{21}}{2}} & \frac{a_{30}}{2} & 0 & 0 & \frac{k_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{32}}{6} & \frac{a_{22}}{2} & \frac{a_{31}}{6} & \frac{a_{12}}{2} & \frac{a_{21}}{2} & 0 & \frac{k_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{2} & \mathbf{\frac{a_{21}}{2}} & 0 & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{a_{14}}{6} & \frac{a_{04}}{6} & \frac{a_{13}}{2} & 0 & 0 & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & \frac{a_{50}}{24} & \frac{a_{31}}{4} & \frac{a_{40}}{6} & 0 & \mathbf{\frac{a_{21}}{6}} & \frac{a_{30}}{4} & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & \frac{a_{42}}{24} & \frac{a_{32}}{6} & \frac{a_{41}}{24} & \frac{a_{22}}{4} & \frac{a_{31}}{6} & 0 & \frac{a_{12}}{6} & \frac{a_{21}}{4} & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{41}}{24} & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & \frac{k_1a_{21}}{2} & 0 & 0 & 0 \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{32}}{4} & \frac{a_{13}}{12} & \frac{a_{22}}{2} & \frac{a_{31}}{4} & 0 & \frac{a_{12}}{4} & \mathbf{\frac{a_{21}}{2}} & \frac{a_{30}}{12} & 0 & \frac{k_1}{6} & 0 & 0 & 0 & \frac{a_{24}}{12} & \frac{a_{14}}{6} & \frac{a_{23}}{4} & \frac{a_{04}}{12} & \frac{a_{13}}{2} & \frac{a_{22}}{4} & 0 & 0 & \frac{a_{12}}{2} & \frac{a_{21}}{12} & 0 & \frac{k_2}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{23}}{12} & \frac{a_{13}}{6} & 0 & \frac{a_{21}}{2} & \frac{k_2a_{21}}{2} & 0 & 0 & 0 \\
\frac{a_{15}}{120} & 0 & \frac{a_{14}}{24} & 0 & 0 & \mathbf{\frac{a_{13}}{12}} & 0 & 0 & 0 & \frac{a_{12}}{12} & 0 & 0 & 0 & 0 & 0 & \frac{a_{06}}{120} & 0 & \mathbf{\frac{a_{05}}{24}} & 0 & 0 & \frac{a_{04}}{12} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{05}}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

By applying the row-addition transformation to this matrix 18 times, we can remove the first 18 rows and columns 3, 5, 8, 10-17, 19, 21, 22, 24, 26-28, yielding the following matrix:

$$
\left(\begin{array}{ccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & & 1 & -k_2 \\
a_{21} & & & & & & & & & & & & & & & & & & & & & & k_2-k_1 & & -a_{12} \\
\frac{a_{31}}{2} & a_{21} & & & & & \frac{a_{21}}{2} & & & & & & \frac{k_2a_{21}}{4} & & & & & \frac{a_{21}}{2} & & & & & * & * & * \\
\frac{a_{13}}{6} & & & & & & & & & & & & & & & & & & & & & & -\frac{a_{12}}{2} & * & -\frac{a_{04}}{6} \\
\frac{a_{41}}{6} & \frac{a_{31}}{2} & \frac{a_{21}}{2} & & & & * & * & & & * & & * & * & & & & * & * & & & & * & * & * \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & & \frac{a_{21}}{2} & & & * & & & & & & * & & & & & * & & & & & * & * & * \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & \frac{a_{31}}{4} & & \frac{a_{21}}{2} & & * & * & * & & * & * & * & * & * & & * & * & * & * & & * & * & * & * \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{13}}{4} & \frac{a_{31}}{4} & & \frac{a_{21}}{2} & * & * & & * & * & & * & * & & * & * & * & * & & * & * & * & * & * \\
\frac{a_{15}}{120} & & & \frac{a_{13}}{12} & & & \frac{a_{05}}{24} & & & & & & -\frac{k_2a_{05}}{240} & & & & & \frac{a_{05}}{120} & & & & & * & * & *
\end{array}\right)
$$

By applying the row-addition transformation again using the fact $a_{21} \neq 0$ and $B_2 \neq 0$, we conclude that this matrix is of full rank if and only if

$$a_{13}(k_1 - k_2) \neq 3a_{21}a_{12} \quad \text{or} \quad (a_{04} - 3k_2^3)(k_1 - k_2) + \tfrac{1}{3}(\tfrac{a_{13}}{a_{21}})^2 \neq 0.$$

5.6 B_3 singularity

We assume that f is $\mathcal{A}$-equivalent to B_3 singularity, that is,

$$a_{21} \neq 0,\ a_{03} = 0,\ a_{05} - \frac{5a_{13}^2}{3a_{21}} = 0 \text{ and } B_3 \neq 0.$$

Remark that B_3 singularity is 7-determined ($m = 7$). With the same arguments as above, the matrix T becomes

$$
\left(\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{720} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{240} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5040} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{60} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{360} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{360} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{31}}{2} & a_{21} & \frac{a_{30}}{2} & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{22}}{2} & a_{12} & \frac{a_{21}}{2} & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{13}}{6} & 0 & \frac{a_{12}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{04}}{6} & 0 & 0 & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{41}}{6} & \frac{a_{31}}{2} & \frac{a_{40}}{6} & \frac{a_{21}}{2} & \frac{a_{30}}{2} & 0 & 0 & \frac{k_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{32}}{6} & \frac{a_{22}}{2} & \frac{a_{31}}{6} & \frac{a_{12}}{2} & \frac{a_{21}}{2} & 0 & \frac{k_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & \frac{a_{22}}{2} & 0 & \frac{a_{12}}{2} & \frac{a_{21}}{2} & 0 & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{14}}{6} & \frac{a_{04}}{6} & \frac{a_{13}}{2} & 0 & 0 & \frac{a_{12}}{2} & 0 & 0 & \frac{k_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{51}}{24} & \frac{a_{41}}{6} & \frac{a_{50}}{24} & \frac{a_{31}}{4} & \frac{a_{40}}{6} & 0 & \frac{a_{21}}{6} & \frac{a_{30}}{4} & 0 & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{42}}{24} & \frac{a_{32}}{6} & \frac{a_{41}}{24} & \frac{a_{22}}{4} & \frac{a_{31}}{6} & 0 & \frac{a_{12}}{6} & \frac{a_{21}}{4} & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{33}}{12} & \frac{a_{23}}{6} & \frac{a_{32}}{4} & \frac{a_{13}}{12} & \frac{a_{22}}{2} & \frac{a_{31}}{4} & 0 & \frac{a_{12}}{4} & \frac{a_{21}}{2} & \frac{a_{30}}{12} & 0 & 0 & 0 & \frac{k_1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{24}}{12} & \frac{a_{14}}{6} & \frac{a_{23}}{4} & \frac{a_{04}}{12} & \frac{a_{13}}{2} & \frac{a_{22}}{4} & 0 & 0 & \frac{a_{12}}{2} & \frac{a_{21}}{12} & 0 & 0 & \frac{k_2}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{15}}{120} & 0 & \frac{a_{14}}{24} & 0 & 0 & \frac{a_{13}}{12} & 0 & 0 & 0 & \frac{a_{12}}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{06}}{120} & 0 & \frac{5a_{13}^2}{72a_{21}} & 0 & 0 & \frac{a_{04}}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{61}}{120} & \frac{a_{51}}{24} & \frac{a_{60}}{120} & \frac{a_{41}}{12} & \frac{a_{50}}{24} & 0 & \frac{a_{31}}{12} & \frac{a_{40}}{12} & 0 & 0 & \frac{a_{21}}{24} & \frac{a_{30}}{12} & 0 & 0 & 0 & 0 & \frac{k_1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{52}}{120} & \frac{a_{42}}{24} & \frac{a_{51}}{120} & \frac{a_{32}}{12} & \frac{a_{41}}{24} & 0 & \frac{a_{22}}{12} & \frac{a_{31}}{12} & 0 & 0 & \frac{a_{12}}{24} & \frac{a_{21}}{12} & 0 & 0 & 0 & \frac{k_2}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{43}}{36} & \frac{a_{33}}{12} & \frac{a_{42}}{12} & \frac{a_{23}}{12} & \frac{a_{32}}{4} & \frac{a_{41}}{12} & \frac{a_{13}}{36} & \frac{a_{22}}{4} & \frac{a_{31}}{4} & \frac{a_{40}}{36} & 0 & \frac{a_{12}}{12} & \frac{a_{21}}{4} & \frac{a_{30}}{12} & 0 & 0 & 0 & 0 & \frac{k_1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{34}}{36} & \frac{a_{24}}{12} & \frac{a_{33}}{12} & \frac{a_{14}}{12} & \frac{a_{23}}{4} & \frac{a_{32}}{12} & \frac{a_{04}}{36} & \frac{a_{13}}{4} & \frac{a_{22}}{4} & \frac{a_{31}}{36} & 0 & 0 & \frac{a_{12}}{4} & \frac{a_{21}}{12} & 0 & 0 & 0 & \frac{k_2}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{25}}{120} & \frac{a_{15}}{120} & \frac{a_{24}}{24} & 0 & \frac{a_{14}}{24} & \frac{a_{23}}{12} & 0 & 0 & \frac{a_{13}}{12} & \frac{a_{22}}{12} & 0 & 0 & 0 & \frac{a_{12}}{12} & \frac{a_{21}}{24} & 0 & 0 & 0 & 0 & 0 & \frac{k_1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{16}}{120} & \frac{a_{06}}{120} & \frac{a_{15}}{24} & 0 & \frac{5a_{13}^2}{72a_{21}} & \frac{a_{14}}{12} & 0 & 0 & \frac{a_{04}}{12} & \frac{a_{13}}{12} & 0 & 0 & 0 & 0 & \frac{a_{12}}{24} & 0 & 0 & 0 & 0 & \frac{k_2}{24} & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{71}}{720} & \frac{a_{61}}{120} & \frac{a_{70}}{720} & \frac{a_{51}}{48} & \frac{a_{60}}{120} & 0 & \frac{a_{41}}{36} & \frac{a_{50}}{48} & 0 & 0 & \frac{a_{31}}{48} & \frac{a_{40}}{36} & 0 & 0 & 0 & \frac{a_{21}}{120} & \frac{a_{30}}{48} & 0 & 0 & 0 & 0 & \frac{k_1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{62}}{720} & \frac{a_{52}}{120} & \frac{a_{61}}{720} & \frac{a_{42}}{48} & \frac{a_{51}}{120} & 0 & \frac{a_{32}}{36} & \frac{a_{41}}{48} & 0 & 0 & \frac{a_{22}}{48} & \frac{a_{31}}{36} & 0 & 0 & 0 & \frac{a_{12}}{120} & \frac{a_{21}}{48} & 0 & 0 & 0 & 0 & \frac{k_2}{720} & 0 & 0 & 0 \\
\frac{a_{53}}{144} & \frac{a_{43}}{36} & \frac{a_{52}}{48} & \frac{a_{33}}{24} & \frac{a_{42}}{12} & \frac{a_{51}}{48} & \frac{a_{23}}{36} & \frac{a_{32}}{8} & \frac{a_{41}}{12} & \frac{a_{50}}{144} & \frac{a_{13}}{144} & \frac{a_{22}}{12} & \frac{a_{31}}{8} & \frac{a_{40}}{36} & 0 & 0 & \frac{a_{12}}{48} & \frac{a_{21}}{12} & \frac{a_{30}}{24} & 0 & 0 & 0 & \frac{k_1}{36} & 0 & 0 & 0 & 0 & 0 & \frac{a_{44}}{144} & \frac{a_{34}}{36} & \frac{a_{43}}{48} & \frac{a_{24}}{24} & \frac{a_{33}}{12} & \frac{a_{42}}{48} & \frac{a_{14}}{36} & \frac{a_{23}}{8} & \frac{a_{32}}{12} & \frac{a_{41}}{144} & \frac{a_{04}}{144} & \frac{a_{13}}{12} & \frac{a_{22}}{8} & \frac{a_{31}}{36} & 0 & 0 & 0 & \frac{a_{12}}{12} & \frac{a_{21}}{24} & 0 & 0 & 0 & \frac{k_2}{48} & 0 & 0 \\
\frac{a_{35}}{240} & \frac{a_{25}}{120} & \frac{a_{34}}{48} & \frac{a_{15}}{240} & \frac{a_{24}}{24} & \frac{a_{33}}{24} & 0 & \frac{a_{14}}{48} & \frac{a_{23}}{12} & \frac{a_{32}}{24} & 0 & 0 & \frac{a_{13}}{24} & \frac{a_{22}}{12} & \frac{a_{31}}{48} & 0 & 0 & 0 & \frac{a_{12}}{24} & \frac{a_{21}}{24} & \frac{a_{30}}{240} & 0 & 0 & \frac{k_1}{120} & 0 & 0 & 0 & 0 & \frac{a_{26}}{240} & \frac{a_{16}}{120} & \frac{a_{25}}{48} & \frac{a_{06}}{240} & \frac{a_{15}}{24} & \frac{a_{24}}{24} & 0 & \frac{5a_{13}^2}{144a_{21}} & \frac{a_{14}}{12} & \frac{a_{23}}{24} & 0 & 0 & \frac{a_{04}}{24} & \frac{a_{13}}{12} & \frac{a_{22}}{48} & 0 & 0 & 0 & 0 & \frac{a_{12}}{24} & \frac{a_{21}}{240} & 0 & 0 & \frac{k_2}{48} & 0 \\
\frac{a_{17}}{5040} & 0 & \frac{a_{16}}{720} & 0 & 0 & \frac{a_{15}}{240} & 0 & 0 & 0 & \frac{a_{14}}{144} & 0 & 0 & 0 & 0 & \frac{a_{13}}{144} & 0 & 0 & 0 & 0 & 0 & \frac{a_{12}}{240} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{08}}{5040} & 0 & \frac{a_{07}}{720} & 0 & 0 & \frac{a_{06}}{240} & 0 & 0 & 0 & \frac{5a_{13}^2}{432a_{21}} & 0 & 0 & 0 & 0 & \frac{a_{04}}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_2}{720}
\end{array}\right)
$$

The matrix W looks like

$$
\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{41}}{24} & \frac{a_{31}}{6} & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k_1a_{21}}{2} & 0 & 0 & 0 & 0 \\
\frac{a_{23}}{12} & \frac{a_{13}}{6} & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{k_2a_{21}}{2} & 0 & 0 & 0 & 0 \\
\frac{a_{13}^2}{72a_{21}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_{51}}{120} & \frac{a_{41}}{24} & \frac{a_{31}}{6} & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & 0 & \frac{a_{21}a_{30}+k_1a_{31}}{6} & \frac{k_1a_{21}}{2} & 0 & 0 & 0 \\
\frac{a_{33}}{36} & \frac{a_{23}}{12} & \frac{a_{13}}{6} & \frac{a_{31}}{6} & 0 & \frac{a_{21}}{2} & 0 & 0 & 0 & \frac{3a_{12}a_{21}+a_{13}k_1+a_{31}k_2}{6} & \frac{k_2a_{21}}{2} & 0 & 0 & 0 \\
\frac{a_{15}}{120} & \frac{a_{13}^2}{72a_{21}} & 0 & \frac{a_{13}}{6} & 0 & 0 & 0 & 0 & 0 & \frac{k_2a_{13}}{6} & 0 & 0 & 0 & 0 \\
\frac{a_{61}}{720} & \frac{a_{51}}{120} & \frac{a_{41}}{24} & 0 & \frac{a_{31}}{6} & 0 & \frac{a_{21}}{2} & 0 & 0 & \frac{4a_{30}a_{31}+3a_{21}a_{40}+3a_{41}k_1}{72} & \frac{a_{21}a_30+a_{31}k_1}{6} & \frac{a_{21}k_1}{2} & 0 & \frac{3a_{21}k_1^2}{8} \\
\frac{a_{43}}{144} & \frac{a_{33}}{36} & \frac{a_{23}}{12} & \frac{a_{41}}{24} & \frac{a_{13}}{6} & \frac{a_{31}}{6} & 0 & \frac{a_{21}}{2} & 0 & \frac{18a_{21}a_{22}+4a_{13}a_{30}+12a_{12}a_{31}+6a_{23}k_1+3a_{41}k_2}{72} & \frac{3a_{12}a_{21}+a_{13}k_1+a_{31}k_2}{6} & \frac{a_{21}k_2}{2} & \frac{a_{21}k_1}{2} & \frac{3a_{21}k_1k_2}{4} \\
\frac{a_{25}}{240} & \frac{a_{15}}{120} & \frac{a_{13}^2}{72a_{21}} & \frac{a_{23}}{12} & 0 & \frac{a_{13}}{6} & 0 & 0 & \frac{a_{21}}{2} & \frac{12a_{12}a_{13}a_{21}+3a_04a_{21}^2+a_{13}^2k_1+6a_{21}a_23k_2}{72a_{21}} & \frac{a_{13}k_2}{6} & 0 & \frac{a_{21}k_2}{2} & \frac{3a_{21}k_2^2}{8} \\
\frac{a_{07}}{5040} & 0 & 0 & \frac{a_{13}^2}{72a_{21}} & 0 & 0 & 0 & 0 & 0 & \frac{a_{13}^2k_2}{72a_{21}} & 0 & 0 & 0 & 0
\end{array}
$$

By applying the row-addition transformation to the matrix M 33 times, we can remove the first 18 rows and columns 3, 5, 8, 10, 12, 14, 17, 19, 21-30, 32, 34, 35, 37, 39, 41, 43, 44, 46, 48, 50-53, and second from last, yielding the following matrix:

$$
\begin{pmatrix}
a_{21} & & & & & & & & & & & & & & & & & & k_2-k_1 & -a_{12} \\
\frac{a_{31}}{2} & a_{21} & & & & & & & & & & & & & & & & & a_{12}-\frac{a_{30}}{2} & -\frac{a_{22}}{2}-\frac{k_2^2(a_{21}-2k_1)}{4} \\
\frac{a_{13}}{6} & & & & & & & & & & & & & & & & & & -\frac{a_{12}}{2} & \frac{k_2^3}{2}-\frac{a_{04}}{6} \\
* & * & \frac{a_{21}}{2} & & & & & & & & & & * & * & * & * & * & * & * & * \\
\frac{a_{23}}{6} & \frac{a_{13}}{6} & & \frac{a_{21}}{2} & & & & & & & & & \frac{a_{13}}{2} & \mathbf{0} & -\frac{k_2a_{13}}{12} & \mathbf{0} & \frac{a_{13}}{6} & \mathbf{0} & \frac{a_{04}}{6}-\frac{a_{22}}{2} & -\frac{a_{14}}{6}-k_2^2(a_{12}-\frac{a_{13}}{12}) \\
* & * & * & & \frac{a_{21}}{6} & & & & & & & & * & * & * & * & * & * & * & * \\
* & * & * & * & & \frac{a_{21}}{2} & & & & & & & \frac{a_{23}}{4} & * & * & * & * & * & * & * \\
\frac{a_{15}}{120} & & & \frac{a_{13}}{12} & & & & & & & & & \frac{5a_{13}^2}{72a_{21}} & \mathbf{0} & \frac{k_2a_{13}^2}{144a_{21}} & \mathbf{0} & \frac{a_{13}^2}{72a_{21}} & \mathbf{0} & -\frac{a_{14}}{24} & \frac{a_{06}}{20}-\frac{k_2^2}{8}(a_{04}-\frac{a_{13}^2}{18}) \\
* & * & * & & * & & \frac{a_{21}}{24} & & & & & & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & & \frac{a_{21}}{4} & & & & & * & * & * & * & * & * & * & * \\
* & * & & \frac{a_{23}}{12} & & \frac{a_{13}}{12} & & & \frac{a_{21}}{24} & & & & \frac{a_{15}}{24} & * & * & * & * & * & * & * \\
* & * & * & & * & & * & & & \frac{a_{21}}{120} & & & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & & * & & \frac{a_{21}}{12} & & * & * & * & * & * & * & * & * \\
* & * & * & * & & * & & * & * & & & \frac{a_{21}}{24} & * & * & * & * & * & * & * & * \\
* & & & \frac{a_{15}}{240} & & & & & \frac{a_{13}}{144} & & & & \frac{a_{07}}{720} & * & * & * & * & * & * & *
\end{pmatrix}
$$

Since $a_{21} \neq 0$ and $B_3 = \frac{a_{07}}{7} - a_{15}\frac{a_{13}}{a_{21}} + \frac{5}{3}a_{23}(\frac{a_{13}}{a_{21}})^2 - \frac{5}{9}a_{31}(\frac{a_{13}}{a_{21}})^3 \neq 0$, we obtain that this matrix is of full rank if and only if

$$
\begin{vmatrix}
a_{12}+\frac{a_{13}(k_2-k_1)}{3a_{21}} & p \\
a_{04}-3k_2^3-\frac{a_{12}a_{13}}{a_{21}} & q
\end{vmatrix} \neq 0.
$$

A Criteria of singularity types of maps

Assume that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $(u, v) \mapsto f(u, v)$, has rank one singularity at 0 and an unit normal vector is extended to $\boldsymbol{\nu}$ on the singular locus. Set $\lambda = \det(f_u\ f_v\ \boldsymbol{\nu})$, $\psi = \det(\boldsymbol{t}\ \eta\boldsymbol{\nu}\ \boldsymbol{\nu})$, where $\boldsymbol{t}$ is a unit tanget vector of the singular locus, and η is a vector field whose restriction is null to the singular locus. We have that $(f, \boldsymbol{\nu}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3 \times \mathbb{R}^3, (0, \boldsymbol{\nu}(0)))$ is an embedding, if and only if $\psi(0) \neq 0$.

Lemma A.1. *The singularity of f is $\mathcal{A}$-equivalent to*

- *cuspidal edge, if $\psi(0) \neq 0$, $\eta\lambda(0) \neq 0$;*

- *swallowtail, if $\psi(0) \neq 0$, $\eta\lambda(0) = 0$, $\eta^2\lambda(0) \neq 0$;*
- *cuspidal cross-cap, if $\psi(0) = 0$, $\eta\lambda(0) \neq 0$, $\psi'(0) \neq 0$.*

Proof. See [8, §1–2] and [6, §1]. □

References

[1] J. W. Bruce, P. J. Giblin, and F. Tari, Families of surfaces: focal sets, ridges and umbilics, Math. Proc. Camb. Phil. Soc., 125 (1999), 243–268.

[2] J. W. Bruce and T. C. Wilkinson, Folding maps and focal sets. Lecture Notes in Math, vol. 1462, pp. 63–72, Springer, Berlin, 1991.

[3] T. Fukui and M. Hasegawa, Singularities of parallel surfaces, Tohoku Mathematical Journal, 64 (2012) 387-408.

[4] T. Fukui and M. Hasegawa, Distance squared functions on singular surfaces parameterized by smooth maps $\mathcal{A}$-equivalnet to S_k, B_k, C_k and F_4, preprint
`http://www.rimath.saitama-u.ac.jp/lab.jp/Fukui/preprint/Distance-squared_functions_on_singular_surfaces.pdf`

[5] T. Fukui and A. Hiramatsu, Versality of rotation unfolding of folding maps for surfaces in $\mathbb{R}^3$, `arxiv.org/abs/2311.15140`.

[6] S. Fujimori, K. Saji, M. Umehara, and K. Yamada, Singularities of maximal surfaces, Math. Z., **259** (2008), 827–848.

[7] S .Izumiya, C. Romero Fuster, M. Ruas and F. Tari, Differential geometry from a singularity theory viewpoint. World Scientific, 2016.

[8] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada: Singularities of flat fronts in hyperbolic space, Pacific. J. Math. 221 (2005), 303-351.

[9] R. Martins and J. J. Nunõ Ballesteros, Finitely determined singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Camb. Phil. Soc., 147 (2009), 701 - 733.

[10] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. London Math. Soc. (3) 50(1985), 333–369.

[11] C. T. C. Wall, Finite determinacy of smooth map-germs, Bull. London Math. Soc., 13 (1981), 481–539.