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Local differential geometry of cuspidal edge and swallowtail

Toshizumi Fukui
Saitama University

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Cuspidal edge

$\gamma(s)$ a regular space curve in $\mathbb{R}^3$
s: arc length parameter
$\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$: Frenet frame

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}$$

Consider the map

$$f : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$$

with Taylor expansion

$$\gamma(s) + \boldsymbol{f}_2(s)\frac{t^2}{2} + \boldsymbol{f}_3(s)\frac{t^3}{6} + \cdots, \qquad \boldsymbol{f}_k \in \langle \boldsymbol{n}, \boldsymbol{b} \rangle_{\mathbb{R}}$$

Arc length parameter

$$f(s,t) = \gamma(s) + \boldsymbol{f}_2(s)\frac{t^2}{2} + \boldsymbol{f}_3(s)\frac{t^3}{6} + \cdots, \qquad \boldsymbol{f}_k \in \langle \boldsymbol{n}, \boldsymbol{b} \rangle_{\mathbb{R}}$$

Assume that $\boldsymbol{f}_2 \neq \boldsymbol{0}$. We can assume that $t^2/2$ is an arc length parameter. i.e.,

$$\langle f_t, f_t \rangle = t^2$$

Proof: Since $f_t = t\boldsymbol{T}$, $\boldsymbol{T} \neq \boldsymbol{0}$,

$$\text{arc length} = \int_0^t t|\boldsymbol{T}|dt = t^2 \times u, \quad u\text{: unit}$$

We are done replacing $t\sqrt{u/2}$ by t.

Frame with respect to singular surface

$$f(s,t) = \gamma(s) + \boldsymbol{f}_2(s)\frac{t^2}{2} + \boldsymbol{f}_3(s)\frac{t^3}{6} + \cdots, \qquad \boldsymbol{f}_k \in \langle \boldsymbol{n}, \boldsymbol{b} \rangle_{\mathbb{R}}$$

Consider the frame defined by

$$\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{t} \\ \boldsymbol{a}_2 &= \boldsymbol{f}_2 \\ \boldsymbol{a}_3 &= \boldsymbol{a}_1 \times \boldsymbol{a}_2 \end{aligned}$$

We define θ by $\boldsymbol{n} = \cos\theta \boldsymbol{a}_2 + \sin\theta \boldsymbol{a}_3$. Then

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}$$

Differential geometric invariants

$$f(s,t) = \gamma(s) + \boldsymbol{f}_2(s)\frac{t^2}{2} + \boldsymbol{f}_3(s)\frac{t^3}{6} + \cdots, \qquad \boldsymbol{f}_k \in \langle \boldsymbol{a}_2, \boldsymbol{a}_3 \rangle_{\mathbb{R}}$$

$$\boldsymbol{f}_k = a_k \boldsymbol{a}_2 + b_k \boldsymbol{a}_3, \quad k = 2, 3, \ldots$$

In particular, $a_2 = 1,\ b_2 = 0$.
The condition $\langle f_t, f_t \rangle = t^2$ decides a_k inductively.

$$a_3 = 0,\ a_4 = -2b_3^2,\ \ldots$$

We thus obtain the differential geometric invariants of cuspidal edge:

$$\kappa,\ \tau,\ \theta,\ b_3,\ b_4,\ \ldots$$

cuspidal edge $\iff b_3(0) \neq 0$
$\iff (f, \boldsymbol{\nu}) : \mathbb{R}^2, 0 \to \mathbb{R}^3 \times S^2$ embedding
cuspidal cross-cap $\iff b_3(0) = 0,\ b_3'(0) \neq 0$

Curvatures at cuspidaledge

$$
\begin{aligned}
K =& \frac{1}{t}\left(\frac{b_3\kappa\sin\theta}{2} + \left[\kappa\left(\frac{b_4\sin\theta}{3} - \frac{b_3^2\cos\theta}{4}\right) - (\tau-\theta')^2\right]t + O(t^2)\right),\\
H =& \frac{1}{t}\left(\frac{b_3}{4} + \left(\frac{b_4}{6} + \frac{\kappa\sin\theta}{2}\right)t + O(t^2)\right)
\end{aligned}
$$

the principal curvatures are given by

$$
\begin{aligned}
\kappa_1 =& \kappa\sin\theta - \frac{b_3^2\kappa\cos\theta + 4(\tau-\theta_s)^2}{2b_3}t + O(t^2),\\
\kappa_2 =& \frac{1}{t}\left(\frac{b_3}{2} + \frac{b_4}{3}t + \frac{2(\tau-\theta_s)^2}{b_3}t^2 + O(t^3)\right).
\end{aligned}
$$

Several geometric invariants for cuspidal edge were already defined.

- normal curvature κ_ν and singular curvature κ_s (K. Saji, M. Umehara and K. Yamada)
- cuspidal curvature κ_c (L. Martins K. Saji, M. Umehara and K. Yamada)
- cusp-directional torsion κ_t and edge inflectional curvature κ_i (K. Saji and L. Martins)

$$\begin{aligned}
\kappa_s &= \kappa \cos\theta, \\
\kappa_\nu &= \kappa \sin\theta, \\
\kappa_c &= b_3, \\
\kappa_t &= \tau - \theta', \\
\kappa_i &= \kappa\tau\cos\theta + \kappa' \sin\theta.
\end{aligned}$$

Swallowtail

$$f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0), \quad (u, v) \mapsto f(u, v), \quad C^\infty$$

$$f(u, v) = \boldsymbol{g}_0(u) + \boldsymbol{g}_1(u)v + \boldsymbol{g}_2(u)\frac{v^2}{2} + \cdots$$

Assume that

- $\Sigma(f) = \{v = 0\}$
- $f(\Sigma(f))$ is of multiplicity 2
- $df|_{\Sigma(f)}$ is rank 1

The curve

$$\Sigma(f) \to f(\Sigma(f)),\ (u, 0) \mapsto f(u, 0)$$

has order 2, and
we may assume that $u^2/2$ is arc-length parameter.

Swallowtail

Since $f(u,v) = \boldsymbol{g}_0(u) + \boldsymbol{g}_1(u)v + \boldsymbol{g}_2(u)\frac{v^2}{2} + \cdots$, we have

$$df|_{\Sigma(f)} = (f_u, f_v)|_{v=0} = (\boldsymbol{g}_0'(u), \boldsymbol{g}_1(u))$$

Because $\boldsymbol{g}_0'(0) = \mathbf{0}$, $\boldsymbol{g}_1(0) \neq \mathbf{0}$, we may assume that

$$|\boldsymbol{g}_1(u)| = 1, \qquad \boldsymbol{g}_0'(u) = u\boldsymbol{g}_1(u).$$

Now we assume that

$$\boldsymbol{g}_1(u) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_{1,1} \\ b_{1,1} \\ 0 \end{pmatrix} u + \begin{pmatrix} a_{1,2} \\ b_{1,2} \\ c_{1,2} \end{pmatrix} \frac{u^2}{2} + \cdots.$$

We have

$$1 = \langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle = 1 + 2a_{1,1}u + (a_{1,2} + 2a_{1,1}^2 + b_{1,1}^2)u^2 + \cdots$$

Claim: $a_{1,k}$ is determined by the lower terms inductively.

Since $\boldsymbol{g}_o' = u\boldsymbol{g}_1$,

$$\boldsymbol{g}_0 = \begin{pmatrix} u^2/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{1,1} \\ 0 \end{pmatrix} \frac{u^3}{3} + \cdots$$

So 2-jet of $f = \boldsymbol{g}_0 + \boldsymbol{g}_1 v + \boldsymbol{g}_2 v^2/2 + \cdots$ is

$$\begin{pmatrix} u^2/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v \\ b_{1,1} uv \\ 0 \end{pmatrix} + \begin{pmatrix} a_{2,0} \\ b_{2,0} \\ c_{2,0} \end{pmatrix} \frac{v^2}{2}$$

Changing (u, v) by $(u + pv, v + v(q_0 u + q_1 v))$, the 2-jet becomes

$$\begin{pmatrix} u^2/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v + (p + q_0)uv \\ b_{1,1}uv \\ 0 \end{pmatrix} + \begin{pmatrix} p^2 + 2q_1 + a_{2,0} \\ b_{1,1}p + b_{2,0} \\ c_{2,0} \end{pmatrix} \frac{v^2}{2}$$

We can send blue parts to zero for suitable choice of q_0 and q_1.
If $b_{1,1} \neq 0$, then we can send red part to zero for suitable choice of p.
If $b_{1,1} \neq 0$, we showed that 2-jet reduces to

$$\begin{pmatrix} v + u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix}$$

Since 1-jets of f_u and f_v are

$$\begin{pmatrix} u \\ b_{1,1}v \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix}$$

we have

$$\begin{aligned}
\langle f_u, f_u \rangle =& u^2 + b_{1,1}^2 v^2 + \cdots \\
\langle f_u, f_v \rangle =& u + b_{1,1} uv + \cdots \\
\langle f_v, f_v \rangle =& 1 + b_{1,1}^2 u^2 + c_{2,0}^2 v^2 + \cdots
\end{aligned}$$

Swallowtail

Thm: If $b_{1,1} \neq 0$, then we can choose (u, v) so that

$$\begin{aligned} \langle f_u, f_u \rangle &= u^2 + v^2 \varphi^2 \\ \langle f_u, f_v \rangle &= u + O(m+1) \\ \langle f_v, f_v \rangle &= 1 \end{aligned}$$

i.e., $ds^2 = \varphi \, du^2 + (u \, du + dv)^2 + O(m+1)$.
Setting

$$f = \sum_{k=0}^{m} \boldsymbol{g}_k \frac{v^k}{k!} + O(v^{m+1}), \quad \boldsymbol{g}_k = \sum_{i=0}^{m} \begin{pmatrix} a_{k,i} \\ b_{k,i} \\ c_{k,i} \end{pmatrix} \frac{u^i}{i!} + O(u^{m+1}),$$

the coefficients $a_{k,i}$ $(k \geq 1)$, $b_{k,i}$ $(k \geq 2)$ are determined by the lower order terms inductively.

We obtain differential geometric invariants:

$$\begin{array}{ccccc} & b_{1,1} & b_{1,2} & b_{1,3} & \dots \\ & & c_{1,2} & c_{1,3} & \cdots \\ c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & \cdots \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$$\varphi = b_{1,1} + b_{1,2}u + \frac{c_{2,0}(c_{2,0} - c_{1,2})}{2b_{1,1}}v + O(2)$$

$$K = -\frac{2\varphi_v + v\varphi_{vv}}{v\varphi} + O(m)$$

Criterion of singularity type and the limiting normal curvature κ_ν, the normalized cuspidal curvature μ_c, and the limiting singular curvature τ_s

$$
\begin{aligned}
f \text{ swallowtail} &\iff c_{2,0} - c_{1,2} \neq 0 \\
&\iff (f, \boldsymbol{\nu}) : \mathbb{R}^2, 0 \to \mathbb{R}^3 \times S^2 \text{: embedding}
\end{aligned}
$$

L. Martins, K. Saji, M. Umehara, and K. Yamda define

- the limiting normal curvature κ_ν,
- the normalized cuspidal curvature μ_c, and
- the limiting singular curvature τ_s

for swallowtail. We have that

$$
\kappa_\nu = -c_{2,0}, \quad \mu_c = \frac{c_{1,2} - c_{2,0}}{b_{1,1}^2}, \quad \tau_s = 2b_{1,1}.
$$

Remark on $\boldsymbol{g}_0$, $\boldsymbol{g}_1$, $\boldsymbol{g}_2$

Recall that $f = \boldsymbol{g}_0(u) + \boldsymbol{g}_1(u)v + \boldsymbol{g}_2(u)\frac{v^2}{2} + \cdots$, $\begin{cases} \boldsymbol{g}_0' = u\boldsymbol{g}_1, \\ |\boldsymbol{g}_1| = 1 \end{cases}$.

If $1 + O(m) = \langle f_v, f_v \rangle = \langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle + \langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle v + \cdots$, then we have

$$\begin{aligned} \langle \boldsymbol{g}_0', \boldsymbol{g}_1 \rangle &= u\langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle = u \\ \langle \boldsymbol{g}_0', \boldsymbol{g}_2 \rangle &= u\langle \boldsymbol{g}_1, \boldsymbol{g}_2 \rangle = O(u^m) \\ \langle \boldsymbol{g}_1', \boldsymbol{g}_1 \rangle &= \frac{1}{2}\langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle' = 0 \\ \langle \boldsymbol{g}_0', \boldsymbol{g}_0' \rangle &= u^2\langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle = u^2 \\ \langle \boldsymbol{g}_0', \boldsymbol{g}_1' \rangle &= u\langle \boldsymbol{g}_1, \boldsymbol{g}_1' \rangle = 0 \end{aligned}$$

and thus

$$\begin{aligned} \langle f_v, f_v \rangle &= \langle \boldsymbol{g}_1, \boldsymbol{g}_1 \rangle + 2\langle \boldsymbol{g}_0, \boldsymbol{g}_1 \rangle v + \cdots \\ \langle f_u, f_v \rangle &= \langle \boldsymbol{g}_0', \boldsymbol{g}_1 \rangle + (\langle \boldsymbol{g}_0', \boldsymbol{g}_2 \rangle + \langle \boldsymbol{g}_1', \boldsymbol{g}_1 \rangle)v + \cdots \\ &= u + O(v^2) \\ \langle f_u, f_u \rangle &= \langle \boldsymbol{g}_0', \boldsymbol{g}_0' \rangle + 2\langle \boldsymbol{g}_0', \boldsymbol{g}_1' \rangle v + (\langle \boldsymbol{g}_0', \boldsymbol{g}_2' \rangle + \langle \boldsymbol{g}_1', \boldsymbol{g}_1' \rangle)v^2 + \cdots \\ &= u^2 + v^2\varphi^2. \end{aligned}$$

Proof

Assume that

$$f = \boxed{\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix} + H_3 + \cdots + H_{k-1}} + H_k + H_{k+1} + \begin{pmatrix} 0 \\ b_{1,1}v^2P_{k-2} \\ 0 \end{pmatrix} + O(k+2)$$

where H_i are homogeneous of degree i and P_{k-2} is a homogeneous of degree $k-2$. Changing (u, v) by $(u + vP_{k-1}, v + vQ_k)$, we have

$$f = \boxed{\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix} + H_3 + \cdots + H_{k-1}} + H_k + H_{k+1} + \begin{pmatrix} 0 \\ b_{1,1}v^2P_{k-2} \\ 0 \end{pmatrix} + \begin{pmatrix} uvP_{k-1} + vQ_k \\ b_{1,1}v^2P_{k-1} \\ c_{2,0}vQ_k \end{pmatrix} + O(k+2)$$

$$
\begin{aligned}
f_u = {} & \boxed{\begin{pmatrix} u \\ b_{1,1} v \\ 0 \end{pmatrix} + (H_3)_u + \cdots + (H_{k-1})_u} + (H_k)_u + (H_{k+1})_u \\
& + \begin{pmatrix} 0 \\ b_{1,1}(v^2 P_{k-2})_u \\ 0 \end{pmatrix} + \begin{pmatrix} (uvP_{k-1} + vQ_k)_u \\ (b_{1,1} v^2 P_{k-1})_u \\ c_{2,0}(vQ_k)_u \end{pmatrix} + O(k+1) \\
f_v = {} & \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{1,1} u \\ c_{2,0} v \end{pmatrix} + (H_3)_v + \cdots + (H_{k-1})_v} + (H_k)_v + (H_{k+1})_v \\
& + \begin{pmatrix} 0 \\ b_{1,1}(v^2 P_{k-2})_v \\ 0 \end{pmatrix} + \begin{pmatrix} (uvP_{k-1} + vQ_k)_v \\ (b_{1,1} v^2 P_{k-1})_v \\ c_{2,0}(vQ_k)_v \end{pmatrix} + O(k+1)
\end{aligned}
$$

Look at the degree k part of $\langle f_v, f_v \rangle$ (degree $< k$ part is not changed)

$$\begin{aligned}
[\langle f_v, f_v \rangle]_k =& 2[\langle \boldsymbol{e}_1, H_{k+1} \rangle + uvP_{k-1} + vQ_k]_v \\
& + 2b_{1,1}^2 u(v^2 P_{k-2})_v + \sum_{i=2}^{k} \langle (H_i)_v, (H_{k+2-i})_v \rangle \\
=& 2 \left[\langle \boldsymbol{e}_1, H_{k+1} \rangle + uvP_{k-1} + vQ_k + b_{1,1}^2 uv^2 P_{k-2} \right]_v \\
& + \sum_{i=2}^{k} \langle (H_i)_v, (H_{k+2-i})_v \rangle
\end{aligned}$$

$$= 2 \left[\langle \boldsymbol{e}_1, H_{k+1} \rangle + uvP_{k-1} + vQ_k + b_{1,1}^2 uv^2 P_{k-2} + R_{k+1} \right]_v$$

whee R_{k+1} is a polynomial so that

$$(R_{k+1})_v = \frac{1}{2} \sum_{i=2}^{k} \langle (H_i)_v, (H_{k+2-i})_v \rangle.$$

$[\langle f_v, f_v \rangle]_k$ is

$$2\left[\langle \boldsymbol{e}_1, H_{k+1}\rangle + uvP_{k-1} + v Q_k + b_{1,1}^2 u v^2 P_{k-2} + R_{k+1}\right]_v$$

We can choose R_{k+1} so that

$$v \text{ divides } \langle \boldsymbol{e}_1, H_{k+1}\rangle + R_{k+1}.$$

Thus we can choose Q_k so that $[\langle f_v, f_v \rangle]_k = 0$.

Look at the degree k part of $\langle f_u, f_v\rangle$ (degree $< k$ part is not changed)

$$[\langle f_u, f_v\rangle]_k = [\langle H_{k+1}, \boldsymbol{e}_1\rangle + uvP_{k-1} + vQ_k]_u$$

$$+ b_{1,1}^2[u(v^2P_{k-2})_u + v(v^2P_{k-2})_v] + \sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle$$

$$=[\langle H_{k+1}, \boldsymbol{e}_1\rangle + uvP_{k-1} + vQ_k]_u$$

$$+b_{1,1}^2 k v^2 P_{k-2} + \sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle$$

$$= b_{1,1}^2 v^2[kP_{k-2} - (uP_{k-2})_u] + \sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle - (R_{k+1})_u$$

Can choose P_{k-2} so that this is zero. $\square$

Gauss curvature and mean curvature

$$
\begin{aligned}
K =& \frac{1}{v}\Big[\frac{c_{2,0}(c_{2,0}-c_{1,2})}{b_{1,1}^2} \\
&+\Big(\frac{3c_{2,0}b_{1,2}(c_{2,0}-c_{1,2})}{b_{11}^3}+\frac{c_{2,0}c_{1,3}+c_{2,1}c_{1,2}-\frac{7}{2}c_{2,0}c_{2,1}}{b_{1,1}^2}\Big)u \\
-\Big(&\frac{c_{2,0}^2(c_{2,0}-c_{1,2})^2}{b_{1,1}^4}-\frac{b_{1,2}c_{2,0}c_{2,1}}{2b_{1,1}^3} \\
&-\frac{c_{2,1}^2-2c_{2,0}c_{2,2}-4c_{1,2}c_{3,0}+6c_{2,0}c_{3,0}}{4b_{1,1}^2}+c_{2,0}^2\Big)v+O(2)\Big], \\
H =& \frac{1}{v}\Big[\frac{c_{2,0}-c_{1,2}}{2b_{1,1}^2}+\Big(\frac{3b_{1,2}(c_{1,2}-c_{2,0})}{2b_{1,1}^3}+\frac{5c_{2,1}-2c_{1,3}}{4b_{1,1}^2}\Big)u \\
&+\Big(\frac{-c_{2,0}(c_{1,2}-c_{2,0})^2}{2b_{1,1}^4}+\frac{b_{1,2}c_{2,1}}{4b_{1,1}^3}+\frac{c_{3,0}-c_{2,2}}{4b_{1,1}^2}-c_{2,0}\Big)v+O(2)\Big].
\end{aligned}
$$

Asymptotic expansion of several invariants nearby swallowtail

The asymptotic expansions of κ, τ, θ, b_3 are given as follows:

$$\begin{aligned}
\kappa =& \frac{1}{|u|}\left[b_{1,1} + b_{1,2}u + \left(b_{1,3} + b_{1,1}^3 + \tfrac{c_{1,2}^2}{b_{1,1}}\right)\frac{u^2}{2} + O(u^3)\right], \\
\tau =& \frac{1}{u}\Big[\frac{c_{1,2}}{b_{1,1}} + \frac{b_{1,1}c_{1,3} - 2b_{1.2}c_{1,2}}{2b_{1,1}^2}u \\
& + \Big(\frac{2c_{1,2}(3b_{1,2}^2 - c_{1,2}^2)}{b_{1,1}^3} - \frac{3(b_{1,3}c_{1,2} + b_{1,2}c_{1,3})}{b_{1,1}^2} + \frac{c_{1,4}}{b_{1,1}} - 2b_{1,1}c_{1,2}\Big)\frac{u^2}{2} \\
& + O(u^3)\Big], \\
\cos\theta =& -1 + \frac{c_{2,0}^2}{b_{1,1}^2}\frac{u^2}{2} - \frac{c_{2,0}(b_{1,2}c_{2,0} - b_{1,1}c_{2,1})}{b_{1,1}^3}u^3 + O(u^4), \\
b_3 =& \frac{-1}{|b_{1,1}u|^{\frac{1}{2}}}\left[\frac{2(c_{1,2} - c_{2,0})}{b_{1,1}} + \left[\frac{5c_{2,1} - 2c_{1,3}}{b_{1,1}} + \frac{b_{1,2}(c_{2,0} - c_{1,2})}{b_{1,1}^2}\right]u + O(u^2)\right].
\end{aligned}$$

ご静聴に感謝します！

Thank you very much for your attention.