A Bifurcation model for nonlinear equations

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Bifurcation problem

to describe bifurcation of solutions to certain nonlinear differential equation.
Example:

$$u'' + \lambda \sin u = 0, \quad u(0) = u(\pi) = 0$$

where u is a function on $[0, \pi]$.

Bifurcation problem

This problem comes back to Euler.
The function $u = 0$ is clearly a solution (trivial solution).
When $\lambda \neq n^2$, $u = 0$ is the only solution nearby trivial solution by inverse function theorem.
When $\lambda = n^2$, Euler's critical load, the solution bifurcate and bifurcation is pitchfork bifurcation.

Set up

Let $L : X \to X$ be a linear self-adjoint operator of a Hilbert space X. We investigate the bifurcation of solutions of the nonlinear equation

$$\Phi(\lambda, u) = Lu - \lambda u + h(\lambda, u) = 0, \quad u \in X, \qquad (1)$$

where $h(\lambda, u) \in C^1(\mathbb{R} \times X, X)$, $h(\lambda, 0) = 0$, $h_u(\lambda, 0) = 0$.

So $\Phi(\lambda, u) = 0$ has trivial solution.

We call $(\lambda^*, 0)$ a bifurcation point, if for any neighborhood U of $(\lambda^*, 0)$, there exists $(\lambda, u) \in U$ so that $\Phi(\lambda, u) = 0$, $u \neq 0$.
It is well-known that if $(\lambda^*, 0)$ is a bifurcation point, then λ^* is an eigenvalue of L, i.e., $V_{\lambda^*} = \mathrm{Ker}(L - \lambda^* I)$ is non zero.
Set $m = \dim_{\mathbb{R}} V_{\lambda^*}$.

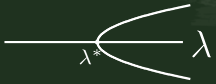
If $m = 1$, and $h(\lambda, u) = a_k(\lambda)u^k + o(u^k)$, $a_k(\lambda^*) \neq 0$, then the bifurcation is described by

$$(\lambda^* - \lambda)u + au^k = 0, \;\; a = a_k(\lambda^*), \qquad (2)$$

and the bifurcation of solutions is decided by k and a, as shown in the following figures.

Transcritical bifurcation (k is even).

Subcritical bifurcation (k is odd, $a < 0$).



Supercritical bifurcation (k is odd, $a > 0$).

Ambrosetti's result

A. Ambrosetti, Branching points for a class of variational operators, Journal d'Analyse Mathématique 76 (1998), 321–335.

Let E be a Hilbert space and consider the equation

$$Lu + H(u) = \lambda u, \quad u \in E \tag{3}$$

where $L : E \to E$ is linear and $H \in C^1(E, E)$ is such that $H(0) = 0$, $H'(0) = 0$. Let $\mu \in \mathbb{R}$ be an eigenvalue of finite multiplicity of L and set $Z = \mathrm{Ker}[\mu I - L]$, where I denotes the identity map in E.

- (A_1) $L \in L(E, E)$ is a symmetric Fredholm operator with index zero.
- (A_2) There exists a functional $h \in C^k(E, \mathbb{R})$, for some $k \geq 3$, such that $H(u) = h'(u)$. Moreover $h(0) = h'(0) = h''(0) = 0$.
- (A_3) there exists an integer $k \geq 3$ and $\tilde{z}$ such that $D^j h(0) = 0$, $j = 1, \dots, k-1$, and $D^k h(0)[\tilde{z}] \neq 0$.

Ambrosetti's result (continued)

For $z \in Z$, set

$$\alpha_k(z) = \frac{1}{k!} D^k h(0)[z]^k.$$

Let T denote the boundary of the unit ball in Z. Let

$$M := \max_T \alpha_k, \quad m := \min_T \alpha_k$$

and let $\xi \in T$, resp. $\eta \in T$, be such that $\alpha_k(\xi) = M$, resp. $\alpha_k(\eta) = m$. We assume

- (A_4) kM and km are not eigenvalues of the matrix $D^2\alpha_k(\xi)$, resp. $D^2\alpha_k(\eta)$.

Theorem Suppose that (A_1), (A_2), (A_3), (A_4) hold and let μ be an isolated eigenvalue of finite multiplicity of L. Then μ is a branching point of (I).

Lyapunov-Schmidt reduction

Let $L : X \to X$ be a self-adjoint operator of a Hilbert space X, and let $\{v_1, \dots, v_m, w_1, w_2, \dots\}$ be an orthonomal basis of X with the following conditions:

- $X = V \oplus W$,
  where
  $V = \mathrm{Ker}(L - \lambda^* I) = \mathrm{span}\{v_1, v_2, \dots, v_m\}$,
  W is the closure of $\mathrm{span}\{w_1, w_2, \dots\}$
  with $Lw_j = \lambda_j w_j$, $\lambda_j \neq \lambda^*$.

Lyapunov-Schmidt reduction (continued)

Recall that $\Phi(\lambda, u) = Lu - \lambda u + h(\lambda, u)$,
V is the λ^*-eigenspace of L
W is complementary subspace to V.
$X = V \oplus W$
Let $P : X \to V$, $Q : X \to W$ denote the projections.
Setting $u = v + w$, $v \in V$, $w \in W$.

$$\Phi(\lambda, u) = 0 \iff \begin{cases} P \circ \Phi(\lambda, v + w) = 0, \\ Q \circ \Phi(\lambda, v + w) = 0 \end{cases}$$

Lyapunov-Schmidt reduction (continued)

Set $v = x_1 v_1 + \cdots + x_m v_m$.
Since

$$D_\xi(Q \circ \Phi(\lambda, v + w))|_{(\lambda^*,0)} = L\xi - \lambda^*\xi$$

$L|_W$ cannot have λ^* as an eigenvalue and $Q \circ \Phi(\lambda, v + w) = 0$ defines w as a function of λ and $x_1, \ldots, x_m$, By implicit function theorem.
We denote this function $W(\lambda, x_1, \ldots, x_m)$.
So

$$P \circ \Phi(\lambda, x_1 v_1 + \cdots + x_m v_m + W(\lambda, x_1, \ldots, x_m)) = 0$$

Bifurcation equation

For $i = 1, \ldots, m$

$$\widehat{F}_i(\lambda, x_1, \ldots, x_m) =$$
$$v_i^*(\Phi(\lambda, x_1 v_1 + \cdots + x_m v_m + W(\lambda, x_1, \ldots, x_m)))$$

$$\Phi(\lambda, u) = 0,\ u = x_1 v_1 + \cdots + x_m v_m + w$$
$$\iff$$
$$P_\circ \Phi(\lambda, x_1 v_1 + \cdots + x_m v_m + W(\lambda, x_1, \ldots, x_m)) = 0$$
$$\iff \quad \widehat{F}_i(\lambda, x_1, \ldots, x_m) = 0, \quad i = 1, \ldots, m.$$

Bifurcation model

Assume that

$$h(\lambda, u) = a_k(\lambda)u^k + o(u^k), \quad a_k(\lambda^*) \neq 0$$

Assume that there exists a linear function $\phi : X \to \mathbb{R}$, such that $v^* x = \phi(vx)$, $v^* \in V^*$, $x \in X$. In many case $\phi(u) = \int_\Omega u$.
Set $F_i = (\lambda^* - \lambda)x_i + H_{x_i}$ $(i = 1, \ldots, m)$ where

$$H = \frac{a_k(\lambda^*)}{k+1}\phi(P(u)^{k+1}), \; P(u) = x_1 v_1 + \cdots + x_m v_m$$

if RHS is not constant on $x_1^2 + \cdots + x_m^2 = 1$.

If $\phi(P(u)^{k+1})$ is constant on $x_1^2 + \cdots + x_m^2 = 1$.

$$H(x) = \begin{cases} \frac{a_2(\lambda^*)^2}{8} \sum_{j=1}^{\infty} \frac{\phi(P(u)^2 w_j)^2}{\lambda_j - \lambda^*} + \frac{a_3(\lambda^*)}{24} \phi(P(u)^4) & (k = 2), \\ \frac{a_{k+1}(\lambda^*)}{(k+2)!} \phi(P(u)^{k+2}) & (k \geq 3). \end{cases}$$

We say the set Z defined by $F_i = 0$ $(i = 1, \ldots, m)$ in $\mathbb{R} \times \mathbb{R}^m$ is the bifurcation model often determined by the initial nonlinear term.

We have

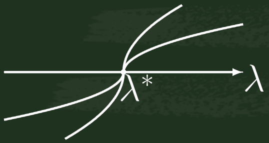
$$F = (F_1, \ldots, F_m) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, \quad (\lambda, x) \mapsto F(\lambda, x)$$

We say that our bifurcation model is non-degenerate if

- the restriction of H to S is a Morse function, and
- 0 is a regular value of the restriction of H to S.

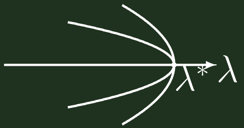
Here S is the sphere defined by $\sum_{i=1}^{m} x_i^2 = k' + 1$ where k' is the degree of H.

k' is even Several transcritical bifurcations take place at the bifurcation point $(\lambda^*, 0)$.
We say such a bifurcation pluritranscritical bifurcation (or multi-transcritical bifurcation).

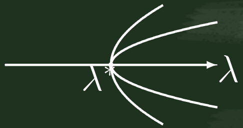


Pluritranscritical bifurcation
(k' is even)

k' is odd The real branches of each non-trivial solution stay in the region $\lambda \leq \lambda^*$ or $\lambda \geq \lambda^*$. We call them **plurisubcritical** (or multi-subcritical) bifurcation, **plurisupercritical** (or multi-supercritical) bifurcation, **mixed critical** bifurcation, respectively.



Plurisubcritical bifurcation (k' is odd)



Plurisupercritical bifurcation (k' is odd)

Mixed critical bifurcation (k' is odd)

Theorem. If the equation (1) is non-degenerate, then the bifurcation equations $\widehat{F}_i = 0$ $(i = 1, \ldots, m)$ are equivalent to the bifurcation model

$$F_i = 0. \quad i = 1, \ldots, m,$$

that is, there is a homeomorphism germ

$$\Xi : (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)) \to (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)),$$

preserving the hyperplane defined by $\lambda = \lambda^*$, with $\Xi(F^{-1}(0)) = \widehat{F}^{-1}(0)$.

Characterization of non-degeneracy

The system (1) is non-degenerate if and only if the following conditions (i) and (ii) hold.

(i) Any irreducible component of $F_i = 0$ $(i = 1, \dots, n)$ is not in the hyperplane defined by $\lambda = \lambda^*$, that is, $\{\lambda = \lambda^*, H_{x_1} = \cdots = H_{x_m} = 0\} = \{0\}$.

(ii) $F_i = 0$ $(i = 1, \dots, m)$ defines curves with an isolated singularity at $(\lambda^*, 0)$, that is, $\mathrm{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x_i x_j}) = m$ if $F_i = 0$ $(i = 1, \dots, n)$ except $(\lambda^*, 0)$.

Dirichlet problem on square $\Omega = [0, \pi]^2$

$$\Delta u + \lambda u + h(u, \lambda) = 0, \quad u|_{\partial\Omega} = 0$$

Eigenvalues of $-\Delta$ are $a^2 + b^2$, $a, b = 1, 2, \dots,$ (with eigenfunction $\sin as \sin bt$) that is,

$$2, 5, 5, 8, 10, 10, 13, 13, 17, 17, 18, 20, 20, 25, 25,$$

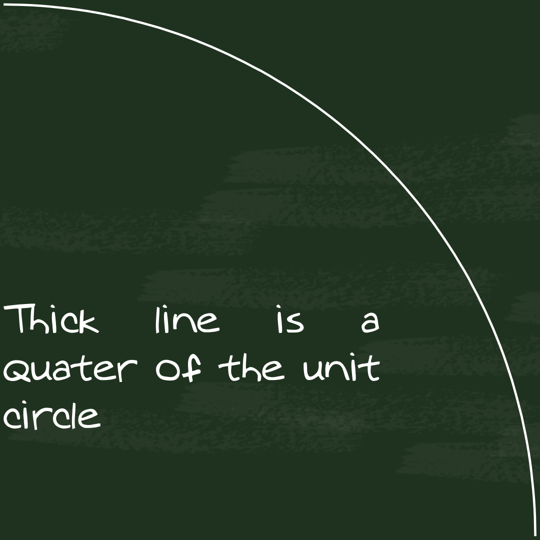
$$26, 26, 29, 29, 32, 34, 34, 37, 37, 40, 40, 41, 41, 45, 45,$$

$$50, 50, 50, 52, 52, 53, 53, 58, 58, 61, 61, 65, 65, 65, 65,$$

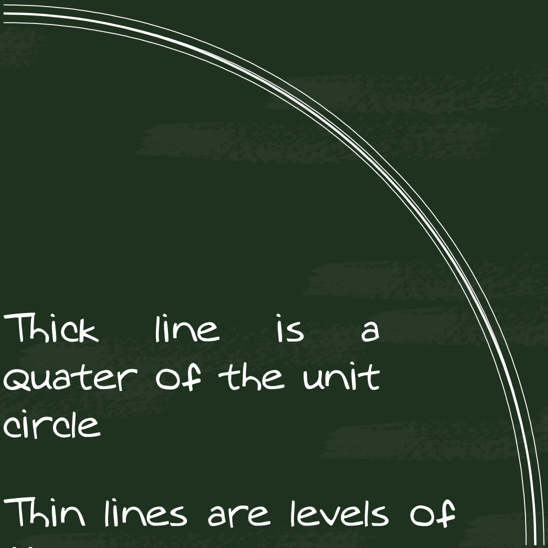
Many eigenvalues are of multiplicity 2, since $a^2 + b^2 = b^2 + a^2$.

$k = 3$ Assume that $k = 3$ and λ^* is an eigenvalue of $-\Delta$ with multiplicity 2. Then the bifurcation model is non-degenerate with

$$H = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4)$$

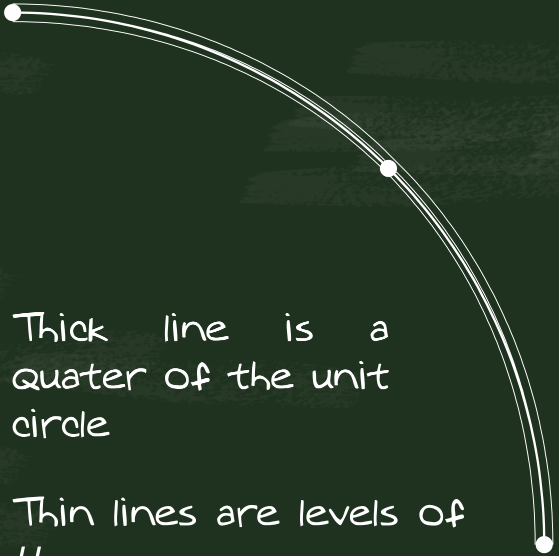


Thick line is a quater of the unit circle



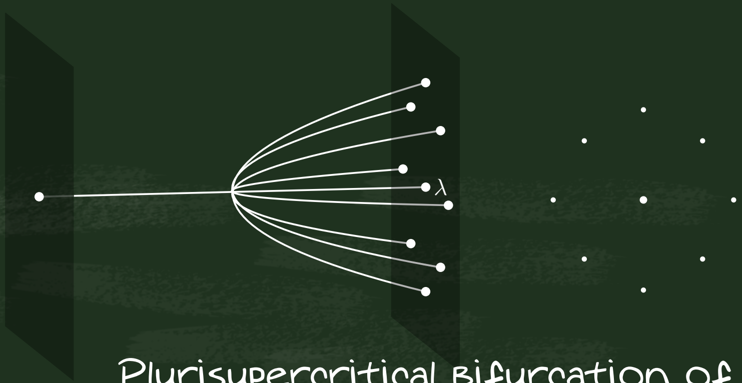
Thick line is a quater of the unit circle

Thin lines are levels of H.



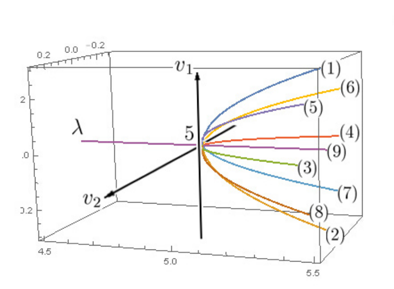
Thick line is a quater of the unit circle

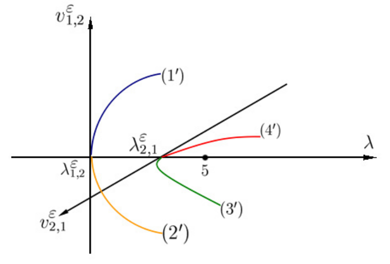
Thin lines are levels of H.



Plurisupercritical Bifurcation of type (1,9)

Collision of Bifurcations





$k = 5$

The Bifurcation is

$(b_-, b_+) = (1, 9)$ if $a_5(\lambda^*) > 0$

$(b_-, b_+) = (9, 1)$ if $a_5(\lambda^*) < 0$

$$\frac{H}{\left(\frac{5\pi}{16}\right)^2 a_5(\lambda^*)} = \begin{cases} (x^2+y^2)\left[\frac{x^4-x^2y^2+y^4}{6} + \frac{3}{2}x^2y^2\right] & \begin{cases} b = 2a \text{ or} \\ a = 2b \end{cases} \\ \left[\frac{x^6+y^6}{6} + \frac{9}{5}x^2y^2(x^2+y^2) + 45x^3y^3\right] & \begin{cases} b = 3a \text{ or} \\ a = 3b \end{cases} \\ (x^2+y^2)\left[\frac{x^4-x^2y^2+y^4}{6} + \frac{9}{5}x^2y^2\right] & \text{(otherwise).} \end{cases}$$

$k=2$ Assume that $k=2$ and $\lambda^*=a^2+b^2$ is an eigenvalue of $-\Delta$ with multiplicity 2. If ab is odd (e.g. $\lambda^*=10=1^2+3^2$), then the bifurcation model is non-degenerate with

$$\frac{H}{16a_2(\lambda^*)}=\frac{1}{27ab}(x^3+y^3)-\frac{ab}{4a^4-17a^2b^2+4b^4}xy(x+y)$$

and $(b_-,b_+)=(4,4)$ transcritical.

If ab is even (e.g. $\lambda^* = 5 = 1^2 + 2^2$), then $\phi(p(x)^3) = 0$ and this is degenerate case.

$$H = \frac{8a_2(\lambda^*)}{3\pi^6}(16a^2b^2)^2 G + \frac{3a_3(\lambda^*)}{4\pi}(3(x^2+y^2)^2+2x^2y^2)$$

where

$$G = \sum_{p\equiv 1(2),\ q\equiv 1(2)} \frac{\left(\begin{array}{l}\frac{1}{pq}\left(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\right)\\ +\frac{2pqxy}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}\end{array}\right)^2}{p^2+q^2-a^2-b^2}$$

if $a+b$ is even; and

$$G = \sum_{p\equiv 1(2), q\equiv 1(2)} \frac{\left(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\right)^2}{(p^2+q^2-a^2-b^2)p^2q^2} + \sum_{p\equiv 0(2), q\equiv 0(2)} \frac{\left(\frac{2pqxy}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}\right)^2}{p^2+q^2-a^2-b^2},$$

if $a+b$ is odd.

Approximations of $(16a^2b^2)^2 G$ are given by the following table:

λ^*	$(16a^2b^2)^2 G$	(b_-, b_+)
$5 = 1^2 + 2^2$	$-0.437133(x^2+y^2)^2 + 0.21458x^2y^2$	$(1,9)$
$13 = 2^2 + 3^2$	$-0.296234(x^2+y^2)^2 + 0.160728x^2y^2$	$(1,9)$
$17 = 1^2 + 4^2$	$-0.112539(x^2+y^2)^2 + 0.638932x^2y^2$	$(5,5)$
$20 = 2^2 + 4^2$	$-0.111457(x^2+y^2)^2 - 0.512649x^2y^2$ $-0.207558xy(x^2+y^2)$	$(1,9)$
$25 = 3^2 + 4^2$	$0.526489(x^2+y^2)^2 - 0.331983x^2y^2$	$(9,1)$
$29 = 2^2 + 5^2$	$-0.12589(x^2+y^2)^2 + 0.614737x^2y^2$	$(5,5)$
$37 = 1^2 + 6^2$	$-0.0548666(x^2+y^2)^2 + 0.215801x^2y^2$	$(1,9)$
$40 = 2^2 + 6^2$	$-0.0595494(x^2+y^2)^2 - 0.158775x^2y^2$ $+0.0276499xy(x^2+y^2)$	$(1,9)$
$41 = 4^2 + 5^2$	$0.0254434(x^2+y^2)^2 - 0.311271x^2y^2$	$(5,5)$
$45 = 3^2 + 6^2$	$-0.00459484(x^2+y^2)^2 - 0.126777x^2y^2$	$(1,9)$
$52 = 4^2 + 6^2$	$-0.22101(x^2+y^2)^2 + 0.106694x^2y^2$ $+0.185669xy(x^2+y^2)$	$(1,5)$

$k=4$ $\lambda^*=a^2+b^2$, $a,b=1,2,\dots$ with $m=2$.
If ab is even, then $\phi(P(u)^5)=0$.
If ab is odd, we have

$$\frac{H}{16^2a_4(\lambda^*)}=\frac{1}{15^2ab}\frac{x^5+y^5}{5}$$
$$+\frac{3^2a^2b^2xy(x^3+y^3)}{(4a^2-b^2)(16a^2-b^2)(a^2-4b^2)(a^2-16b^2)}$$
$$+\frac{4ab(5b^2-2a^2)(5a^2-2b^2)x^2y^2(x+y)}{9(4b^2-a^2)(9a^2-4b^2)(a+2b)(4a^2-b^2)(9b^2-4a^2)(2a+b)}.$$

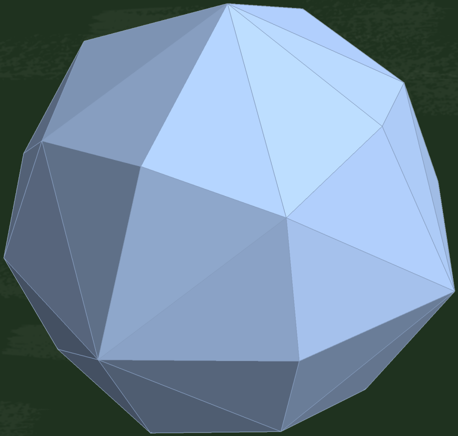
$(b_-,b_+)=(4,4)$ if
$\lambda^*=10,26,34,58,74,82,90,106,122,146$
$(b_-,b_+)=(6,6)$ if $\lambda^*=178$

The first eigenvalue with multiplicity 3 is 50. Note that $50 = 1^2 + 7^2 = 2 \times 5^2$. Here is the data for bifurcation model.

k	$\int_0^\pi \int_0^\pi (x \sin s \sin 7t + y \sin 7s \sin t + z \sin 5s \sin 5t)^{k+1} ds\, dt$	(b_-, b_+)
2	$\frac{16}{675}\begin{pmatrix}(x+y)(\frac{25(x^2+y^2)}{7} - \frac{374}{91}xy) + 7z(\frac{26250}{20449}xy - \frac{x^2+y^2}{19}) \\ +\frac{15625}{1309}(x+y)z^2 + z^3\end{pmatrix}$	$(8, 8)$
3	$\frac{3\pi^2}{256}(3(x^4+y^4+z^4) + 8(x^2y^2 + x^2z^2 + y^2z^2))$	$(1, 27)$
4	$\frac{256}{441}\begin{pmatrix}7(x+y)(\frac{x^4+y^4}{125} - \frac{99964xy(x^2+y^2)}{13996125} + \frac{421500766}{103977212625}x^2y^2) \\ +(\frac{16807(x^4+y^4)}{1081575} + \frac{790130684xy(x^2+y^2)}{1173261375} + \frac{4049858x^2y^2}{946785675})z \\ +2(x+y)(\frac{15625(x^2+y^2)}{63767} - \frac{58360350722xy}{135847622625})z^2 \\ +2(\frac{19531250xy}{97357689} - \frac{42189(x^2+y^2)}{3044275})z^3 + \frac{390625(x+y)}{1247103}z^4 + \frac{49z^5}{3125}\end{pmatrix}$	$(8, 8)$
5	$\frac{5\pi^2}{1536}\begin{pmatrix}5(x^6+y^6+z^6) + 27(x^4(y^2+z^2) + y^4(x^2+z^2) + z^4(x^2+y^2)) \\ +72x^2y^2z^2 - 9xy(x+y)z^3\end{pmatrix}$	$(1, 27)$

Here b_- (resp. b_+) is the number of semi-branches, with $\lambda < \lambda_*$ (resp. $\lambda > \lambda_*$).

Convex hull of 26 nontrivial solutions



Neumann problem on square $[0, \pi]^2$

$$\Delta u + \lambda u + h(u, \lambda) = 0, \quad D_n u|_{\partial\Omega} = 0$$

The eigenvalues of $-\Delta$ are

$$\lambda^* = a^2 + b^2, \quad a, b = 0, 1, 2, \ldots,$$

that is,

$$0, 1, 1, 2, 4, 4, 5, 5, 8, 9, 9, 10, 10, 13, 13, 16, 16, 17, 17,$$

$$18, 20, 20, 25, 25, 25, 25, 26, 26, 29, 29, 32, 34, 34, 36, 36,$$

$$37, 37, 40, 40, 41, 41, 45, 45, 49, 49, 50, 50, 50, \ldots$$

Neuman problem on $[0, \pi]^2$

$(m, k) = (2, 3)$ Similar to Dirichlet with
$(m, k) = (2, 3)$.
$(m, k) = (2, 5)$

(a, b)	$H/a_5(\lambda^*)$	(b_-, b_+)
$(1, 2)$	$\frac{5}{512}(x^2+y^2)(10x^4+53x^2y^2+10y^4)$	$(1, 9)$
$(1, 3)$	$\frac{5}{512}(5x^6+27x^4y^2+9x^3y^3+27x^2y^4+5y^6)$	$(1, 9)$
$(1, 4)$	$\frac{5}{512}(x^2+y^2)(5x^4+22x^2y^2+5y^4)$	$(1, 9)$
$(2, 3)$	$\frac{5}{512}(x^2+y^2)(5x^4+22x^2y^2+5y^4)$	$(1, 9)$
$(2, 4)$	$\frac{5}{512}(x^2+y^2)(10x^4+53x^2y^2+10y^4)$	$(1, 9)$

Rectangle $\Omega = [0, \ell_1 \pi] \times [0, \ell_2 \pi]$ with $(m, k) = (2, 3)$

Dirichlet Problem

$$\lambda^* = \left(\frac{b_1}{\ell_1}\right)^2 + \left(\frac{b_2}{\ell_2}\right)^2, \quad b_i = 1, 2, \ldots$$

$$H = C(3x^4 + 8x^2y^2 + 3y^4)$$

Neumann Problem

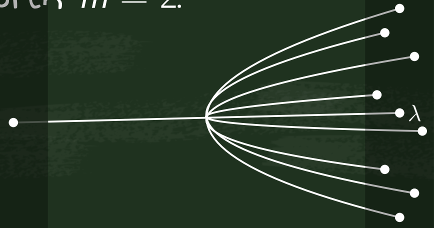
$$\lambda^* = \left(\frac{b_1}{\ell_1}\right)^2 + \left(\frac{b_2}{\ell_2}\right)^2, \quad b_i = 0, 1, 2, \ldots$$

$H = C(3x^4 + 8x^2y^2 + 3y^4)$ if $b_i \neq 0$

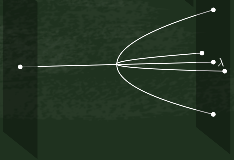
$H = C(x^4 + 4x^2y^2 + y^4)$ if some $b_i = 0$.

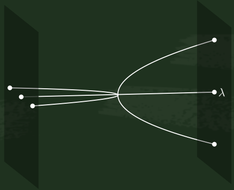
Main conclusion

The following bifurcation is common for nonlinear Dirichlet (Neuman) problems with $m = 2$.



Not hilltop bifurcatioin!





Thank you very much
for your attention.

ご清聴ありがとう
ございました