Multilinear Restriction Theory

Ioan Bejenaru

Setup

For $n \geq 1$, let $U \subset \mathbb{R}^n$ be an open, bounded neighborhood of the origin and let $\Sigma : U \to \mathbb{R}^{n+1}$ be a smooth parametrization of the n-dimensional submanifold $S = \Sigma(U)$ of $\mathbb{R}^{n+1}$ - a hypersurface. Define

$$\mathcal{E}f(x) = \int_U e^{ix\cdot\Sigma(\xi)} f(\xi) d\xi.$$

Assume we have k such operators generated by the hypersurfaces $S_1, .., S_k$. We assume the transversality assumption :

$$vol(N_1(\zeta_1), .., N_k(\zeta_k)) \geq \nu.$$

for all choices $\zeta_i \in \Sigma_i(U_i)$. Here by $vol(N_1(\zeta_1), .., N_k(\zeta_k))$ we mean the volume of the k-dimensional parallelepiped spanned by the vectors $N_1(\zeta_1), .., N_k(\zeta_k)$.

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Multilinear restriction. Assuming that $S_1, .., S_k$ are transversal hypersurfaces in $\mathbb{R}^{n+1}$, the following is conjectured to be true

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^{\frac{2}{k-1}}(\mathbb{R}^{n+1})} \leq C\Pi_{i=1}^k \|f_i\|_{L^2(U_i)}.$$

No curvature assumptions are needed ; in fact, curvature complicates things.

Multilinear Kakeya. Given k families of tubes $\mathcal{T}_1, .., \mathcal{T}_k$ such that each tube $T \in \mathcal{T}_i$ has the property that its core makes an angle $\ll 1$ with the vector e_i. We allow tubes in the same family to be parallel. The multilinear Kakeya conjecture is the following :

$$\|\Pi_{i=1}^k (\sum_{T_i \in \mathcal{T}_i} \chi_{T_i})\|_{L^{\frac{1}{k-1}}(\mathbb{R}^{n+1})} \leq C\Pi_{i=1}^k \# \mathcal{T}_i.$$

where $\# T_i$ is the cardinality of the family of tubes T_i.

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For $k = n+1$ the two estimates are "morally" equivalent, with the nonlinear Kakeya being the weaker one : the nonlinear restriction implies Kakeya, but Kakeya implies the nonlinear restriction with losses of R^ϵ !

For $k < n+1$, the Rademacher type argument does not allow for the nonlinear Kakeya to be obtained from the nonlinear restriction ; however one can obtain the nonlinear restriction from the nonlinear Kakeya.

The original argument of Bennett, Carbery, Tao proves the nonlinear Kakeya ; the later argument by Guth does the same.

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The effect of small support.

Assume that $\Sigma_1(suppf_1) \subset B(\mathcal{H}, \mu)$, where $B(\mathcal{H}, \mu)$ is the neighborhood of size μ of the k-dimensional affine subspace $\mathcal{H}$. Assume that $|N_1(\zeta_1) - \pi_{\mathcal{H}} N_1(\zeta_1)| \lesssim \mu, \forall \zeta_1 \in \Sigma_1(suppf_1)$, where $\pi_{\mathcal{H}} : \mathbb{R}^{n+1} \to \mathcal{H}$ is the projection onto $\mathcal{H}$. In addition assume that if $N_i, i = k+1, .., n+1$ is a basis of the normal space $\mathcal{H}^\perp$ to $\mathcal{H}$, then $N_1(\zeta_1), .., N_k(\zeta_k), N_{k+1}, .., N_{n+1}$ are transversal for any choice $\zeta_i \in \Sigma_i$.

Theorem

In addition to transversality conditions, assume that f_1 satisfies the above. Then for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^{\frac{2}{k-1}}(B(0,R))} \leq C(\epsilon) \mu^{\frac{n+1-k}{2}} R^\epsilon \Pi_{i=1}^k \|f_i\|_{L^2(U_i)}.$$

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A few ideas about proof - inspired by the argument of Guth.

Given some $0 < \delta \ll 1$ we split each domain U_i, thus S_i into smaller pieces of diameter $\leq \delta$. It suffices to prove the multilinear estimate for such S_i. Moreover, we can assume that the normals $N_i(\zeta_i^0) = e_i$.
We define $A(R)$ to be the best constant for which the estimate

$$\|\Pi_{i=1}^{n+1}\mathcal{E}_i f_i\|_{L^{\frac{2}{n}}(Q)} \leq A(R)\Pi_{i=1}^{n+1}\|f_i\|_{L^2}$$

holds true for all cubes Q of size R.
Then we use an induction on scale argument to prove

$$A(\delta^{-1}R) \leq CA(R)$$

where C is independent on δ and R.

Induction combined with localization reduces the problem to a discreet Loomis-Whitney inequality.
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The point is to bring the problem to a scale small enough, say δ^{-1}, where the estimate is trivial.

The same problem mentioned in the second lecture occurs here : localization on the physical space collides with the frequency localization ; we need to use the margin concept.

The above proof sees no difference between $k = n + 1$ and $k < n + 1$.

As for the result with localization, this require a careful definition of the localization operators so as to not touch at all the frequency localization in the direction where we have smallness.

The point is to bring the problem to a scale small enough, say δ^{-1}, where the estimate is trivial by using L^∞ bounds in for $\mathcal{E}_1 f_1$ in some directions :

$$\|\mathcal{E}_1 f_1\|_{L^2 L^\infty} \lesssim \mu^{\frac{n+1-k}{2}} \|f_1\|_{L^2}.$$

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If $k < n+1$, then the exponent $\frac{2}{k-1}$ is sharp in the generic case ; if one assumes curvature hypothesis, then it should improve.
Anticipated condition

$$vol(N_1(\zeta_1), .., N_k(\zeta_k), S_{N_j(\zeta_j)}v) \geq \nu > 0 \tag{1}$$

for any choices $\zeta_i \in S_i, i = 1, .., k$, for any $j \in \{1, .., k\}$, for any choice $\zeta \in S_j$ and for any choice of unit vector v in the tangent space of specific $n-k+1$-dimensional submanifolds $S \subset S_j$.
Under such hypothesis the conjecture is that

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^p(\mathbb{R}^{n+1})} \leq C \Pi_{i=1}^k \|f_i\|_{L^2(U_i)}.$$

for any $p(k) \leq p \leq \infty$ were $p(k) = \frac{2(n+1+k)}{k(n+k-1)} < \frac{2}{k-1}$.
This conjecture is mostly open.

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Theorem

(B.) If $S_i, i = 1, .., k$ are $k-1$-conical hypersurfaces and satisfy (1) then

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^p} \lesssim \Pi_{i=1}^k \|f_i\|_{L^2}.$$

holds true for all $p > p(k)$.

Guth formulates a weaker version of the above conjecture and proves it for subsets of the paraboloid using the polynomial partition method. He then uses it to further improve the linear restriction theory in higher dimensions.

Our proof is entirely analytical in nature. The $k-1$-conical surfaces have a simplified wave packet structure that reduces the geometry of the problem. They have curvature in the necessary directions and they are flat in the unnecessary directions.

To keep things simple, we sketch the argument for $k = 3$. At the end we will highlight the obstacles that occur for $k \geq 4$.

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Geometry

Given a surface S_i we let $\mathcal{N}_i = \{N_i(\zeta_i) : \zeta_i \in S_i\}$ be the set of normals at S_i. By *dspan*$\mathcal{N}_i$ we denote the following subset of the classical span of $\mathcal{N}_i$:

$$dspan\mathcal{N}_i = \{\alpha N_\alpha + \beta N_\beta : N_\alpha, N_\beta \in \mathcal{N}_i, \alpha, \beta \in \mathbb{R}\}.$$

dspan$\mathcal{N}_i$ is the set of linear combinations of two vectors in $\mathcal{N}_i$; it is not a linear subspace.

With these notation in place, we claim the following result.

Lemma

For any $N \in dspan\mathcal{N}_1$ and any $N_2 \in \mathcal{N}_2, N_3 \in \mathcal{N}_3$ the following holds true for all real numbers a, b, c :

$$|aN + bN_2 + cN_3| \gtrsim \max(|a||N|, |b|, |c|). \quad (2)$$

The statement is symmetric with respect to S_1, S_2, S_3.

The occurrence of $|N|$ in $\max(|a||N|, |b|, |c|)$ is motivated by the fact that vectors in *dspan*$\mathcal{N}_1$ are not normalized, but vectors in $\mathcal{N}_2$ and $\mathcal{N}_3$ are

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For any $N \in dspan\mathcal{N}_1$ and any $N_2 \in \mathcal{N}_2, N_3 \in \mathcal{N}_3$ the following holds true for all real numbers a, b, c :

$$|aN + bN_2 + cN_3| \gtrsim \max(|a||N|, |b|, |c|). \tag{2}$$

The statement is symmetric with respect to S_1, S_2, S_3.

The occurrence of $|N|$ in $\max(|a||N|, |b|, |c|)$ is motivated by the fact that vectors in *dspan*$\mathcal{N}_1$ are not normalized, but vectors in $\mathcal{N}_2$ and $\mathcal{N}_3$ are.

Wave packet theory

The construction is sensitive to the 2-conical character of the hypersurface. The construction of wave packets starts with defining two lattices : $\mathcal{L} = r^{-1}\mathbb{Z}^n \cap D$, for frequencies, and $L = r\mathbb{Z}^n$, on the spatial side. Here we modify the frequency lattice $\mathcal{L}$ to account for the 2-conical structure.

With $x_T \in L, \xi_T \in \mathcal{L}$, define the tube $T := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_T + \nabla\varphi(\xi_T)t| \leq r\}$; denote by $\mathcal{T}$ the set of such tubes. For $T \in \mathcal{T}$, define the cut-off $\tilde{\chi}_T$ on $\mathbb{R}^{n+1}$ by

$$\tilde{\chi}_T(x,t) = \tilde{\chi}_{D(x_T - \nabla\varphi(\xi_T)t, t; r)}(x).$$

Let Q be a cube of radius $R \gg 1$ and ϕ be a free wave. For each $T \in \mathcal{T}$ there is a free wave ϕ_T, with $\hat{\phi}_T$ supported in a parallelepiped of size roughly $1 \times 1 \times R^{-\frac{1}{2}} \times ... \times R^{-\frac{1}{2}}$. The map $\phi \to \phi_T$ is linear and

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and the following estimates hold true

$$\sum_{T} \sup_{q \in Q_J(Q)} \tilde{\chi}_T(x_q, t_q)^{-N} \|\phi_T\|^2_{L^2(q)} \lesssim rM(\phi)$$

and

$$\left(\sum_{q_0} M(\sum_{T} m_{q_0,T}\phi_T) \right)^{\frac{1}{2}} \lesssim M(\phi),$$

provided that the coefficients $m_{q_0,T} \geq 0$ satisfy

$$\sum_{q_0} m_{q_0,T} = 1, \qquad \forall T \in \mathcal{T}. \tag{3}$$

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Localization of the trilinear estimate.

Assume that $\Sigma_1(suppf_1) \subset B(\mathcal{H}, \mu)$, where $B(\mathcal{H}, \mu)$ is the neighborhood of size μ of the k-dimensional affine subspace $\mathcal{H}$. Assume that $|N_1(\zeta_1) - \pi_{\mathcal{H}} N_1(\zeta_1)| \lesssim \mu, \forall \zeta_1 \in \Sigma_1(suppf_1)$, where $\pi_{\mathcal{H}} : \mathbb{R}^{n+1} \to \mathcal{H}$ is the projection onto $\mathcal{H}$. In addition assume that if $N_i, i = k+1, .., n+1$ is a basis of the normal space $\mathcal{H}^\perp$ to $\mathcal{H}$, then $N_1(\zeta_1), .., N_k(\zeta_k), N_{k+1}, .., N_{n+1}$ are transversal for any choice $\zeta_i \in \Sigma_i$.

Under these hypotheses the improved trilinear estimate states that :

$$\|\mathcal{E}_1 f_1 \mathcal{E}_2 f_2 \mathcal{E}_3 f_3\|_{L^1(B(0,r))} \leq C(\epsilon) \mu^{\frac{n-2}{2}} r^\epsilon \|f_1\|_{L^2(U_1)} \|f_2\|_{L^2(U_2)} \|f_3\|_{L^2(U_3)}. \quad (4)$$

We need the following version of this : assume q is a cube of size r and that $\mu \gtrsim r^{-1}$ then

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Proof of the main result

Main strategy - induction on scales. Define $A(R)$ to be the best constant in the estimate

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2 \cdot \mathcal{E}_3 f_3\|_{L^p(B(x,R))} \leq A(R)\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2}.$$

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$$A(CR) \leq (1 + c(R))A(R).$$

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$$A(CR) \leq (1 + c(R))A(R).$$

Proposition

Let Q be a cube of size $R \gg 1$. Assume $\phi_i = \mathcal{E}_i f_i, i \in \{1,2,3\}$ have positive margin. Then there is a free wave table $\Phi_1 = \Phi(\phi_1, \phi_2, Q)$ on Q with depth C_0 such that the following properties hold true :

$$\phi_1 = \sum_{q \in \mathcal{Q}_{C_0}(Q)} \Phi_1^{(q)}, \tag{6}$$

$$M(\Phi) \lesssim M(\phi), \tag{7}$$

and for any $q', q'' \in \mathcal{Q}_{C_0}(Q), q' \neq q''$

$$\|\Phi_1^{(q')} \phi_2 \phi_3\|_{L^1(q'')} \lesssim_\epsilon R^{-\frac{n-2}{4}+\epsilon} \Pi_{i=1}^3 M^{\frac{1}{2}}(\phi_i). \tag{8}$$

This "morally" suffices ; the following estimate is trivial

$$\begin{aligned}\|\Phi_1^{(q')}\phi_2\phi_3\|_{L^{\frac{2}{3}}(Q)} &\lesssim R^{\frac{3}{2}}\|\Phi_1^{(q')}\phi_2\phi_3\|_{L_t^\infty L_x^{\frac{2}{3}}(Q)} \\ &\lesssim R^{\frac{3}{2}}\|\Phi_1^{(q')}\|_{L_t^\infty L_x^2(Q)}\|\phi_2\|_{L_t^\infty L_x^2(Q)}\|\phi_3\|_{L_t^\infty L_x^2(Q)} \\ &\lesssim R^{\frac{3}{2}}M(\Phi_1^{(q')})^{\frac{1}{2}}M(\phi_1)^{\frac{1}{2}}M(\phi_3)^{\frac{1}{2}}.\end{aligned}$$

Interpolating between the above $L^{\frac{2}{3}}$ estimate and the improved L^1 estimate so as to cancel the power of R reveals

$$\|\Phi_1^{(q')}\phi_2\phi_3\|_{L^p(Q)} \lesssim M(\Phi_1^{(q')})^{\frac{1}{2}}M(\phi_2)^{\frac{1}{2}}M(\phi_3)^{\frac{1}{2}}.$$

with $p > p(3)$.

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Wave packet decomposition for ϕ_1,

$$\phi_1 = \sum_{T_1 \in \mathcal{T}_1} \phi_{1,T_1}.$$

For any $q_0 \in \mathcal{Q}_{C_0}(Q)$ and $T_1 \in \mathcal{T}_1$ we define

$$m_{q_0,T_1} := \|\tilde{\chi}_{T_1}\phi_2\|^2_{L^2(q_0)}$$

and

$$m_{T_1} := \sum_{q_0 \in \mathcal{Q}_{C_0}(Q)} m_{q_0,T_1}.$$

Based on this we define

$$\Phi_1^{(q_0)} := \sum_{T_1 \in \mathcal{T}_1} \frac{m_{q_0,T_1}}{m_{T_1}} \phi_{1,T_1}.$$

It is obvious that :

$$\phi_1 = \sum_{q_0 \in \mathcal{Q}_{C_0}(Q)} \Phi_1^{(q_0)}$$

thus justifying (6).

All that is left to prove is (8), which is equivalent to

$$\sum_{q\in\mathcal{Q}_j(Q):d(q,q_0)\gtrsim cR} \|\Phi_1^{(q_0)}\phi_2\phi_3\|_{L^1(q)} \lesssim_\epsilon c^{-C} r^{-\frac{n-2}{2}+\epsilon}\Pi_{i=1}^3 M(\phi_i).$$

Suffices to prove

$$\sum_{q\in\mathcal{Q}_j(Q):d(q,q_0)\gtrsim cR} \| \sum_{T_1\cap q\neq\emptyset} \frac{m_{q_0,T_1}}{m_{T_1}}\phi_{1,T_1}\phi_2\phi_3\|_{L^1(q)}$$

The localized form of the trilinear estimate (5) gives

$$\|\Phi_1^{(q_0)}\phi_2\phi_3\|_{L^1(q)} \lesssim_\epsilon r^{-\frac{n+1}{2}+\epsilon}\sum_{T_1}\frac{m_{q_0,T_1}}{m_{T_1}}\|\phi_{1,T_1}\tilde{\chi}_q\|_{L^2}\|\phi_2\tilde{\chi}_q\|_{L^2}\|\phi_3\tilde{\chi}_q\|_{L^2}.$$

Using the obvious inequality $\frac{m_{q_0,T_1}}{m_{T_1}} \leq \frac{m_{q_0,T_1}^{\frac{1}{2}}}{m_{T_1}^{\frac{1}{2}}}$, we obtain :

$$\sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0,T_1}}{m_{T_1}} \|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}$$

$$\lesssim \left(\sum_{T_1} \frac{\|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \right)^{\frac{1}{2}} \left(\sum_{T_1} m_{q_0,T} \tilde{\chi}_{T_1}(x_q, t_q) \right)^{\frac{1}{2}} .$$

Next we claim the following estimate

$$\sum_{T_1 \in \mathcal{T}_1} m_{q_0,T_1} \tilde{\chi}_{T_1}(x_q, t_q) \lesssim \|\tilde{\chi}_{S(q)} \phi_2\|_{L^2}^2 .$$

Using the definition of m_{q_0,T_1} we identify the function

$$\tilde{\chi}_{S(q)} = (\sum_{T_1 \in \mathcal{T}_1} \tilde{\chi}(x_q, t_q) \tilde{\chi}_{T_1}) \chi_{q_0}$$

which makes the above hold true.

Then we note that $\tilde{\chi}_{S(q)}$ has the following decay property :

$$\tilde{\chi}_{S(q)}(x,t) \lesssim \left(1 + \frac{d((x,t),S(q))}{r}\right)^{-N}.$$

Here the surface $S(q)$ is the translate by $c(q)$ of the neighborhood of size r of cone of normals at S_1, $\mathcal{CN}_1 := \{\alpha N_1(\zeta), \zeta \in S_1, \alpha \in \mathbb{R}\}$.
It suffices to prove the following :

$$\sum_q \|\tilde{\chi}_{S(q)}\phi_2\|_{L^2} \left(\sum_{T_1 \cap q \neq \emptyset} \frac{\|\phi_{1,T_1}\tilde{\chi}_q\|_{L^2}^2}{m_{T_1}\tilde{\chi}_{T_1}(x_q,t_q)} \right)^{\frac{1}{2}} \|\phi_2\tilde{\chi}_q\|_{L^2}\|\phi_3\tilde{\chi}_q\|_{L^2}$$
$$\lesssim r^{\frac{3}{2}}\Pi_{i=1}^3 M(\phi_i).$$

This can be broken down into the following two claims :

$$\sum_q \|\phi_2\tilde{\chi}_{S(q)}\|_{L^2}^2 \|\phi_3\tilde{\chi}_q\|_{L^2}^2 \lesssim r^2 M(\phi_2)M(\phi_3) \qquad (9)$$

and

$$\sum_q \left(\sum_{T_1 \cap q \neq \emptyset} \frac{\|\phi_{1,T_1}\tilde{\chi}_q\|_{L^2}^2}{m_{T_1}\tilde{\chi}_{T_1}(x_q,t_q)} \right) \|\phi_2\tilde{\chi}_q\|_{L^2}^2 \lesssim rM(\phi_1). \qquad (10)$$

Proof of (10) : By rearranging the sum, it suffices to show

$$\sum_{T_1} \sum_{q \cap T_1 \neq \emptyset} \frac{\|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2 \|\phi_2 \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim rM(\phi_1).$$

The inner sum is estimated as follows :

$$\sum_{q \cap T_1 \neq \emptyset} \frac{\|\phi_2 \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim \frac{\|\phi_2 \tilde{\chi}_{T_1}\|_{L^2}^2}{m_{T_1}} \lesssim 1,$$

and the outer one is estimated by

$$\sum_{T_1} \sup_q \|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2 \lesssim r \sum_{T_1} M(\phi_{1,T_1}) \lesssim rM(\phi_1),$$

which is obvious given the size of q in the x_1-direction is $\approx r$ and the mass of ϕ_{1,T_1} is constant across slices in space with $x_1 = \mathit{constant}$.

We continue with the proof of (9). Using the fast decay of $\tilde{\chi}_{S(q)}$ away from $S(q)$ and of $\tilde{\chi}_q$ away from q, it suffices to show that

$$\sum_q \|\chi_{S(q)}\phi_2\|_{L^2}^2 \|\chi_q \phi_3\|_{L^2}^2 \lesssim r^2 M(\phi_2) M(\phi_3), \tag{11}$$

where by χ_A is the characteristic function of the set A.

We define the following relation : $q' \sim q$ if $q' \cap S(q) \neq \emptyset$ and note that this is equivalent to saying that there is a tube $T_1 \in \mathcal{T}_1$ intersecting both q and q' and that $d(q, q') \gtrsim cR$. We start from the obvious inequality

$$\|\chi_{S(q)}\phi_2\|_{L^2}^2 \lesssim \sum_{q' \sim q} \|\chi_{q'}\phi_2\|_{L^2}^2$$

which implies

$$\sum_q \|\chi_{S(q)}\phi_2\|_{L^2}^2 \|\chi_q \phi_3\|_{L^2}^2 \lesssim \sum_q \sum_{q' \sim q} \|\chi_{q'}\phi_2\|_{L^2}^2 \|\chi_q \phi_3\|_{L^2}^2$$

We are tacitly using again at this point the full dispersion property of the set of normals $\mathcal{N}_1$: the tubes T_1 passing through q separate inside q_0 ; in the absence of this property, the above inequality would fail, as we would encounter large number of tubes $T_1 \in \mathcal{T}_1$ passing through both q and q'.

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Next we bring the wave packet decomposition for ϕ_2 and ϕ_3. This reduces (11) to the following

$$\sum_q \sum_{q' \sim q} \left(\sum_{T_2 \cap q' \neq \emptyset} M(\phi_{T_2}) \right) \left(\sum_{T_3 \cap q \neq \emptyset} M(\phi_{T_3}) \right) \lesssim M(\phi_2) M(\phi_3). \quad (12)$$

The key point in justifying (12) is that, as we vary (q, q') with $q' \sim q$, the number of occurrences of a pair of tubes (T_2, T_3) on the left hand-side is bounded by a universal constant. Indeed, if that is the case we bound the left hand side by

$$\lesssim \sum_{T_2 \in \mathcal{T}_2} \sum_{T_3 \in \mathcal{T}_3} M(\phi_{T_2}) M(\phi_{T_3}) \lesssim M(\phi_2) M(\phi_3).$$

We finish the argument by establishing an upper bound on the number of occurrences of a pair of tubes (T_2, T_3) on the left-hand side of (12).

Assume that a pair (T_2, T_3) shows up multiple times. That means that there are $(q, q'), (\tilde{q}, \tilde{q}')$ such that $q \sim q', \tilde{q} \sim \tilde{q}'$ and $q' \cap T_2 \neq \emptyset, \tilde{q}' \cap T_2 \neq \emptyset$, $q \cap T_3 \neq \emptyset, \tilde{q} \cap T_3 \neq \emptyset$. We tolerate repeated occurrences coming from the setup $d(q, \tilde{q}) \lesssim r$ and $d(q', \tilde{q}') \lesssim r$, which are bounded by a universal constant, but rule out all the others.

Consider the case $d(q, \tilde{q}), d(q', \tilde{q}') \gg r$. This implies the following :
$c(q) - c(\tilde{q}) = \alpha_3 N_3 + O(r)$ for some $N_3 = N_3(\zeta_3), \zeta_3 \in S_3$, $|\alpha_3| \gg r$,
$c(q') - c(\tilde{q}') = \alpha_2 N_2 + O(r)$ for some $N_2 = N_2(\zeta_2), \zeta_2 \in S_2$, $|\alpha_2| \gg r$,
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$|\alpha_1|, |\tilde{\alpha}_1| \gg r$.

Since $c(q) - c(q') - (c(\tilde{q}) - c(\tilde{q}')) = c(q) - c(\tilde{q}) - (c(q') - c(\tilde{q}'))$, this implies

$$\alpha_1 N_1 - \tilde{\alpha}_1 \tilde{N}_1 = \alpha_3 N_3 - \alpha_2 N_2 + O(r)$$

Since $\alpha_1 N_1 - \tilde{\alpha}_1 \tilde{N}_1 \in dspan \mathcal{N}_1$, the geometric Lemma 3 gives

$$|\alpha_1 N_1 - \tilde{\alpha}_1 \tilde{N}_1 - \alpha_3 N_3 + \alpha_2 N_2| \geq max(|\alpha_2|, |\alpha_3|) \gg r$$

which is in contradiction with the previous statement.

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Challenges in the case $k \geq 4$.

The last bit of the argument cannot rely anymore on l^2/L^2 techniques. Instead a true multilinear estimate needs to come into play. This makes the geometry of the problem far more complicated.

We made use of the triangle inequality at multiple levels, and we could do that since the key inequality is an L^1 improved estimate. If $k \geq 4$ the key inequality requires an interpolation with an $L^{\frac{2}{k-1}}$ improved estimate and $\frac{2}{k-1} < 1$. This creates real difficulties in running parts of the argument.

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