A Fourier restriction estimate for a surface of finite type

Stefan Buschenhenke

University of Birmingham
joint work with Detlef Müller and Ana Vargas

Interactions between Harmonic Analysis and Geometric Analysis
Samita/Tokyo
November 2016

The restriction operator

Let S be a smooth, compact hypersurface in $\mathbb{R}^n$ with surface measure σ.

The restriction operator

Let S be a smooth, compact hypersurface in $\mathbb{R}^n$ with surface measure σ.
The restriction operator is $R(f) = \hat{f}|_S$,
where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \mathrm{d}x$$

is the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

The restriction operator

Let S be a smooth, compact hypersurface in $\mathbb{R}^n$ with surface measure σ.
The restriction operator is $R(f) = \hat{f}|_S$,
where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \mathrm{d}x$$

is the Fourier transform of $f \in L^1(\mathbb{R}^n)$.
For which $1 \leq p, q \leq \infty$ does

$$R : L^p(\mathbb{R}^n) \to L^q(S, \sigma)$$

give a bounded Operator?

The restriction operator

Let S be a smooth, compact hypersurface in $\mathbb{R}^n$ with surface measure σ.
The restriction operator is $R(f) = \hat{f}|_S$,
where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \mathrm{d}x$$

is the Fourier transform of $f \in L^1(\mathbb{R}^n)$.
For which $1 \leq p, q \leq \infty$ does

$$R : L^p(\mathbb{R}^n) \to L^q(S, \sigma)$$

give a bounded Operator?
Adjoint Operator: $R^*(g)(x) = \widehat{g\mathrm{d}\sigma}(x) = \int_S g(\xi) e^{-ix\cdot\xi} \mathrm{d}\sigma(\xi)$.

Trivial Results

We always have

$$\|\widehat{g\mathrm{d}\sigma}\|_\infty \leq \|g\|_{L^1(S,\sigma)}$$

Trivial Results

We always have

$$\|\widehat{g\mathrm{d}\sigma}\|_\infty \leq \|g\|_{L^1(S,\sigma)}$$

i.e. $R^* : L^q(S,\sigma) \to L^p(\mathbb{R}^n)$ is bounded for $p = \infty$, $q = 1$

Trivial Results

We always have

$$\|\widehat{g\mathrm{d}\sigma}\|_\infty \le \|g\|_{L^1(S,\sigma)} \lesssim \|g\|_{L^q(S,\sigma)},$$

i.e. $R^* : L^q(S,\sigma) \to L^p(\mathbb{R}^n)$ is bounded for $p=\infty$, $q \ge 1$

Trivial Results

We always have

$$\|\widehat{g\mathrm{d}\sigma}\|_\infty \leq \|g\|_{L^1(S,\sigma)} \lesssim \|g\|_{L^q(S,\sigma)},$$

i.e. $R^* : L^q(S,\sigma) \to L^p(\mathbb{R}^n)$ is bounded for $p = \infty$, $q \geq 1$

If $S \subset \mathbb{R}^{n-1}$ this is sharp.

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary
- $n = 1$ solved (Zygmund 74).

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary
- $n = 1$ solved (Zygmund 74).
- $n \geq 2$: $q = 2$ solved (Stein, Tomas 75).

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary
- $n = 1$ solved (Zygmund 74).
- $n \geq 2$: $q = 2$ solved (Stein, Tomas 75).
- Other values of q : improvements in the 90's by Bourgain's work (then Wolff, Moyua, Vargas, Vega, Tao (Bilinear approach)).

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary
- $n = 1$ solved (Zygmund 74).
- $n \geq 2 : q = 2$ solved (Stein, Tomas 75).
- Other values of q : improvements in the 90's by Bourgain's work (then Wolff, Moyua, Vargas, Vega, Tao (Bilinear approach)).
- Confirmed for $p > 2\frac{n+3}{n+1}$ (Tao 2003).

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature. Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$
- Both conditions are known to be necessary
- $n = 1$ solved (Zygmund 74).
- $n \geq 2 : q = 2$ solved (Stein, Tomas 75).
- Other values of q : improvements in the 90's by Bourgain's work (then Wolff, Moyua, Vargas, Vega, Tao (Bilinear approach)).
- Confirmed for $p > 2\frac{n+3}{n+1}$ (Tao 2003).
- Further progress by multilinear approach (Bourgain and Guth 2011), "polynomial method" (Guth 2015)

Surfaces with non-vanishing Gaussian curvature

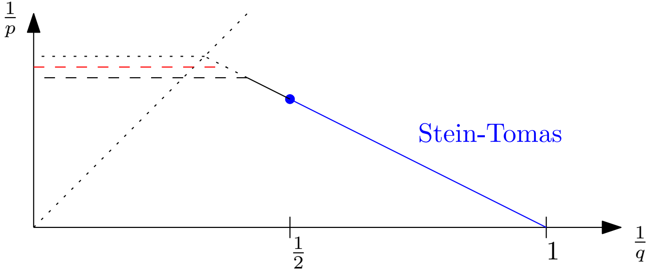
Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

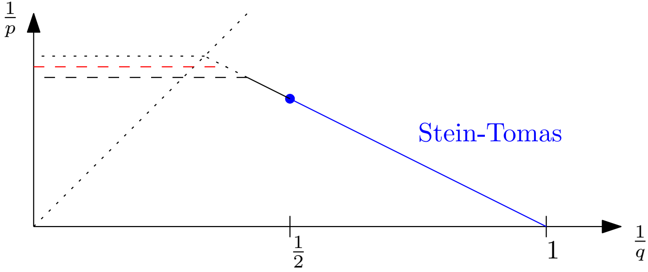
- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$



Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature. Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$

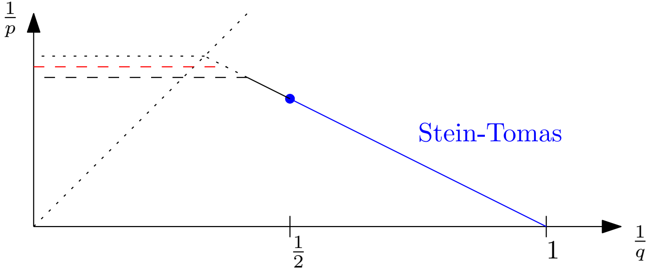


- Hölder: (q,p) implies $(\tilde{q},p)$ for $q \leq \tilde{q}$

Surfaces with non-vanishing Gaussian curvature

Let S be a compact hypersurface with non-vanishing Gaussian curvature.
Boundedness of $R^* : L^q(S) \to L^p(\mathbb{R}^{n+1})$:

- Conjecture: $\frac{1}{q'} \geq \frac{n+2}{np}$ and $p > \frac{2n+2}{n}$



- Hölder: (q, p) implies $(\tilde{q}, p)$ for $q \leq \tilde{q}$
- Nikishin-Maurey-Pisier factorisation: (∞, p) implies (p, p) for $S = S^n$

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Sjölin 74, Ruiz 1983, Barcelo 1986

$R^* : L^{q,p}(\gamma) \to L^p(\mathbb{R}^2)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Sjölin 74, Ruiz 1983, Barcelo 1986

$R^* : L^{q,p}(\gamma) \to L^p(\mathbb{R}^2)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} | \ \frac{x}{z} \in \gamma, \ 1 \leq z \leq 2\}$.

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Sjölin 74, Ruiz 1983, Barcelo 1986

$R^* : L^{q,p}(\gamma) \to L^p(\mathbb{R}^2)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} | \frac{x}{z} \in \gamma,\ 1 \leq z \leq 2\}$.

Barcelo 1986, B. 2015

$R^* : L^{q,p}(\Gamma) \to L^p(\mathbb{R}^3)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Sjölin 74, Ruiz 1983, Barcelo 1986

$R^* : L^{q,p}(\gamma) \to L^p(\mathbb{R}^2)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} | \frac{x}{z} \in \gamma,\ 1 \leq z \leq 2\}$.

Barcelo 1986, B. 2015

$R^* : L^{q,p}(\Gamma) \to L^p(\mathbb{R}^3)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, |x|^m) \in \mathbb{R}^n \times \mathbb{R}\}$.

Curves and surfaces of finite type

Let $\gamma = \{(x, x^m) | x \in [0,1]\}$, $m > 2$.

Sjölin 74, Ruiz 1983, Barcelo 1986

$R^* : L^{q,p}(\gamma) \to L^p(\mathbb{R}^2)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} | \frac{x}{z} \in \gamma,\ 1 \leq z \leq 2\}$.

Barcelo 1986, B. 2015

$R^* : L^{q,p}(\Gamma) \to L^p(\mathbb{R}^3)$ for $p > 4$ and $\frac{1}{q'} \geq \frac{m+1}{p}$.

Let $\Gamma = \{(x, |x|^m) \in \mathbb{R}^n \times \mathbb{R}\}$.

Stovall 2014

$R^* : L^{q,p}(\Gamma) \to L^p(\mathbb{R}^{n+1})$ for $\frac{1}{q'} \geq \frac{m+n}{np}$ and p such that the restriction conjecture holds.

The proof involves affine arclength measure.

Known results

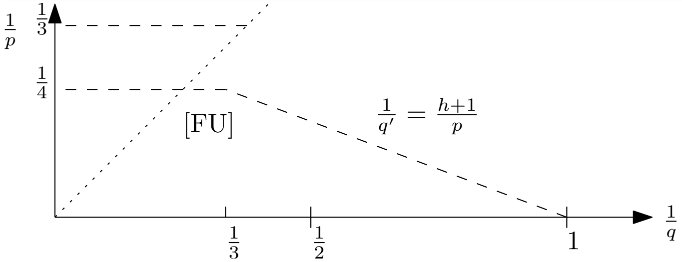
Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0, 1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

Known results

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

Ferreyra and Urciuolo 2009

$R^* : L^q(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} > \frac{h+1}{p}$ and $p > 4$.

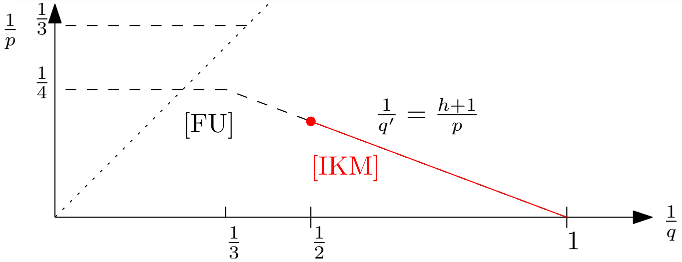


Known results

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

Ikromov, Kempe and Müller 2010

$R^* : L^2(S, \sigma) \to L^p(\mathbb{R}^3)$ if $p \geq 2h + 2$.

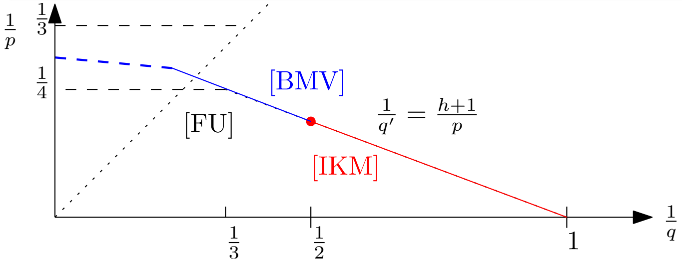


Known results

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.



Necessary conditions

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

Necessary conditions

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

B., Müller and Vargas 2014

Assume $R^* : L^q(S,\sigma) \to L^p(\mathbb{R}^3)$. Then $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{3, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

Lorentz space?

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

Lorentz space?

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

- By trivial considerations this gives the $L^q(S) \to L^p(\mathbb{R}^3)$ result, for $\frac{1}{q'} > \frac{h+1}{p}$.

Lorentz space?

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

- By trivial considerations this gives the $L^q(S) \to L^p(\mathbb{R}^3)$ result, for $\frac{1}{q'} > \frac{h+1}{p}$.
- When $\frac{1}{p} \leq \frac{1}{q}$, the strong type result follows by interpolation.

Lorentz space?

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

- By trivial considerations this gives the $L^q(S) \to L^p(\mathbb{R}^3)$ result, for $\frac{1}{q'} > \frac{h+1}{p}$.
- When $\frac{1}{p} \leq \frac{1}{q}$, the strong type result follows by interpolation.
- If $\frac{1}{q'} = \frac{h+1}{p}$ and $\frac{1}{p} \geq \frac{1}{q}$, $L^q(S, \sigma) \to L^p(\mathbb{R}^3)$ fails.

Different situations

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

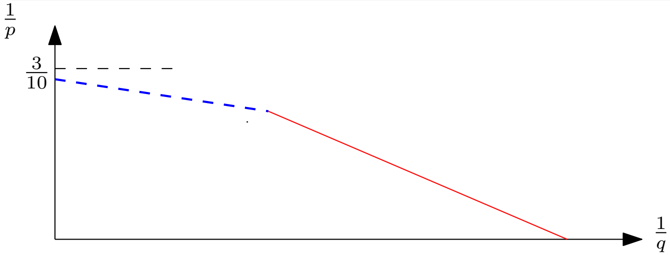
$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

Different situations

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

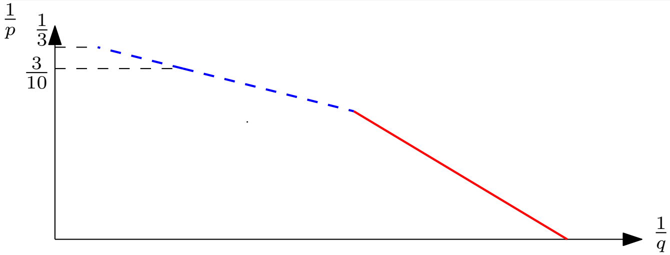


Different situations

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

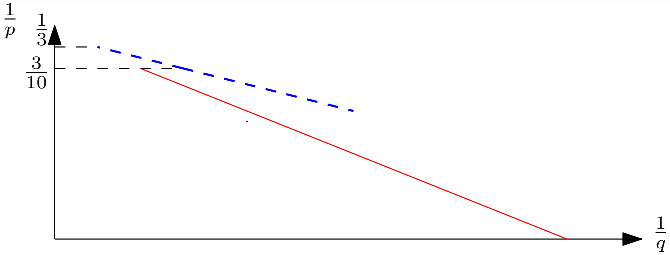


Different situations

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

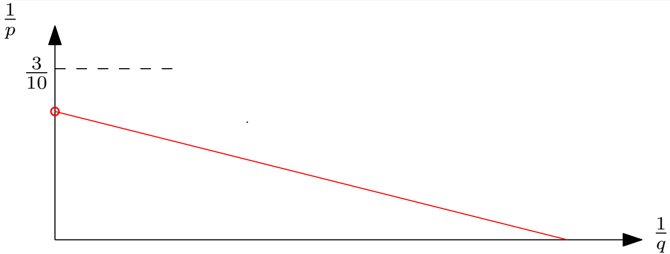


Different situations

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

B., Müller and Vargas 2014

$R^* : L^{q,p}(S,\sigma) \to L^p(\mathbb{R}^3)$ if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.

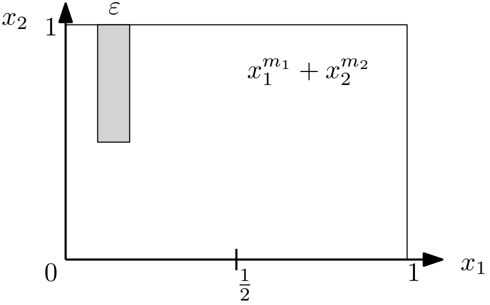


Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Consider subsurface $S_\varepsilon = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | \varepsilon \leq x_1 \leq 2\varepsilon, \frac{1}{2} \leq x_2 \leq 1\}$, where the principle curvatures are more or less constant

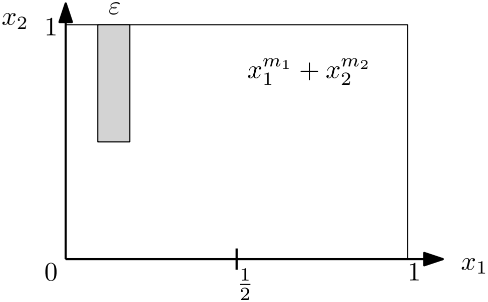
Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Consider subsurface $S_\varepsilon = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | \varepsilon \leq x_1 \leq 2\varepsilon, \frac{1}{2} \leq x_2 \leq 1\}$, where the principle curvatures are more or less constant



Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Consider subsurface $S_\varepsilon = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | \varepsilon \le x_1 \le 2\varepsilon, \frac{1}{2} \le x_2 \le 1\}$, where the principle curvatures are more or less constant



Rescaling: $S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3)) | x_1 \in [0,1], x_2 \in [0, K]\}$, $K = \varepsilon^{-\frac{m_1}{2}}$ and

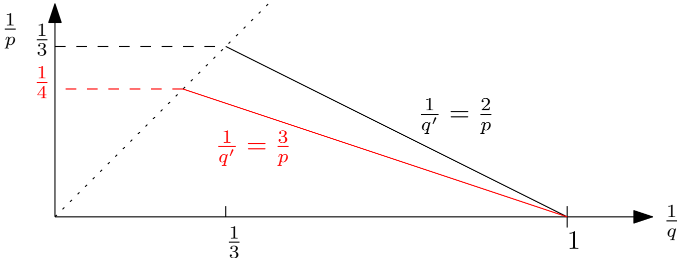
$$\|R^*\|_{L^q(S(\varepsilon^{-\frac{m_1}{2}})) \to L^p(\mathbb{R}^3)} \gtrsim \varepsilon^{\frac{1}{p}\left(\frac{3m_1}{2}+1\right) - \frac{1}{q'}\left(\frac{m_1}{2}+1\right)}.$$

Long-stretched paraboloid

$$S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3))|x_1 \in [0,1], x_2 \in [0,K]\}$$

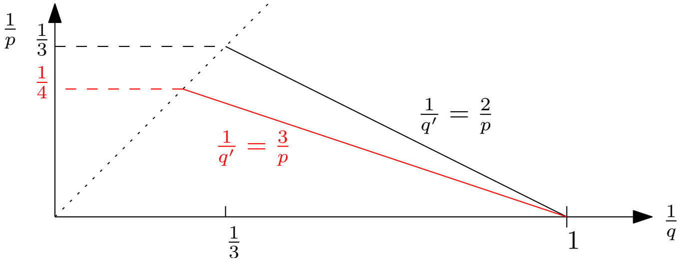
Long-stretched paraboloid

$$S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3))|x_1 \in [0,1], x_2 \in [0,K]\}$$



Long-stretched paraboloid

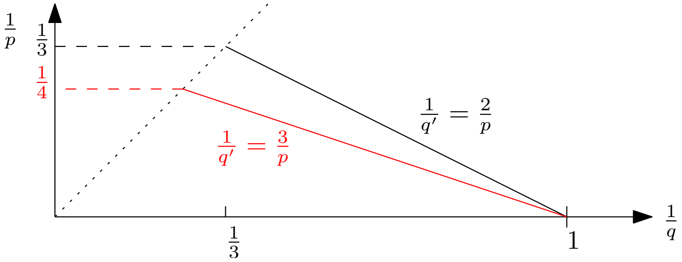
$$S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3))|x_1 \in [0,1], x_2 \in [0,K]\}$$



On $\frac{1}{q'} = \frac{2}{p}$: Estimates invariant under parabolic scaling.

Long-stretched paraboloid

$$S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3)) | x_1 \in [0,1], x_2 \in [0,K]\}$$

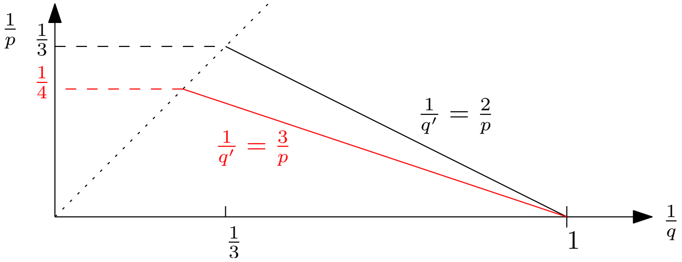


On $\frac{1}{q'} = \frac{2}{p}$: Estimates invariant under parabolic scaling.

On $\frac{1}{q'} = \frac{3}{p}$: Apply estimates for the Parabola.

Long-stretched paraboloid

$$S(K) = \{(x_1, x_2, x_1^2 + x_2^2 + \mathcal{O}(|x|^3)) | x_1 \in [0,1], x_2 \in [0,K]\}$$



On $\frac{1}{q'} = \frac{2}{p}$: Estimates invariant under parabolic scaling.

On $\frac{1}{q'} = \frac{3}{p}$: Apply estimates for the Parabola.

Conjecture:

$$\|R^*\|_{L^q(S(K)) \to L^p(\mathbb{R}^3)} \lesssim K^{\left(\frac{1}{p} - \frac{1}{q}\right)_+}.$$

Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Conjecture:

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \lesssim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Conjecture:

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \lesssim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

We could prove that

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \gtrsim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

Necessary condition $\frac{1}{q}+\frac{2m_1+1}{p}<\frac{m_1+2}{2}$

Conjecture:

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \lesssim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

We could prove that

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \gtrsim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

Thus

$$\varepsilon^{\left(\frac{1}{q}-\frac{1}{p}\right)\frac{m_1}{2}} \lesssim \|R^*\|_{L^q(S(\varepsilon^{-\frac{m_1}{2}}))\to L^p(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{1}{p}\left(\frac{3m_1}{2}+1\right)-\frac{1}{q'}\left(\frac{m_1}{2}+1\right)}.$$

Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Conjecture:

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \lesssim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

We could prove that

$$\|R^*\|_{L^q(S(K))\to L^p(\mathbb{R}^3)} \gtrsim K^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

Thus

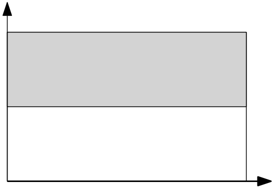
$$\varepsilon^{\left(\frac{1}{q}-\frac{1}{p}\right)\frac{m_1}{2}} \lesssim \|R^*\|_{L^q(S(\varepsilon^{-\frac{m_1}{2}}))\to L^p(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{1}{p}\left(\frac{3m_1}{2}+1\right)-\frac{1}{q'}\left(\frac{m_1}{2}+1\right)}.$$

For $\varepsilon \to 0$:

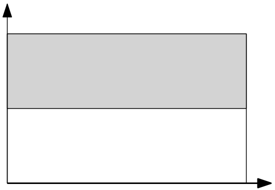
$$\frac{1}{q} + \frac{2m_1+1}{p} \le \frac{m_1+2}{2}$$

Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$

Consider the subsurface where $0.5 \leq x_2 \leq 1$.



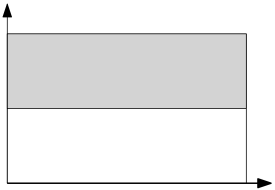
Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$



Consider the subsurface where $0.5 \leq x_2 \leq 1$.

Test the restriction operator for $f(x) = 1 \in L^q(S)$:

Necessary condition $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$



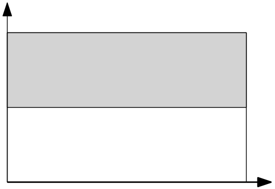
Consider the subsurface where $0.5 \le x_2 \le 1$.

Test the restriction operator for $f(x) = 1 \in L^q(S)$:

$$\widehat{f\,\mathrm{d}\sigma}(\xi) = \int_0^1 e^{i(\xi_1 x_1 - \xi_3 x_1^{m_1})}\mathrm{d}x_1 \int_0^1 e^{i(\xi_2 x_2 - \xi_3 [x_2^2 + \mathcal{O}(x_2^3)])}\mathrm{d}x_2$$

Necessary condition $\frac{1}{q}+\frac{2m_1+1}{p}<\frac{m_1+2}{2}$

Consider the subsurface where $0.5 \leq x_2 \leq 1$.



Test the restriction operator for $f(x) = x_1^{-\frac{1}{q}} \log(x_1)^s \in L^q(S)$:

$$\widehat{f\,\mathrm{d}\sigma}(\xi) = \int_0^1 e^{i(\xi_1 x_1 - \xi_3 x_1^{m_1})} x_1^{-\frac{1}{q}} \log(x_1)^s \mathrm{d}x_1 \int_0^1 e^{i(\xi_2 x_2 - \xi_3 [x_2^2 + \mathcal{O}(x_2^3)])} \mathrm{d}x_2$$

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

$$\|\widehat{f\mathrm{d}\sigma}\|_{2p}^2 = \|\widehat{f\mathrm{d}\sigma}\widehat{f\mathrm{d}\sigma}\|_p$$

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

$$\begin{aligned} \|\widehat{f\mathrm{d}\sigma}\|_{2p}^2 =& \|\widehat{f\mathrm{d}\sigma}\widehat{f\mathrm{d}\sigma}\|_p \\ &\leq \left(\sum_{(k,l)} \|\widehat{f\mathrm{d}\sigma_k}\widehat{f\mathrm{d}\sigma_l}\|_p^{p*} \right)^{\frac{1}{p^*}} \leq \dots , \end{aligned}$$

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

$$\begin{aligned} \|\widehat{f\mathrm{d}\sigma}\|_{2p}^2 =& \|\widehat{f\mathrm{d}\sigma}\widehat{f\mathrm{d}\sigma}\|_p \\ & \leq \left(\sum_{(k,l)} \|\widehat{f\mathrm{d}\sigma_k}\widehat{f\mathrm{d}\sigma_l}\|_p^{p*} \right)^{\frac{1}{p^*}} \leq \dots \ , \end{aligned}$$

where $p^* = \min\{p, p'\}$.

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

$$\begin{aligned} \|\widehat{f\mathrm{d}\sigma}\|_{2p}^2 &= \|\widehat{f\mathrm{d}\sigma}\widehat{f\mathrm{d}\sigma}\|_p \\ &\leq \left(\sum_{(k,l)} \|\widehat{f\mathrm{d}\sigma_k}\widehat{f\mathrm{d}\sigma_l}\|_p^{p*} \right)^{\frac{1}{p^*}} \leq \dots \ , \end{aligned}$$

where $p^* = \min\{p, p'\}$.
Goal: Establish a bilinear estimate

$$\|\widehat{f\mathrm{d}\sigma_k g\mathrm{d}\sigma_l}\|_p \leq C(k,l,p)\|f\|_{L^q(\sigma_k)}\|g\|_{L^q(\sigma_l)}.$$

The bilinear method

The "bilinear implies linear" argument due to Tao-Vargas-Vega 1997:
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

$$\begin{aligned} \|\widehat{f\mathrm{d}\sigma}\|_{2p}^2 =& \|\widehat{f\mathrm{d}\sigma}\widehat{f\mathrm{d}\sigma}\|_p \\ & \leq \left(\sum_{(k,l)} \|\widehat{f\mathrm{d}\sigma_k}\widehat{f\mathrm{d}\sigma_l}\|_p^{p*} \right)^{\frac{1}{p^*}} \leq \dots , \end{aligned}$$

where $p^* = \min\{p, p'\}$.
Goal: Establish a bilinear estimate

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_p \leq C(k,l,p)\|f\|_{L^q(\sigma_k)}\|g\|_{L^q(\sigma_l)}.$$

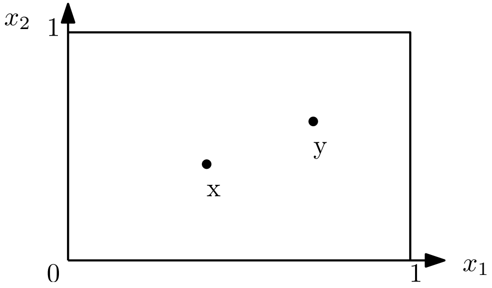
Important: How does the constant depend on the specific pair of subsurfaces over (τ_k, τ_l)?

Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

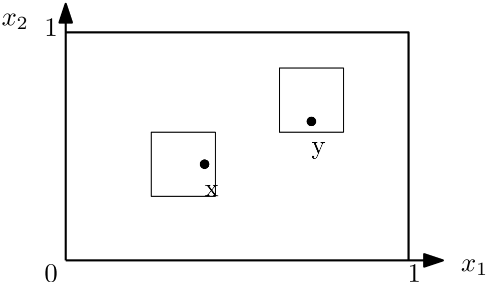
Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.



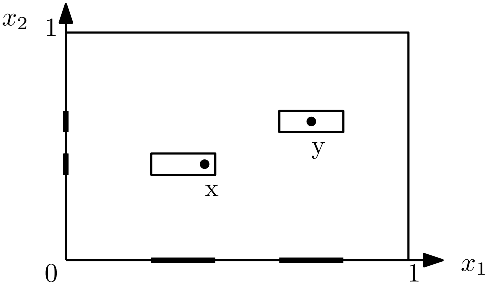
Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.



Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.



Bidyadic decompostiton

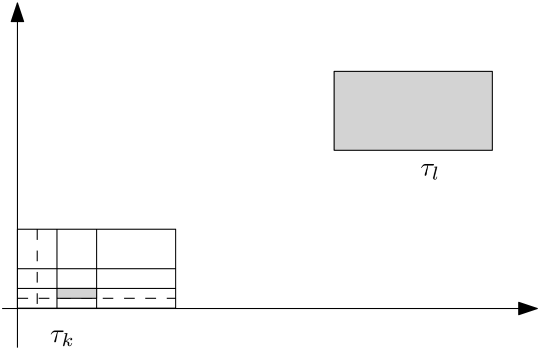
Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

In some cases, we need to decompose further:

Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

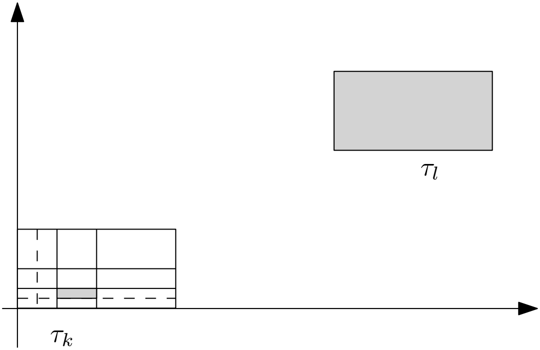
In some cases, we need to decompose further:



Bidyadic decompostiton

Let $Q = [0,1] \times [0,1]$ be the unit square.
Decompose $Q \times Q = \bigcup_{(k,l)} \tau_k \times \tau_l$ in a "suitable" way.

In some cases, we need to decompose further:



The main reason for this decomposition is that the curvatures are essentially constant.

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.
- Find a suitable rescaling to a "nice" situation ("isotropic" wave packets)

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.
- Find a suitable rescaling to a "nice" situation ("isotropic" wave packets)
- Decompose into wave packets (well localised in position and momentum space)

The bilinear method

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.
- Find a suitable rescaling to a "nice" situation ("isotropic" wave packets)
- Decompose into wave packets (well localised in position and momentum space)
- Geometric argument

Some troubleshooters

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

Some troubleshooters

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

- We need a quantitative version of the classical ε-removal argument, being sensitive to the dependence of the constant $C(p,k,l)$.

Some troubleshooters

- Global bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(\mathbb{R}^3)} \leq C(p,k,l)\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)},$$

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

- We need a quantitative version of the classical ε-removal argument, being sensitive to the dependence of the constant $C(p,k,l)$.
- For instance, the ε-removal uses some decay of the Fourier transform: $|\widehat{\mathrm{d}\sigma}_k(x)| \leq C'(k)(1+|x|)^{-s}$.

Some troubleshooters

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon \|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

- Induction on scales: reduce ε step by step.

Some troubleshooters

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon \|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.

- Induction on scales: reduce ε step by step.
- Start of the induction:

$$\begin{aligned}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} &\leq |Q_{k,l}(R)|^{\frac{1}{p}}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_\infty \\ &\leq A(p,k,l)R^N\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.\end{aligned}$$

Some troubleshooters

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon \|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.
- Start of the induction:

$$\begin{aligned}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} &\leq |Q_{k,l}(R)|^{\frac{1}{p}}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_\infty \\ &\leq A(p,k,l)R^N\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.\end{aligned}$$

- However, $A(p,k,l) \gg C(p,k,l)$.

Some troubleshooters

- Local bilinear estimates:

$$\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq C(p,k,l)C_\varepsilon R^\varepsilon \|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.$$

  for $\varepsilon > 0$, $R \geq 1$ and certain cuboids $Q_{k,l}(R) \to \mathbb{R}^3$, $R \to \infty$.
- Induction on scales: reduce ε step by step.
- Start of the induction:

$$\begin{aligned}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_{L^p(Q_{k,l}(R))} \leq& |Q_{k,l}(R)|^{\frac{1}{p}}\|\widehat{f\mathrm{d}\sigma_k}\widehat{g\mathrm{d}\sigma_l}\|_\infty \\ \leq& A(p,k,l)R^N\|f\|_{L^2(S_k,\sigma_k)}\|g\|_{L^2(S_l,\sigma_l)}.\end{aligned}$$

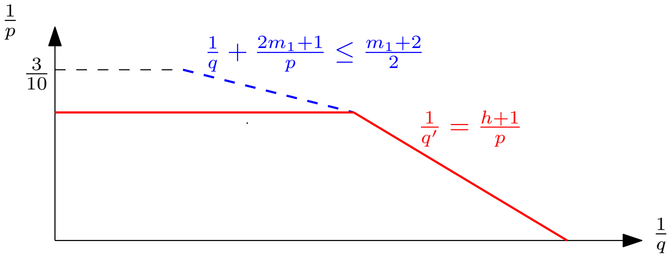
- However, $A(p,k,l) \gg C(p,k,l)$.
- Similar problem for "Schwartz tail" contributions of wave packets

Completing the proof

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.

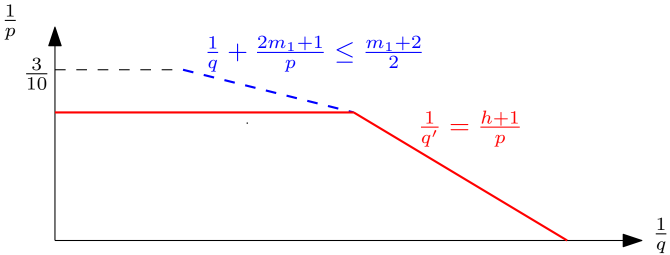
Completing the proof

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.
The initial analysis only gives a partial result:



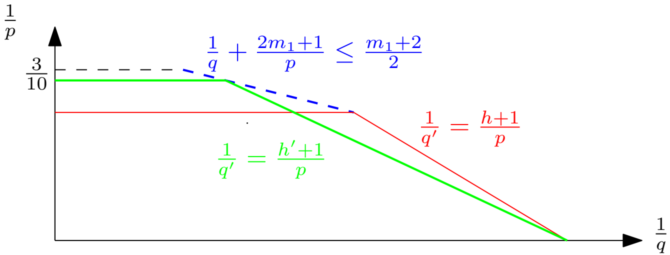
Completing the proof

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.
The initial analysis only gives a partial result.
Fix m_1 and choose $m_2' > m_2$, i.e. $h' > h$.



Completing the proof

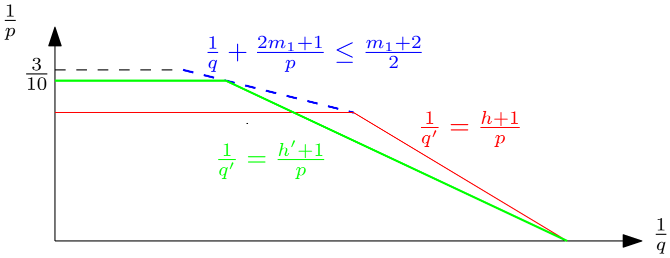
Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.
The initial analysis only gives a partial result.
Fix m_1 and choose $m_2' > m_2$, i.e. $h' > h$.



Completing the proof

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.
The initial analysis only gives a partial result.
Fix m_1 and choose $m_2' > m_2$, i.e. $h' > h$.
Observe

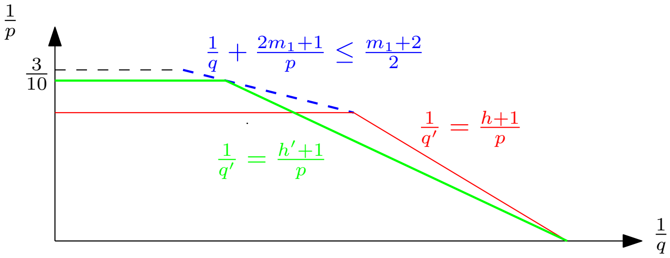
$$\|R^*_{BL}\|_{L^2(S_k,\sigma_k)\times L^2(S_l,\sigma_l)\to L^p(\mathbb{R}^3)} \leq C_{m_1,m_2}(p,k,l)$$



Completing the proof

Let $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | x_1, x_2 \in [0,1]\}$, $m_1 \geq m_2 \geq 2$.
By h denote the height of S, given by $\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}$.
The initial analysis only gives a partial result.
Fix m_1 and choose $m_2' > m_2$, i.e. $h' > h$.
Observe

$$\|R^*_{BL}\|_{L^2(S_k,\sigma_k)\times L^2(S_l,\sigma_l)\to L^p(\mathbb{R}^3)} \leq C_{m_1,m_2}(p,k,l) \leq C_{m_1,m_2'}(p,k,l).$$



Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$

Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$
- Curvature with different signs: $\Phi(x_1, x_2) = x_1^{m_1} - x_2^{m_2}$

Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$
- Curvature with different signs: $\Phi(x_1, x_2) = x_1^{m_1} - x_2^{m_2}$
- Work in progress for $\Phi(x_1, x_2) = x_1^2 - x_2^2 + \mathcal{O}(|x|^3)$.

Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$
- Curvature with different signs: $\Phi(x_1, x_2) = x_1^{m_1} - x_2^{m_2}$
- Work in progress for $\Phi(x_1, x_2) = x_1^2 - x_2^2 + \mathcal{O}(|x|^3)$.
- Results in higher dimensions

Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$
- Curvature with different signs: $\Phi(x_1, x_2) = x_1^{m_1} - x_2^{m_2}$
- Work in progress for $\Phi(x_1, x_2) = x_1^2 - x_2^2 + \mathcal{O}(|x|^3)$.
- Results in higher dimensions
  However, any reasonable result would imply progress for the paraboloid as
  $\phi_m(x) = x_1^2 + x_2^2 + x_3^m$

Related questions

- Up to now, we can only handle functions of the form $\Phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$
- Curvature with different signs: $\Phi(x_1, x_2) = x_1^{m_1} - x_2^{m_2}$
- Work in progress for $\Phi(x_1, x_2) = x_1^2 - x_2^2 + \mathcal{O}(|x|^3)$.
- Results in higher dimensions
  However, any reasonable result would imply progress for the paraboloid as
  $\phi_m(x) = x_1^2 + x_2^2 + x_3^m \overset{m\to\infty}{\longrightarrow} \phi(x) = x_1^2 + x_2^2$

Thank you for your attention!