Complex Interpolation of Morrey Spaces

Denny Ivanal Hakim

Tokyo Metropolitan University

Joint work with Yoshihiro Sawano (Tokyo Metropolitan University)

Interactions Between Harmonic and Geometric Analysis
Saitama University Satellite Campus
Tokyo Station College,
28 November-1 December, 2016

Outline

1. The Riesz-Thorin interpolation theorem and Calderon's complex interpolation method
2. Previous results on complex interpolation of Morrey spaces
3. Complex interpolation method for quasi-Banach spaces
4. Main theorem
5. Complex interpolation of closed subspaces of Morrey spaces

The Riesz-Thorin interpolation theorem

Theorem

Let $\theta \in (0,1)$, $1 \le p_0, p_1 \le \infty$, *and* $1 \le r_0, r_1 \le \infty$. *Suppose that* T *is a linear operator from* $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ *to* $L^{r_0}(\mathbb{R}^n) + L^{r_1}(\mathbb{R}^n)$ *for which*

$$\|Tf\|_{L^{r_0}} \le C_0 \|f\|_{L^{p_0}(\mathbb{R}^n)} \quad \text{and} \quad \|Tf\|_{L^{r_1}} \le C_1 \|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

Define p *and* r *by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{r} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Then T *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$.

Calderón's first complex interpolation method

A couple (X_0, X_1) of Banach spaces is said to be compatible if X_0 and X_1 can be embedded into a Hausdorff topological vector space Z. Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\overline{S}$ be its closure.

Definition (Calderón's first complex interpolation functor)

Let (X_0, X_1) be a compatible couple of Banach spaces. The space $\mathcal{F}(X_0, X_1)$ is defined to be the set of all continuous functions $F : \overline{S} \to X_0 + X_1$ such that

1. $\sup_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$;
2. F is holomorphic on S;
3. For each $j = 0, 1$, the function $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ is continuous;
4. $\|F\|_{\mathcal{F}(X_0,X_1)} := \max_{j=0,1} \sup_{t\in\mathbb{R}} \|F(j + it)\|_{X_j} < \infty$.

Calderón's first complex interpolation method (cont.)

Definition (Calderón's first complex interpolation space)

Let $\theta \in (0,1)$. Define

$$[X_0, X_1]_\theta := \{F(\theta) : F \in \mathcal{F}(X_0, X_1)\}.$$

The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

Theorem (Calderón, 1964)

Let $\theta \in (0,1)$. Suppose that T is a bounded linear operator from X_j to Y_j for $j = 0, 1$. Then, T is bounded from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$.

Example

Let $\theta \in (0,1)$, $1 \le p_0, p_1 \le \infty$, and $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then $[L^{p_0}, L^{p_1}]_\theta = L^p$

Morrey spaces

Definition

Let $0 < q \le p < \infty$. The Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(a,r)} |f(x)|^q \, dx\right)^{1/q} < \infty.$$

Remark: If $p = q$, then $\mathcal{M}^p_q = L^p$.

Example

Let $0 < q < p < \infty$. Then $f(x) := |x|^{-n/p} \in \mathcal{M}^p_q$.

Prevous results

Theorem (Stampacchia, 1964)

Let $\theta \in (0,1)$, $1 \le p_0, p_1 < \infty$, $1 \le r_0 \le s_0 < \infty$, and $1 \le r_1 \le s_1 < \infty$. Define p, r, and s by

$$\left(\frac{1}{p}, \frac{1}{r}, \frac{1}{s}\right) := (1-\theta)\left(\frac{1}{p_0}, \frac{1}{r_0}, \frac{1}{s_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{r_1}, \frac{1}{s_1}\right).$$

If T is a bounded linear operator from L^{p_0} to $\mathcal{M}^{r_0}_{s_0}$ and from L^{p_1} to $\mathcal{M}^{r_1}_{s_1}$, then T is bounded from L^p to $\mathcal{M}^r_s$.

Theorem (Ruiz and Vega, 1995)

Let $\theta \in (0,1)$ and $n > 1$. There exist $1 \le p_0, p_1 < \infty$, a bounded linear operator T from $\mathcal{M}^{p_0}_q$ to L^1 and from $\mathcal{M}^{p_1}_q$ to L^1, but T is not bounded from $\mathcal{M}^p_q$ to L^1 where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

The case $n = 1$ can be seen in [Blasco, Ruiz, and Vega, 1999].

Previous results (cont.)

Theorem (Cobos, Peetre, and Persson, 1998)

Let $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$ Define p and q by

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$.

Theorem

Keep the notations of the previous theorem. Assume $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Then

1. (Lu, Yang, and Yuan, 2014) $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$
2. (H. and Sawano, 2016)
$$[\mathcal{M}^{p_0}_{q_0}, M^{p_1}_{q_1}]_\theta = \left\{ f \in \mathcal{M}^p_q : \lim_{N\to\infty} \left\| f - f\chi_{\{\frac{1}{N} \le |f| \le N\}} \right\|_{\mathcal{M}^p_q} = 0 \right\}.$$

Calderon's second complex interpolation method

Definition (Calderon's second complex interpolation functor)

Let (X_0, X_1) be a compatible couple of Banach spaces. $\mathcal{G}(X_0, X_1)$ is defined to be the set of all continuous functions $G : \overline{S} \to X_0 + X_1$ such that:

1. $\sup_{z \in \bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$ and G is holomorphic in S;
2. For every $j = 0, 1$ and $t \in \mathbb{R}$, $G(j + it) - G(j) \in X_j$;
3. $\|G\|_{\mathcal{G}(X_0,X_1)} := \max_{j=0,1} \sup_{-\infty<t<s<\infty} \frac{\|G(j+it)-G(j+is)\|_{X_j}}{|t-s|} < \infty.$

Definition (Calderon's second complex interpolation space)

For $\theta \in (0, 1)$, define

$$[X_0, X_1]^\theta = \{G'(\theta) : G \in \mathcal{G}(X_0, X_1)\}.$$

and $\|f\|_{[X_0,X_1]^\theta} := \inf_{f=G'(\theta)} \|G\|_{\mathcal{G}(X_0,X_1)}$.

The second complex interpolation of Morrey spaces

Theorem (Lemarié-Rieusset, 2014)

Let $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, *and* $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. *Define* p *and* q *by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q$.

Complex interpolation of quasi-Banach spaces

Definition

Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\overline{S}$ be its closure. Let X be a quasi-Banach space.

1. A map $f : S \to X$ is said to be analytic, if for any $z_0 \in S$, there exist $\eta \in (0, \infty)$ and $\{h_j\}_{j=0}^{\infty} \subset X$ such that the disk $\overline{\Delta}(z_0, \eta) \subset S$ and for all $z \in \Delta(z_0, \eta)$

$$f(z) = \sum_{j=0}^{\infty} h_j (z - z_0)^j \text{ in } X.$$

2. A quasi-Banach space X is called analytically convex if there exists a positive constant C such that, for any continuous and bounded function $f : \overline{S} \to X$ which is analytic in S,

$$\sup_{z \in S} \|f(z)\|_X \le C \sup_{z \in \overline{S} \setminus S} \|f(z)\|_X.$$

The first complex interpolation method

Let (X_0, X_1) be a compatible couple of quasi-Banach spaces such that $X_0 + X_1$ is analytically convex.

Definition (The first complex interpolation functor)

The space $\mathcal{F}(X_0, X_1)$ is defined to be the set of all continuous functions $F : \overline{S} \to X_0 + X_1$ such that

1. $\sup_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$ and F is analytic in S;
2. for $j = 0, 1$, the function $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ is continuous.
3. $\|F\|_{\mathcal{F}(X_0,X_1)} := \max_{j=0,1} \sup_{t \in \mathbb{R}} \|F(j + it)\|_{X_j} < \infty$.

Definition (The first complex interpolation space)

For $\theta \in (0, 1)$, define

$$[X_0, X_1]_\theta := \{F(\theta) : F \in \mathcal{F}(X_0, X_1)\}$$

and $\|f\|_{[X_0,X_1]_\theta} := \inf_{f=F(\theta)} \|F\|_{\mathcal{F}(X_0,X_1)}$.

The second complex interpolation method

Definition (The second complex interpolation functor)

Let (X_0, X_1) be a compatible couple. Denote by $\mathcal{G}(X_0, X_1)$ the set of all continuous functions $G : \bar{S} \to X_0 + X_1$ such that:

1. $\sup_{z \in \bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$ and G is analytic in S;
2. for every $j = 0, 1$ and $t \in \mathbb{R}$, $G(j + it) - G(j) \in X_j$;
3. $\|G\|_{\mathcal{G}(X_0,X_1)} := \max_{j=0,1} \sup_{-\infty<s<t<\infty} \frac{\|G(j+is)-G(j+it)\|_{X_j}}{|t-s|} < \infty$.

Definition (The second complex interpolation space)

For $\theta \in (0, 1)$, define

$$[X_0, X_1]^\theta := \{G'(\theta) : G \in \mathcal{G}(X_0, X_1)\}.$$

and $\|f\|_{[X_0,X_1]^\theta} := \inf_{f=G'(\theta)} \|G\|_{\mathcal{G}(X_0,X_1)}$.

Main theorem

Theorem (H. and Sawano, 2016)

Let $\theta \in (0,1)$, $0 < q_0 \le p_0 < \infty$, and $0 < q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let $A := \{f \in \mathcal{M}^p_q : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0\}$. Then

1. *If* $\min(q_0, q_1) < 1$, *then*
$$A \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q.$$
2. $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q.$

Remark: This theorem is also valid for Morrey spaces on a metric measure space $(\mathcal{X}, \mu)$ equipped with a σ-finite measure μ.

Proof of $A \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$

We may assume that $q_0 > q_1$. Suppose that $f \in \mathcal{M}^p_q$ satisfies

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^p_q} = 0 \tag{1}$$

For every $z \in \overline{S}$, define

$$F(z) := \mathrm{sgn}(f)|f|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}.$$

Since $f = F(\theta)$, once we show that $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$, we can conclude that $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$.

Proof of $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$: Note that our assumptions yield $\frac{p_0}{q_0} = \frac{p_1}{q_1} = \frac{p}{q}$. By using the decomposition $F_0(z) := \chi_{\{|f|\le 1\}} F(z)$ and $F_1(z) := F(z) - F_0(z)$, we have $F(z) \in \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ and

$$\sup_{z\in\overline{S}} \|F(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} \le \|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q} < \infty.$$

The continuity of F on $\overline{S}$ and $t \in \mathbb{R} \mapsto F(j+it) \in \mathcal{M}^{p_0}_{q_0}$ can be checked by utilizing (1).

Proof of $A \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ (cont.)

Proof of $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ (cont.): For the proof of F is analytic in S, it suffices to show that $F|_{S_\varepsilon}$ is analytic where $\varepsilon \in (0, 1/2)$ and $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}$. For a fixed $z_0 \in S_\varepsilon$, we set

$$\eta := \frac{\min(\mathrm{Re}(z_0 - \varepsilon), \mathrm{Re}(1 - \varepsilon - z_0))}{2}$$

and $h_j := \frac{F(z_0)}{n!}\left(\left(\frac{p}{p_1} - \frac{p}{p_0}\right)\log|f|\right)^j$. Then, the disk $\overline{\Delta}(z_0, \eta) \subseteq S_\varepsilon$ and for all $z \in \Delta(z_0, \eta)$

$$\sum_{j=0}^{\infty} h_j (z - z_0)^j = F(z) \text{ in } \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}.$$

Finally, by using $\frac{p_0}{q_0} = \frac{p_1}{q_1} = \frac{p}{q}$ again, we have

$$\max_{j=0,1} \sup_{t \in \mathbb{R}} \|F(j + it)\|_{\mathcal{M}^{p_j}_{q_j}} = \max_{j=0,1} \|f\|^{\frac{p}{p_j}}_{\mathcal{M}^p_q} < \infty.$$

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$

Let $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$. Then, there exists $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ such that

$$f = F(\theta) \text{ and } \|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta}.$$

For a fixed ball $B = B(a, r) \subseteq \mathbb{R}^n$ and $z \in \overline{S}$, define

$$G_B(z) := |B|^{\frac{1-z}{p_0} + \frac{z}{p_1} - \left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)} \chi_B F(z).$$

By using the properties of $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$, we can check that $G_B \in \mathcal{F}(L^{q_0}, L^{q_1})$ and

$$\|G_B\|_{\mathcal{F}(L^{q_0}, L^{q_1})} \le \|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta}. \tag{2}$$

If we can prove that

$$\|G_B(\theta)\|_{L^q} \le \|G_B\|_{\mathcal{F}(L^{q_0}, L^{q_1})}, \tag{3}$$

then by combining (2) and (3), we have $f \in \mathcal{M}^p_q$.

The proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$ (cont.)

Proof of $\|G_B(\theta)\|_{L^q} \le \|G_B\|_{\mathcal{F}(L^{q_0},L^{q_1})}$: Let $u \in (0, \min(q_0, q_1))$. Set $r_0 := \frac{q_0}{u}$, $r_1 := \frac{q_1}{u}$, and $r := \frac{q}{u}$. Then we have

$$\|G_B(\theta)\|_{L^q}^u = \|\,|G_B(\theta)|^u\,\|_{L^r} = \sup_{\|g\|_{L^{r'}}=1} \int_X |G_B(\theta, x)|^u g(x)\,dx.$$

Let $g = \sum_{k=1}^N a_j \chi_{E_k}$ where $a_k \ge 0$. For every $z \in \overline{S}$, we define

$$\tilde{G}_B(z,x) = \sum_{k=1}^N \left(\frac{1}{|E_k|} \int_{E_j} |G_B(z,y)|^u\,dy \right) \chi_{E_k}(x) \quad (z \in \bar{S}, x \in \mathbb{R}^n).$$

Then we have

$$\int_{\mathbb{R}^n} |G_B(\theta,x)|^u g(x)\,dx = \int_{\mathbb{R}^n} \tilde{G}_B(\theta,x) g(x)\,dx \le \|\tilde{G}_B(\theta)\|_{L^r}. \quad (4)$$

The proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$ (cont.)

Proof of $\|G_B(\theta)\|_{L^q} \le \|G_B\|_{\mathcal{F}(L^{q_0},L^{q_1})}$ (cont.): Note that $\tilde{G}_B(\cdot, x)$ is subharmonic on S and continuous on $\overline{S}$, because

$$z \in \bar{S} \mapsto \frac{1}{|E_k|} \int_{E_k} |G_B(z,x)|^u \, dx$$

have the same property. Therefore, $\log \tilde{G}_B(\cdot, x)$ is subharmonic on S. Consequently

$$\log \tilde{G}_B(\theta, x) \le \sum_{j=0}^{1} \int_{\mathbb{R}} P_j(\theta,t) \log \tilde{G}_B(j+it,x) \; dt,$$

where $P_j(\theta,t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t)+(-1)^{j+1}\cos(\pi\theta))}$. By using Jensen's inequality, we have

$$\tilde{G}_B(\theta,x) \le f_0(\theta,x)^{1-\theta} f_1(\theta,x)^{\theta}$$

where $f_j(\theta,x) := \frac{1}{1+(-1)^{j+1}\theta-j} \int_{\mathbb{R}} \tilde{G}_B(j+it,x) P_j(\theta,t) \; dt$ ($\forall j = 0,1$).

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$ (cont.)

Proof of $\|G_B(\theta)\|_{L^q} \le \|G_B\|_{\mathcal{F}(L^{q_0},L^{q_1})}$ (cont.): By using Hölder's inequality, we have

$$\|\tilde{G}_B(\theta)\|_{L^r} \le \|f_0(\theta,\cdot)\|_{L^{r_0}}^{1-\theta} \|f_1(\theta,\cdot)\|_{L^{r_1}}^{\theta}. \tag{5}$$

We use Hölder's inequality again to obtain

$$\frac{1}{|E_k|}\int_{E_k} |G_B(j+it,y)|^u \, dy \le \frac{1}{|E_k|^{\frac{1}{r_j}}} \left(\int_{E_k} |G_B(j+it,y)|^{q_j} \, dy\right)^{\frac{1}{r_j}},$$

so $\|\tilde{G}_B(j+it,\cdot)\|_{L^{r_j}} \le \|G_B(j+it,\cdot)\|_{L^{q_j}}^u$, for all $t \in \mathbb{R}$. Consequently,

$$\|f_j(\theta,\cdot)\|_{L^{r_j}} \le \frac{1}{1+(-1)^{j+1}\theta - j} \int_{\mathbb{R}} \|G_B(j+it)\|_{L^{q_j}}^u P_j(\theta,t) \, dt. \tag{6}$$

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$ (cont.)

Proof of $\|G_B(\theta)\|_{L^q} \le \|G_B\|_{\mathcal{F}(L^{q_0}, L^{q_1})}$ (cont.): We combine the previous inequalities to obtain

$$\|G_B(\theta)\|_{L^q}^u \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|G_B(it)\|_{L^{q_0}}^u P_0(\theta,t)\ dt\right)^{1-\theta} \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|G_B(1+it)\|_{L^{q_1}}^u P_1(\theta,t)\ dt\right)^{\theta}. \qquad (7)$$

Since $G_B \in \mathcal{F}(L^{q_0}, L^{q_1})$, $\|P_0(\theta,\cdot)\|_{L^1} = 1-\theta$, and $\|P_1(\theta,\cdot)\|_{L^1} = \theta$, we have

$$\|G_B(\theta,\cdot)\|_{L^q} \le \left(\sup_{t\in\mathbb{R}} \|G_B(it,\cdot)\|_{L^{q_0}}\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|G_B(1+it,\cdot)\|_{L^{q_1}}\right)^{\theta} \le \|G_B\|_{\mathcal{F}(L^{q_0}, L^{q_1})},$$

as desired.

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \subseteq \mathcal{M}^p_q$

Let $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta$. Then $f = G'(\theta)$ for some $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ and

$$\|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta}. \tag{8}$$

For $z \in \overline{S}$ and $j \in \mathbb{N}$, write $f_j(z) := \frac{G(z+ij^{-1})-G(z)}{ij^{-1}}$. Then $f_j(\theta) \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ with

$$\|f_j(\theta)\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta} \le \|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \tag{9}$$

By the first part of main theorem, we have $f_j(\theta) \in \mathcal{M}^p_q$, and combining this with (8) and (9) yield

$$\|f_j(\theta)\|_{\mathcal{M}^p_q} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta} \tag{10}$$

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \subseteq \mathcal{M}^p_q$ (cont.)

Since $\lim_{j\to\infty} f_j(\theta) = f$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$, $\exists \{f_{j_k}\}_{k=1}^\infty \subseteq \{f_j\}_{j=1}^\infty$ such that

$$\lim_{k\to\infty} f_{j_k}(\theta)(x) = f(x) \text{ a.e.}$$

Thus, by the Fatou lemma and (10), we obtain $f \in \mathcal{M}^p_q$ with

$$\|f\|_{\mathcal{M}^p_q} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta}.$$

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \supseteq \mathcal{M}^p_q$

Assume that $q_0 > q_1$. Let $f \in \mathcal{M}^p_q$. For $z \in \overline{S}$, we define

$$F(z) := \operatorname{sgn}(f)|f|^{p\left(\frac{1-w}{p_0}+\frac{w}{p_1}\right)} \text{ and } G(z) := \int_\theta^z F(w)\, dw.$$

Since $G'(\theta) = F(\theta) = f$, the proof of $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta$ is complete, once we can show that $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$.

Proof of $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$: Let

$$G_0(z) := \chi_{\{|f|\le 1\}} G(z) \text{ and } G_1(z) := \chi_{\{|f|>1\}} G(z).$$

Since $|G_j(z)| \le (1+|z|)|f|^{\frac{p}{p_j}}$ for $z \in \overline{S}$ and $j \in \{0,1\}$, we have

$$\|G(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} \le (1+|z|)\sum_{j=0}^{1} \||f|^{p/p_j}\|_{\mathcal{M}^{p_j}_{q_j}} \le (1+|z|)\sum_{j=0}^{1} \|f\|_{\mathcal{M}^p_q}^{\frac{p}{p_j}},$$

so $G(z) \in \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ and $\sup_{z\in\overline{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} < \infty.$

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \supseteq \mathcal{M}^p_q$ (cont.)

Proof of $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ (cont.): The continuity of G on $\overline{S}$ follows from

$$|G_j(z+h) - G_j(z)| \lesssim |h||f|^{\frac{p}{p_j}}$$

for every $j = 0, 1$, $z \in \overline{S}$, and $h \in \mathbb{C}$ with $z + h \in \overline{S}$.
Let $\varepsilon \in (0, 1/2)$ and $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}$. Given $z_0 \in S_\varepsilon$, by letting

$$\eta := \frac{1}{2} \min(\mathrm{Re}(z_0) - \varepsilon, 1 - \varepsilon - \mathrm{Re}(z_0)),$$

we have $\overline{\Delta}(z_0, \eta) \subseteq S_\varepsilon$ and for all $z \in \Delta(z_0, \eta)$

$$G(z) = G(z_0) + \sum_{j=0}^{\infty} \frac{F(z_0)\left(\left(\frac{p}{p_1} - \frac{p}{p_0}\right)\log|f|\right)^j}{(j+1)!}(z - z_0)^{j+1}$$

in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$.

Proof of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \supseteq \mathcal{M}^p_q$ (cont.)

Proof of $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ (cont.): Finally, since $|F(j+it)| = |f|^{\frac{p}{p_j}}$ $(\forall t \in \mathbb{R}, \forall j \in \{0,1\})$, we have

$$\|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} = \max_{j=0,1} \sup_{-\infty<t<s<\infty} \frac{\| \int_t^s F(j+i\tilde{t})\, d\tilde{t}\|_{\mathcal{M}^{p_j}_{q_j}}}{|t-s|}$$

$$\leq \max_{j=0,1} \left\| |f|^{\frac{p}{p_j}} \right\|_{\mathcal{M}^{p_j}_{q_j}}$$

$$\leq \max_{j=0,1} \|f\|^{\frac{p}{p_j}}_{\mathcal{M}^p_q} < \infty.$$

Closed subspaces of Morrey spaces

Definition

Let $0 < q \le p < \infty$.

1. The tilde space $\widetilde{\mathcal{M}^p_q}$ is defined to be $\widetilde{\mathcal{M}^p_q} := \overline{L^\infty_{\rm c}}^{\mathcal{M}^p_q}$;
2. The star space $\overset{*}{\mathcal{M}}{}^p_q$ is defined to be $\overset{*}{\mathcal{M}}{}^p_q := \overline{L^0_{\rm c} \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q}$, where $L^0_{\rm c}$ is the set of all compactly supported functions;
3. The bar space $\overline{\mathcal{M}^p_q}$ is defined to be $\overline{\mathcal{M}^p_q} := \overline{L^\infty \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q}$.

Theorem

Let $\theta \in (0,1)$, $0 < q_0 \le p_0 < \infty$, $0 < q_1 \le p_1 < \infty$, and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define p and q by $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then

1. $[\widetilde{\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}}]_\theta = \widetilde{\mathcal{M}^p_q}$;
2. $[\overline{\mathcal{M}^{p_0}_{q_0}}, \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = \widetilde{\mathcal{M}^p_q}$.

References

- A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), no. 2, 113–190.
- N. Kalton and M. Mitrea, Stability results on interpolation scales of quasi-Banach spaces and applications, Transactions of the American Mathematical Society 350.10, 1998, 3903–3922.
- D.I. Hakim and Y. Sawano, Calderón's First and Second Complex Interpolations of Closed Subspaces of Morrey Spaces, JFAA online.
- P. G. Lemarié-Rieusset, Erratum to: Multipliers and Morrey spaces, Potential Anal. **41**, 2014, 1359–1362.
- Y. Lu, D. Yang, and W. Yuan, Interpolation of Morrey Spaces on Metric Measure Spaces, Canad. Math. Bull. **57**, 2014, 598–608.
- W. Yuan, W. Sickel and D. Yang, Interpolation of Morrey-Campanato and Related Smoothness Spaces, Sci. China Math. **58**, no. 9, 2015, 1835–1908.

Acknowledgement

Thank you very much for your attention.