Ill-posedness results for the 1d Dirac-Klein-Gordon system

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Dirac-Klein-Gordon system

The Cauchy problem for Dirac-Klein-Gordon system (DKG):

$$\text{(DKG)} \quad \begin{cases} i\gamma^\mu D_\mu \psi = m\psi + \phi\psi, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ (\partial_t^2 - \partial_x^2 + M^2)\phi = \psi^* \gamma^0 \psi, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ \psi(0,x) = \psi_0(x), \quad \phi(0,x) = \phi_0(x),\ \partial_t \phi(0,x) = \phi_1(x), \quad x \in \mathbb{R}. \end{cases}$$

Solution (Unknown functions)

$$\psi = \psi(t,x) = \begin{pmatrix} \psi_1(t,x) \\ \psi_2(t,x) \end{pmatrix} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2,$$

$$\phi = \phi(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Initial data (Given funtions)

$$\psi_0(x) = \begin{pmatrix} \psi_{01}(x) \\ \psi_{02}(x) \end{pmatrix} : \mathbb{R} \to \mathbb{C}^2,$$

$$\phi_0(x),\ \phi_1(x) : \mathbb{R} \to \mathbb{R}.$$

$$
\text{(DKG)} \quad \begin{cases} i\gamma^\mu D_\mu \psi = m\psi + \phi\psi, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ (\partial_t^2 - \partial_x^2 + M^2)\phi = \psi^* \gamma^0 \psi, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ \psi(0,x) = \psi_0(x), \quad \phi(0,x) = \phi_0(x),\ \partial_t \phi(0,x) = \phi_1(x), \quad x \in \mathbb{R}. \end{cases}
$$

where

$$
\begin{aligned}
&m,\ M \geq 0 : \ \text{constants} \\
&\gamma^\mu D_\mu := \gamma^0 \partial_t + \gamma^1 \partial_x, \\
&\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

Yukawa type interaction

$$
\phi\psi = \begin{pmatrix} \phi\psi_1 \\ \phi\psi_2 \end{pmatrix},
$$

$$
\psi^* \gamma^0 \psi = (\overline{\psi}_1, \overline{\psi}_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = |\psi_1|^2 - |\psi_2|^2.
$$

Rewrite DKG

Set $u_\pm := \psi_1 \mp \psi_2$ for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Rewrite DKG as

$$\begin{cases} (\partial_t + \partial_x) u_+ = i(m - \phi) u_-, \\ (\partial_t - \partial_x) u_- = i(m - \phi) u_+, \\ (\partial_t^2 - \partial_x^2)\phi = -M^2 \phi + 2\Re(u_+ \overline{u_-}), \\ u_\pm(0,x) = u_{\pm,0}(x),\ \phi(0,x) = \phi_0(x),\ \partial_t \phi(0,x) = \phi_1(x). \end{cases}$$

Solutions

$$(\psi, \phi) \longleftrightarrow (u_\pm, \phi).$$

Well-posedness for Cauchy problem

Cauchy problem is called well-posed if the three conditions hold

- Solution exists for $t > 0$ in the same space as the initial data
- Solution is unique in the space (and additional spaces)
- Solution is continuous with respect to the initial data

$$\|\psi_0^{(n)} - \psi_0\|_H \to 0 \quad \Longrightarrow \quad \|\psi^{(n)} - \psi\|_X \to 0.$$

If one (or more) condition fails, the problem is called ill-posed.

Time local well-posedness (TLW) means well-posedness on the interval $[0, T]$ for some $T > 0$. Time global well-posedness (TGW) means well - posedness on the infinite interval $[0, \infty)$.

Fourier transform and Sobolev space

Fourier transform:

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^2} e^{-ix\xi - it\tau} u(t,x)dtdx.$$

Sobolev space norm: $s \in \mathbb{R}$,

$$\|f\|_{H^s} = \|\langle\xi\rangle^s \widehat{f}\|_{L^2}$$

Other norms: $a, b, \alpha \in \mathbb{R}$,

$$\|u\|_{Z^{a,b}} = \|\langle\tau+\xi\rangle^a \langle\tau-\xi\rangle^b \tilde{u}\|_{L^2_\tau L^2_\xi}$$

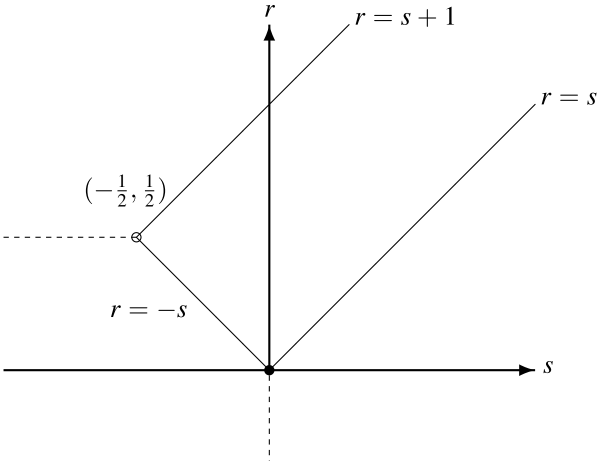
$$\|u\|_{Y^{\alpha,a,b}} = \|\langle\xi\rangle^\alpha \langle\tau+\xi\rangle^a \langle\tau-\xi\rangle^b \tilde{u}\|_{L^2_\xi L^1_\tau}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$.

Well-posed results in Sobolev spaces

$$(u_{\pm,0}, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1} \quad \Longrightarrow \quad (u_\pm, \phi, \partial_t \phi) \in C(I : H^s \times H^r \times H^{r-1}).$$

- M., Nakanishi and Tsugawa (2010),
  TLW $s > -\frac{1}{2},\ |s| \le r \le 1 + s$. TGW $s \ge 0$.



Earlirer works

$(u_\pm, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$.

- Chadam (1973), Chadam and Glassey (1974), TGW $s = r = 1$.
- Bournaveas (2000), TGW $s = 0, r = 1$.
- Bournaveas and Gibbeson (2006), TGW $s = 0,\ \frac{1}{4} \le r \le 1$.
- Fang (2004, 2008), TLW $-\frac{1}{4} < s \le 0,\ \frac{1}{2} < r \le 1 + 2s$, TGW $s = 0$.
- Pecher (2006), TLW $s > -\frac{1}{4}, r > 0, |s| \le r \le 1 + s, r < 1 + 2s$, TGW $s = 0$.
- M. (2007), TLW $s > -\frac{1}{4}, r > 0, 2|s| \le r \le 1 + 2s, r \le 1 + s$. TGW $s = 0$.
- Selberg and Tesfahun (2008, 2010), TLW $s > -\frac{1}{4}, r > 0, |s| \le r \le 1 + s$.

More works for TGW

The charge conservation law (L^2 conservation):

$$\|u_+(t)\|_{L^2}^2 + \|u_-(t)\|_{L^2}^2 = \|u_+(0)\|_{L^2}^2 + \|u_-(0)\|_{L^2}^2 \qquad {}^\forall t > 0.$$

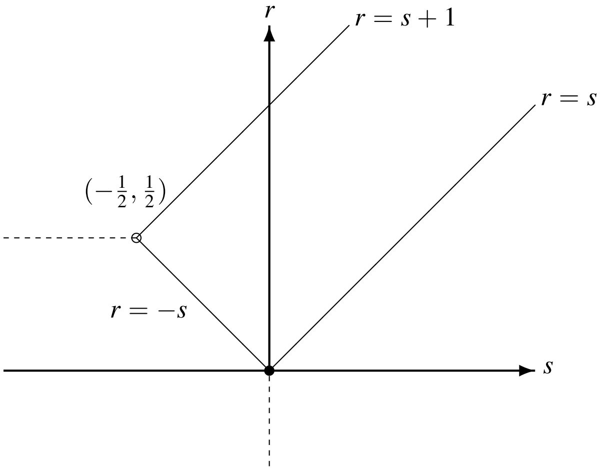
TGW results below L^2:

- Selberg (2007) $-\frac{1}{8} < s < 0,\ -s + \sqrt{s^2 - s} < r \leq 1 + s$.
- Tesfahun (2009) $-\frac{1}{8} < s < 0,\ s + \sqrt{s^2 - s} < r \leq 1 + s$.
- Candy (2013) $-\frac{1}{6} < s < 0,\ s - \frac{1}{4} + \sqrt{(s - \frac{1}{4})^2 - s} < r \leq 1 + s$.

Why the rectangle is relevant?

$$(u_\pm, \phi) \in H^s \times H^r, \quad s > -\frac{1}{2},\ |s| \le r \le 1 + s.$$

$$(\partial_t \pm \partial_x) u_\pm = imu_\mp - i\phi u_\mp, \qquad (\partial_t^2 - \partial_x^2)\phi = -M^2\phi + \Re(u_+ \overline{u_-}).$$



Proof of well-posedness

Integral equation system corresponds to (rewritten) DKG:

$$u_+ = u_+^F + I_+((m-\phi)u_-),$$
$$u_- = u_-^F + I_-((m-\phi)u_+),$$
$$\phi = \phi^F + I_+I_-(M^2\phi - 2\Re(u_+u_-)),$$

where F denotes the free solutions

$$u_\pm^F(t,x) = u_{\pm,0}(x \mp t),$$
$$\phi^F(t,x) = \frac{\phi_0(x+t)+\phi_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}\phi_1(y)dy,$$

and $I_\pm$ stands for Duhamel term

$$I_\pm(v) = -i\int_0^t v(\tau, x \mp (t-\tau))d\tau.$$

Contraction mapping principle

$$\begin{aligned}\|\chi_T(t)u_+\|_{Z^{b,s}\cap Y^{s,0,0}} &\lesssim \|u_{+,0}\|_{H^s} + \|u_-\|_{Z^{s,b}\cap Y^{s,-1,0}} + \|\phi\|_{Z^{r,r}}\|u_-\|_{Z^{s,b}},\\ \|\chi_T(t)u_-\|_{Z^{s,b}\cap Y^{s,0,0}} &\lesssim \|u_{-,0}\|_{H^s} + \|u_+\|_{Z^{b,s}\cap Y^{s,0,-1}} + \|\phi\|_{Z^{r,r}}\|u_+\|_{Z^{b,s}},\\ \|\chi_T(t)\phi\|_{Z^{r,r}\cap Y^{r,0,0}} &\lesssim \|\phi_0\|_{H^r} + \|\phi_1\|_{H^{r-1}} + \|\phi\|_{Z^{r-1,r-1}\cap Y^{r-1,0,0}}\\ &\qquad + \|u_+\|_{Z^{b,s}}\|u_-\|_{Z^{s,b}}\end{aligned}$$

implies well-posedness. Moreover, fixed $R>0$, for any data $u_{+,0}, u_{-,0}$, ϕ_0, ϕ_1 satisfying

$$\|u_{+,0}\|_{H^s},\ \|u_{+,0}\|_{H^s},\ \|\phi_0\|_{H^r} + \|\phi_1\|_{H^{r-1}} \le R,$$

the solution satisfies

$$\|u_+\|_{Z^{b,s}\cap Y^{s,0,0}},\ \|u_-\|_{Z^{s,b}\cap Y^{s,0,0}},\ \|\phi\|_{Z^{r,r}\cap Y^{r,0,0}} \le CR.$$

Remark the embedding

$$Y^{\alpha,0,0} \hookrightarrow L_t^\infty H_x^\alpha.$$

Ill-posed results

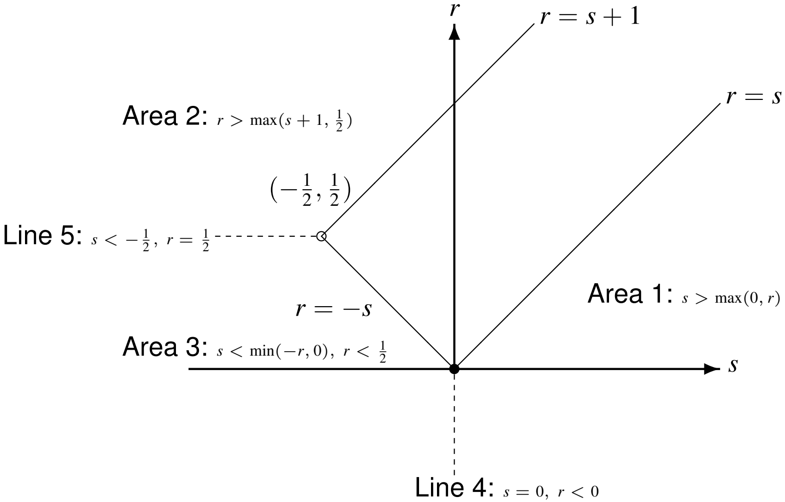
The Cauchy problem for DKG $(\psi, \phi, \partial_t \phi) \in H^s \times H^r \times H^{r-1}$ is Ill-posed if

1. $s > \max(0, r)$,
2. $r > \max(s + 1, \frac{1}{2})$,
3. $s < \min(-r, 0),\ r < \frac{1}{2}$,
4. $s = 0,\ r < 0$,
5. $s < -\frac{1}{2},\ r = \frac{1}{2}$.

1, 2 were shown in 2010 by M., Nakanishi and Tsugawa. 3 was shown in 2014 and 4, 5 were shown in 2015 by M. and Okamoto.

Areas for Ill-posedness

$(u_\pm, \phi) \in H^s \times H^r$:



Ill-posedness in the Area 1

In the case $0 \leq r < s$,

$$(u_{\pm,0}, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1} \hookrightarrow H^r \times H^r \times H^{r-1}$$

Well-posedness in $(u_\pm, \phi, \phi_t) \in H^r \times H^r \times H^{r-1}$.
Split into the terms

$$\begin{aligned} Z^{b,r} \ni u_+ &= u_+^F + u_+^D := u_+^F + I_+(\phi u_-), \\ Z^{r,b} \ni u_- &= u_-^F + u_-^D := u_-^F + I_-(\phi u_+), \\ Z^{r,r} \ni \quad \phi &= \phi^F + \phi^D := \phi^F + I_+ I_-(u_+ u_-), \end{aligned}$$

where $b > \frac{1}{2}$ and moreover

$$\phi^D \in Z^{r+1,r+1}, \quad I_+(\phi^D u_-) \in Z^{r+1,b}, \quad I_-(\phi^D u_+) \in Z^{b,r+1}.$$

u_+ instantaneously exits the space H^s

Observe that $u_+ \notin H^s, t > 0$ although $u_{+,0} \in H^s$.

$$\begin{aligned} u_+ &= u_+^F + u_+^D \\ &= u_+^F + I_+(\phi u_-) \\ &= u_+^F + I_+(\phi^F u_-) + I_+(\phi^D u_-) \end{aligned}$$

where the first term and the third term have enough regularity

$$u_+^F \in H^s, \qquad I_+(\phi^D u_-) \in Z^{r+1,b} \hookrightarrow L_t^\infty H_x^{r+1}.$$

Therefore $\quad I_+(\phi^F u_-) \notin H^s \quad$ implies $\quad u_+ \notin H^s$.

With some initial data ϕ_0, ϕ_1, the free part of the solution takes the form $\phi^F(t,x) = \varphi(x-t)$ where $\varphi \in H^r$. So the integrabilities hold

$$u_+ \in Z^{b,r} \hookrightarrow L^p_{x-t} L^\infty_{x+t},$$
$$\phi = \phi^F + \phi^D \in Z^{2,r} + Z^{r+1,r+1} \hookrightarrow L^p_{x-t} L^\infty_{x+t} + L^\infty_{t,x}$$

where $\frac{1}{p} = \frac{1}{2} - r$, and then

$$\phi u_+ = \phi^F u_+ + \phi^D u_+ \in L^{\frac{p}{2}}_{x-t} L^\infty_{x+t} + L^p_{x-t} L^\infty_{x+t}.$$

Therefore

$$|u^D_-| = |I_-(\phi u_+)| \lesssim |t|^{1-\frac{2}{p}}.$$

$$u_- = u_-^F + u_-^D, \qquad |u_-^D| \lesssim |t|^{1-\frac{2}{p}}.$$

Choose $u_{-,0}$ smooth and $u_{-,0}(x) = 1$ for $|x| < 1$ to have

$$\Re(u_-(t,x)) > \frac{1}{2} \qquad \text{if} \quad |t| + |x| < 1, \ |t| \ll 1.$$

From $\phi^F(t,x) = \varphi(x-t)$,

$$I_+(\phi^F u_-) = \varphi(x-t) I_+(u_-)$$

where $I_+(u_-)$ is smooth and bounded from below

$$I_+(u_-) \in Z^{r+1,b} \hookrightarrow L_t^\infty H_x^b, \qquad |I_+(u_-)| > \frac{t}{2}$$

to conclude

$$I_+(\phi^F u_-) \notin H^s \qquad \text{if} \quad (H^r \ni) \ \varphi \notin H^s.$$

In the case $r < 0 < s$.

For $0 < r' < s$, the initial data belongs to

$$(u_{\pm,0}, \phi_0, \phi_1) \in H^s \times H^{r'} \times H^{r'-1} \hookrightarrow H^s \times H^r \times H^{r-1},$$

but the solution exits instantaneously

$$u_+(t, \cdot) \notin H^s \qquad \text{for} \quad t > 0.$$

The special condition for the initial data

Chadam and Glassey observed

$$\int_{\mathbb{R}} |\psi_1(t,x) - \overline{\psi}_2(t,x)|^2 dx = \int_{\mathbb{R}} |\psi_{01}(x) - \overline{\psi}_{02}(x)|^2 dx, \quad t > 0.$$

The condition $\psi_{01} = \overline{\psi}_{02}$ gives

$$\psi_1 = \overline{\psi}_2 \quad \text{and so} \quad \psi^* \gamma^0 \psi = |\psi_1|^2 - |\psi_2|^2 = 0,$$

this means, the condition $\Re u_{+,0} = \Im u_{-,0} = 0$ gives

$$\Re u_+ = \Im u_- = 0 \quad \text{and so} \quad \Re(u_+ \overline{u}_-) = \Re u_+ \Re u_- + \Im u_+ \Im u_- = 0.$$

Ill-posedness in Area 3

Follow the argument by Bejenaru-Tao and Kishimoto-Tsugawa.
Consider Area 3: $s+r<0, s<0, r<\frac{1}{2}$, massless case $m=M=0$,

$$(\partial_t \pm \partial_x)u_\pm = -i\phi u_\mp, \quad (\partial_t^2 - \partial_x^2)\phi = \Re(u_+\overline{u_-}),$$
$$u_\pm(0,x) = u_{\pm,0}(x), \quad \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x).$$

The first and the second iteration terms are given by

$$u_\pm^{(1)}(t,x) := u_{\pm,0}(x \mp t),$$
$$\phi^{(1)}(t,x) := \frac{\phi_0(x+t)+\phi_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y)dy,$$
$$u_\pm^{(2)}(t,x) := -i\int_0^t (\phi^{(1)}u_\mp^{(1)})(t', x \mp (t-t'))dt'.$$

The condition $\Re u_{+,0} = \Im u_{-,0} = 0$ gives $\phi = \phi^{(1)}$.

With $s < s_0 < -r$ and $\delta > 0$, take the initial data

$$\widehat{u}_{+,0} = \delta N^{-s_0} \chi_{[N-1,N+1]}, \quad u_{-,0} = 0,$$
$$\widehat{\phi}_0 = \delta N^{s_0} \chi_{[-N-1,-N+1]}, \quad \phi_1 = 0.$$

Then

$$\|u_{+,0}\|_{H^s} \sim \delta N^{s-s_0} \to 0, \quad \|\phi_0\|_{H^r} \sim \delta N^{r+s_0} \to 0.$$

Estimate on

$$\mathcal{F}u_-^{(2)}(t,\xi) = -\frac{i}{2} e^{it\xi} \Big(\int \widehat{\phi}_0(\xi - \eta) \frac{e^{-2it\eta} - 1}{-2i\eta} \widehat{u_{+,0}}(\eta) d\eta + \frac{e^{-2it\xi} - 1}{-2i\xi} \int \widehat{\phi}_0(\xi - \eta) \widehat{u_{+,0}}(\eta) d\eta \Big)$$

to obtain

$$\|u_-^{(2)}(t,\cdot)\|_{H^s} \gtrsim t\delta^2 \|\langle\cdot\rangle^s \chi_{[-1,1]}\|_{L^2} \sim t\delta^2$$

which immediately means that the solution map is not C^2.

Consider the function $v_\pm := u_\pm - u_\pm^{(1)} - u_\pm^{(2)}$, that is

$$v_\pm(t,x) = -i\int_0^t (\phi^{(1)} v_\mp + \phi^{(1)} u_\mp^{(2)})(t', x \mp (t-t'))dt'.$$

Apply the proof of well-posedness with $s_0 + (-s_0) = 0$ for the initial data above

$$u_{-,0} = \phi_1 = 0, \quad \|u_{+,0}\|_{H^{s_0}} \sim \|\phi_0\|_{H^{-s_0}} \sim \delta$$

to obtain

$$\|v_\pm\|_{L_T^\infty H^{s_0}} \lesssim t\delta^3.$$

Since $u_-^{(1)} = 0$ and so $v_- = u_- - u_-^{(2)}$,

$$\begin{aligned}\|u_-\|_{L_T^\infty H^s} &\geq \|u_-^{(2)}\|_{L_T^\infty H^s} - \|v_-\|_{L_T^\infty H^s} \\ &\geq \|u_-^{(2)}\|_{L_T^\infty H^s} - \|v_-\|_{L_T^\infty H^{s_0}} \gtrsim t\delta^2 - t\delta^3 \sim t\delta^2 > 0\end{aligned}$$

which establishes the discontinuity of the solution map.

Review the argument

Area 3: $s + r < 0$.
The sequence of initial data with $s < s_0 < -r$

$$\widehat{u}_{+,0} = \delta N^{-s_0} \chi_{[N-1,N+1]}, \quad \widehat{\phi}_0 = \delta N^{s_0} \chi_{[-N-1,-N+1]}$$

for $N = 1, 2, 3, \ldots$, gives

$$|\mathcal{F} u_-^{(2)}(t,\xi)| \gtrsim t\delta^2 \quad \text{for} \quad |\xi| < 1.$$

This is an interaction of the type " Hign $\times$ High $\to$ Low ". This allows

$$R \geq \|u_{+,0}\|_{H^{s_0}} \geq \|u_{+,0}\|_{H^s} \to 0,$$
$$R \geq \|\phi_0\|_{H^{-s_0}} \geq \|\phi_0\|_{H^r} \to 0,$$
$$CR \geq \|u_-\|_{L_T^\infty H^{s_0}} \geq \|u_-\|_{L_T^\infty H^s} > c > 0.$$

Thank you for your attention.