Weighted maximal operators and related topics.

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Introduction

Introduction: Lebesgue differentiation theorem

Theorem

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy = f(x) \quad \textit{a.e. } x \in \mathbb{R}^n.$$

- When $n = 1$, $f \in C(\mathbb{R})$,
  $$\lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(y)\,dy = \lim_{r \to 0} \frac{F(x+r) - F(x-r)}{2r} = f(x).$$
  The Fundamental Theorem of Calculus.

Introduction: Hardy-Littlewood maximal function

It is often proved by using the following Hardy-Littlewood maximal operator:

- $B_r = B(0, r)$: n-dim ball centered at origin, radius r.
- $Mf(x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|\, dy$
  Hardy-Littlewood maximal operator

Theorem

(1) $|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, dx$

(2) $\int_{\mathbb{R}^n} Mf(x)^p\, dx \leq C_{n,p} \int_{\mathbb{R}^n} |f(x)|^p\, dx \qquad (1 < p \leq \infty)$

In this talk, it is called "maximal theorem".

Covering Lemma

"Covering lemma" plays important role to prove the maximal theorems:

Theorem

$\forall\{B(x_j,r_j)\}_{j=1}^N$ *balls,* $\exists\{B(x_k,r_k)\}_{k=1}^M \subset \{B(x_j,r_j)\}_{j=1}^N$ *such that*

(a) $\{B(x_k,r_k)\}_{k=1}^M$*: mutually disjoint.*

(b) $\left|\bigcup_{j=1}^N B(x_j,r_j)\right| \leq 3^n \sum_{k=1}^M |B(x_k,r_k)|$

Proof of the maximal theorem

$$K \subset \{x \in \mathbb{R}^n : Mf(x) > \lambda\} : \text{ compact}$$

There exists $\{B(x_j, r_j)\}_{j=1}^N$ such that

$$K \subset \bigcup_{j=1}^N B(x_j, r_j), \quad \frac{1}{|B(x_j, r_j)|} \int_{B(x_j, r_j)} |f(y)|\, dy > \lambda.$$

$$|K| \leq \left| \bigcup_{j=1}^N B(x_j, r_j) \right| \leq 3^n \sum_{k=1}^M |B(x_k, r_k)|$$

$$\leq \frac{3^n}{\lambda} \sum_{k=1}^M \int_{B(x_j, r_j)} |f(y)|\, dy \leq \frac{3^n}{\lambda} \|f\|_{L^1}$$

Proof of LDT

Proposition

For $f \in L^p(\mathbb{R}^n)$, $(1 \leq p < \infty)$. *Let*

$$f_t(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy, \quad (t > 0).$$

Then

$$\lim_{t \to 0} \|f - f_t\|_{L^p(\mathbb{R}^n)} = 0.$$

We see that there exists $t_k \to 0$ such that

$$\lim_{k \to \infty} f_{t_k}(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

It suffices to show that $\exists \lim_{t \to 0} f_t(x)$.

Proof of LDT

For $f \in L^1(\mathbb{R}^n)$, we define its oscillation:

$$\Omega f(x) := \limsup_{t \to 0} f_t(x) - \liminf_{t \to 0} f_t(x)$$

- We want to see $|\{x \in \mathbb{R}^n : \Omega f(x) > \lambda\}| = 0$ for all $\lambda > 0$.
- $g \in C_0(\mathbb{R}^n) \implies \Omega g(x) \equiv 0$.
- $\Omega g(x) \leq 2Mg(x)$.

Proof of LDT

For $f \in L^1(\mathbb{R}^n)$, there exists $g \in C_0(\mathbb{R}^n)$ such that

$$\|f - g\|_{L^1} < \varepsilon.$$

Let $h := f - g$. Then

$$\Omega f(x) \leq \Omega g(x) + \Omega h(x) = \Omega h(x).$$

By the maximal theorem,

$$|\{\Omega f > \lambda\}| \leq |\{\Omega h > \lambda\}| \leq |\{2Mh > \lambda\}| \leq \frac{C}{\lambda}\|h\|_{L^1}.$$

It is not difficult to consider the cubes instead of balls.
Next, we will try to consider the same problem for rectangles.

Why not rectangles?

Can we get the following theorem?

Theorem (??)

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\lim_{\varepsilon \to 0} \frac{1}{|R_\varepsilon|} \int_{R_\varepsilon} f(y)\, dy = f(x) \quad \textit{a.e. } x \in \mathbb{R}^n.$$

where R_ε *is any rectangle of length and width less than* ε.

Why not rectangles?

Can we get the following theorem?

Theorem (??)

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\lim_{\varepsilon \to 0} \frac{1}{|R_\varepsilon|} \int_{R_\varepsilon} f(y)\, dy = f(x) \quad \textit{a.e. } x \in \mathbb{R}^n.$$

where R_ε *is any rectangle of length and width less than* ε.

- The answer is negative.
- If we had demanded that R have bounded eccentricity the answer would be positive.

Maximal operators

The strong maximal operator

- $\mathcal{R}$ is the set of all rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.
- The strong maximal operator:

$$M_{\mathcal{R}} f(x) := \sup_{R \in \mathcal{R}} \frac{1}{|R|} \int_R |f(y)| \, dy \, \mathbf{1}_R(x).$$

Problem.

Does the analogue hold for $M_{\mathcal{R}}$?

(1) $|\{x \in \mathbb{R}^n : M_{\mathcal{R}} f(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, dx$

(2) $\int_{\mathbb{R}^n} M_{\mathcal{R}} f(x)^p \, dx \leq C_{n,p} \int_{\mathbb{R}^n} |f(x)|^p \, dx \qquad (1 < p \leq \infty)$

Elementary properties

- $\|M_{\mathcal{R}}f\|_\infty \leq \|f\|_\infty$.

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Proof.

$$\begin{aligned} &\frac{1}{|R|}\int_R |f(y)|\,dy, \quad R \in \mathcal{R} \\ \leq &\|f\|_\infty \cdot \frac{1}{|R|}\int_R 1\,dy = \|f\|_\infty, \end{aligned}$$

we have

$$\|M_{\mathcal{R}} f\|_\infty \leq \|f\|_\infty.$$

Elementary properties

- $\|M_{\mathcal{R}}f\|_\infty \leq \|f\|_\infty$.
- $M_{\mathcal{R}}f(x) \leq M_1 M_2 \cdots M_n f(x)$,
  where M_j is the 1-dim. HL maximal operator.

Elementary properties

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- $M_{\mathcal{R}} f(x) \le M_1 M_2 \cdots M_n f(x)$,
  where M_j is the 1-dim. HL maximal operator.

Proof. Let $n = 2$ and $R = I_1 \times I_2$. Then

$$
\begin{aligned}
&\frac{1}{|R|} \int_R |f(y)|\, dy \mathbf{1}_R(x) \\
=&\frac{1}{|I_1| \cdot |I_2|} \int_{I_1 \times I_2} |f(y_1, y_2)|\, dy \mathbf{1}_{I_1}(x_1) \mathbf{1}_{I_2}(x_2) \\
=&\frac{1}{|I_1|} \int_{I_1} \left(\frac{1}{|I_2|} \int_{I_2} |f(y_1, y_2)|\, dy_2 \cdot \mathbf{1}_{I_2}(x_2) \right) dy_1 \cdot \mathbf{1}_{I_1}(x_1) \\
\le&\frac{1}{|I_1|} \int_{I_1} M_2 f(y_1, x_2)\, dy_1 \cdot \mathbf{1}_{I_1}(x_1) \le M_1(M_2 f)(x_1, x_2)
\end{aligned}
$$

Elementary properties

- $\|M_{\mathcal{R}}f\|_\infty \leq \|f\|_\infty$.
- $M_{\mathcal{R}}f(x) \leq M_1 M_2 \cdots M_n f(x)$, where M_j is the 1-dim. HL maximal function.
- For $1 < p < \infty$, the boundedness of $M_{\mathcal{R}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ follows from that of M_j.

Elementary properties

- $\|M_{\mathcal{R}} f\|_\infty \le \|f\|_\infty$.
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Proof. By using the boundedness of HL: $\|M_j f\|_p \le C\|f\|_p$ repeatedly

$$\|M_1(M_2 f)\|_p \le C\|M_2 f\|_p \le C'\|f\|_p.$$

We obtain the L^p boundedness of $M_{\mathcal{R}}$ for $1 < p < \infty$:

$$\|M_{\mathcal{R}} f\|_{L^p} \le C(p,n)\|f\|_{L^p}.$$

Elementary properties

- $\|M_{\mathcal{R}}f\|_\infty \le \|f\|_\infty$.
- $M_{\mathcal{R}}f(x) \le M_1 M_2 \cdots M_n f(x)$,
  where M_j is the 1-dim. HL maximal function.
- For $1 < p < \infty$, the boundedness of $M_{\mathcal{R}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ follows from that of M_j.
- This iteration argument fails when $p = 1$!
  Because we know only the weak type inequality

$$\|M_1 f\|_{L^{1,\infty}} \le C\|f\|_{L^1},$$

  and M_2 need not preserve L^1, we cannot deduce

$$\|M_1(M_2 f)\|_{L^{1,\infty}} \le C\|M_2 f\|_{L^1}.$$

- It is known $M_j : L \log L \to L^1$.

Geometrical proof of $M_{\mathcal{R}} : L^p \to L^p$, $(1 < p < \infty)$

To get the $L \log L$ type inequality, we try to consider the geometrical proof. To do this, we introduce the covering lemma corresponding to the strong maximal operator.

Theorem

$p > 1$. $\forall \{R_j\}_{j \in J} \subset \mathcal{R}$, $\exists \{\widetilde{R}_k\} \subset \{R_j\}_{j \in J}$ *such that*

(a) $$\left| \bigcup_{j \in J} R_j \right| \le c(n) \left| \bigcup_k \widetilde{R}_k \right|$$

(b) $$\left\| \sum_k \mathbf{1}_{\widetilde{R}_k} \right\|_{L^p(\mathbb{R}^n)}^p \le C(n) \left| \bigcup_{j \in J} R_j \right|$$

Geometrical proof of $M_{\mathcal{R}} : L^p \to L^p$, $(1 < p < \infty)$

$$E_\lambda := \{x \in \mathbb{R}^n : M_{\mathcal{R}} f(x) > \lambda\}$$

$$\Rightarrow \quad \exists \{R_j\} : E_\lambda = \bigcup_j R_j, \quad \frac{1}{|R_j|} \int_{R_j} |f(y)|\, dy > \lambda.$$

$$\begin{aligned} |E_\lambda| &= \left| \bigcup_j R_j \right| \le c \left| \bigcup_k \widetilde{R}_k \right| \le c \sum_k |\widetilde{R}_k| \\ &\le \frac{c}{\lambda} \sum_k \int_{\widetilde{R}_k} |f(y)|\, dy = \frac{c}{\lambda} \int_{\mathbb{R}^n} \left(\sum_k \mathbf{1}_{\widetilde{R}_k} \right) |f(y)|\, dy \\ &\le \frac{c}{\lambda} \left\| \sum_k \mathbf{1}_{\widetilde{R}_k} \right\|_q \|f\|_p \le \frac{C(n)}{\lambda} \left| \bigcup_j R_j \right|^{\frac{1}{q}} \|f\|_p \end{aligned}$$

Geometrical proof of $M_{\mathcal{R}} : L^p \to L^p$, $(1 < p < \infty)$

$$E_\lambda := \{x \in \mathbb{R}^n : M_{\mathcal{R}} f(x) > \lambda\}$$

$$|E_\lambda| \leq \frac{C}{\lambda} \left| \bigcup_j R_j \right|^{\frac{1}{q}} \|f\|_p = \frac{C}{\lambda} |E_\lambda|^{\frac{1}{q}} \|f\|_p$$

Thus, we get the weak type inequality

$$|\{x \in \mathbb{R}^n : M_{\mathcal{R}} f(x) > \lambda\}|^{\frac{1}{p}} \leq \frac{C}{\lambda} \|f\|_{L^p}$$

Remark. For $p = 1$, we have an another covering lemma due to Cordoba and R. Fefferman.

Kakeya maximal operator

- $N \gg 1$, $a > 0$.
- $\mathcal{B}_{N,a}$: is all rectangles whose size $a \times a \times \cdots \times aN$ in $\mathbb{R}^n$.
- $\mathcal{B}_N = \bigcup_{a>0} \mathcal{B}_{N,a}$
- Kakeya(Nikodým) maximal operators

$$K_N^a f(x) = \sup_{R \in \mathcal{B}_{N,a}} \frac{1}{|R|} \int_R |f(y)|\, dy\, \mathbf{1}_R(x)$$

$$K_N f(x) = \sup_{R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)|\, dy\, \mathbf{1}_R(x)$$

- $K_N f(x) \leq C N^{n-1} M f(x)$ $\quad M$: H-L maximal operator
- $\|K_N f\|_{L^p(\mathbb{R}^n)} \leq C N^{n-1} \|f\|_{L^p(\mathbb{R}^n)}$.

But it is conjectured more sharp estimate...

Kakeya maximal operator

Conjecture

$$\|K_N f\|_{L^n(\mathbb{R}^n)} \le C(\log N)^{\alpha_n} \|f\|_{L^n(\mathbb{R}^n)}.$$

- **Córdoba (1977)** proved for $n = 2$, $\alpha_2 = 2$.
- **Strömberg (1978)** reproved for $n = 2$ $\alpha_2 = 1$.
- **Open!** $n > 2$.
- There have been partial results (explained Monday and Tuesday!).
- Product functions and radial functions (**Igari, Tanaka, etc**)

Related topics

Theorem

(1) *Bochner-Riesz implies Kakeya conjecture (T.Tao, 1999).*

(2) *Restriction conjecture implies Kakeya (J. Bourgain, 1991).*

(3) L^p*-estimate for Kakeya maximal implies Kakeya conjecture (J. Bourgain, 1991).*

Directional maximal operator

We consider the directional maximal operator in $n = 2$.

- S^1: a unit circle on the plane. Let $\Omega \subset S^1$
- $\mathcal{B}_\Omega$: rectangles $R \subset \mathbb{R}^2$ with longer side forming an angle θ with the x-axis, for some $\theta \in \Omega$.
- Directional maximal operator

$$M_\Omega f(x) := \sup_{R \in \mathcal{B}_\Omega} \frac{1}{|R|} \int_R |f(y)|\, dy\, \mathbf{1}_R(x)$$

- If Ω is equidistributed and $|\Omega| = N$, then

$$K_N f(x) \leq C M_\Omega f(x).$$

Directional maximal operator

- **(Strömberg (1978))** Ω: equidistributed. $|\Omega| = N$,

$$(*) \quad \|M_\Omega f\|_{L^2(\mathbb{R}^2)} \leq C \log N \|f\|_{L^2(\mathbb{R}^2)}$$

- **(Katz(1999))** $(*)$ is true whenever $|\Omega| = N$.
- **(Alfonseca, Soria and Vargas(2003))** Reproved $(*)$ by using an almost orthogonality principle in $L^2(\mathbb{R}^2)$, when $|\Omega| = N$.

General maximal operator

- Basis: $\mathcal{B}$, collection of open and bounded sets $B \subset \mathbb{R}^n$.
- $M_{\mathcal{B}} f(x) = \sup_{B \in \mathcal{B}} \frac{1}{|B|} \int_B |f(y)|\, dy\, \mathbf{1}_B(x), \quad x \in \bigcup_{B \in \mathcal{B}} B$

- $\mathcal{B} = \mathcal{Q} \implies M_{\mathcal{B}} = M$. (Hardy-Littlewood maximal operator)
- $\mathcal{B} = \mathcal{R} \implies M_{\mathcal{B}} = M_{\mathcal{R}}$. (strong maximal operator)
- $\mathcal{B} = \mathcal{B}_N \implies M_{\mathcal{B}} = K_N$. (Kakeya maximal operator)
- $\mathcal{B} = \mathcal{B}_\Omega \implies M_{\mathcal{B}} = M_\Omega$. (directional maximal operator)

Weight theory

Weight theory

- w: weight $\Leftrightarrow$ non-negative locally integrable function on $\mathbb{R}^n$.
- $w(E) := \int_E w(x)\,dx$.
- $$\|f\|_{L^p(\mathbb{R}^n,w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

Weight theory

- w: weight $\Leftrightarrow$ non-negative locally integrable function on $\mathbb{R}^n$.
- $w(E) := \int_E w(x)\,dx$.
- $$\|f\|_{L^p(\mathbb{R}^n,w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$
- Weighted inequalities arise naturally in Fourier analysis.
- For example, the theory of weights plays an important role in the study of boundary value problems for Laplace's equation on Lipschitz domains.
- Other applications of weighted inequalities include extrapolation theory, vector-valued inequalities, and estimates for certain classes of nonlinear PDEs.

$A_{p,\mathcal{Q}}$ condition

We consider the condition for $M : L^p(w) \to L^p(w)$.

$w \in A_{p,\mathcal{Q}} \overset{\text{def}}{\Longleftrightarrow}$

- $[w]_{A_{p,\mathcal{Q}}} = \sup_{Q \in \mathcal{Q}} \frac{w(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q w^{\frac{-1}{p-1}}(x)\, dx \right)^{p-1} < \infty, \ 1 < p < \infty,$
- $[w]_{A_{1,\mathcal{Q}}} = \sup_{Q \in \mathcal{Q}} \frac{w(Q)}{|Q|} \cdot \frac{1}{\operatorname*{ess\,inf}_{x \in Q} w(x)} < \infty,$
- $A_{\infty,\mathcal{Q}} := \bigcup_{p>1} A_p.$

$w \in A_{p,\mathcal{Q}},\ 1 < p < \infty \Longleftrightarrow M : L^p(w) \to L^p(w).$

i.e.,

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx$$

Problem

Can we get the following estimates?

- $w(\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \lambda\}) \le \dfrac{C}{\lambda} \displaystyle\int_{\mathbb{R}^n} |f(x)| w(x)\, dx$
- $\displaystyle\int_{\mathbb{R}^n} M_{\mathcal{B}} f(x)^p w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx$

Why not,

- $M_{\mathcal{B},w} f(x) = \sup\limits_{B \in \mathcal{B}} \dfrac{1}{w(B)} \displaystyle\int_B |f(y)| w(y)\, dy\, \mathbf{1}_B(x)$

Can we get the following estimates?

- $\displaystyle\int_{\mathbb{R}^n} M_{\mathcal{B}} f(x)^p w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p M_{\mathcal{B}} w(x)\, dx$

This type inequality is called Fefferman-Stein type inequality.

Fefferman-Stein type inequality

- Given a suitable operator $\mathcal{T}$, $p > 1$.
- We consider the inequalities typically take the form

$$\int_{\mathbb{R}^n} |\mathcal{T} f(x)|^p w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p \mathcal{M}_{\mathcal{T}} w(x)\, dx. \qquad (1)$$

- $\mathcal{M}_{\mathcal{T}}$: some maximal operator. w: weight.

Determine some maximal operator $\mathcal{M}_{\mathcal{T}}$ capturing certain geometric characteristics of $\mathcal{T}$.

Duality argument

Under such circumstances a simple duality argument generally allows the previous inequality to transfer bounds on $\mathcal{M}$ to bounds on $\mathcal{T}$. For $q, \widetilde{q} \geq p$,

$$\begin{aligned}
\|\mathcal{T}f\|_{L^q(\mathbb{R}^n)} &= \sup_{\|w\|_{(q/p)'}=1} \left(\int_{\mathbb{R}^n} |\mathcal{T}f|^p w\right)^{1/p} \\
&\leq \sup_{\|w\|_{(q/p)'}=1} \left(\int_{\mathbb{R}^n} |f|^p \mathcal{M}_{\mathcal{T}} w\right)^{1/p} \\
&\leq \sup_{\|w\|_{(q/p)'}=1} \|\mathcal{M}_{\mathcal{T}} w\|_{(\widetilde{q}/p)'}^{1/p} \|f\|_{\widetilde{q}},
\end{aligned}$$

and so $\|\mathcal{T}\|_{\widetilde{q}\to q} \leq \|\mathcal{M}_{\mathcal{T}}\|_{(q/p)'\to(\widetilde{q}/p)'}^{1/p}$.

Example

We got many successful results for several important operators $\mathcal{T}$ such as

- maximal averaging operators,
- fractional integral operators,
- square functions,
- Calderón-Zygmund singular integral operators.

Córdoba and C. Fefferman (1976)

Let $p, r > 1$. If T: CZ, then

$$\int_{\mathbb{R}^n} |Tf|^p w \leq C_{p,r} \int_{\mathbb{R}^n} |f|^p M_r w,$$

where $M_r w := (Mw^r)^{1/r}$ and M is the Hardy-Littlewood maximal operator.

Results: Weighted maximal operators

Previous works (General maximal operator)

For general basis $\mathcal{B}$ and associated weights w, we do not know so much concerning the boundedness of $M_{\mathcal{B}}$ and $M_{\mathcal{B},w}$ on $L^p(w)$.

Theorem (Jawerth (1986))

$1 < p < \infty$, $\sigma = w^{1/(p-1)}$. *Then*

$$\begin{array}{l} M_{\mathcal{B}} : L^p(w) \to L^p(w) \\ M_{\mathcal{B}} : L^{p'}(\sigma) \to L^{p'}(\sigma) \end{array} \iff \begin{array}{l} w \in A_{p,\mathcal{B}} \\ M_{\mathcal{B},\sigma} : L^p(\sigma) \to L^p(\sigma) \\ M_{\mathcal{B},w} : L^{p'}(w) \to L^{p'}(w) \end{array}$$

This theorem was reproved by Lerner with better understanding of dependency of the constant.

- A. K. Lerner, *An elementary approach to several results on the Hardy-Littlewood maximal operator*, Proc. Amer. Math. Soc., **136** (2008), no. 8, 2829–2833.

Previous works (General maximal operator)

The following abstract theorem gives a necessary and sufficient condition for the boundedness of $M_{\mathcal{B},w}$ in terms of $M_{\mathcal{B}}$ in the special case $w \in A_{\infty,\mathcal{B}}$.

Theorem (Pérez (1989))

$$\begin{array}{c} 1 < p < \infty, w \in A_{p,\mathcal{B}} \\ M_{\mathcal{B}} : L^p(w) \to L^p(w) \end{array} \iff \begin{array}{c} 1 < p < \infty, w \in A_{\infty,\mathcal{B}} \\ M_{\mathcal{B},w} : L^p(\sigma) \to L^p(\sigma) \end{array}$$

For the Kakeya maximal conjecture, the main interest is to determine the factor N appearing in its operator norms. This theorem is quite abstract and thus cannot apply to the Kakeya maximal operator.

Weight classes: $\mathrm{RH}_{s,\mathcal{B}}$

Definition

$w \in \mathrm{RH}_{s,\mathcal{B}} \overset{def}{\Longleftrightarrow}$

$$[w]_{\mathrm{RH}_{s,\mathcal{B}}} = \sup_{B\in\mathcal{B}} \left(\frac{w(B)}{|B|}\right)^{-1} \left(\frac{1}{|B|}\int_B w(x)^s\,dx\right)^{1/s} < \infty, \ 1 < s < \infty,$$

$$[w]_{\mathrm{RH}_{\infty,\mathcal{B}}} = \sup_{B\in\mathcal{B}} \left(\frac{w(B)}{|B|}\right)^{-1} \cdot \operatorname*{ess\,sup}_{x\in B} w(x) < \infty.$$

- $1 < s < r < \infty$, $1 \leq [w]_{\mathrm{RH}_{s,\mathcal{B}}} \leq [w]_{\mathrm{RH}_{r,\mathcal{B}}} < \infty$.
- $1 < s < r < \infty$, $\mathrm{RH}_{s,\mathcal{B}} \supset \mathrm{RH}_{r,\mathcal{B}}$
- $\mathrm{RH}_{1,\mathcal{B}} := \bigcup_{s>1} \mathrm{RH}_{s,\mathcal{B}}$.

Weight classes: $\mathrm{RH}_{s,\mathcal{B}}$

Cruz-Uribe and Neugebauer investigated carefully the relationship between $A_{p,\mathcal{Q}}$ and $\mathrm{RH}_{s,\mathcal{Q}}$. Due to Cruz-Uribe and Neugebauer, we see some properties of $RH_{\infty,\mathcal{B}}$ weights.

- D. Cruz-Uribe and C. J. Neugebauer, *The structure of the Reverse Hölder classes*, Trans. Amer. Math. Soc., **347** (1995), no. 8, 2941–2960.

Theorem (A)

(1) $w \in \mathrm{RH}_{\infty,\mathcal{B}} \Longrightarrow w^r \in \mathrm{RH}_{\infty,\mathcal{B}}$ *for* $r \geq 1$.

(2) $w \in A_{\infty,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}} \Longrightarrow w^r \in \mathrm{RH}_{\infty,\mathcal{B}}$ *for* $0 < r < 1$.

Theorem (B)

$u, v \in A_{\infty,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}} \Longrightarrow uv \in \mathrm{RH}_{\infty,\mathcal{B}}$.

Weight classes: $\mathrm{RH}_{s,\mathcal{B}}$

Theorem (C)

Let $p > 1$*.*

(1) $w \in A_{1,\mathcal{B}} \Longrightarrow w^{1-p} \in A_{p,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}}$.

(2) $w \in A_{p,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}} \Longrightarrow w^{1-p'} \in A_{1,\mathcal{B}}$.

- Suppose that $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^n)$ for some $p > 1$.
- Then using the well-known Rubio de Francia algorithm we can obtain many $A_{1,\mathcal{B}}$ weights $\mathcal{R}u := \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k u}{2^k \|M_{\mathcal{B}}\|_{L^p}}$.
- Theorem (C) asserts that, if $w \in A_{1,\mathcal{B}}$ then $w^{-1} \in A_{\infty,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}}$.)

So, we have many weights belonging to $A_{\infty,\mathcal{B}} \cap \mathrm{RH}_{\infty,\mathcal{B}}$.

Previous results

- **S. and Tanaka (2013)** If $w(x) = |x|^\lambda, \lambda > 0$, then

$$\|M_{\Omega,w} f\|_{L^2(\mathbb{R}^2,w)} \leq C \log N \|f\|_{L^2(\mathbb{R}^2,w)}.$$

- **S. and Sawano (2015)** If w is radial $(w(x) = w_0(|x|))$ and $w_0 \in \mathrm{RH}_{\infty,\mathcal{B}}$, then

$$\|K^a_{N,w} f\|_{L^2(\mathbb{R}^2,w)} \leq C \sqrt{\log N} \|f\|_{L^2(\mathbb{R}^2,w)}$$

We introduce a theorem concerning general maximal operators which implies those two estimates as a corollary.

Weighted area formula

Lemma (S. and Tanaka (2013))

If w is radial and $w_0 \in \mathrm{RH}_\infty$. Let $R \subset \mathbb{R}^2$ be a rectangle. Then

$$w(R) \sim \frac{|R|}{|\operatorname{rad}(R)|} \int_{\operatorname{rad}(R)} w_0(r)\, dr.$$

Here,

$$\operatorname{rad}(R) := \left[\inf_{x \in R} |x|, \sup_{x \in R} |x| \right].$$

Remark. It is useful to see

$$\frac{w(R)}{|R|} \sim \frac{w_0(I)}{|I|}, \quad I := \operatorname{rad}(R).$$

Figure of Lemma

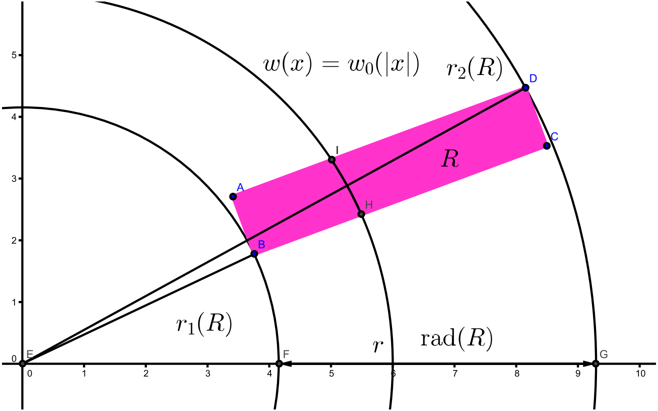

Figure: $r_1(R) = \inf_{x \in R} |x|,\ r_2(R) = \sup_{x \in R} |x|$

Results

Theorem (1)

- $1 < p, s \leq \infty$
- $w \in \mathrm{RH}_{s,\mathcal{B}}$ *with* $[w]_{\mathrm{RH}_{s,\mathcal{B}}} = C_1$
- $\|M_{\mathcal{B}}\|_{L^t(\mathbb{R}^n)} = C_2$ *with* $t = p - (p-1)/s > 1$
- $\exists C_3 > 0$ *s.t.*

$$\sum_j \left(\frac{w(B_j)}{|B_j|} \right)^{-1} \cdot w(E(B_j)) \leq C_3 \left| \bigcup_j B_j \right| \qquad (2)$$

for $\{B_j\} \subset \mathcal{B}$ *and p.w. disjoint* $E(B_j) \subset B_j$.

$\Longrightarrow$

$$\|M_{\mathcal{B},w}\|_{L^p(w)} \leq 2 C_1^{1/p'} C_2^{1/(p's)'} C_3^{1/p}.$$

Results

When $s = \infty$, i.e., $w \in \mathrm{RH}_{\infty,\mathcal{B}}$, the condition (2) can be checked easily. Let $[w]_{\mathrm{RH}_{\infty,\mathcal{B}}} = C_1$.

$$\sum_j \left(\frac{w(B_j)}{|B_j|}\right)^{-1} \cdot w(E(B_j)) \le \sum_j |E(B_j)| \left(\frac{w(B_j)}{|B_j|}\right)^{-1} \operatorname*{ess\,sup}_{x \in B_j} w(x)$$

$$\le [w]_{\mathrm{RH}_{\infty,\mathcal{B}}} \sum_j |E(B_j)| \le C_1 \left| \bigcup_j B_j \right|.$$

Corollary (2)

$1 < p \le \infty$. *Let* $w \in \mathrm{RH}_{\infty,\mathcal{B}}$ *with* $C_1 = [w]_{\mathrm{RH}_{\infty,\mathcal{B}}}$. *Let* $C_2 = \|M_{\mathcal{B}}\|_{L^p(\mathbb{R}^n)}$. *Then*

$$\|M_{\mathcal{B},w}\|_{L^p(w)} \le 2C_1C_2.$$

Results

Now we can obtain the estimate for the Kakeya maximal operators.

- $n = 2$. $\mathcal{B} = \mathcal{B}_N$ or $= \mathcal{B}_{N,a}$.
- $w(x) = w_0(|x|)$ and $w_0 \in \mathrm{RH}_\infty$.
- $$\frac{w(R)}{|R|} \sim \frac{w_0(I)}{|I|} \gtrsim \sup_{r \in I} w_0(r) = \sup_{x \in R} w(x), \quad I = \mathrm{rad}\,(R).$$
- So we find $w \in \mathrm{RH}_{\infty,\mathcal{B}_N}$ or $w \in \mathrm{RH}_{\infty,\mathcal{B}_{N,a}}$
- $\|K_N\|_{L^2(\mathbb{R}^2)} = O(\log N)$, $\|K_N^a\|_{L^2(\mathbb{R}^2)} = O(\sqrt{\log N})$

Corollary (3)

$$\|K_{N,w}\|_{L^2(\mathbb{R}^2,w)} \leq 2C[w]_{\mathrm{RH}_{\infty,\mathcal{B}_N}} \log N.$$

$$\|K_{N,w}^a\|_{L^2(\mathbb{R}^2,w)} \leq 2C[w]_{\mathrm{RH}_{\infty,\mathcal{B}_{N,a}}} \sqrt{\log N}.$$

$1 < s < \infty$

As I have shown when $s = \infty$, the sufficient condition (2) can be checked quite easily. But, when $1 < s < \infty$, we cannot say any more. The next theorem characterize the boundedness of $M_{\mathcal{B},w}$ on $L^p(w)$ in terms of so-called Tauberian condition.

Definition

w satisfies Tauberian condition (A)
$\overset{def}{\Longleftrightarrow} 0 < \exists\lambda < 1,\ 0 < \exists c = c(\lambda) < \infty$ *s.t. for all m'ble* E

$$w(\{x \in \mathbb{R}^n : M_{\mathcal{B}}(\mathbf{1}_E)(x) > \lambda\}) \le cw(E). \tag{A}$$

Theorem (4)

- $w \in \mathrm{RH}_{s,\mathcal{B}},\ 1 < s < \infty$
- *w satisfy the condition (A)*
- $M_{\mathcal{B}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *for all* $p > 1$.

Then $M_{\mathcal{B},w} : L^p(w) \to L^p(w)$ *for all* $p > 1$.

Volume formula

We introduced the formula

$$w(R) \sim \frac{|R|}{\mathrm{rad}\,(R)} \int_{\mathrm{rad}\,(R)} w_0 \, dr$$

for any rectangle R in $\mathbb{R}^2$. We can prove this formula for any rectangle R in $\mathbb{R}^n$.

- $\mathcal{R}$: rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.
- $\mathcal{B}_N$: rectangles in $\mathbb{R}^n$ with eccentricity N.

Theorem (5)

Let $w(x) = w_0(|x|)$ be the radial weight and $w_0 \in \mathrm{RH}_\infty$. Then $w \in \mathrm{RH}_{\infty,\mathcal{R}}$ and also $w \in \mathrm{RH}_{\infty,\mathcal{B}_N}$.

$$\frac{w(R)}{|R|} \sim \frac{w_0(I)}{|I|} \gtrsim \sup_{r \in I} w_0(r) = \sup_{x \in R} w(x), \quad I = \mathrm{rad}\,(R).$$

Higher dimension

- $M_{\mathcal{R}}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$.
- K_N is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq (n+2)/2$

$$\|K_N\|_{L^p(\mathbb{R}^n)} \leq C N^{n/p-1} (\log N)^{\alpha_n}.$$

This is proved by **H. Tanaka (2004)**.

Corollary (6)

If $w(x) = w_0(|x|)$ *and* $w_0 \in \mathrm{RH}_\infty$. *Then*

(1) $1 < p \leq \infty$,

$$\|M_{\mathcal{R},w} f\|_{L^p(\mathbb{R}^n,w)} \leq C \|f\|_{L^p(\mathbb{R}^n,w)}.$$

(2) $1 < p \leq (n+2)/2$,

$$\|K_{N,w} f\|_{L^p(\mathbb{R}^n,w)} \leq C N^{n/p-1} (\log N)^{\alpha_n} \|f\|_{L^p(\mathbb{R}^n,w)}$$

Results: Fefferman-Stein type inequalities

Previous works (Hardy-Littlewood maximal operator)

C. Fefferman-Stein (1971)

M: Hardy-Littlewood maximal operator. $\Rightarrow$

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p Mw(x)\,dx$$

for an arbitrary weight w and $p > 1$.

$$\sup_{t>0} t\, w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \le C \int_{\mathbb{R}^n} |f(x)|\, Mw(x)\,dx.$$

Previous works (Nikodým maximal operator)

It is conjectured that for all $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} K_N f(x)^n w(x)\, dx \leq C N^{\varepsilon} \int_{\mathbb{R}^n} |f(x)|^n K_N w(x)\, dx.$$

Theorem

- **(Müller and Soria (1995))** It is true when $n = 2$.
- **(H. Tanaka (2004))**

$$\int_{\mathbb{R}^n} K_N f(x)^p w(x)\, dx \leq C N^{\frac{n}{p}-1} (\log N)^{\alpha} \int_{\mathbb{R}^n} |f(x)|^p K_N w(x)\, dx$$

for an arbitrary weight w and $1 < p \leq \frac{n+2}{2}$.

Previous works (strong maximal operator)

$w \in A_p^* \overset{\text{def}}{\Longleftrightarrow}$

- $[w]_{A_p^*} = \sup\limits_{R \in \mathcal{R}} \dfrac{w(R)}{|R|} \left(\dfrac{1}{|R|} \displaystyle\int_R w^{\frac{-1}{p-1}}(x)\,dx \right)^{p-1} < \infty, \; 1 < p < \infty,$
- $[w]_{A_1^*} = \sup\limits_{R \in \mathcal{R}} \dfrac{w(R)}{|R|} \cdot \dfrac{1}{\operatorname{ess\,inf}\limits_{x \in R} w(x)} < \infty,$
- $A_\infty^* := \bigcup_{p>1} A_p^*$.

- $w \in A_p^*$, $1 < p < \infty \Longleftrightarrow M_{\mathcal{R}} : L^p(w) \to L^p(w)$.
- **Kai-Ching Lin (1984), Pérez (1993)** $w \in A_\infty^* \Longrightarrow$

$$\int_{\mathbb{R}^n} (M_{\mathcal{R}} f)^p w \le C_{w,n} \int_{\mathbb{R}^n} |f|^p M_{\mathcal{R}} w, \quad 1 < p < \infty.$$

Previous works (strong maximal operator)

The endpoint ($p = 1$) behavior of $M_{\mathcal{R}}$ was studied as follows:

- **Bagby and Kurtz (1985)** $w \in A_1^* \Longrightarrow$

$$w(M_{\mathcal{R}}f > \lambda) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + (\log^+ \frac{|f(x)|}{\lambda})^{n-1}\right) w(x)\, dx.$$

- **Mitsis (2006)** $n = 2$, $w \in A_\infty^* \Longrightarrow$

$$w(M_{\mathcal{R}}f > \lambda) \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) M_{\mathcal{R}}w(x)\, dx.$$

Previous works (strong maximal operator)

- **R. Fefferman (1981)** $w \in A_\infty^* \implies$

$$\|M_{S,w}f\|_{L^p(\mathbb{R}^n,w)} \leq C_{w,n}c_{p,n}\|f\|_{L^p(\mathbb{R}^n,w)}, \quad 1 < p \leq \infty.$$

  Furthermore we have the asymptotic estimate

$$c_{p,n} = O_n((p-1)^{-n}), \quad \text{as} \quad p \to 1^+.$$

- **Jawerth and Torchinsky (1984)** $w \in A_\infty^* \implies$

$$w(M_{S,w}f > \lambda) \leq C_{w,n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + (\log^+ \frac{|f(x)|}{\lambda})^{n-1}\right) w(x)\, dx.$$

Previous works (strong maximal operator)

Luque and Parissis (2014)

$w \in A_\infty^* \implies$

$$w(M_S f > \lambda) \leq C_{w,n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + (\log^+ \frac{|f(x)|}{\lambda})^{n-1}\right) M_S w(x)\, dx.$$

- By interpolation, this implies the strong L^p FS inequality (Kai-Ching Lin, Pérez).
- Since every A_1^*-weight is an A_∞^*-weight, this recover the Jawerth and Torchinsky's result.

Fefferman-Stein type inequality for the strong maximal operator

Theorem (S. and Tanaka)

Let w *be any weight on* $\mathbb{R}^2$*. Let* $W = M_{\mathcal{R}} M w$*. Then*

$$w(\{x \in \mathbb{R}^2 : M_{\mathcal{R}} f(x) > t\}) \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left(1 + \log^+ \frac{|f(x)|}{t}\right) W(x)\, dx$$

Main results

Theorem (S. and Tanaka)

Let $N > 10$ and w be any weight on $\mathbb{R}^2$. Let $W = M_\Omega M w$. There exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{R}^2, W)$ we have

$$\sup_{t>0} t\, w(\{x \in \mathbb{R}^2 : M_\Omega f(x) > t\})^{1/2} \leq C(\log N)^{1/2} \|f\|_{L^2(\mathbb{R}^2,W)}.$$

- By interpolation, we get $\|M_\Omega f\|_{L^p(\mathbb{R}^2,w)} \leq C_p(\log N)^{1/p} \|f\|_{L^p(\mathbb{R}^2,W)}$ for $2 < p < \infty$.
- Here $\|f\|^p_{L^p(\mathbb{R}^n,W)} = \int_{\mathbb{R}^n} |f(x)|^p M_\Omega M w(x)\, dx$

Thank you for your attention!