Saitama Math. J.
Vol. 26 (2009), 49–65



# 2-Weierstrass points of certain plane curves of genus three

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(*Received October 28, 2009; Revised November 30, 2009*)

**Abstract**

In this paper, we completely determine the 2-gap sequences of the 2-Weierstrass points on cyclic coverings of genus 3 with four branch points in the projective line.

## 1. Introduction

Let $C_{n,m_1,m_2,m_3,\lambda}$ be the algebraic curves of genus $g = 3$ defined by the equation:

$$C_{n,m_1,m_2,m_3,\lambda} : y^n = x^{m_1}(x-1)^{m_2}(x-\lambda)^{m_3}, \qquad n \geq 4,\ \lambda \in \mathbb{C}\backslash\{0,1\},$$

such that $1 \leq m_i \leq n-1$, $\Sigma_i m_i$ and $n$ are relatively prime. Then, $C_{n,m_1,m_2,m_3,\lambda}$ is isomorphic to one of the following plane curves [6]:

$$\begin{aligned} C_{1,a} &: y^4 = x(x-1)(x-a), \\ C_{2,a} &: y^6 = x^3(x-1)^2(x-a)^2, \\ C_{3,a} &: y^4 = x^3(x-1)(x-a), \\ C_{4,a} &: y^6 = x^3(x-1)^3(x-a). \end{aligned}$$

The 1-Weierstrass points of $C_{1,a}$ and $C_{2,a}$ are classified as follows ([9] and [6]).

**Proposition 1.** *We can classify the 1-Weierstrass points of $C_{1,a}$ as follows:*

| | *ordinary flex* | *hyperflex* |
|---|---|---|
| $a = -1, 2, 1/2$ | 0 | 12 |
| *otherwise* | 16 | 4 |

**Proposition 2.** *We can classify the 1-Weierstrass points of $C_{2,a}$ as follows:*

| | *ordinary flex* | *hyperflex* |
|---|---|---|
| $a = -1$ | 16 | 4 |
| $P(a) = 0$ | 10 | 7 |
| *otherwise* | 22 | 1 |

*2000 Mathematics Subject Classification.* Primary 14H55; Secondary 14H10
*Key words and phrases.* Weierstrass points, quartic curves, sextactic points.

*where* $P(a) = 11a^4 - 1036a^3 + 1794a^2 - 1036a + 11$.

**Remark 1.** The curves $C_{3,a}$ and $C_{4,a}$ are hyperelliptic (see subsection 2.3 below). So they have eight 1-Weierstrass points whose 1-gap sequences are $\{1, 3, 5\}$.

In this paper, we compute the 2-gap sequences of the 2-Weierstrass points on $C_{i,a}$, $i = 1, \cdots, 4$. We note that $C_{1,a}$ is a smooth plane quartic and $C_{2,a}$ is isomorphic to the smooth plane quartic curve $C'_{2,b}$ which is defined by the equation (see subsection 2.3 below)

$$C'_{2,b} : y^3 = x^4 - bx^2 - 1, \qquad b^2 + 4 \neq 0.$$

Our main results on $C_{1,a}$ and $C'_{2,b}$ are stated as follows:

**Theorem 1.** *We can classify the 2-Weierstrass points of* $C_{1,a}$ *as follows:*

| | *ordinary flex* | *hyperflex* | *1-sextactic* | *2-sextactic* | *3-sextactic* |
|---|---|---|---|---|---|
| $a = -1, 2, 1/2$ | 0 | 12 | 48 | 0 | 0 |
| $P(a) = 0$ | 16 | 4 | 40 | 16 | 0 |
| $Q(a) = 0$ | 16 | 4 | 48 | 0 | 8 |
| *otherwise* | 16 | 4 | 72 | 0 | 0 |

*where* $P(a) = (a^2 + a + 1)(a^2 - 3a + 3)(3a^2 - 3a + 1)$ *and*
$Q(a) = (a^2 - 6a + 1)(a^2 + 4a - 4)(4a^2 - 4a - 1)$.

**Theorem 2.** *We can classify the 2-Weierstrass points of* $C'_{2,b}$ *as follows:*

| | *ordinary flex* | *hyperflex* | *1-sextactic* | *2-sextactic* | *3-sextactic* |
|---|---|---|---|---|---|
| $b = 0$ | 16 | 4 | 72 | 0 | 0 |
| $P(b) = 0$ | 10 | 7 | 63 | 0 | 0 |
| $Q(b) = 0$ | 22 | 1 | 69 | 6 | 0 |
| $R(b) = 0$ | 22 | 1 | 72 | 0 | 3 |
| *otherwise* | 22 | 1 | 81 | 0 | 0 |

*where* $P(b) = 11b^4 + 1080b^2 + 3888$, $\quad R(b) = b^4 + 18b + 54$ *and*
$Q(b) = 119532070599991b^{48} - 1170934255940539104b^{46} + \cdots$
$+ 849437234182329111530108544142540800000000000$.

Our main results on $C_{3,a}$ and $C_{4,a}$ are stated as follows:

**Theorem 3.** *We can classify the 2-Weierstrass points of* $C_{3,a}$ *as follows:*

| *2-gap sequence* | $\{1,2,3,4,5,7\}$ | $\{1,2,3,4,5,8\}$ | $\{1,2,3,4,5,9\}$ | $\{1,2,3,5,7,9\}$ |
|---|---|---|---|---|
| $a = 3/4, 4/3$ | 24 | 0 | 12 | 8 |
| $P(a) = 0$ | 16 | 16 | 4 | 8 |
| *otherwise* | 48 | 0 | 4 | 8 |

*where* $P(a) = 16a^2 - 17a + 16$.

**Theorem 4.** *We can classify the 2-Weierstrass points of* $C_{4,a}$ *as follows:*

| *2-gap sequence* | $\{1,2,3,4,5,7\}$ | $\{1,2,3,4,5,9\}$ | $\{1,2,3,5,7,9\}$ |
|---|---|---|---|
| $a = 1/9, 8/9$ | 24 | 12 | 8 |
| $P(a) = 0$ | 42 | 6 | 8 |
| *otherwise* | 60 | 0 | 8 |

*where* $P(a) = 5103a^4 - 10206a^3 + 33183a^2 - 28080a - 64$.

## 2. Preliminaries

Let $C$ be a non-singular projective curve of genus $g \geq 2$. Let $f(x, y) = 0$ be the defining equation of $C$. Take a divisor $qK$, where $K$ is a *canonical divisor* and $q = 1, 2$. Let $\dim |qK| = r \geq 0$. We denote by $L(qK)$ the $\mathbb{C}$-vector space of all meromorphic functions $f$ such that $\mathrm{div}(f) + qK \geq 0$ and by $\ell(qK)$ the dimension of $L(qK)$ over $\mathbb{C}$.

For a point $P$ on $C$, if $n$ is a positive integer such that $\ell(qK - (n-1)P) > \ell(qK - nP)$, we call this integer $n$ a *"q-gap"* at $P$. There are exactly $r+1$ $q$-gaps and the sequence of $q$-gaps $\{n_1, n_2, \cdots, n_{r+1}\}$ such that $n_1 < n_2 < \cdots < n_{r+1}$ is called *the q-gap sequence at P*. Assume that $\{f_1, \cdots, f_{r+1}\}$ is a basis for $L(qK)$. The Wronskian $W(f_1, \cdots, f_{r+1})$ of $\{f_1, \cdots, f_{r+1}\}$ is given by

$$W(f_1, \cdots, f_{r+1}) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{r+1}(x) \\ f_1'(x) & f_2'(x) & \cdots & f_{r+1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(r)}(x) & f_2^{(r)}(x) & \cdots & f_{r+1}^{(r)}(x) \end{vmatrix},$$

here all the derivatives have taken with respect to $x$. Consider the divisor $E$ :

$$E = (r+1)qK + \mathrm{div}(W(f_1, \cdots, f_{r+1})) + \frac{r(r+1)}{2}\mathrm{div}(dx).$$

Then the multiplicity of $E$ at a point $P$ can be computed as $\sum_{i=1}^{r+1}(n_i - i)$ (see Miranda [10]). This integer is called *q-weight at P* and denoted by $w^{(q)}(P)$. If $w^{(q)}(P) > 0$, we call the point $P$ a *q-Weierstrass point*.

Let $\Omega^{(q)}(C)$ be the $\mathbb{C}$-vector space of holomorphic $q$-differentials of $C$. It is known that $\Omega^{(q)}(C) \cong L(qK)$, therefore we have

$$\dim_{\mathbb{C}}\Omega^{(q)}(C) = \begin{cases} g, & q = 1 \\ 3g-3, & q = 2 \end{cases}$$

and the number of $q$-Weierstrass points $N^{(q)}(C)$ counted according to their $q$-weight is given by

$$N^{(q)}(C) = \begin{cases} g(g^2-1), & q=1 \\ 9g(g-1)^2, & q=2 \end{cases}$$

**Lemma 1.** *An integer $n$ is contained in $q$-gap sequence at $P$ if and only if there is a holomorphic $q$-differential $\omega \in \Omega^{(q)}(C)$ such that $ord_P(\omega) = n-1$.*

**Lemma 2.** *Let $P$ be a point in a plane curve $C$ of genus 3. Then we can choose a basis $\{\omega_1, \cdots, \omega_6\}$ of $\Omega^{(2)}(C)$ in such a way that:*

$$0 = ord_P(\omega_1) < ord_P(\omega_2) < \cdots < ord_P(\omega_6) < 9.$$

*Therefore we see that the 2-gap sequence at $P$ is*

$$\{1, ord_P(\omega_2)+1, ord_P(\omega_3)+1, \cdots, ord_P(\omega_6)+1\}.$$

**Lemma 3** (Duma [3])**.** *Let $\sigma$ be an involution of $C$. If the number of fixed points of $\sigma$ is $\geq 3$, then every fixed point is a $q$-Weierstrass point ($q \geq 2$).*

Let $W_q(C)$ be the set of all $q$-Weierstrass points on a curve $C$. We denote by $G^{(q)}(P)$ the $q$-gap sequence at the point $P \in C$.

**Lemma 4.** *Let $\Phi : C \longrightarrow C'$ be a birational transformation between the non-singular algebraic curves $C$ and $C'$. Then we have*

$$\Phi(W_q(C)) = W_q(C') \text{ and } G^{(q)}(\Phi(P)) = G^{(q)}(P).$$

**Remark 2.** We have the following facts:

**(i)** Let $C$ be a plane curve of genus 3. Then for any $P \in C$ we have $w^{(2)}(P) \leq 6$. Furthermore, equality occurs if and only if $C$ is hyperelliptic and $P$ is a 1-Weierstrass point [5].

**(ii)** Let $C$ be a plane curve of genus 3. Let $P$ be a point on $C$ such that $P$ is a 2-Weierstrss point and $P$ is not a 1-Weierstrss point. Then we obtain $w^{(2)}(P) \leq 4$ [3].

Using Remark 2, we obtain the following lemma.

**Lemma 5.** *The 2-gap sequences of the 2-Weierstrass points of a plane curve of genus three are as follows:*

| *2-weight* | *2-gap sequence* |
|---|---|
| 1 | $\{1,2,3,4,5,7\}$ |
| 2 | $\{1,2,3,4,5,8\}$<br>$\{1,2,3,4,6,7\}$ |
| 3 | $\{1,2,3,4,5,9\}$<br>$\{1,2,3,5,6,7\}$ |
| 4 | $\{1,2,3,4,6,9\}$ |
| 6 | $\{1,2,3,5,7,9\}$ |

We use the following notation to describe the repeated roots of a polynomial.

**Notation.** Let $f(x)$ be a polynomial. We write $T(f) = (n_\alpha, m_\beta, \cdots)$, $n, m \in \mathbb{Z}^+$, if $f(x)$ has $\alpha$ roots of multiplicities $n$, $\beta$ roots of multiplicities $m$, and so on. For instance the polynomial $f(x) = x^3(x-1)^2(x+1)^2(x^3-2)$ is of type $T(f) = (3, 2_2, 1_3)$.

### 2.1 Subresultant Method

To determine the multiplicities of the repeated roots of a polynomial with a parameter, we use the subresultant method [6].

We denote by $R^{(k)}(f(x), g(x); x)$ to the *subresultant of degree k* for the polynomials $f(x)$ and $g(x)$.

**Lemma 6.** *The polynomials $f(x)$ and $g(x)$ have a non-constant common factor of multiplicity at least $k$ if and only if*

$$R^{(i)}(f(x), g(x); x) = 0, \qquad i = 1, 2, \cdots, k.$$

**Definition.** For a polynomial $f(x)$, we define $s := s(f)$, if the subresultant of degree $i$, $R^{(i)}(f(x), f'(x); x) = 0$, for all $i = 1, \cdots, s$ and $R^{(s+1)}(f(x), f'(x); x) \neq 0$.

**Lemma 7.** *Take a polynomial $f(x) = c\prod_{i=1}^{k}(x-a_i)^{n_i}$, where $a_i \neq a_i$ if $i \neq j$ and $c$ is a complex number. Then $s(f) = \sum_{i=1}^{k}(n_i - 1)$.*

### 2.2 Smooth Plane Quartics

Let $P$ be a point on a smooth plane curve $C$ of degree $d \geq 3$. Then there is an unique irreducible conic $D_P$ with $I_P(C, D_P) \geq 5$ unless $P$ is a flex. Such the unique irreducible conic $D_P$ is called the *osculating conic of $C$ at $P$.*

**Definition** ([2])**.** A point $P$ on a smooth plane curve $C$ is said to be a *sextactic point* if the osculating conic $D_P$ meets $C$ at $P$ with contact order at least six. A sextactic point $P$ is called *$i$-sextactic*, if $i = I_P(C, D_P) - 5$.

In particular, let $C$ be a smooth plane quartic curve and $P$ be a point on $C$. It is well known that the 1-Weierstrass points on $C$ are nothing but flexes [12] and divided into two types ordinary flex and hyperflex.

| $w^{(1)}(P)$ | 1-gap sequence | Geometry |
|---|---|---|
| 1 | $\{1, 2, 4\}$ | ordinary flex |
| 2 | $\{1, 2, 5\}$ | hyperflex |

A flex $P$ on $C$ is called a *hyperflex* if the contact order with the tangent line $L_P$ at $P$ is equal to four, i.e., $I(C, L_p) = 4$. It is well known that the 2-Weierstrass points on $C$ are divided into two types flexes and sextactic points. F. Sakai in [2] gave the following classification of the 2-Weierstrass points on a smooth plane quartic $C$.

**Proposition 3** ([2])**.** *The 2-Weierstrass points on a smooth plane quartic can be classified as follows:*

| $w^{(2)}(P)$ | *2-gap sequence* | *geometry* |
|---|---|---|
| 1 | $\{1,2,3,4,5,7\}$ | *ordinary flex* |
| 5 | $\{1,2,3,5,6,9\}$ | *hyperflex* |
| 1 | $\{1,2,3,4,5,7\}$ | *1-sextactic* |
| 2 | $\{1,2,3,4,5,8\}$ | *2-sextactic* |
| 3 | $\{1,2,3,4,5,9\}$ | *3-sextactic* |

### 2.3 Isomorphisms

In this section, we summarize some isomorphisms on the curves $C_{i,a}$ $(i = 1,\cdots,4)$. On $C_{1,a}$, we have the following proposition [6].

**Proposition 4.** $(i)$ *The curve $C_{1,a}$ is isomorphic to the curve $C_{1,a'}$ if and only if $a'$ is equal to one of the following [11]:*

$$a,\ 1/a,\ 1-a,\ 1/(1-a),\ (a-1)/a,\ a/(a-1).$$

$(ii)$ *The curve $C_{1,a}$ is isomorphic to the Fermat curve $F_4 : x^4+y^4=1$ if and only if $a=2,1/2$ or $-1$.*

$(iii)$ *If $a$ is a root of the polynomial $a^2-a+1$ then the curve $C_{1,a}$ is isomorphic to the curve $C'_{2,0}$.*

On $C_{2,a}$, we have the following proposition [6].

**Proposition 5.** $(i)$ *If $a=a'$ or $1/a'$ then the curve $C_{2,a}$ is isomorphic to the curve $C_{2,a'}$.*

$(ii)$ *If $b=-i(a+1)/\sqrt{a}$ then the curve $C'_{2,b}$ is isomorphic to the curve $C_{2,a}$.*

$(iii)$ *Let $P_{a,b}$ be the curve defined by the equation $y^3=x(x-1)(x-a)(x-b)$[1]. If $a=(2c-1)^2$ then the curve $C_{2,a}$ is isomorphic to the curve $P_{c,1-c}$.*

On $C_{3,a}$, we have the following proposition [6].

**Proposition 6.** $(i)$ *If $a=a'$ or $1/a'$, then the curve $C_{3,a}$ is isomorphic to the curve $C_{3,a'}$.*

$(ii)$ *Let $H_{1,a}$ be the curve defined by the equation $y^2=x^8+2(a+1)x^4+(a-1)^2$. The curve $C_{3,a}$ is isomorphic to the curve $H_{1,a}$.*

[1]The curve $P_{a,b}$ is called a Picard curve. M. Kawasaki and F. Sakai completely determine the 1-gap sequences of the 1-Weierstrass points on $P_{a,b}$. ( [7], see also [8]).

*Proof of (ii).* Applying the birational transformation

$$\phi_1 : \begin{cases} x = x' \\ y = x'y' \end{cases}$$

to the curve $C_{3,a} : y^4 = x^3(x-1)(x-a)$, we obtain the curve $\phi_1(C_{3,a})$ defined by the equation

$$\big(2x' - (a+1+y'^4)\big)^2 = y'^8 + 2(a+1)y'^4 + (a-1)^2.$$

Now, applying the birational transformation

$$\phi_2 : \begin{cases} x' = (Y + a + 1 + X^4)/2 \\ y' = X \end{cases}$$

to the curve $\phi_1(C_{3,a})$, we obtain the curve $H_{1,a}$. □

On $C_{4,a}$, we have the following proposition [6].

**Proposition 7.** *(i) If $a = a'$ or $1 - a'$, then the curve $C_{4,a}$ is isomorphic to the curve $C_{4,a'}$.*

*(ii) Let $H_{2,a}$ be the curve defined by the equation $y^2 = x(x^3 + a)(x^3 + a - 1)$. The curve $C_{4,a}$ is isomorphic to the curve $H_{2,a}$.*

**Remark 3.** (**1**) The curve $H_{1,a}$ has the following automorphisms:

$$\sigma : (x,y) \to (ix, y), \qquad \tau : (x,y) \to (x, -y).$$

If the point $P = (x,y) \in H_{1,a}$ is $q$-Weierstrass points, then all the points $(\pm x, \pm y)$, $(\pm ix, \pm y)$ in the orbit of $P$ are $q$-Weierstrass points of the same $q$-gap sequences.

(**2**) The curve $H_{2,a}$ has the following automorphisms:

$$\sigma' : (x,y) \to (\omega x, \eta y), \qquad \tau : (x,y) \to (x, -y),$$

where $\omega = exp(2\pi i/3) = \eta^2$. If the point $P = (x,y) \in H_{2,a}$ is $q$-Weierstrass points, then all the points $(x, \pm y)$, $(\omega x, \pm \eta y)$ and $(\omega^2 x, \pm \eta^2 y)$ in the orbit of $P$ are $q$-Weierstrass points of the same $q$-gap sequences.

(**3**) The curve $H_{1,a}$ is not isomorphic to the curve $H_{2,a'}$ for any $a$ and $a'$.

### 2.4 Matrix Rank Method

Suppose that $C$ is a plane curve of genus 3 which is defined by the equation $f(x,y) = 0$. Let $P$ be a point on $C$. Let $\{\omega_1, \cdots, \omega_6\}$ be a basis of $\Omega^{(2)}(C)$. Let $t$ be a local parameter around $P$. Then, locally, we can write $\omega_i$ as the following power series:

$$\omega_i = (\sum_{j=0}^{l} a_{j,i}t^j + o[t^{l+1}])dt^2 \qquad (i = 1, \cdots, 6, \text{ and } l \in \mathbb{Z}_{\geq \mathbf{0}}).$$

Consider the $6 \times (l+1)$ matrix

$$M_l := \begin{pmatrix} a_{0,1} & a_{1,1} & \cdots & a_{l,1} \\ a_{0,2} & a_{1,2} & \cdots & a_{l,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{0,6} & a_{1,6} & \cdots & a_{l,6} \end{pmatrix}.$$

By using the rank of the matrix $M_l$, we can determine the 2-gap sequence $G^{(2)}(P)$ at $P$.

**Lemma 8.** *(i) Suppose that $w^{(2)}(P) = 2$. Then we obtain*

$$G^{(2)}(P) = \begin{cases} \{1,2,3,4,5,8\}, & \textit{if rank } M_4 = 5 \\ \{1,2,3,4,6,7\}, & \textit{if rank } M_4 = 4 \end{cases}$$

*(ii) Suppose that $w^{(2)}(P) = 3$. Then we obtain*

$$G^{(2)}(P) = \begin{cases} \{1,2,3,4,5,9\}, & \textit{if rank } M_3 = 4 \\ \{1,2,3,5,6,7\}, & \textit{if rank } M_3 = 3 \end{cases}$$

## 3. Proof of Theorems

Now, let us prove our main results.

### 3.1 Proof of Theorem 1

Let $C_{1,a}$ be a smooth plane quartic curve defined by the equation

$$C_{1,a} : y^4 = x(x-1)(x-a), \qquad a \neq 0, 1.$$

Then

$$\omega_1 = dx/y^2, \quad \omega_2 = dx/y^3, \quad \omega_3 = xdx/y^3$$

is a basis of the holomorphic 1-differential space $\Omega^{(1)}(C_{1,a})$. We can prove Proposition 1 as follows:

*Proof.* We can use the Wronskian of holomorphic 1-differentials or the Hessian method. Let $f_1(x, y)$ be the defining equation of $C_{1,a}$. Let $H_{f_1}$ be its associated Hessian curve. We compute the resultant

$$Res\,(f_1, H_{f_1}; y) = const.x^2(x-1)^2(x-a)^2h(x,a),$$

where

$$(1)\qquad h(x,a) = 3x^4 - 4(1+a)x^3 + 2(2+a+2a^2)x^2 - 4a(1+a)x + 3a^2.$$

The discriminant of $h(x,a)$ shows that $h(x,a)$ has repeated roots if and only if $a = -1, 2, 1/2$. It is easy to describe the repeated roots of $h(x,a)$ as follows:

$$T(h) = \begin{cases} (2_2), & \text{if } a = -1, 2, 1/2 \\ (1_4), & \text{otherwise.} \end{cases}$$

This means that $h(x,a)$ has two repeated roots of multiplicities two if $a = -1, 2, 1/2$, otherwise $h(x,a)$ has four distinct complex roots. Now the result is clear. □

We now pass to study the 2-Weierstrass points on $C_{1,a}$. The Wronskian $W(x,a)$ of $\{1, x, y, xy, x^2, y^2\}$ can be written as

$$W(x,a) = const. f(x,a) \cdot h(x,a) \cdot g(x,a)/y^{40},$$

where $h(x,a)$ is as in (1),

$$\begin{aligned} f(x,a) &= (x^2-a)(x^2-2ax+a)(x^2-2x+a), \\ g(x,a) &= -7a^4\left(52x^2 - 2ax(2+15x) + a^2\left(1-4x+52x^2\right)\right) \\ &\quad +5a^2x^3\left(a^2(220-1173x) + a(96-528x) + 48x + 48a^4(2+x)\right. \\ &\quad \left.-44a^3(-5+12x)\right) + \cdots + 14x^{10}\left(-26+15a-26a^2+2x+2ax\right) - 7x^{12}. \end{aligned}$$

The polynomial $f(x,a)$ has six distinct roots for any $a \neq 0, 1$. The resultants of $f(x,a)$, $h(x,a)$ and $g(x,a)$ are given by

$$\begin{aligned} Res\,(f,h;x) &= const.a^6(a-1)^6(a-2)^2(a+1)^2(2a-1)^2, \\ Res\,(f,g;x) &= const.a^{18}(a-1)^{18}(a-2)^2(a+1)^2(2a-1)^2 Q(a), \\ Res\,(g,h;x) &= const.a^{12}(a-1)^{12}(a-2)^4(a+1)^4(2a-1)^4. \end{aligned}$$

where

$$Q(a) = (a^2-6a+1)(a^2+4a-4)(4a^2-4a-1).$$

At $a = -1, 2, 1/2$, we have very special cases.

$$\begin{aligned} W(x,-1) &= const.\left(1+x^2\right)^5\left(-1-2x+x^2\right)\left(-1+2x+x^2\right) \\ &\quad \times\left(1+132x^2-250x^4+132x^6+x^8\right)/y^{40}, \end{aligned}$$

$$\begin{aligned} W(x,2) &= const.\left(-2+x^2\right)\left(2-4x+x^2\right)\left(2-2x+x^2\right)^5 (16-64x+ \\ &\quad 640x^2-1696x^3+1800x^4-848x^5+160x^6-8x^7+x^8\right)/y^{40}, \end{aligned}$$

$$W(x,1/2) = const.\left(-1+2x^2\right)\left(1-4x+2x^2\right)\left(1-2x+2x^2\right)^5\left(1-8x+\right.$$
$$\left.160x^2-848x^3+1800x^4-1696x^5+640x^6-64x^7+16x^8\right)/y^{40}.$$

In these cases, the polynomial $f(x,a)\cdot h(x,a)\cdot g(x,a)$ has two repeated roots of multiplicities five and the other roots are distinct. Therefore, if $a=-1,2,1/2$, then we have 12 hyperflexes and 48 ordinary sextactic points.

Now, if $a \neq -1,2,1/2$, then the number of sextactic points counted according to their 2-weight is equal to 72 and the repeated roots of the polynomial $g(x,a)$ have multiplicities $\leq 3$. Moreover, the discriminant of $g(x,a)$ shows that $g(x,a)$ has repeated roots if and only if $P(a)Q(a)=0$, where

$$P(a) = (a^2+a+1)(a^2-3a+3)(3a^2-3a+1).$$

The resultants of $g(x,a)$, $g_x(x,a)$ and $g_{xx}(x,a)$ show that $g(x,a)$ does not have repeated roots of multiplicity 3 for any $a \neq 0,1$. By using subresultant method (Lemma 7), we find

(**1**) If $P(a)=0$, then $s(g)=4$. Therefore $T(g)=(2_4,1_4)$.

(**2**) If $Q(a)=0$, then $s(g)=2$. Therefore $T(g)=(2_2,1_8)$. Here the two repeated roots of multiplicity 2 will be common roots with $f(x,a)$.

(**3**) Otherwise, then $s(g)=0$. Therefore $T(g)=(1_{12})$.

Now, we can describe the repeated roots of the polynomial $h(x,a)\cdot f(x,a)\cdot g(x,a)$ as follows:

(**1**)$'$ If $P(a)=0$, then we have

$$T(h)=(1_4),\ T(f)=(1_6),\ T(g)=(2_4,1_4).$$

Hence we have

$$T(h\cdot f\cdot g)=(2_4,1_{14}).$$

(**2**)$'$ If $Q(a)=0$, then we have

$$T(h)=(1_4),\ T(f)=(1_6),\ T(g)=(2_2,1_8).$$

Here note that the two repeated roots of $g$ are common roots with $f$. Hence we have

$$T(h\cdot f\cdot g)=(3_2,1_{16}).$$

(**3**)$'$ Otherwise, then we have

$$T(h)=(1_4),\ T(f)=(1_6),\ T(g)=(1_{12}).$$

Hence we have

$$T(h \cdot f \cdot g) = (1_{22}).$$

Summarizing above, we can prove Theorem 1 as follows:

*Proof.* Let $P_\infty$ be the point on $C_{1,a}$ lying over $\infty$. Consider the divisor

$$E = 6D_{P_\infty} + \operatorname{div}(W(x,a)) + 15\operatorname{div}(dx),$$

where $D_{P_\infty} = 8P_\infty$. Then $w^{(2)}(P) =$ the multiplicity of $P$ in the divisor $E$. Note that $C_{1,a}$ can be express as 4-sheeted covering of $\mathbb{P}^1(\mathbb{C})$. Putting everything together and consider the discussion before the theorem yield to the result. □

**Example 1.** Consider the curve:

$$C_{1,(-2+2\sqrt{2})} : y^4 = x\,(x-1)\left(x-(-2+2\sqrt{2})\right).$$

Note that $a = -2+2\sqrt{2}$ is a root of $Q(a)$. At the points $P = \left(2-\sqrt{2}, \sqrt{3\sqrt{2}-4}\right)$ and $P' = \left(\sqrt{2}, \sqrt{2-\sqrt{2}}\right)$ there exists a conic $D$ (resp. $D'$) which meets $C$ only at $P$ (resp. $P'$). The equations of $D$ and $D'$ are the following

$$D\,:\,2\left(10-7\sqrt{2}\right)-4\left(3-2\sqrt{2}\right)x-2\left(2+\sqrt{2}\right)\left(-4+3\sqrt{2}\right)^{3/2}y+$$
$$\left(2-\sqrt{2}\right)x^2-4\left(4-3\sqrt{2}\right)y^2+2\left(1+\sqrt{2}\right)\left(-4+3\sqrt{2}\right)^{3/2}xy=0$$

$$D'\,:\,2\left(2-\sqrt{2}\right)+4\left(1-\sqrt{2}\right)x+2\sqrt{2}\left(2-\sqrt{2}\right)^{3/2}y+$$
$$\left(2-\sqrt{2}\right)x^2+4\left(2-\sqrt{2}\right)y^2-2\left(2-\sqrt{2}\right)^{3/2}\left(1+\sqrt{2}\right)xy=0$$

### 3.2 Proof of Theorem 2

Using Kawasaki [6] and Proposition 5, we have the following proposition:

**Proposition 8.** *We can classify the 1-Weierstrass points of $C'_{2,b}$ as follows:*

| | *ordinary flex* | *hyperflex* |
|---|---|---|
| $b=0$ | 16 | 4 |
| $P(b)=0$ | 10 | 7 |
| *otherwise* | 22 | 1 |

*where $P(b) = 11b^4 + 1080b^2 + 3888$.*

**Remark 4.** Let $Q_i^{(0)}$ $(i = 1, 2, 3)$ be the points on $C'_{2,b}$ lying over 0. These points are sextactic points for any $b \neq 0$. Since they are the fixed points of the involution $\sigma \in Aut(C'_{2,b})$ which assigns $(x, y) \mapsto (-x, y)$, then either they are flexes or sextactic points (Lemma 3). Using Proposition 8, the points $Q_i^{(0)}$ are hyperflexes only if $b = 0$.

In a similar manner as in the proof of Theorem 1, we can prove Theorem 2 (for more details, see Alwaleed [1]).

**Example 2.** Take $b = i\sqrt{3\left(3+\sqrt{3}\right)}$ as a root of $R(b) = 0$. Then at the point $Q_1^{(0)} = (0,-1)$, there is a conic $D_1$ such that $C'_{2,b} \cap D_1 = \{Q_1\}$. The equation of $D_1$ is given by

$$D_1 : 6x^2 + i\sqrt{3+\sqrt{3}}(1+y)\left(3 - 2\sqrt{3} + \sqrt{3}y\right) = 0.$$

### 3.3 Proof of Theorem 3

As we have seen in Proposition 6, the curve $C_{3,a}$ is isomorphic to the hyperelliptic curve $H_{1,a}$ defined by the equation $f_3(x,y) = y^2 - x^8 - 2(a+1)x^4 - (a-1)^2 = 0$. The curve $H_{1,a}$ has eight 1-Weierstrass points which are the ramification points of $H_{1,a}$ whose 1-gap sequences are $\{1,3,5\}$. Therefore using Remark 2 (i), $H_{1,a}$ has eight 2-Weierstrass points whose 2-gap sequences are $\{1,2,3,5,7,9\}$. Let $P_i^{\infty}$, $P_i^0$ $(i = 1,2)$ be the points on $H_{1,a}$ lying over $\infty$ and 0, respectively. Then

$$\begin{aligned}\omega_1 &= dx^2/y,\ \omega_2 = dx^2/y^2,\ \omega_3 = x dx^2/y^2,\\ \omega_4 &= x^2 dx^2/y^2,\ \omega_5 = x^3 dx^2/y^2,\ \omega_6 = x^4 dx^2/y^2\end{aligned}$$

is a basis of $\Omega^{(2)}(H_{1,a}) \cong L(D)$, where $D = \operatorname{div}(dx^2/y^2) = 4(P_1^{\infty} + P_2^{\infty})$. The Wronskian $W(x,a)$ of $\{x^4, x^3, x^2, x, y, 1\}$ can be written as

$$W(x,a) = const.x^3 \cdot h(x,a) \cdot g(x,a)/y^9,$$

where

$$h(x,a) = \left(1 - a + x^4\right)\left(-1 + a + x^4\right),$$

$$\begin{aligned}g(x,a) = {} & 7 - 28a + 42a^2 - 28a^3 + 7a^4 - 36x^4\\ & +36ax^4 + 36a^2x^4 - 36a^3x^4 - 86x^8\\ & +220ax^8 - 86a^2x^8 - 36x^{12} - 36ax^{12} + 7x^{16}.\end{aligned}$$

Now, consider the divisor

$$E = 6D + \operatorname{div}(W) + 15\operatorname{div}(dx).$$

Then $w^{(2)}(P) =$ the multiplicity of $P$ in the divisor $E$. The discriminant of the polynomial $h(x,a)$ with respect to $x$ shows that $h(x,a)$ does not have repeated roots for any $a \neq 0,1$. The discriminant of the polynomial $g(x,a)$ with respect to

$x$ shows that $g(x,a)$ has repeated roots if and only if $(-4+3a)(-3+4a)P(a)=0$, where

$$P(a) = 16 - 17a + 16a^2.$$

Moreover, we have

$$\begin{aligned} Res\,(x, h(x,a); x) &= (a-1)^2, \\ Res\,(x, g(x,a); x) &= 7(a-1)^4, \\ Res\,(g(x,a), h(x,a); x) &= const.a^4(a-1)^{16}(-4+3a)^4(-3+4a)^4. \end{aligned}$$

At $a = 3/4, 4/3$, we have very special cases

$$\begin{aligned} W(x,3/4) =& \, const.x^3\left(-1+2x^2\right)^3\left(1+2x^2\right)^3\left(1-4x+2x^2\right) \\ & \times\left(1-2x+2x^2\right)\left(1+2x+2x^2\right)\left(1+4x+2x^2\right)\left(1+12x^2+4x^4\right)/y^9, \\ W(x,4/3) =& \, const.x^3\left(-1+3x^4\right)^3\left(1+3x^4\right)\left(1-102x^4+9x^8\right)/y^9. \end{aligned}$$

Hence, if $a = 3/4, 4/3$ we have twelve 2-Weierstrass points of 2-weight 3 and twenty-four 2-Weierstrass points of 2-weight 1. Now, let $a \neq 3/4, 4/3$. The resultants with respect to $x$ of $g(x,a)$, $g_x(x,a)$ and $g_{xx}(x,a)$ show that there is no common factors of $g(x,a)$, $g_x(x,a)$ and $g_{xx}(x,a)$. Thus $g(x,a)$ has repeated roots of multiplicities $< 3$. Using Lemma 7, we can describe the repeated roots of $g(x,a)$ as follows:

(**1**) If $P(a) = 0$, then $s(g) = 8$. Therefore $T(g) = (2_8)$.

(**2**) Otherwise, $s(g) = 0$. Therefore $T(g) = (1_{16})$.

So, we have the following table:

| 2-weight | 1 | 2 | 3 | 6 | $N^{(2)}(H_{1,a})$ |
|---|---|---|---|---|---|
| $a = 3/4, 4/3$ | 24 | 0 | 12 | 8 | 44 |
| $P(a) = 0$ | 16 | 16 | 4 | 8 | 44 |
| Otherwise | 48 | 0 | 4 | 8 | 60 |

Now, we compute the 2-gap sequences of the 2-Weierstrass points on $H_{1,a}$. Firstly, note that $H_{1,a}$ has four 2-Weierstrass points of 2-weight 3 for any $a \neq 0, 1$. We shall see that these points are nothing but the ramification points of $C_{3,a}$. Let $P_\infty$, $A$, $B$ and $C$ be the points on $C_{3,a}$ lying over $\infty$, 0, 1 and $a$, respectively. Then we have

$$\begin{array}{ll}
\omega'_1 = dx^2/y^2, & \mathrm{div}(\omega'_1) = 4(B+C), \\
\omega'_2 = xdx^2/y^3, & \mathrm{div}(\omega'_2) = 3(B+C)+A+P_\infty, \\
\omega'_3 = x^2dx^2/y^4, & \mathrm{div}(\omega'_3) = 2(A+B+C+P_\infty), \\
\omega'_4 = x^3dx^2/y^5, & \mathrm{div}(\omega'_4) = 3(A+P_\infty)+B+C, \\
\omega'_5 = x^3dx^2/y^6, & \mathrm{div}(\omega'_5) = 8P_\infty, \\
\omega'_6 = x^4dx^2/y^6, & \mathrm{div}(\omega'_6) = 4(A+P_\infty), \\
\omega'_7 = x^5dx^2/y^6, & \mathrm{div}(\omega'_7) = 8A, \\
\omega'_8 = x^3(x-1)^2dx^2/y^6, & \mathrm{div}(\omega'_8) = 8B, \\
\omega'_9 = x^3(x-a)^2dx^2/y^6, & \mathrm{div}(\omega'_9) = 8C.
\end{array}$$

Using Lemma 2, we obtain

$$G^{(2)}(P) = \{1,2,3,4,5,9\}, \qquad P \in \{A,B,C,P_\infty\}.$$

Putting $\phi := \phi_2 \circ \phi_1$ (here $\phi_1$, $\phi_2$ are as in the proof of Proposition 6), we find

$$\phi(\{A,P_\infty\}) = \{P_1^\infty, P_2^\infty\}, \qquad \phi(\{B,C\}) = \{P_1^0, P_2^0\}.$$

Therefore, we have (by Lemma 4)

$$G^{(2)}(P) = \{1,2,3,4,5,9\}, \qquad P \in \{P_1^0, P_2^0, P_1^\infty, P_2^\infty\}.$$

Now, we consider the cases in which $a = 3/4, 4/3$ and $P(a) = 0$.

### 3.3.1 The case $a = 3/4, 4/3$

Using Proposition 6 (i), it is enough to consider $a = 3/4$. In this case, the remainder of 2-Weierstrass points of 2-weight 3 are the 8 points $(\pm 1/\sqrt{2}, \pm 1)$, $(\pm i/\sqrt{2}, \pm 1)$. Moreover, these points are conjugate under $Aut(H_{1,3/4})$ (Remark 3, (1)).

Let $t := x - 1/\sqrt{2}$ be the local parameter around the point $P = (1/\sqrt{2}, 1)$. Then we can write $\omega_1, \cdots, \omega_6$ as follows:

$$\begin{aligned}
\omega_1 &= (1 - 2\sqrt{2}t + 5t^2 - 5\sqrt{2}t^3 + o[t]^4)dt^2, \\
\omega_2 &= (1 - 4\sqrt{2}t + 18t^2 - 30\sqrt{2}t^3 + o[t]^4)dt^2, \\
\omega_3 &= (1/\sqrt{2} - 3t + 5\sqrt{2}t^2 - 12t^3 + o[t]^4)dt^2, \\
\omega_4 &= (1/2 - \sqrt{2}t + 2t^2 - \sqrt{2}t^3 + o[t]^4)dt^2, \\
\omega_5 &= (1/2\sqrt{2} - (1/2)\,t + t^3 + o[t]^4)dt^2, \\
\omega_6 &= (1/4 - (1/2)\,t^2 + (1/\sqrt{2})t^3 + o[t]^4)dt^2.
\end{aligned}$$

Consider the matrix

$$M_3 := \begin{pmatrix} 1 & -2\sqrt{2} & 5 & -5\sqrt{2} \\ 1 & -4\sqrt{2} & 18 & -30\sqrt{2} \\ 1/\sqrt{2} & -3 & 5\sqrt{2} & -12 \\ 1/2 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1/2\sqrt{2} & -1/2 & 0 & 1 \\ 1/4 & 0 & -1/2 & 1/\sqrt{2} \end{pmatrix}.$$

Then we see that the rank of $M_3$ is 4. Using Lemma 8 (ii), we obtain $G^{(2)}(P) = \{1, 2, 3, 4, 5, 9\}$.

### 3.3.2 The case $P(a) = 0$

The polynomial $P(a)$ has two roots $a = (17 + 7i\sqrt{15})/32$ and $\bar{a}$. We here consider the root $a$. The polynomial $g(x, a)$ has 8 distinct repeated roots $\{\alpha_1, \alpha_2, \cdots, \alpha_8\}$. For $\alpha_i$ the polynomial $f_3(\alpha_i, y)$ has two roots $\{\pm\beta_i\}$. Take

$$\begin{aligned} \alpha_1 &= \sqrt[4]{\frac{63}{32} + \frac{7\sqrt{6}}{8} + \frac{3}{32} i \sqrt{5\left(59 + 24\sqrt{6}\right)}}, \\ \alpha_2 &= i\sqrt[4]{\frac{63}{32} - \frac{7\sqrt{6}}{8} - \frac{3}{32} i \sqrt{5\left(59 - 24\sqrt{6}\right)}}. \end{aligned}$$

Then, we see that there are sixteen 2-Weierstrass points whose 2-weight are 2: $(\pm\alpha_j, \pm\beta_j)$, $(\pm i\alpha_j, \pm\beta_j)$ $(j = 1, 2)$. For each $j = 1, 2$, these 8 points are conjugate to each other (Remark 3, (1)).

Let $t := x - \alpha_1$ be the local parameter around the point $P = (\alpha_1, \beta_1)$. Then we can write $\omega_1, \cdots, \omega_6$ as follows:

$$\begin{aligned} \omega_1 = \;& ((0.142705 - 0.078956i) - (0.28827 - 0.210933i)t \\ & +(0.331175 - 0.313533i)t^2 - (0.288179 - 0.351726i)t^3 \\ & +(0.231468 - 0.369136i)t^4 + o[t]^5)dt^2, \\ \omega_2 = \;& ((0.0141305 - 0.0225348i) - (0.0489658 - 0.105724i)t \\ & +(0.0836161 - 0.263393i)t^2 - (0.0853732 - 0.466369i)t^3 \\ & +(0.0369109 - 0.673932i)t^4 + o[t]^5)dt^2, \\ \omega_3 = \;& ((0.02482 - 0.0302932i) - (0.0770041 - 0.122803i)t \\ & +(0.122031 - 0.263481i)t^2 - (0.127604 - 0.401956i)t^3 \\ & +(0.0935695 - 0.511142i)t^4 + o[t]^5)dt^2, \\ \omega_4 = \;& ((0.0418728 - 0.0396422i) - (0.110437 - 0.134789i)t \\ & +(0.150119 - 0.239403i)t^2 - (0.138922 - 0.29995i)t^3 \\ & +(0.103894 - 0.32727i)t^4 + o[t]^5)dt^2, \end{aligned}$$

$$\begin{aligned}\omega_5 &= ((0.0685147 - 0.0501338i) - (0.14444 - 0.136745i)t \\ &\quad +(0.153245 - 0.187037i)t^2 - (0.108441 - 0.172937i)t^3 \\ &\quad +(0.0735446 - 0.158792i)t^4 + o[t]^5)dt^2, \\ \omega_6 &= ((0.109375 - 0.0605154i) - (0.167826 - 0.122802i)t \\ &\quad +(0.114093 - 0.108015i)t^2 - (0.0372301 - 0.0454398i)t^3 \\ &\quad +(0.0284395 - 0.0453541i)t^4 + o[t]^5)dt^2.\end{aligned}$$

Consider the matrix $M_4$. Then we find the rank of $M_4$ is 5 (See Appendix). Using Lemma 8 (i), we have $G^{(2)}(P) = \{1,2,3,4,5,8\}$. In a similar manner, we can conclude that the 2-gap sequence at the point $(\alpha_2, \beta_2)$ is $\{1,2,3,4,5,8\}$.

In a similar manner to that in the proof of Theorem 3, we can prove Theorem 4 (for more details, see Alwaleed [1]).

## 4. Appendix

To compute the rank of the matrix $M_l$, one can use Mathematica. For example, we consider the curve $H_{1,\alpha_1}$. Around the point $P = (\alpha_1, \beta_1)$, we can compute the rank of $M_4$ as follows:

**In**[**1**] $:= f := x^8 + 2(a+1)x^4 + (a-1)^2;$

**In**[**2**] $:= f_1 := f/.\left\{a \to \left(17 + 7i\sqrt{15}\right)/32\right\};$

**In**[**3**] $:=\alpha 1 := \sqrt[4]{\frac{63}{32} + \frac{7\sqrt{6}}{8} + \frac{3}{32}i\sqrt{5(59+24\sqrt{6})}}$

**In**[**4**] $:= y_1 := (f_1/.\{x \to t + \alpha_1\})^{1/2}$

**In**[**5**] $:= s_1 := \mathit{Series}[1/y_1, \{t, 0, 4\}];$

**In**[**6**] $:= s_2 := \mathit{Series}[1/y_1^2, \{t, 0, 4\}];$

**In**[**7**] $:= s_3 := \mathit{Series}[(t+\alpha_1)/y_1^2, \{t, 0, 4\}];$

**In**[**8**] $:= s_4 := \mathit{Series}[(t+\alpha_1)^2/y_1^2, \{t, 0, 4\}];$

**In**[**9**] $:= s_5 := \mathit{Series}[(t+\alpha_1)^3/y_1^2, \{t, 0, 4\}];$

**In**[**10**] $:= s_6 := \mathit{Series}[(t+\alpha_1)^4/y_1^2, \{t, 0, 4\}];$

**In**[**11**] $:= c_1 = \mathit{CoefficientList}[s_1, t];$

**In**[**12**] $:= c_2 = \mathit{CoefficientList}[s_2, t];$

**In**[**13**] $:= c_3 = \mathit{CoefficientList}[s_3, t];$

**In**[**14**] $:= c_4 = \mathit{CoefficientList}[s_4, t];$

**In**[**15**] $:= c_5 = \mathit{CoefficientList}[s_5, t];$

**In**[**16**] $:= c_6 = \mathit{CoefficientList}[s_6, t];$

**In**[**17**] $:= M_4 := \{c_1, c_2, c_3, c_4, c_5, c_6\};$

**In**[**18**] $:= \mathit{MatrixRank}[M_4]$

**Out**[**18**] $:= 5$

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