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Asymptotic distribution of eigenvalues of Hill's equation with integrable potential under periodic condition

Takeyuki Nagasawa* and Hirotaka Ohrui

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Abstract

Hill's equation with periodic and integrable potential is considered. In particular asymptotic distribution of eigenvalues is proved, which improves Borg's result.

1. Introduction

In this short note we deal with Hill's equation

$$y'' + (\lambda + Q(x))\,y = 0, \tag{1.1}$$

where the potential Q is a periodic, integrable and real-valued function with period π and λ is a parameter. We consider the asymptotic distribution of eigenvalues $\{\lambda_j\}$ for (1.1) under the periodic condition

$$y(0) = y(\pi), \quad y'(0) = y'(\pi).$$

It is well-known that the $2n-1$-st and $2n$-th eigenvalues, denoted by λ_{2n-1} and λ_{2n}, are near $2n$ for large $n \in \mathbb{N}$ (see Theorem 2.1 below). Borg [1, 2] showed a more precise asymptotics (see also [6, Theorem 2.11]).

Theorem 1.1 *Let Q be a real-valued $L^2(\mathbb{R}/\pi\mathbb{Z})$-function and*

$$\int_0^\pi Q(x)\,dx = 0.$$

Then it holds for any integer n greater than $\|Q\|_{L^1}/(2\pi)$ (not $\|Q\|_{L^2}/(2\pi)$) that

$$\left|\sqrt{\lambda_{2n-1}} - 2n\right| \leqq \frac{\|Q\|_{L^1}}{4n\pi}, \quad \left|\sqrt{\lambda_{2n}} - 2n\right| \leqq \frac{\|Q\|_{L^1}}{4n\pi}.$$

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We improve this result as follows.

Theorem 1.2 *Let Q be a real valued $L^1(\mathbb{R}/\pi\mathbb{Z})$-function and*

$$\int_0^\pi Q(x)\,dx = 0.$$

Then it holds that

$$\left|\sqrt{\lambda_{2n-1}} - 2n\right| = o(n^{-1}), \quad \left|\sqrt{\lambda_{2n}} - 2n\right| = o(n^{-1}) \quad \textit{as} \quad n \to \infty.$$

Indeed, we can show the following, from which Theorem 1.2 is easily derived.

Theorem 1.3 *Let λ be λ_{2n-1} or λ_{2n}, and put $\sqrt{\lambda} - 2n = d$. Under the same assumption as in Theorem 1.2, it holds that*

$$4nd + d^2 = o(1), \quad (4nd + d^2)^2 - |\bar{Q}_{4n}|^2 = O(n^{-1}) \quad \textit{as} \quad n \to \infty,$$

where

$$\bar{Q}_n = \frac{1}{\pi}\int_0^\pi Q(x) e^{-inx} dx.$$

Remark 1.1 The advantage of our theorem is appeared in the case $\displaystyle\int_0^\pi Q(x)\,dx \neq 0$. Put

$$\bar{Q}(= \bar{Q}_0) = \frac{1}{\pi}\int_0^\pi Q(x)\,dx, \quad \tilde{Q}(x) = Q(x) - \bar{Q}.$$

Since our equation is

$$y'' + (\lambda + \bar{Q} + \tilde{Q}(x))y = 0 \quad \text{with} \quad \int_0^\pi \tilde{Q}(x)\,dx = 0,$$

Theorem 1.1 implies that the eigenvalue near $2n$ behaves like

$$\sqrt{\lambda + \bar{Q}} - 2n = O(n^{-1}) \quad \text{as} \quad n \to \infty.$$

Combining this with

$$\sqrt{\lambda + \bar{Q}} = \sqrt{\lambda}\left(1 + \frac{\bar{Q}}{2\lambda} + O(\lambda^{-2})\right),$$

we get

$$\sqrt{\lambda} - 2n = -\frac{\bar{Q}}{2\sqrt{\lambda}} + O(n^{-1}).$$

Both terms in the right hand side are $O(n^{-1})$, and therefore we know only $\sqrt{\lambda} = 2n + O(n^{-1})$. By using Theorem 1.2, we can determine the next term to $2n$ as

$$\sqrt{\lambda} = 2n - \frac{\bar{Q}}{4n} + o(n^{-1}).$$

Remark 1.2 Under higher regularity of Q, the more precise asymptotics have been already known. For example, see [6, Theorems 2.12 and 2.13]. The readers can refer [4, 5, 7] and references cited therein for asymptotic formulae of eigenvalues for Hill's equation with discontinuous coefficient.

2. On $\sqrt{\lambda_{2n-1}}$, $\sqrt{\lambda_{2n}} \sim 2n$

In this section we comment on the fact

$$\sqrt{\lambda_{2n-1}} = 2n + o(1), \quad \sqrt{\lambda_{2n}} = 2n + o(1) \quad \text{as} \quad n \to \infty. \tag{2.1}$$

This was shown in [6] under the assumption

$$Q \in L^\infty(\mathbb{R}/\pi\mathbb{Z}), \quad \int_0^\pi Q(x)\, dx = 0.$$

Borg [2, pp.88ff.] had investigated the problem under $Q \in L^2(\mathbb{R}/\pi\mathbb{Z})$. Here we assert that

Theorem 2.1 *The asymptotics* (2.1) *holds under*

$$Q \in L^1(\mathbb{R}/\pi\mathbb{Z}), \quad \int_0^\pi Q(x)\, dx = 0.$$

Since Hill's equation has very long history, we find about 600 papers on MathSciNet, some of them are not easy to access now. The authors could not study all of them, and are not sure whether the above theorem has been already known or not. Even if it is not new, they would like to give the proof here for the sake of completeness. Since it is almost parallel with that in [6], we mention only its outline.

Proof. Let $y_1(\cdot, \lambda)$ and $y_2(\cdot, \lambda)$ be solutions of (1.1) satisfying

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \tag{2.2}$$

(2.3) $$y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1.$$

Put

$$\Delta(\lambda) = y_1(\pi,\lambda) + y_2'(\pi,\lambda),$$

then λ is an eigenvalue if and only if $\Delta(\lambda) = 2$ ([6, 3]). Put

$$\theta(\lambda) = \frac{-\Delta(\lambda)+2}{2(1-\cos\pi\sqrt{\lambda})}.$$

Since both numerator and denominator are analytic in λ, in order to show (2.1) we show

(2.4) $$\frac{1}{2\pi i}\int_{C_n} \frac{d}{d\lambda}\log\theta(\lambda)\,d\lambda = 0$$

for a closed curve $C_n \subset \mathbb{C}$ containing

$$\{\lambda \in \mathbb{C} \,|\, |\lambda| \leqq (2n+1)^2,\ \lambda \in \mathbb{R}\}$$

in its interior, provided n is sufficient large. It is easy to see

$$\theta(\lambda) - 1 = \frac{1}{2(1-\cos\pi\sqrt{\lambda})}\int_0^\pi Y(x,\lambda)Q(x)\,dx,$$

where

$$Y(x,\lambda) = \frac{\sin\sqrt{\lambda}(\pi-x)}{\sqrt{\lambda}} y_1(x,\lambda) + \left\{\cos\sqrt{\lambda}(\pi-x)\right\} y_2(x,\lambda).$$

Take

$$C_n = \left\{\lambda \in \mathbb{C} \,\middle|\, \sqrt{\lambda} \in C_{1,n} \cup C_{2,n} \cup C_{3,n}\right\},$$

where

$$C_{1,n} = \{\xi - i(2n+1) \in \mathbb{C} \,|\, 0 \leqq \xi \leqq 2n+1\},$$
$$C_{2,n} = \{2n+1+i\eta \in \mathbb{C} \,|\, |\eta| \leqq 2n+1\},$$
$$C_{3,n} = \{\xi + i(2n+1) \in \mathbb{C} \,|\, 2n+1 \geqq \xi \geqq 0\}.$$

From now on λ is on C_n. Then we have

$$2e^{-\pi|\Im\sqrt{\lambda}|}\left(1-\cos\pi\sqrt{\lambda}\right) = 1 + o(1) \quad \text{as} \quad n \to \infty.$$

Therefore we know

(2.5) $$\theta(\lambda) - 1 = o(1) \quad \text{as} \quad n \to \infty$$

provided

$$\int_0^\pi Y(x,\lambda)Q(x)\,dx = o\left(e^{\pi|\Im\sqrt{\lambda}|}\right) \quad \text{as} \quad n\to\infty \tag{2.6}$$

holds. (2.5) implies (2.4) for large n. Thus what we should show is (2.6).

By successive approximation, the functions y_i are given by

$$y_i(x,\lambda) = \sum_{j=0}^\infty y_{i,j}(x,\lambda), \quad y_{1,0}(x,\lambda) = \cos\sqrt{\lambda}x, \quad y_{2,0}(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$y_{i,j}(x,\lambda) = -\frac{1}{\sqrt{\lambda}}\int_0^x \left\{\sin\sqrt{\lambda}(x-x_1)\right\} Q(x_1)y_{i,j-1}(x_1,\lambda)\,dx_1 \quad (j \geqq 1).$$

We can prove by induction

$$|y_{i,j}(x,\lambda)| \leqq \frac{e^{|\Im\sqrt{\lambda}|x}}{j!\left|\sqrt{\lambda}\right|^{i+j-1}}\left(\int_0^x |Q(x_1)|\,dx_1\right)^j$$

for $x \geqq 0$. It follows from this that $\sum_{j=0}^\infty y_{i,j}(x,\lambda)$ converges uniformly in $x\in[0,\pi]$ for every $\lambda \in C_n$. Therefore

$$\int_0^\pi Y(x,\lambda)Q(x)\,dx = \sum_{j=0}^\infty \int_0^\pi Y_j(x,\lambda)Q(x)\,dx$$

holds, where

$$Y_j(x,\lambda) = \frac{\sin\sqrt{\lambda}(\pi-x)}{\sqrt{\lambda}}y_{1,j}(x,\lambda) + \left\{\cos\sqrt{\lambda}(\pi-x)\right\} y_{2,j}(x,\lambda).$$

Since Y_0 is independent of x,

$$\int_0^\pi Y_0(x,\lambda)Q(x)\,dx = 0.$$

From above estimates we get

$$\left|\int_0^\pi Y_j(x,\lambda)Q(x)\,dx\right| \leqq \frac{2e^{\pi|\Im\sqrt{\lambda}|}\|Q\|_{L^1}^{j+1}}{(j+1)!\left|\sqrt{\lambda}\right|^{j+1}}$$

for $j \geqq 1$. Hereafter L^p means $L^p(0,\pi)$. Consequently

$$\left|\int_0^\pi Y(x,\lambda)Q(x)\,dx\right| \leqq 2e^{\pi|\Im\sqrt{\lambda}|}\left\{\exp\left(\frac{\|Q\|_{L^1}}{\left|\sqrt{\lambda}\right|}\right) - 1 - \frac{\|Q\|_{L^1}}{\left|\sqrt{\lambda}\right|}\right\},$$

and (2.6) follows. □

3. Proof of Theorem 1.3

In this section λ is λ_{2n-1} or λ_{2n}, and put

$$\sqrt{\lambda} - 2n = d.$$

The functions y_i satisfy

$$y_i'' + (2n)^2 y_i = -(4nd + d^2 + Q(x))y_i.$$

Taking their initial conditions into account, we have

$$y_1 = \cos 2nx + Ky_1, \quad y_2 = \frac{\sin 2nx}{2n} + Ky_2, \tag{3.1}$$

where K is a bounded operator from $L^\infty(0,\pi)$ into itself defined by

$$(Ky)(x) = -\frac{1}{2n}\int_0^x \{\sin 2n(x-x_1)\}(4nd + d^2 + Q(x_1))y(x_1)\,dx_1.$$

Since $d = o(1)$ as $n \to \infty$, it is easy to see

$$\|K\|_{L^\infty \to L^\infty} \leqq \frac{1}{2n}\left\{\pi|4nd + d^2| + \|Q\|_{L^1}\right\} = o(1).$$

For n so large that $\|K\|_{L^\infty \to L^\infty} < 1$, we have

$$\|y_1\|_{L^\infty} \leqq \frac{1}{1 - \|K\|_{L^\infty \to L^\infty}}, \quad \|y_2\|_{L^\infty} \leqq \frac{1}{2n(1 - \|K\|_{L^\infty \to L^\infty})}.$$

The derivative of y_2 is

$$y_2' = \cos 2nx - \int_0^x \{\cos 2n(x-x_1)\}\,(4nd + d^2 + Q(x_1))y_2(x_1)\,dx_1.$$

Therefore the condition $\Delta(\lambda) = 2$ is equivalent to

$$\int_0^\pi U_n(x)(4nd + d^2 + Q(x))\,dx = 0,$$

where

$$U_n(x) = \frac{\sin 2nx}{2n} y_1(x) - (\cos 2nx)\, y_2(x).$$

Putting (3.1) into this twice, we have

$$\begin{aligned} U_n(x) = {} & \frac{\sin 2nx}{2n}(Ky_1)(x) - (\cos 2nx)\,(Ky_2)(x) \\ = {} & -\frac{1}{4n^2}\int_0^x \left\{\sin^2 2n(x-x_1)\right\}(4nd + d^2 + Q(x_1))\,dx_1 \\ & + \frac{\sin 2nx}{2n}(K^2 y_1)(x) - (\cos 2nx)\,(K^2 y_2)(x). \end{aligned}$$

Therefore it holds that

$$
\begin{aligned}
0 &= \int_0^\pi U_n(x)(4nd + d^2 + Q(x))\,dx \\
&= -\frac{1}{4n^2}\int_0^\pi\int_0^x \left\{\sin^2 2n(x-x_1)\right\}(4nd+d^2+Q(x))(4nd+d^2+Q(x_1))\,dx_1dx \\
&\quad + \int_0^\pi \left\{\frac{\sin 2nx}{2n}(K^2y_1)(x) - (\cos 2nx)(K^2y_2)(x)\right\}(4nd+d^2+Q(x))\,dx.
\end{aligned}
$$

It is easy to see

$$
\int_0^\pi \int_0^x \sin^2 2n(x-x_1)\,dx_1dx = \frac{\pi^2}{4}.
$$

Using the condition $\int_0^\pi Q(x)\,dx = 0$, we have

$$
\int_0^\pi \int_0^x \left\{\sin^2 2n(x-x_1)\right\}(Q(x_1)+Q(x))\,dx_1dx = 0,
$$

$$
\int_0^\pi \int_0^x \left\{\sin^2 2n(x-x_1)\right\}Q(x_1)Q(x)\,dx_1dx = -\frac{\pi^2}{4}|\bar{Q}_{4n}|^2.
$$

Hence d satisfies

$$
\begin{aligned}
&\left|(4nd+d^2)^2 - |\bar{Q}_{4n}|^2\right| \\
&= \left|\frac{16n^2}{\pi^2}\int_0^\pi\left\{\frac{\sin 2nx}{2n}(K^2y_1)(x) - (\cos 2nx)(K^2y_2)(x)\right\}(4nd+d^2+Q(x))\,dx\right| \\
&\leqq \frac{16n^2}{\pi^2}\|K\|^2_{L^\infty\to L^\infty}\left(\frac{\|y_1\|_{L^\infty}}{2n} + \|y_2\|_{L^\infty}\right)\left(\pi|4nd+d^2| + \|Q\|_{L^1}\right) \\
&\leqq \frac{16}{n\pi^2}\frac{\left(\pi|4nd+d^2| + \|Q\|_{L^1}\right)^3}{1-\|K\|_{L^\infty\to L^\infty}} \\
&\leqq \frac{C\left(\pi|4nd+d^2| + \|Q\|_{L^1}\right)}{n}\left\{(4nd+d^2)^2+1\right\}.
\end{aligned}
\tag{3.2}
$$

Since we have already known $d = o(1)$, it holds

$$
\pi|4nd+d^2| + \|Q\|_{L^1} = o(n).
$$

Combining this and $\bar{Q}_{4n} = o(1)$ (by Riemann-Lebesgue's Lemma) with (3.2), we obtain the assertion of Theorem 1.3 □

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Takeyuki Nagasawa
Hirotaka Ohrui
Department of Mathematics
Graduate School of Science and Engineering
Saitama University
255 Shimo-Okubo
Sakura-ku, Saitama 338-8570, Japan
e-mail: tnagasaw@rimath.saitama-u.ac.jp
s08mm101@mail.saitama-u.ac.jp