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Positivity and Hierarchical Structure of 16 Green Functions Corresponding to a Bending Problem of a Beam

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Abstract

We consider 2-point boundary value problem for 4-th order linear ordinary differential equation, which represents bending of a beam on an elastic foundation under a tension. A tension is relatively stronger than a spring constant of elastic foundation. We here treat 4 self-adjoint boundary conditions, clamped, Dirichlet, Neumann and free edges, on each side. Hence we have investigated $4 \times 4 = 16$ boundary value problems. We show the positivity and the hierarchical structure of 16 Green functions.

1. Conclusion

A beam is supported by uniformly distributed springs with spring constant $q > 0$ on a fixed floor and is exerted a tension $p > 0$ on both sides. Under a density of a load $f(x)$, a bending of a beam $u(x)$ satisfies the following boundary value problem [1, 3, 4]:

$$\begin{cases} u^{(4)} - pu'' + qu = f(x) & (-1 < x < 1) \\ u_{\alpha_i}(-1) = u_{\beta_i}(1) = 0 & (i = 0, 1), \end{cases} \tag{1.1}$$

where $u_0 = u,\ u_1 = u',\ u_2 = u'',\ u_3 = u''',\ u_{\widetilde{3}} = u_3 - pu_1$. We here treat only four self-adjoint cases $\alpha = (\alpha_0, \alpha_1),\ \beta = (\beta_0, \beta_1) = (0,1), (0,2), (1,3), (2,\widetilde{3})$, which have engineering importance and correspond to clamped, Dirichlet(simply-supported), Neumann(sliding) and free edge, respectively [2, Chap. 2]. Therefore, 16 kinds of boundary value problems can be considered. We assume that a tension is relatively stronger than a spring constant. That is to say, we impose the following two equivalent assumptions:

Assumption 1.1. $(p/2)^2 > q > 0, \quad p > 0.$

Assumption 1.2. $p = a^2 + b^2, \quad q = a^2 b^2, \quad a > b > 0.$

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Under these assumptions, (1.1) is solved as

$$u(x) = \int_{-1}^{1} G(\alpha, \beta; x, y) f(y) dy \qquad (-1 < x < 1), \tag{1.2}$$

where $G(\alpha, \beta; x, y)$ is an impulse response, that is Green function. The above bending problem of a beam is important in the field of classical mechanics of materials. The purpose of this paper is to give a mathematical foundation of this problem. We here present main conclusion in this paper.

Theorem 1.1. 16 *Green functions are positive valued and satisfy the hierarchical structure shown in Figures.* 1 *and* 2 *in the cases* $b^{-2}\cosh(2b) < a^{-2}\cosh(2a)$ *and* $b^{-2}\cosh(2b) \geq a^{-2}\cosh(2a)$, *respectively. In these figures,* $\boxed{\alpha_0\alpha_1\beta_0\beta_1}$ *means* $G(\alpha, \beta; x, y) = G(\alpha_0, \alpha_1, \beta_0, \beta_1; x, y)$, $\boxed{0}$ *means* 0, $A \to B$ *means* $A \leq B$ $(-1 \leq x, y \leq 1)$ *and* $A \times B$ *means that* $B - A$ *takes both positive and negative values.*

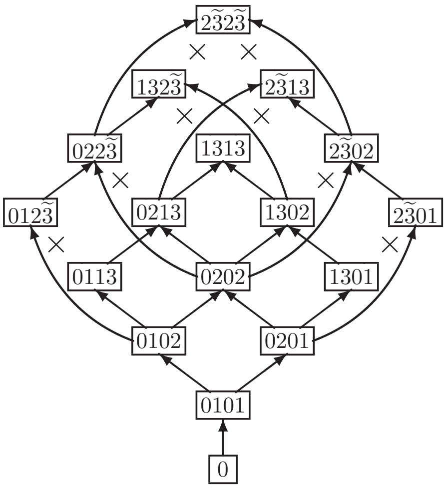

Figure 1 $b^{-2}\cosh(2b) < a^{-2}\cosh(2a)$.

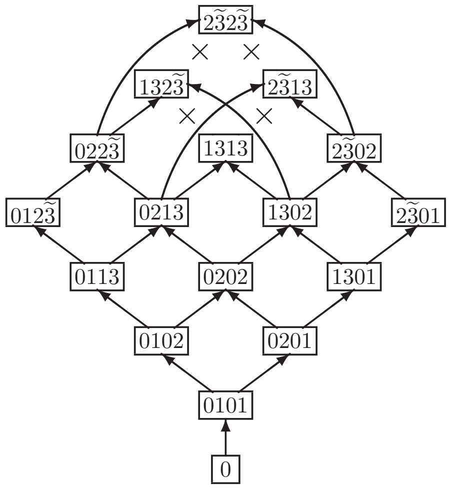

Figure 2 $b^{-2}\cosh(2b) \geq a^{-2}\cosh(2a)$.

This theorem shows that if boundary condition becomes looser as $(0,1) \to (0,2) \to (1,3) \to (2,\widetilde{3})$, the impulse response gets larger in most cases.

In the previous paper [1], we derived 9 self-adjoint Green functions $G(\alpha, \beta; x, y)$ for $\alpha, \beta = (0,1),\ (0,2),\ (1,3)$ [1, Theorem 4.1] and showed their positivity and hierarchical structure [1, Theorem 7.1]. Here we have one more condition $\alpha, \beta = (2,\widetilde{3})$ and have obtained a more detailed hierarchical structure among 16 self-adjoint Green functions. Moreover, although we have proved the positivity of Green function $G(0,1,0,1;x,y)$ in [1, Theorem 6.1], the proof requires tedious calculations. In this paper, we give another simpler proof.

2. Green function $G(x,y)$, function $K_j(x)$ and fundamental solution $A_i(x), B_i(x)$

We introduce a key function

$$K_0(x) = d^{-1}\left(a^{-1}\mathrm{sh}(ax) - b^{-1}\mathrm{sh}(bx)\right),$$

where $\mathrm{ch}(x) = \cosh(x),\ \mathrm{sh}(x) = \sinh(x)$ and $d = a^2 - b^2$. $u = K_0(x)$ satisfies the following initial value problem:

$$\begin{cases} u^{(4)} - pu'' + qu = 0 \qquad (0 < x < \infty) \\ u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 1. \end{cases}$$

We also define $K_j(x) = (d/dx)^j K_0(x)$ $(j = 1, 2, \cdots)$. $K_j(x)$ satisfies the recurrence relation

$$K_{j+4}(x) - pK_{j+2}(x) + qK_j(x) = 0 \qquad (-\infty < x < \infty).$$

Employing $K_j(x)$, we also introduce functions $\widetilde{K}_j(x),\ \widetilde{\widetilde{K}}_j(x),\ K(i,j,k,l;x)$ defined by

$$\widetilde{K}_j(x) = K_j(x) - pK_{j-2}(x), \quad \widetilde{\widetilde{K}}_j(x) = \widetilde{K}_j(x) - p\widetilde{K}_{j-2}(x),$$
$$K(i,j,k,l;x) = K_i(x)K_j(x) - K_k(x)K_l(x).$$

In particular, we give an explicit form of $\widetilde{K}_3(x)$,

$$\widetilde{K}_3(x) = -qd^{-1}\left(a^{-2}\mathrm{ch}(ax) - b^{-2}\mathrm{ch}(bx)\right),$$

which is important in the sense that the hierarchical structure stated in Theorem 1.1 differs according to the positivity of $\widetilde{K}_3(2)$. From Taylor series of $K_0(x)$ as

$$K_0(x) = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}\,\frac{a^{2j} - b^{2j}}{a^2 - b^2}\,x^{2j+1}$$

and $a > b > 0$, we have

$$K_j(x) > 0 \qquad (j = 0, 1, 2, \cdots, 0 < x < \infty),$$
$$\widetilde{K}_{j+4}(x) = K_{j+4}(x) - pK_{j+2}(x) = -qK_j(x) < 0 \quad (j = 0, 1, 2, \cdots, 0 < x < \infty).$$

We here use abbreviations $K_j = K_j(2),\ \widetilde{K}_j = \widetilde{K}_j(2),\ \widetilde{\widetilde{K}}_j = \widetilde{\widetilde{K}}_j(2),\ K(i,j,k,l) =$

$K(i,j,k,l;2)$. Using the functions $K_j(x)$, we have concrete forms of Green functions.

Theorem 2.1. *For any bounded continuous function $f(x)$ on an interval $-1<x<1$, the boundary value problem*

$$\mathrm{BVP}(\alpha,\beta)$$
$$\begin{cases} u^{(4)}-pu''+qu=f(x) & (-1<x<1) \\ u_{\alpha_i}(-1)=a_i, \quad u_{\beta_i}(1)=b_i & (i=0,1) \end{cases}$$

has a unique classical solution given by

$$u(x)=\int_{-1}^{1} G(x,y)f(y)dy+\sum_{i=0}^{1}\Big[a_iA_i(x)+b_iB_i(x)\Big] \qquad (-1<x<1), \quad (2.1)$$

where $G(x,y)=G(\alpha,\beta;x,y)$ are Green functions and $A_i(x)=A_i(\alpha,\beta;x)$, $B_i(x)=B_i(\alpha,\beta;x)$ $(i=0,1)$ are fundamental solutions satisfying the prescribed boundary conditions. We enumerate the concrete forms of Green functions and fundamental solutions. Note that $x\vee y=\max\{x,y\}$ and $x\wedge y=\min\{x,y\}$.

(i) $(\alpha,\beta)=(0,1,0,1)$

$$G(x,y)=-\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}K_0 & K_1\\ K_1 & K_2\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_1\end{pmatrix}(1-x\vee y)=$$
$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_0\\ K_1\end{pmatrix}(1-x\vee y)=\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$
$$\big(B_0,\,B_1\big)(x)=\big(K_0,\,K_1\big)(1+x)\begin{pmatrix}K_0 & K_1\\ K_1 & K_2\end{pmatrix}^{-1}$$
$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_0 & K_1\\ K_1 & K_2\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_1\end{pmatrix}(1-x)$$

(ii) $(\alpha,\beta)=(0,1,0,2)$

$$G(x,y)=-\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}K_0 & K_1\\ K_2 & K_3\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x\vee y)=$$
$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x\vee y)=\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$
$$\big(B_0,\,B_1\big)(x)=\big(K_0,\,K_1\big)(1+x)\begin{pmatrix}K_0 & K_1\\ K_2 & K_3\end{pmatrix}^{-1}$$
$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_0 & K_1\\ K_2 & K_3\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x)$$

(iii) $(\alpha,\beta)=(0,1,1,3)$

$$G(x,y)=-\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}K_1 & K_2\\ K_3 & K_4\end{pmatrix}^{-1}\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\,B_1\big)(x)=\big(K_0,\,K_1\big)(1+x)\begin{pmatrix}K_1 & K_2\\ K_3 & K_4\end{pmatrix}^{-1}$$

$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_1 & K_2\\ K_3 & K_4\end{pmatrix}^{-1}\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x)$$

(iv) $(\alpha,\beta)=(0,1,2,\widetilde{3})$

$$G(x,y)=-\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}K_2 & K_3\\ \widetilde{K}_3 & \widetilde{K}_4\end{pmatrix}^{-1}\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x\vee y)=\big(K_0,\,K_1\big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\,B_1\big)(x)=\big(K_0,\,K_1\big)(1+x)\begin{pmatrix}K_2 & K_3\\ \widetilde{K}_3 & \widetilde{K}_4\end{pmatrix}^{-1}$$

$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_2 & K_3\\ \widetilde{K}_3 & \widetilde{K}_4\end{pmatrix}^{-1}\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x)$$

(v) $(\alpha,\beta)=(0,2,0,2)$

$$G(x,y)=-\big(K_0,\,K_2\big)(1+x\wedge y)\begin{pmatrix}K_0 & K_2\\ K_2 & K_4\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x\vee y)=-\big(K_0,\,K_2\big)(1+x\wedge y)\begin{pmatrix}A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\,B_1\big)(x)=\big(K_0,\,K_2\big)(1+x)\begin{pmatrix}K_0 & K_2\\ K_2 & K_4\end{pmatrix}^{-1}$$

$$\begin{pmatrix}A_0\\ A_1\end{pmatrix}(x)=\begin{pmatrix}K_0 & K_2\\ K_2 & K_4\end{pmatrix}^{-1}\begin{pmatrix}K_0\\ K_2\end{pmatrix}(1-x)$$

(vi) $(\alpha,\beta)=(0,2,1,3)$

$$G(x,y)=-\big(K_0,\,K_2\big)(1+x\wedge y)\begin{pmatrix}K_1 & K_3\\ K_3 & K_5\end{pmatrix}^{-1}\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=-\big(K_0,\,K_2\big)(1+x\wedge y)\begin{pmatrix}A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\, B_1\big)(x) = \big(K_0,\, K_2\big)(1+x)\begin{pmatrix} K_1 & K_3 \\ K_3 & K_5 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x) = \begin{pmatrix} K_1 & K_3 \\ K_3 & K_5 \end{pmatrix}^{-1}\begin{pmatrix} K_1 \\ K_3 \end{pmatrix}(1-x)$$

(vii) $(\alpha,\beta) = (0,2,2,\widetilde{3})$

$$G(x,y) = -\big(K_0,\, K_2\big)(1+x\wedge y)\begin{pmatrix} K_2 & K_4 \\ \widetilde{K}_3 & \widetilde{K}_5 \end{pmatrix}^{-1}\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x\vee y) =$$

$$-\big(B_0,\, B_1\big)(x\wedge y)\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x\vee y) = -\big(K_0,\, K_2\big)(1+x\wedge y)\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x\vee y)$$

$$\big(B_0,\, B_1\big)(x) = \big(K_0,\, K_2\big)(1+x)\begin{pmatrix} K_2 & K_4 \\ \widetilde{K}_3 & \widetilde{K}_5 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x) = \begin{pmatrix} K_2 & K_4 \\ \widetilde{K}_3 & \widetilde{K}_5 \end{pmatrix}^{-1}\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x)$$

(viii) $(\alpha,\beta) = (1,3,1,3)$

$$G(x,y) = -\big(K_1,\, K_3\big)(1+x\wedge y)\begin{pmatrix} K_2 & K_4 \\ K_4 & K_6 \end{pmatrix}^{-1}\begin{pmatrix} K_1 \\ K_3 \end{pmatrix}(1-x\vee y) =$$

$$-\big(B_0,\, B_1\big)(x\wedge y)\begin{pmatrix} K_1 \\ K_3 \end{pmatrix}(1-x\vee y) = \big(K_1,\, K_3\big)(1+x\wedge y)\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x\vee y)$$

$$\big(B_0,\, B_1\big)(x) = \big(K_1,\, K_3\big)(1+x)\begin{pmatrix} K_2 & K_4 \\ K_4 & K_6 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x) = -\begin{pmatrix} K_2 & K_4 \\ K_4 & K_6 \end{pmatrix}^{-1}\begin{pmatrix} K_1 \\ K_3 \end{pmatrix}(1-x)$$

(ix) $(\alpha,\beta) = (1,3,2,\widetilde{3})$

$$G(x,y) = -\big(K_1,\, K_3\big)(1+x\wedge y)\begin{pmatrix} K_3 & K_5 \\ \widetilde{K}_4 & \widetilde{K}_6 \end{pmatrix}^{-1}\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x\vee y) =$$

$$-\big(B_0,\, B_1\big)(x\wedge y)\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x\vee y) = \big(K_1,\, K_3\big)(1+x\wedge y)\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x\vee y)$$

$$\big(B_0,\, B_1\big)(x) = \big(K_1,\, K_3\big)(1+x)\begin{pmatrix} K_3 & K_5 \\ \widetilde{K}_4 & \widetilde{K}_6 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}(x) = -\begin{pmatrix} K_3 & K_5 \\ \widetilde{K}_4 & \widetilde{K}_6 \end{pmatrix}^{-1}\begin{pmatrix} K_2 \\ \widetilde{K}_3 \end{pmatrix}(1-x)$$

(x) $(\alpha,\beta)=(2,\widetilde{3},1,3)$

$$G(x,y)=-\Big(K_2,\,\widetilde{K}_3\Big)(1+x\wedge y)\begin{pmatrix}K_3 & \widetilde{K}_4\\ K_5 & \widetilde{K}_6\end{pmatrix}^{-1}\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x\vee y)=\Big(K_2,\,\widetilde{K}_3\Big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\,B_1\big)(x)=\Big(K_2,\,\widetilde{K}_3\Big)(1+x)\begin{pmatrix}K_3 & \widetilde{K}_4\\ K_5 & \widetilde{K}_6\end{pmatrix}^{-1}$$

$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_3 & \widetilde{K}_4\\ K_5 & \widetilde{K}_6\end{pmatrix}^{-1}\begin{pmatrix}K_1\\ K_3\end{pmatrix}(1-x)$$

(xi) $(\alpha,\beta)=(2,\widetilde{3},2,\widetilde{3})$

$$G(x,y)=-\Big(K_2,\,\widetilde{K}_3\Big)(1+x\wedge y)\begin{pmatrix}K_4 & \widetilde{K}_5\\ \widetilde{K}_5 & \widetilde{\widetilde{K}}_6\end{pmatrix}^{-1}\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x\vee y)=$$

$$-\big(B_0,\,B_1\big)(x\wedge y)\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x\vee y)=\Big(K_2,\,\widetilde{K}_3\Big)(1+x\wedge y)\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(x\vee y)$$

$$\big(B_0,\,B_1\big)(x)=\Big(K_2,\,\widetilde{K}_3\Big)(1+x)\begin{pmatrix}K_4 & \widetilde{K}_5\\ \widetilde{K}_5 & \widetilde{\widetilde{K}}_6\end{pmatrix}^{-1}$$

$$\begin{pmatrix}A_0\\ -A_1\end{pmatrix}(x)=\begin{pmatrix}K_4 & \widetilde{K}_5\\ \widetilde{K}_5 & \widetilde{\widetilde{K}}_6\end{pmatrix}^{-1}\begin{pmatrix}K_2\\ \widetilde{K}_3\end{pmatrix}(1-x)$$

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1. *Green functions and fundamental solutions satisfy the following symmetric properties.*

$$G(\alpha,\beta;x,y)=G(\beta,\alpha;-x,-y)$$

$$A_0(0,2,0,1;x)=B_0(0,1,0,2;-x)\qquad A_1(0,2,0,1;x)=B_1(0,1,0,2;-x)$$
$$B_0(0,2,0,1;x)=A_0(0,1,0,2;-x)\qquad B_1(0,2,0,1;x)=-A_1(0,1,0,2;-x)$$

$$A_0(1,3,0,1;x)=-B_0(0,1,1,3;-x)\qquad A_1(1,3,0,1;x)=-B_1(0,1,1,3;-x)$$
$$B_0(1,3,0,1;x)=A_0(0,1,1,3;-x)\qquad B_1(1,3,0,1;x)=-A_1(0,1,1,3;-x)$$

$$A_0(1,3,0,2;x)=-B_0(0,2,1,3;-x)\qquad A_1(1,3,0,2;x)=-B_1(0,2,1,3;-x)$$
$$B_0(1,3,0,2;x)=A_0(0,2,1,3;-x)\qquad B_1(1,3,0,2;x)=A_1(0,2,1,3;-x)$$

$$A_0(2,\widetilde{3},0,1;x)=B_0(0,1,2,\widetilde{3};-x)\qquad A_1(2,\widetilde{3},0,1;x)=-B_1(0,1,2,\widetilde{3};-x)$$
$$B_0(2,\widetilde{3},0,1;x)=A_0(0,1,2,\widetilde{3};-x)\qquad B_1(2,\widetilde{3},0,1;x)=-A_1(0,1,2,\widetilde{3};-x)$$

$$A_0(2,\widetilde{3},0,2;x) = B_0(0,2,2,\widetilde{3};-x) \qquad A_1(2,\widetilde{3},0,2;x) = -B_1(0,2,2,\widetilde{3};-x)$$
$$B_0(2,\widetilde{3},0,2;x) = A_0(0,2,2,\widetilde{3};-x) \qquad B_1(2,\widetilde{3},0,2;x) = A_1(0,2,2,\widetilde{3};-x)$$

$$A_0(2,\widetilde{3},1,3;x) = B_0(1,3,2,\widetilde{3};-x) \qquad A_1(2,\widetilde{3},1,3;x) = -B_1(1,3,2,\widetilde{3};-x)$$
$$B_0(2,\widetilde{3},1,3;x) = -A_0(1,3,2,\widetilde{3};-x) \qquad B_1(2,\widetilde{3},1,3;x) = -A_1(1,3,2,\widetilde{3};-x)$$

So it is enough to treat 10 Green functions. In order to show the hierarchical structure, however, we also need concrete forms of $G(2,\widetilde{3},1,3;x,y)$ as well as $G(1,3,2,\widetilde{3};x,y)$.

The uniqueness of the solution to BVP(α,β) and the concrete form of Green function $G(x,y)$ and fundamental solution $A_i(x)$, $B_i(x)$ are shown by taking similar procedures as [1, Theorem 2.1, 3.1 and 4.1]. The existence of the solution to BVP(α,β) is shown by using the next Lemma 2.1.

Lemma 2.1. (1) *Green function $G(x,y) = G(\alpha,\beta;x,y)$ satisfies the following properties:*

$$\left[\partial_x^4 - p\partial_x^2 + q\right]G(x,y) = 0 \qquad (-1 < x,y < 1, \quad x \neq y),$$
$$\partial_x^{\alpha_i}G(x,y)\Big|_{x=-1} = \partial_x^{\beta_i}G(x,y)\Big|_{x=1} = 0 \qquad (i=0,1, \quad -1<y<1),$$
$$\partial_x^{i}G(x,y)\Big|_{y=x-0} - \partial_x^{i}G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (i=0,1,2) \\ 1 & (i=3) \end{cases} \qquad (-1<x<1).$$

Note that $\partial_x^{\widetilde{3}} = \partial_x^3 - p\partial_x$.
(2) *The fundamental solutions $A_i = A_i(x) = A_i(\alpha,\beta;x)$ and $B_i = B_i(x) = B_i(\alpha,\beta;x)$ $(i=0,1)$ satisfies the following boundary value problems, respectively:*

$$\begin{cases} A_i^{(4)} - pA_i'' + qA = 0 & (-1<x<1) \\ A_i^{(\alpha_j)}(-1) = \delta_{ij}, \quad A_i^{(\beta_j)}(1) = 0 & (j=0,1), \end{cases}$$
$$\begin{cases} B_i^{(4)} - pB_i'' + qB = 0 & (-1<x<1) \\ B_i^{(\alpha_j)}(-1) = 0, \quad B_i^{(\beta_j)}(1) = \delta_{ij} & (j=0,1), \end{cases}$$

where $\delta_{ij} = 0 (i \neq j),\ 1 (i=j)$. Note that $A_1^{(\widetilde{3})}(-1) = (A_1''' - pA_1')(-1)$, $B_1^{(\widetilde{3})}(1) = (B_1''' - pB_1')(1)$.

The proof of the above lemma is done by calculating derivatives of $G(x,y)$, $A_i(x)$ and $B_i(x)$ given in Theorem 2.1 directly.

Next, we present some lemmas concerning the functions $K_j(x)$.

Lemma 2.2. *The following functions are expressed as follows and are positive for $0 < x < \infty$:*

$$K(1,1,0,2;x) = \frac{1}{ab}\left[\left(\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 - \left(\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2\right] > 0$$

$$K(1,2,0,3;x) = q^{-1}K(3,\widetilde{4},1,\widetilde{6};x) = \frac{1}{2ab}\left[\frac{\mathrm{sh}((a+b)x)}{a+b} - \frac{\mathrm{sh}((a-b)x)}{a-b}\right] > 0$$

$$K(2,2,0,4;x) = q^{-1}K(4,4,2,6;x) = \frac{1}{2ab}\Big[\mathrm{ch}((a+b)x) - \mathrm{ch}((a-b)x)\Big] > 0$$

$$K(1,\widetilde{3},0,\widetilde{4};x) = \left(\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 + \left(\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2 > 0$$

$$K(2,3,1,4;x) = K(2,\widetilde{3},0,\widetilde{5};x) = \frac{1}{2}\left[\frac{\mathrm{sh}((a+b)x)}{a+b} + \frac{\mathrm{sh}((a-b)x)}{a-b}\right] > 0$$

$$K(3,3,1,5;x) = \frac{1}{2}\Big[\mathrm{ch}((a+b)x) + \mathrm{ch}((a-b)x)\Big] > 0$$

$$K(3,4,1,6;x) = K(3,\widetilde{4},1,\widetilde{6};x) + pK(2,3,1,4;x) > 0$$

$$K(1,3,0,4;x) = q^{-1}K(\widetilde{3},\widetilde{5},2,\widetilde{\widetilde{6}};x) =$$
$$\frac{1}{ab}\left[(a^2+ab+b^2)\left(\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 - (a^2-ab+b^2)\left(\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2\right] > 0$$

$$K(2,3,0,5;x) = q^{-1}K(\widetilde{4},5,3,\widetilde{6};x) =$$
$$\frac{1}{2ab}\left[(a^2+ab+b^2)\frac{\mathrm{sh}((a+b)x)}{a+b} - (a^2-ab+b^2)\frac{\mathrm{sh}((a-b)x)}{a-b}\right] > 0$$

$$K(3,\widetilde{3},2,\widetilde{4};x) =$$
$$(a^2+ab+b^2)\left(\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 + (a^2-ab+b^2)\left(\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2 + 1 > 0$$

$$K(\widetilde{3},4,2,\widetilde{5};x) = \frac{1}{2}\left[(a^2+ab+b^2)\frac{\mathrm{sh}((a+b)x)}{a+b} + (a^2-ab+b^2)\frac{\mathrm{sh}((a-b)x)}{a-b}\right] > 0$$

$$K(\widetilde{3},5,2,\widetilde{6};x) = \frac{1}{2}\left[\left(\frac{a^2+ab+b^2}{a+b}\right)^2\mathrm{ch}((a+b)x) + \left(\frac{a^2-ab+b^2}{a-b}\right)^2\mathrm{ch}((a-b)x)\right]$$
$$-\frac{a^2b^2(a^2+b^2)}{(a^2-b^2)^2} > 0$$

$$K(\widetilde{5},\widetilde{5},4,\widetilde{\widetilde{6}};x) =$$
$$ab\left[\left((a^2+ab+b^2)\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 - \left((a^2-ab+b^2)\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2\right] > 0.$$

Lemma 2.3. *The following functions are monotone decreasing in the interval* $0 < x < \infty$*:*

$$\left(\frac{K_1}{K_0}\right)'(x) = -\frac{K(1,1,0,2;x)}{\{K_0(x)\}^2} < 0, \qquad \left(\frac{K_2}{K_0}\right)'(x) = -\frac{K(1,2,0,3;x)}{\{K_0(x)\}^2} < 0,$$

$$\left(\frac{K_3}{K_1}\right)'(x) = -\frac{K(2,3,1,4;x)}{\{K_1(x)\}^2} < 0, \qquad \left(\frac{\widetilde{K}_3}{K_2}\right)'(x) = -\frac{K(3,\widetilde{3},2,\widetilde{4};x)}{\{K_2(x)\}^2} < 0.$$

Lemmas 2.2 and 2.3 are proved through simple calculations and so we omit the proof.

Lemma 2.4. *For $j = 0,1,2,3$, the following functions are nonnegative for $0 < x \leq 2$:*

$$K_jK_1(x) - K_{j+1}K_0(x) \geq 0, \qquad K_jK_2(x) - K_{j+2}K_0(x) \geq 0,$$

$$K_{j+1}K_3(x) - K_{j+3}K_1(x) \geq 0, \qquad K_{j+2}\widetilde{K}_3(x) - \widetilde{K}_{j+3}K_2(x) \geq 0.$$

Proof of Lemma 2.4 Using Lemma 2.2 and Lemma 2.3, we have the following inequality for $0 < x \leq 2$. From the relations $K_j = K_j(2) > 0$ $(j = 0,1,2,\cdots)$ and $K(0,1,0,1) = 0$, $K(1,1,0,2)$, $K(1,2,0,3)$, $K(1,3,0,4) > 0$, we have

$$\frac{K_jK_1(x) - K_{j+1}K_0(x)}{K_0(x)} = K_j\left(\frac{K_1}{K_0}\right)(x) - K_{j+1} \geq \frac{K(1,j,0,j+1)}{K_0} \geq 0.$$

From $K(0,2,0,2) = 0$, $K(1,2,0,3)$, $K(2,2,0,4)$, $K(2,3,0,5) > 0$, we have

$$\frac{K_jK_2(x) - K_{j+2}K_0(x)}{K_0(x)} = K_j\left(\frac{K_2}{K_0}\right)(x) - K_{j+2} \geq \frac{K(2,j,0,j+2)}{K_0} \geq 0.$$

From $K(1,3,1,3) = 0$, $K(2,3,1,4)$, $K(3,3,1,5)$, $K(3,4,1,6) > 0$, we have

$$\frac{K_{j+1}K_3(x) - K_{j+3}K_1(x)}{K_1(x)} = K_{j+1}\left(\frac{K_3}{K_1}\right)(x) - K_{j+3} \geq \frac{K(3,j+1,1,j+3)}{K_1} \geq 0.$$

From $K(2,\widetilde{3},2,\widetilde{3}) = 0$, $K(3,\widetilde{3},2,\widetilde{4})$, $K(\widetilde{3},4,2,\widetilde{5})$, $K(\widetilde{3},5,2,\widetilde{6}) > 0$, we have

$$\frac{K_{j+2}\widetilde{K}_3(x) - \widetilde{K}_{j+3}K_2(x)}{K_2(x)} = K_{j+2}\left(\frac{\widetilde{K}_3}{K_2}\right)(x) - \widetilde{K}_{j+3} \geq \frac{K(\widetilde{3},j+2,2,\widetilde{j+3})}{K_2} \geq 0.$$

This shows Lemma 2.4. ■

Lemma 2.5. *The following double-angle formulae of $K_0(x)$, $K_2(x)$ and $K_4(x)$ hold:*

$$K_0(2x) = 2\Big[\, K_0\widetilde{K}_3 + K_1K_2 \,\Big](x), \qquad K_2(2x) = 2\Big[\, K_1\widetilde{K}_4 + K_2K_3 \,\Big](x),$$

$$K_4(2x) = 2\Big[\, K_2\widetilde{K}_5 + K_3K_4 \,\Big](x).$$

Proof of Lemma 2.5 We introduce matrices

$$\boldsymbol{u}=\begin{pmatrix}u_0\\u_1\\u_2\\u_{\widetilde{3}}\end{pmatrix},\qquad \boldsymbol{P}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&p&0&1\\-q&0&0&0\end{pmatrix},\qquad \boldsymbol{I}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

where u_i is the same as Section 1. Since $u^{(4)}-pu''+qu=0$, we have $\boldsymbol{u}'=\boldsymbol{P}\boldsymbol{u}$. The fundamental solution $\boldsymbol{E}(x)$ to $\boldsymbol{u}'=\boldsymbol{P}\boldsymbol{u}$ is $\boldsymbol{E}(x)=\boldsymbol{K}(x)\boldsymbol{K}(0)^{-1}$, where

$$\boldsymbol{K}(x)=\begin{pmatrix}K_0&K_1&K_2&\widetilde{K}_3\\K_1&K_2&K_3&\widetilde{K}_4\\K_2&K_3&K_4&\widetilde{K}_5\\\widetilde{K}_3&\widetilde{K}_4&\widetilde{K}_5&\widetilde{\widetilde{K}}_6\end{pmatrix}(x),\qquad \boldsymbol{K}(0)=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=\boldsymbol{K}^{-1}(0).$$

Since $\boldsymbol{E}(x)$ $(-\infty<x<\infty)$ satisfies $\boldsymbol{E}'(x)=\boldsymbol{P}\boldsymbol{E}(x)$ and $\boldsymbol{E}(0)=\boldsymbol{I}$, we have $\boldsymbol{E}(x)=\exp(\boldsymbol{P}x)$ and the addition formula $\boldsymbol{E}(x+y)=\boldsymbol{E}(x)\boldsymbol{E}(y)$ that is

$$\begin{pmatrix}\widetilde{K}_3&K_2&K_1&K_0\\\widetilde{K}_4&K_3&K_2&K_1\\\widetilde{K}_5&K_4&K_3&K_2\\\widetilde{\widetilde{K}}_6&\widetilde{K}_5&\widetilde{K}_4&\widetilde{K}_3\end{pmatrix}(x+y)=\begin{pmatrix}\widetilde{K}_3&K_2&K_1&K_0\\\widetilde{K}_4&K_3&K_2&K_1\\\widetilde{K}_5&K_4&K_3&K_2\\\widetilde{\widetilde{K}}_6&\widetilde{K}_5&\widetilde{K}_4&\widetilde{K}_3\end{pmatrix}(x)\begin{pmatrix}\widetilde{K}_3&K_2&K_1&K_0\\\widetilde{K}_4&K_3&K_2&K_1\\\widetilde{K}_5&K_4&K_3&K_2\\\widetilde{\widetilde{K}}_6&\widetilde{K}_5&\widetilde{K}_4&\widetilde{K}_3\end{pmatrix}(y)$$

holds. Putting $y=x$ in $(0,3),(1,2),(2,1)$-element, we obtain double-angle formulae. ∎

Next, we show the properties of fundamental solutions $A_i(x)$ and $B_i(x)$, including their positive- (or negative-)definiteness.

Lemma 2.6. (1) *For $-1<x<1$, the following inequalities hold:*

(α,β)				
$(0,1,0,1)$:	$A_0(x)>0$	$A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(0,1,0,2)$:	$A_0(x)>0$	$A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(0,1,1,3)$:	$A_0(x)>0$	$A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(0,1,2,\widetilde{3})$:	$A_0(x)>0$	$A_1(x)>0$		$-B_1(x)>0$
$(0,2,0,2)$:	$A_0(x)>0$	$-A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(0,2,1,3)$:	$A_0(x)>0$	$-A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(0,2,2,\widetilde{3})$:	$A_0(x)>0$	$-A_1(x)>0$		$-B_1(x)>0$
$(1,3,1,3)$:	$-A_0(x)>0$	$A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(1,3,2,\widetilde{3})$:	$-A_0(x)>0$	$A_1(x)>0$		$-B_1(x)>0$
$(2,\widetilde{3},1,3)$:		$A_1(x)>0$	$B_0(x)>0$	$-B_1(x)>0$
$(2,\widetilde{3},2,\widetilde{3})$:		$A_1(x)>0$		$-B_1(x)>0.$

(2) *If* $\widetilde{K}_3 \geq 0$*, or equivalently* $a^{-2}\mathrm{ch}(2a) \leq b^{-2}\mathrm{ch}(2b)$*, we have*

$$B_0(0,1,2,\widetilde{3};x) > 0, \quad B_0(0,2,2,\widetilde{3};x) > 0 \qquad (-1 < x < 1).$$

If $\widetilde{K}_3 < 0$*, or equivalently* $a^{-2}\mathrm{ch}(2a) > b^{-2}\mathrm{ch}(2b)$*, then we have the following relations:*

$$B_0(0,1,2,\widetilde{3};x)\begin{cases} <0 & (-1<x<x_0) \\ =0 & (x=x_0) \\ >0 & (x_0<x<1), \end{cases} \qquad B_0(0,2,2,\widetilde{3};x)\begin{cases} <0 & (-1<x<x_1) \\ =0 & (x=x_1) \\ >0 & (x_1<x<1), \end{cases}$$

where x_0, x_1 *are the unique solution of the equation*

$$(K_1/K_0)\,(1+x_0) = \widetilde{K}_4/\widetilde{K}_3, \qquad (K_2/K_0)\,(1+x_1) = \widetilde{K}_5/\widetilde{K}_3,$$

respectively.

(3) *We have the following inequalities:*

$$B_0(1,3,2,\widetilde{3};x)\begin{cases} <0 & (-1<x<x_2) \\ =0 & (x=x_2) \\ >0 & (x_2<x<1), \end{cases}$$

$$B_0(2,\widetilde{3},2,\widetilde{3};x)\begin{cases} <0 & (-1<x<x_3) \\ =0 & (x=x_3) \\ >0 & (x_3<x<1), \end{cases} \qquad A_0(2,\widetilde{3},2,\widetilde{3};x)\begin{cases} >0 & (-1<x<-x_3) \\ =0 & (x=-x_3) \\ <0 & (-x_3<x<1), \end{cases}$$

where x_2, x_3 *are the unique solution of the equation*

$$(K_3/K_1)\,(1+x_2) = \widetilde{K}_6/\widetilde{K}_4, \qquad \left(\widetilde{K}_3/K_2\right)(1+x_3) = \widetilde{\widetilde{K}}_6/\widetilde{K}_5,$$

respectively.

Proof of Lemma 2.6 From concrete forms of $A_i(x)$ and $B_i(x)$ in Theorem 2.1, it is shown that all the denominators are positive from Lemma 2.2 and that numerators of fundamental solutions listed in (1) are positive from Lemma 2.4.

Next we show (2). In the case $(\alpha,\beta) = (0,1,2,\widetilde{3})$, we set $B_0(x) = B_0(0,1,2,\widetilde{3};x)$ and $\gamma_0 = K(3,\widetilde{3},2,\widetilde{4})$. If $\widetilde{K}_3 \geq 0$, then we have

$$\frac{\gamma_0 B_0(x)}{K_0(1+x)} = \widetilde{K}_3\left(\frac{K_1}{K_0}\right)(1+x) - \widetilde{K}_4 \geq \frac{K(1,\widetilde{3},0,\widetilde{4})}{K_0} > 0 \qquad (-1<x<1).$$

Next we consider the case $\widetilde{K}_3 < 0$. We put

$$h_0(x) := \frac{\gamma_0 B_0(x)}{-\widetilde{K}_3 K_0(1+x)} = \frac{\widetilde{K}_4}{\widetilde{K}_3} - \left(\frac{K_1}{K_0}\right)(1+x) \qquad (-1<x<1).$$

The equation $h_0(x) = 0$ has a unique solution $x = x_0$, because

$$\lim_{x\to-1+0} h_0(x) = -\infty, \quad h_0'(x) = -\left(\frac{K_1}{K_0}\right)'(1+x) > 0, \quad h_0(1) = \frac{\widetilde{K}_4}{\widetilde{K}_3} - \frac{K_1}{K_0} > 0,$$

where we have used the fact $K(1, \widetilde{3}, 0, \widetilde{4}) > 0$ and $\widetilde{K}_3 < 0$.

In the case $(\alpha, \beta) = (0, 2, 2, \widetilde{3})$, we set $B_0(x) = B_0(0, 2, 2, \widetilde{3}; x)$ and $\gamma_1 = K(\widetilde{3}, 4, 2, \widetilde{5})$. If $\widetilde{K}_3 \geq 0$, then we have

$$\frac{\gamma_1 B_0(x)}{K_0(1+x)} = \widetilde{K}_3 \frac{K_2(1+x)}{K_0(1+x)} - \widetilde{K}_5 \geq \frac{K(2, \widetilde{3}, 0, \widetilde{5})}{K_0} > 0 \qquad (-1 < x < 1).$$

If $\widetilde{K}_3 < 0$, we put

$$h_1(x) := \frac{\gamma_1 B_0(x)}{-\widetilde{K}_3 K_0(1+x)} = \frac{\widetilde{K}_5}{\widetilde{K}_3} - \left(\frac{K_2}{K_0}\right)(1+x) \qquad (-1 < x < 1).$$

The equation $h_1(x) = 0$ has a unique solution $x = x_1$, because

$$\lim_{x\to-1+0} h_1(x) = -\infty, \quad h_1'(x) = -\left(\frac{K_2}{K_0}\right)'(1+x) > 0, \quad h_1(1) = \frac{\widetilde{K}_5}{\widetilde{K}_3} - \frac{K_2}{K_0} > 0,$$

where we have used the fact $K(2, \widetilde{3}, 0, \widetilde{5}) > 0$ and $\widetilde{K}_3 < 0$. Thus we have (2).

Finally we show (3). In the case $(\alpha, \beta) = (1, 3, 2, \widetilde{3})$, we set $B_0(x) = B_0(1, 3, 2, \widetilde{3}; x)$ and $\gamma_2 = K(\widetilde{4}, 5, 3, \widetilde{6})$. We put

$$h_2(x) := \frac{\gamma_2 B_0(x)}{-\widetilde{K}_4 K_1(1+x)} = \frac{\widetilde{K}_6}{\widetilde{K}_4} - \left(\frac{K_3}{K_1}\right)(1+x) \qquad (-1 < x < 1).$$

The equation $h_2(x) = 0$ has a unique solution $x = x_2$, because

$$\lim_{x\to-1+0} h_2(x) = -\infty, \quad h_2'(x) = -\left(\frac{K_3}{K_1}\right)'(1+x) > 0, \quad h_2(1) = \frac{\widetilde{K}_6}{\widetilde{K}_4} - \frac{K_3}{K_1} > 0,$$

where we have used the fact $K(3, \widetilde{4}, 1, \widetilde{6}) > 0$ and $\widetilde{K}_4 < 0$.

In the case $(\alpha, \beta) = (2, \widetilde{3}, 2, \widetilde{3})$, we set $B_0(x) = B_0(2, \widetilde{3}, 2, \widetilde{3}; x)$ and $\gamma_3 = K(\widetilde{5}, \widetilde{5}, 4, \widetilde{\widetilde{6}})$. We put

$$h_3(x) := \frac{\gamma_3 B_0(x)}{-\widetilde{K}_5 K_2(1+x)} = \frac{\widetilde{\widetilde{K}}_6}{\widetilde{K}_5} - \left(\frac{\widetilde{K}_3}{K_2}\right)(1+x) \qquad (-1 < x < 1).$$

The equation $h_3(x) = 0$ has a unique solution $x = x_3$ because

$$\lim_{x\to-1+0} h_3(x) = -\infty, \quad h_3'(x) = -\left(\frac{\widetilde{K}_3}{K_2}\right)'(1+x) > 0, \quad h_3(1) = \frac{\widetilde{\widetilde{K}}_6}{\widetilde{K}_5} - \frac{\widetilde{K}_3}{K_2} > 0,$$

where we have used the fact $K(\widetilde{3},\widetilde{5},2,\widetilde{\widetilde{6}})>0$ and $\widetilde{K}_5<0$. For $A_0(2,\widetilde{3},2,\widetilde{3};x)$, we remark $A_0(2,\widetilde{3},2,\widetilde{3};x)=B_0(2,\widetilde{3},2,\widetilde{3};-x)$. So we have (3). ■

The following three Lemmas 2.7, 2.8 and 2.9 play important roles in investigating difference of two Green functions in section 4.

Lemma 2.7.

$$\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,0,2;x)-\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,0,1;x)=$$

$$\frac{1}{K(1,2,0,3)}\begin{pmatrix}-K_1\\K_0\end{pmatrix}(-B_1(0,1,0,1;x))=\frac{1}{K(1,1,0,2)}\begin{pmatrix}-K_1\\K_0\end{pmatrix}(-B_1(0,1,0,2;x)) \tag{2.2}$$

$$\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,1,3;x)-\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,0,2;x)=$$

$$\frac{1}{K(2,3,1,4)}\left[\begin{pmatrix}-K_2\\K_1\end{pmatrix}B_0+\begin{pmatrix}-\widetilde{K}_4\\\widetilde{K}_3\end{pmatrix}(-B_1)\right](0,1,0,2;x)=$$

$$\frac{1}{K(1,2,0,3)}\left[\begin{pmatrix}-K_1\\K_0\end{pmatrix}B_0+\begin{pmatrix}-\widetilde{K}_3\\\widetilde{K}_2\end{pmatrix}(-B_1)\right](0,1,1,3;x) \tag{2.3}$$

$$\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,2,\widetilde{3};x)-\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,0,2;x)=$$

$$\frac{1}{K(3,\widetilde{3},2,\widetilde{4})}\begin{pmatrix}-K_3\\K_2\end{pmatrix}B_0(0,1,0,2;x)=\frac{1}{K(1,2,0,3)}\begin{pmatrix}-K_3\\K_2\end{pmatrix}(-B_1(0,1,2,\widetilde{3};x)) \tag{2.4}$$

$$\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,2,\widetilde{3};x)-\begin{pmatrix}-A_0\\A_1\end{pmatrix}(0,1,1,3;x)=$$

$$\frac{1}{K(3,\widetilde{3},2,\widetilde{4})}\begin{pmatrix}-\widetilde{K}_4\\\widetilde{K}_3\end{pmatrix}(B_0+pB_1)(0,1,1,3;x)=\frac{1}{K(2,3,1,4)}\begin{pmatrix}-\widetilde{K}_4\\\widetilde{K}_3\end{pmatrix}B_0(0,1,2,\widetilde{3};x) \tag{2.5}$$

$$\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,1,3;x)-\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,0,2;x)=$$

$$-\frac{1}{K(3,3,1,5)}\left[\begin{pmatrix}-K_3\\K_1\end{pmatrix}B_0+\begin{pmatrix}-\widetilde{K}_5\\\widetilde{K}_3\end{pmatrix}(-B_1)\right](0,2,0,2;x)=$$

$$-\frac{1}{K(2,2,0,4)}\left[\begin{pmatrix}-K_2\\K_0\end{pmatrix}B_0+\begin{pmatrix}-\widetilde{K}_4\\\widetilde{K}_2\end{pmatrix}(-B_1)\right](0,2,1,3;x) \tag{2.6}$$

$$\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,2,\widetilde{3};x)-\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,0,2;x)=$$
$$-\frac{1}{K(\widetilde{3},4,2,\widetilde{5})}\begin{pmatrix}-K_4\\K_2\end{pmatrix}B_0(0,2,0,2;x)=-\frac{1}{K(2,2,0,4)}\begin{pmatrix}-K_4\\K_2\end{pmatrix}(-B_1(0,1,2,\widetilde{3};x)) \tag{2.7}$$

$$\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,2,\widetilde{3};x)-\begin{pmatrix}A_0\\A_1\end{pmatrix}(0,2,1,3;x)=$$
$$-\frac{1}{K(\widetilde{3},4,2,\widetilde{5})}\begin{pmatrix}-\widetilde{K}_5\\\widetilde{K}_3\end{pmatrix}(B_0+pB_1)(0,2,1,3;x)=$$
$$-\frac{1}{K(3,3,1,5)}\begin{pmatrix}-\widetilde{K}_5\\\widetilde{K}_3\end{pmatrix}B_0(0,2,2,\widetilde{3};x) \tag{2.8}$$

$$\begin{pmatrix}A_0\\A_1\end{pmatrix}(1,3,2,\widetilde{3};x)-\begin{pmatrix}A_0\\A_1\end{pmatrix}(1,3,1,3;x)=$$
$$\frac{1}{K(\widetilde{4},5,3,\widetilde{6})}\begin{pmatrix}-\widetilde{K}_6\\\widetilde{K}_4\end{pmatrix}(B_0+pB_1)(1,3,1,3;x)=\frac{1}{K(4,4,2,6)}\begin{pmatrix}-\widetilde{K}_6\\\widetilde{K}_4\end{pmatrix}B_0(1,3,2,\widetilde{3};x) \tag{2.9}$$

$$\big(B_0,\ B_1\big)(0,2,0,2;x)-\big(B_0,\ B_1\big)(0,1,0,2;x)=$$
$$\frac{1}{K(2,2,0,4)}(K_2,-K_0)A_1(0,1,0,2;x)=\frac{1}{K(1,2,0,3)}(K_2,-K_0)(-A_1(0,2,0,2;x)) \tag{2.10}$$

$$\big(B_0,\ B_1\big)(0,2,1,3;x)-\big(B_0,\ B_1\big)(0,1,1,3;x)=$$
$$\frac{1}{K(3,3,1,5)}(K_3,-K_1)A_1(0,1,1,3;x)=\frac{1}{K(2,3,1,4)}(K_3,-K_1)(-A_1(0,2,1,3;x)) \tag{2.11}$$

$$\big(B_0,\ B_1\big)(1,3,1,3;x)-\big(B_0,\ B_1\big)(0,2,1,3;x)=$$
$$\frac{1}{K(4,4,2,6)}\Big[(K_4,-K_2)A_0+(\widetilde{K}_6,-\widetilde{K}_4)(-A_1)\Big](0,2,1,3;x)=$$
$$\frac{-1}{K(3,3,1,5)}\Big[(K_3,-K_1)A_0+(\widetilde{K}_5,-\widetilde{K}_3)(-A_1)\Big](1,3,1,3;x) \tag{2.12}$$

$$\big(B_0,\ B_1\big)(2,\widetilde{3},1,3;x)-\big(B_0,\ B_1\big)(0,2,1,3;x)=$$
$$\frac{1}{K(\widetilde{4},5,3,\widetilde{6})}(K_5,-K_3)A_1(0,2,1,3;x)=\frac{1}{K(3,3,1,5)}(K_5,-K_3)A_1(2,\widetilde{3},1,3;x) \tag{2.13}$$

$$\big(B_0,\, B_1\big)(0,2,2,\widetilde{3};x)-\big(B_0,\, B_1\big)(0,1,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{3},4,2,\widetilde{5})}(\widetilde{K}_3,-K_2)A_1(0,1,2,\widetilde{3};x)=\frac{1}{K(3,\widetilde{3},2,\widetilde{4})}(\widetilde{K}_3,-K_2)(-A_1(0,2,2,\widetilde{3};x)) \tag{2.14}$$

$$\big(B_0,\, B_1\big)(1,3,2,\widetilde{3};x)-\big(B_0,\, B_1\big)(0,2,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{4},5,3,\widetilde{6})}\Big[(\widetilde{K}_4,-K_3)A_0+(\widetilde{\widetilde{K}}_6,-\widetilde{K}_5)(-A_1)\Big](0,2,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{3},4,2,\widetilde{5})}(\widetilde{K}_3,-K_2)(-A_0)+(\widetilde{\widetilde{K}}_5,-\widetilde{K}_4)A_1\Big](1,3,2,\widetilde{3};x) \tag{2.15}$$

$$\big(B_0,\, B_1\big)(2,\widetilde{3},2,\widetilde{3};x)-\big(B_0,\, B_1\big)(0,2,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{5},\widetilde{5},4,\widetilde{\widetilde{6}})}(\widetilde{K}_5,-K_4)A_0(0,2,2,\widetilde{3};x)=\frac{1}{K(\widetilde{3},4,2,\widetilde{5})}(\widetilde{K}_5,-K_4)A_1(2,\widetilde{3},2,\widetilde{3};x) \tag{2.16}$$

$$\big(B_0,\, B_1\big)(2,\widetilde{3},2,\widetilde{3};x)-\big(B_0,\, B_1\big)(1,3,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{5},\widetilde{5},4,\widetilde{\widetilde{6}})}(\widetilde{\widetilde{K}}_6,-\widetilde{K}_5)(-A_0-pA_1)(1,3,2,\widetilde{3};x)=$$
$$\frac{1}{K(\widetilde{4},5,3,\widetilde{6})}(\widetilde{\widetilde{K}}_6,-\widetilde{K}_5)A_0(2,\widetilde{3},2,\widetilde{3};x) \tag{2.17}$$

Proof of Lemma 2.7 We only prove the 0-th element in (2.2). We start with the fact that the homogeneous equation $u^{(4)}-pu''+qu=0$ $(-1<x<1)$ has a unique classical solution as follows:

$$\begin{aligned} u(x)=&A_0(\alpha,\beta;x)u_{\alpha_0}(-1)+A_1(\alpha,\beta;x)u_{\alpha_1}(-1)+\\ &B_0(\alpha,\beta;x)u_{\beta_0}(1)+B_1(\alpha,\beta;x)u_{\beta_1}(1) \qquad (-1<x<1), \end{aligned} \tag{2.18}$$

which is obtained by applying Theorem 2.1 to $f(x)=0$.

Putting $u(x)=(-A_0(0,1,0,2;x))-(-A_0(0,1,0,1;x))$, $(\alpha,\beta)=(0,1,0,1)$ in (2.18) and considering that

$$u(-1)=(-1)-(-1)=0, \qquad u'(-1)=0-0=0,$$
$$u(1)=0-0=0, \qquad u'(1)=(-A_0'(0,1,0,2;1))-0=\frac{K_1}{K(1,2,0,3)},$$

we have

$$u(x)=(-A_0(0,1,0,2;x))-(-A_0(0,1,0,1;x))=\frac{K_1}{K(1,2,0,3)}B_1(0,1,0,1;x).$$

Other relations can be shown by taking similar procedures. ■

Lemma 2.8.

$$\min_{|x|\le 1}\left(\frac{B_0}{-B_1}\right)(0,1,0,2;x)=\left(\frac{B_0}{-B_1}\right)(0,1,0,2;-1)=\frac{K_2}{K_0}>0$$
$$\min_{|x|\le 1}\left(\frac{B_0}{-B_1}\right)(0,1,1,3;x)=\left(\frac{B_0}{-B_1}\right)(0,1,1,3;-1)=\frac{K_3}{K_1}>0$$
$$\min_{|x|\le 1}\left(\frac{B_0}{-B_1}\right)(0,2,0,2;x)=\left(\frac{B_0}{-B_1}\right)(0,2,0,2;-1)=\frac{K_2}{K_0}>0$$
$$\min_{|x|\le 1}\left(\frac{B_0}{-B_1}\right)(0,2,1,3;x)=\left(\frac{B_0}{-B_1}\right)(0,2,1,3;-1)=\frac{K_3}{K_1}>0$$
$$\min_{|x|\le 1}\left(\frac{A_0}{-A_1}\right)(0,2,1,3;x)=\left(\frac{A_0}{-A_1}\right)(0,2,1,3;1)=\frac{K_3}{K_1}>0$$
$$\min_{|x|\le 1}\left(\frac{-A_0}{A_1}\right)(1,3,1,3;x)=\left(\frac{-A_0}{A_1}\right)(1,3,1,3;1)=\frac{K_4}{K_2}>0$$
$$\min_{|x|\le 1}\left(\frac{A_0}{-A_1}\right)(0,2,2,\widetilde{3};x)=\left(\frac{A_0}{-A_1}\right)(0,2,2,\widetilde{3};1)=\frac{K_4}{K_2}>0$$
$$\min_{|x|\le 1}\left(\frac{-A_0}{A_1}\right)(1,3,2,\widetilde{3};x)=\left(\frac{-A_0}{A_1}\right)(1,3,2,\widetilde{3};1)=\frac{K_5}{K_3}>0$$

Proof of Lemma 2.8 Through direct calculations, the following inequalities hold on $-1<x<1$:

$$\left(\frac{B_0}{-B_1}\right)'(0,1,0,2;x)=\frac{K(1,2,0,3)K(1,1,0,2;1+x)}{\left(K_0K_1(1+x)-K_1K_0(1+x)\right)^2}>0$$
$$\left(\frac{B_0}{-B_1}\right)'(0,1,1,3;x)=\frac{K(2,3,1,4)K(1,1,0,2;1+x)}{\left(K_1K_1(1+x)-K_2K_0(1+x)\right)^2}>0$$
$$\left(\frac{B_0}{-B_1}\right)'(0,2,0,2;x)=\frac{K(2,2,0,4)K(1,2,0,3;1+x)}{\left(K_0K_2(1+x)-K_2K_0(1+x)\right)^2}>0$$
$$\left(\frac{B_0}{-B_1}\right)'(0,2,1,3;x)=\frac{K(3,3,1,5)K(1,2,0,3;1+x)}{\left(K_1K_2(1+x)-K_3K_0(1+x)\right)^2}>0$$
$$\left(\frac{A_0}{-A_1}\right)'(0,2,1,3;x)=-\frac{K(3,3,1,5)K(2,3,1,4;1-x)}{\left(K_1K_3(1-x)-K_3K_1(1-x)\right)^2}<0$$
$$\left(\frac{-A_0}{A_1}\right)'(1,3,1,3;x)=-\frac{K(4,4,2,6)K(2,3,1,4;1-x)}{\left(K_2K_3(1-x)-K_4K_1(1-x)\right)^2}<0$$
$$\left(\frac{A_0}{-A_1}\right)'(0,2,2,\widetilde{3};x)=-\frac{K(\widetilde{3},4,2,\widetilde{5})K(3,\widetilde{3},2,\widetilde{4};1-x)}{\left(K_2\widetilde{K}_3(1-x)-\widetilde{K}_3K_2(1-x)\right)^2}<0$$
$$\left(\frac{-A_0}{A_1}\right)'(1,3,2,\widetilde{3};x)=-\frac{K(\widetilde{4},5,3,\widetilde{6})K(3,\widetilde{3},2,\widetilde{4};1-x)}{\left(K_3\widetilde{K}_3(1-x)-\widetilde{K}_4K_2(1-x)\right)^2}<0.$$

This shows Lemma 2.8. ■

Lemma 2.9. *The following functions are positive if* $-1 \le x, y \le 1$*:*

$$\begin{aligned}
&\begin{pmatrix} \frac{B_0}{-B_1} \end{pmatrix}(0,1,1,3;x) + \begin{pmatrix} \frac{B_0}{-B_1} \end{pmatrix}(0,1,0,2;y) - p \ge \frac{K_0(4)}{2K_0K_1} > 0 \\
&\begin{pmatrix} \frac{B_0}{-B_1} \end{pmatrix}(0,2,1,3;x) + \begin{pmatrix} \frac{B_0}{-B_1} \end{pmatrix}(0,2,0,2;y) - p \ge \frac{K_0(4)}{2K_0K_1} > 0 \\
&\begin{pmatrix} \frac{-A_0}{A_1} \end{pmatrix}(1,3,1,3;x) + \begin{pmatrix} \frac{A_0}{-A_1} \end{pmatrix}(0,2,1,3;y) - p \ge \frac{K_2(4)}{2K_1K_2} > 0 \\
&\begin{pmatrix} \frac{-A_0}{A_1} \end{pmatrix}(1,3,2,\widetilde{3};x) + \begin{pmatrix} \frac{A_0}{-A_1} \end{pmatrix}(0,2,2,\widetilde{3};y) - p \ge \frac{K_4(4)}{2K_2K_3} > 0.
\end{aligned}$$

Proof of Lemma 2.9 This lemma follows from Lemma 2.5 and Lemma 2.8. ■

3. Positivity of Green function $G(0,1,0,1;x,y)$

In this section, we prove the lowest part of the hierarchy in the main Theorem 1.1, that is, $G(0,1,0,1;x,y) > 0$. Here we give a simpler proof based on the positivity of the diagonal value of Green function $G(0,1,0,1;x,x)$. For the sake of simplicity, we here put $G(x,y) = G(0,1,0,1;x,y)$, $A_i(x) = A_i(0,1,0,1;x)$, $B_i(x) = B_i(0,1,0,1;x)$ $(i = 0,1)$ in this section. We rewrite the goal of this section in the following theorem:

Theorem 3.1.

$$(1) \quad G(x,y) \ge \frac{K_0(1+x\wedge y)}{K_0(1+x\vee y)} G(x\vee y, x\vee y) \qquad (-1 < x, y < 1)$$

$$(2) \quad 0 = G(1,1) \le G(x,x) \qquad (0 \le x \le 1)$$

In order to prove this theorem, we prepare two lemmas.

Lemma 3.1.

$$\begin{pmatrix} K_1, & K_2 \end{pmatrix}(1+x) = \begin{pmatrix} A_0, & A_1 \end{pmatrix}(x) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} B_0, & B_1 \end{pmatrix}(x) \begin{pmatrix} K_1 & K_2 \\ K_2 & K_3 \end{pmatrix}$$

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}(1-x) = \begin{pmatrix} K_1 & K_2 \\ K_2 & K_3 \end{pmatrix} \begin{pmatrix} A_0 \\ -A_1 \end{pmatrix}(x) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_0 \\ -B_1 \end{pmatrix}(x)$$

Proof of Lemma 3.1 Since $u = K_j(x)$ satisfies the homogeneous equation $u^{(4)} - pu'' + qu = 0$, we have Lemma 3.1 from by applying (2.18) to $u = K_j(x)$. ■

Lemma 3.2.

$$(|B_1|-|A_1|)(x)=(-B_1-A_1)(x)=$$
$$\frac{1}{K(1,2,0,3;1)}\Big[K_0(1)K_2(x)-K_2(1)K_0(x)\Big]>0 \qquad (0<x<1)$$

Proof of Lemma 3.2 First equality is obvious from Lemma 2.6 (1). Since

$$(-B_1-A_1)(-x)=(A_1+B_1)(x)=-(-B_1-A_1)(x) \qquad (0<x<1),$$

$(-B_1-A_1)(x)$ is an odd function in x. Furthermore, since $u=(-B_1-A_1)(x)$ satisfy $u^{(4)}-pu''+qu=0$, we can apply (2.18) and obtain

$$(-B_1-A_1)(x)=C_0K_0(x)+C_1K_2(x). \tag{3.1}$$

Differentiating both sides 0 and 1 times and inserting $x=1$, we have $C_0=-\frac{K_2(1)}{K(1,2,0,3;1)}$, $C_1=\frac{K_0(1)}{K(1,2,0,3;1)}$. Putting these values into (3.1), we obtain the second equality.

Next, we show the positivity. Setting $\gamma=K(1,2,0,3;1)$, then we have

$$\frac{\gamma(-B_1-A_1)(x)}{K_0(x)}=K_0(1)\left(\frac{K_2}{K_0}\right)(x)-K_2(1)\geq\frac{K(0,2,0,2;1)}{K_0(1)}=0 \quad (0<x\leq 1),$$

where we have used Lemma 2.3. This completes the proof of Lemma 3.2. ■

Proof of Theorem 3.1 Green function $G(x,y)$ can be expressed as

$$G(x,y)=\left|\begin{array}{cc|c} K_0 & K_1 & K_0(1-x\vee y) \\ K_1 & K_2 & K_1(1-x\vee y) \\ \hline K_0(1+x\wedge y) & K_1(1+x\wedge y) & 0 \end{array}\right| \Bigg/ \left|\begin{array}{cc} K_0 & K_1 \\ K_1 & K_2 \end{array}\right|.$$

We may assume $-1<y<x<1$ without loss of generality. $G(x,y)$ is expressed as

$$\frac{K(1,1,0,2)G(x,y)}{K_0(1+y)}=-\left|\begin{array}{cc|c} K_0 & K_1 & K_0(1-x) \\ K_1 & K_2 & K_1(1-x) \\ \hline 1 & (K_1/K_0)(1+y) & 0 \end{array}\right|=$$
$$(K_1/K_0)(1+y)\left|\begin{array}{cc} K_0 & K_0(1-x) \\ K_1 & K_1(1-x) \end{array}\right|-\left|\begin{array}{cc} K_1 & K_0(1-x) \\ K_2 & K_1(1-x) \end{array}\right|\geq$$
$$(K_1/K_0)(1+x)\left|\begin{array}{cc} K_0 & K_0(1-x) \\ K_1 & K_1(1-x) \end{array}\right|-\left|\begin{array}{cc} K_1 & K_0(1-x) \\ K_2 & K_1(1-x) \end{array}\right|=$$
$$-\left|\begin{array}{cc|c} K_0 & K_1 & K_0(1-x) \\ K_1 & K_2 & K_1(1-x) \\ \hline 1 & (K_1/K_0)(1+x) & 0 \end{array}\right|=\frac{K(1,1,0,2)G(x,x)}{K_0(1+x)}$$

where we have used the properties $(K_1/K_0)(1+y)$ is monotone decreasing in $y \in (-1,1)$ and

$$\begin{vmatrix} K_0 & K_0(1-x) \\ K_1 & K_1(1-x) \end{vmatrix} = K_0K_1(1-x) - K_1K_0(1-x) > 0 \qquad (-1<x<1),$$

which are shown in Lemma 2.3 and 2.4, respectively. Hence we have (1). Next, we show (2). $G(1,1)=0$ is obvious. Considering that

$$G(x,x) = -\big(K_0,\, K_1\big)(1+x)\begin{pmatrix} K_0 & K_1 \\ K_1 & K_2 \end{pmatrix}^{-1}\begin{pmatrix} K_0 \\ K_1 \end{pmatrix}(1-x) \qquad (-1<x<1)$$

is an even function, so we investigate the behavior of $G(x,x)$ on an interval $0<x<1$. It is enough to show $-(d/dx)G(x,x)>0$ $(0<x<1)$. Differentiating $G(x,x)$, we have

$$\begin{aligned}
&-\frac{d}{dx}G(x,x) = \\
&\big(K_1,\, K_2\big)(1+x)\begin{pmatrix} K_0 & K_1 \\ K_1 & K_2 \end{pmatrix}^{-1}\begin{pmatrix} K_0 \\ K_1 \end{pmatrix}(1-x) - \\
&\big(K_0,\, K_1\big)(1+x)\begin{pmatrix} K_0 & K_1 \\ K_1 & K_2 \end{pmatrix}^{-1}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}(1-x) = \\
&\big(K_1,\, K_2\big)(1+x)\begin{pmatrix} A_0 \\ -A_1 \end{pmatrix}(x) - \big(B_0,\, B_1\big)(x)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}(1-x) = \\
&B_1^2(x) - A_1^2(x) = \left(|B_1|^2 - |A_1|^2\right)(x) > 0 \qquad (0<x<1),
\end{aligned}$$

where we have used the matrix form of $A_i(x)$ and $B_i(x)$ (Theorem 2.1) in the second equality, Lemma 3.1 in the third equality and Lemma 3.2 in the last inequality. Thus $G(x,x)$ is monotone decreasing function in $x \in (0,1)$. Hence we have (2). This completes the proof of Theorem 3.1. ■

4. Hierarchical structure of Green functions

This section is devoted to a proof of the hierarchical structure among 16 Green functions in Theorem 1.1. Together with the inequality $G(0,1,0,1;x,y)>0$, which is proved in the previous section, it is concluded that all the 16 Green functions are positive. It is enough to prove the following lemma.

Lemma 4.1. *Differences of the following two Green functions are expressed as follows and are positive if $-1<x,y<1$.*

$$G(0,1,0,2;x,y) - G(0,1,0,1;x,y) = \begin{cases} (-B_1(0,1,0,2;x\wedge y))(-B_1(0,1,0,1;x\vee y)) > 0 \\ (-B_1(0,1,0,2;x\vee y))(-B_1(0,1,0,1;x\wedge y)) > 0 \end{cases}$$

$$G(0,1,2,\widetilde{3};x,y)-G(0,1,0,2;x,y)=\begin{cases}(-B_1(0,1,2,\widetilde{3};x\wedge y))B_0(0,1,0,2;x\vee y)>0\\(-B_1(0,1,2,\widetilde{3};x\vee y))B_0(0,1,0,2;x\wedge y)>0\end{cases}$$

$$G(0,2,2,\widetilde{3};x,y)-G(0,2,0,2;x,y)=\begin{cases}(-B_1(0,2,2,\widetilde{3};x\wedge y))B_0(0,2,0,2;x\vee y)>0\\(-B_1(0,2,2,\widetilde{3};x\vee y))B_0(0,2,0,2;x\wedge y)>0\end{cases}$$

$$G(0,2,0,2;x,y)-G(0,1,0,2;x,y)=\begin{cases}(-A_1(0,2,0,2;x\wedge y))A_1(0,1,0,2;x\vee y)>0\\(-A_1(0,2,0,2;x\vee y))A_1(0,1,0,2;x\wedge y)>0\end{cases}$$

$$G(0,2,1,3;x,y)-G(0,1,1,3;x,y)=\begin{cases}(-A_1(0,2,1,3;x\wedge y))A_1(0,1,1,3;x\vee y)>0\\(-A_1(0,2,1,3;x\vee y))A_1(0,1,1,3;x\wedge y)>0\end{cases}$$

$$G(2,\widetilde{3},1,3;x,y)-G(0,2,1,3;x,y)=\begin{cases}A_1(2,\widetilde{3},1,3;x\wedge y)A_0(0,2,1,3;x\vee y)>0\\A_1(2,\widetilde{3},1,3;x\vee y)A_0(0,2,1,3;x\wedge y)>0\end{cases}$$

$$G(0,2,2,\widetilde{3};x,y)-G(0,1,2,\widetilde{3};x,y)=\begin{cases}(-A_1(0,2,2,\widetilde{3};x\wedge y))A_1(0,1,2,\widetilde{3};x\vee y)>0\\(-A_1(0,2,2,\widetilde{3};x\vee y))A_1(0,1,2,\widetilde{3};x\wedge y)>0\end{cases}$$

$$G(2,\widetilde{3},2,\widetilde{3};x,y)-G(0,2,2,\widetilde{3};x,y)=\begin{cases}A_1(2,\widetilde{3},2,\widetilde{3};x\wedge y)A_0(0,2,2,\widetilde{3};x\vee y)>0\\A_1(2,\widetilde{3},2,\widetilde{3};x\vee y)A_0(0,2,2,\widetilde{3};x\wedge y)>0\end{cases}$$

$$G(0,1,1,3;x,y)-G(0,1,0,2;x,y)=$$
$$\begin{cases}(-B_1(0,1,1,3;x\wedge y))\,(-B_1(0,1,0,2;x\vee y))\Big[\\ \left(\dfrac{B_0}{-B_1}\right)(0,1,1,3;x\wedge y)+\left(\dfrac{B_0}{-B_1}\right)(0,1,0,2;x\vee y)-p\Big]>0\\ (-B_1(0,1,1,3;x\vee y))\,(-B_1(0,1,0,2;x\wedge y))\Big[\\ \left(\dfrac{B_0}{-B_1}\right)(0,1,1,3;x\vee y)+\left(\dfrac{B_0}{-B_1}\right)(0,1,0,2;x\wedge y)-p\Big]>0\end{cases}$$

$$G(0,2,1,3;x,y)-G(0,2,0,2;x,y)=$$
$$\begin{cases}(-B_1(0,2,1,3;x\wedge y))\,(-B_1(0,2,0,2;x\vee y))\Big[\\ \left(\dfrac{B_0}{-B_1}\right)(0,2,1,3;x\wedge y)+\left(\dfrac{B_0}{-B_1}\right)(0,2,0,2;x\vee y)-p\Big]>0\\ (-B_1(0,2,1,3;x\vee y))\,(-B_1(0,2,0,2;x\wedge y))\Big[\\ \left(\dfrac{B_0}{-B_1}\right)(0,2,1,3;x\vee y)+\left(\dfrac{B_0}{-B_1}\right)(0,2,0,2;x\wedge y)-p\Big]>0\end{cases}$$

$$G(1,3,1,3;x,y)-G(0,2,1,3;x,y)=$$

$$\begin{cases} A_1(1,3,1,3;x{\wedge}y)\,(-A_1(0,2,1,3;x{\vee}y))\Big[\\ \left(\dfrac{-A_0}{A_1}\right)(1,3,1,3;x{\wedge}y)+\left(\dfrac{A_0}{-A_1}\right)(0,2,1,3;x{\vee}y)-p\Big]>0 \\ A_1(1,3,1,3;x{\vee}y)\,(-A_1(0,2,1,3;x{\wedge}y))\Big[\\ \left(\dfrac{-A_0}{A_1}\right)(1,3,1,3;x{\vee}y)+\left(\dfrac{A_0}{-A_1}\right)(0,2,1,3;x{\wedge}y)-p\Big]>0 \end{cases}$$

$$G(1,3,2,\widetilde{3};x,y)-G(0,2,2,\widetilde{3};x,y)=$$

$$\begin{cases} A_1(1,3,2,\widetilde{3};x{\wedge}y)\left(-A_1(0,2,2,\widetilde{3};x{\vee}y)\right)\Big[\\ \left(\dfrac{-A_0}{A_1}\right)(1,3,2,\widetilde{3};x{\wedge}y)+\left(\dfrac{A_0}{-A_1}\right)(0,2,2,\widetilde{3};x{\vee}y)-p\Big]>0 \\ A_1(1,3,2,\widetilde{3};x{\vee}y)\left(-A_1(0,2,2,\widetilde{3};x{\wedge}y)\right)\Big[\\ \left(\dfrac{-A_0}{A_1}\right)(1,3,2,\widetilde{3};x{\vee}y)+\left(\dfrac{A_0}{-A_1}\right)(0,2,2,\widetilde{3};x{\wedge}y)-p\Big]>0 \end{cases}$$

$$G(0,1,2,\widetilde{3};x,y)-G(0,1,1,3;x,y)=$$
$$\frac{K(3,\widetilde{3},2,\widetilde{4})}{K(2,3,1,4)}B_0(0,1,2,\widetilde{3};x{\wedge}y)B_0(0,1,2,\widetilde{3};x{\vee}y),$$

which is positive if $\widetilde{K}_3\geq 0$. *If* $\widetilde{K}_3<0$, *we have*

$$(\text{r.h.s.})\begin{cases} >0 & (-1<x,y<x_0 \quad \text{or} \quad x_0<x,y<1) \\ =0 & (x=x_0 \quad \text{or} \quad y=x_0) \\ <0 & (-1<x<x_0<y<1 \quad \text{or} \quad -1<y<x_0<x<1). \end{cases}$$

$$G(0,2,2,\widetilde{3};x,y)-G(0,2,1,3;x,y)=$$
$$\frac{K(\widetilde{3},4,2,\widetilde{5})}{K(3,3,1,5)}B_0(0,2,2,\widetilde{3};x{\wedge}y)B_0(0,2,2,\widetilde{3};x{\vee}y),$$

which is positive if $\widetilde{K}_3\geq 0$. *If* $\widetilde{K}_3<0$, *we have*

$$(\text{r.h.s.})\begin{cases} >0 & (-1<x,y<x_1 \quad \text{or} \quad x_1<x,y<1) \\ =0 & (x=x_1 \quad \text{or} \quad y=x_1) \\ <0 & (-1<x<x_1<y<1 \quad \text{or} \quad -1<y<x_1<x<1). \end{cases}$$

$$
\begin{aligned}
&G(1,3,2,\widetilde{3};x,y)-G(1,3,1,3;x,y)=\\
&\frac{K(\widetilde{4},5,3,\widetilde{6})}{K(4,4,2,6)}B_0(1,3,2,\widetilde{3};x\wedge y)B_0(1,3,2,\widetilde{3};x\vee y)\\
&\begin{cases} >0 & (-1<x,y<x_2 \quad \text{or} \quad x_2<x,y<1)\\ =0 & (x=x_2 \quad \text{or} \quad y=x_2)\\ <0 & (-1<x<x_2<y<1 \quad \text{or} \quad -1<y<x_2<x<1)\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&G(2,\widetilde{3},2,\widetilde{3};x,y)-G(1,3,2,\widetilde{3};x,y)=\\
&\frac{K(\widetilde{5},\widetilde{5},4,\widetilde{\widetilde{6}})}{K(\widetilde{4},5,3,\widetilde{6})}A_0(2,\widetilde{3},2,\widetilde{3};x\wedge y)A_0(2,\widetilde{3},2,\widetilde{3};x\vee y)\\
&\begin{cases} >0 & (-1<x,y<-x_3 \quad \text{or} \quad -x_3<x,y<1)\\ =0 & (x=-x_3 \quad \text{or} \quad y=-x_3)\\ <0 & (-1<x<-x_3<y<1 \quad \text{or} \quad -1<y<-x_3<x<1)\end{cases}
\end{aligned}
$$

Note that x_0, x_1, x_2, x_3 *are introduced in Lemma* 2.6 (2) *and* (3).

Proof of Lemma 4.1 We only treat $G(0,1,2,\widetilde{3};x,y)-G(0,1,1,3;x,y)$. Taking a difference between Green functions (iv) and (iii) in Theorem 2.1 and using (2.5) in Lemma 2.7, we have

$$
\begin{aligned}
&G(0,1,2,\widetilde{3};x,y)-G(0,1,1,3;x,y)=\\
&\big(K_0,\,K_1\big)\,(1+x\wedge y)\left\{\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(0,1,2,\widetilde{3};x\vee y)-\begin{pmatrix}-A_0\\ A_1\end{pmatrix}(0,1,1,3;x\vee y)\right\}=\\
&\big(K_0,\,K_1\big)\,(1+x\wedge y)\frac{1}{K(2,3,1,4)}\begin{pmatrix}-\widetilde{K}_4\\ K_3\end{pmatrix}B_0(0,1,2,\widetilde{3};x\vee y)=\\
&\frac{K(3,\widetilde{3},2,\widetilde{4})}{K(2,3,1,4)}B_0(0,1,2,\widetilde{3};x\wedge y)B_0(0,1,2,\widetilde{3};x\vee y),
\end{aligned}
\tag{4.1}
$$

where we have used fundamental solution (iv) in Theorem 2.1 at the last equality. From Lemma 2.6 (2), the right hand side of (4.1) is positive if $\widetilde{K}_3=-qd^{-1}(a^{-2}\mathrm{ch}(2a)-b^{-2}\mathrm{ch}(2b))\geq 0$ and takes both positive and negative values if $\widetilde{K}_3=-qd^{-1}(a^{-2}\mathrm{ch}(2a)-b^{-2}\mathrm{ch}(2b))<0$. We can show other cases in the similar way. Thus we proved Lemma 4.1. ■

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