Saitama Math. J.
Vol. 29 (2012), 25–53

Genus two Heegaard splittings of exteriors of 1-genus 1-bridge knots II

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(*Received 5 September, 2011; Revised 24 January, 2012; Accepted 26 January, 2012*)

Abstract

A knot K is called a 1-genus 1-bridge knot in a 3-manifold M if (M, K) has a Heegaard splitting $(V_1, t_1) \cup (V_2, t_2)$ where V_i is a solid torus and t_i is a boundary parallel arc properly embedded in V_i. If the exterior of a knot has a genus 2 Heegaard splitting, we say that the knot has an unknotting tunnel. Naturally the exterior of a 1-genus 1-bridge knot K allows a genus 2 Heegaard splitting, i.e., K has an unknotting tunnel. But, in general, there are unknotting tunnels which are not derived form this procedure. Some of them may be levelled with the torus $\partial V_1 = \partial V_2$, whose case was studied in our previous paper [4]. In this paper, we consider the remaining case.

1. Introduction

This paper is a sequel to our previous paper [4]. We will use the same notations and terminology.

A properly embedded arc t in a solid torus V is called *trivial* if it is boundary parallel, that is, there is a disk C embedded in V such that $t \subset \partial C$ and $C \cap \partial V = \mathrm{cl}\,(\partial C - t)$. We call C a *canceling disk* of t. Let M be a closed connected orientable 3-manifold, and K a knot in M. We call K a 1-*genus* 1-*bridge knot* in M if M is a union of two solid tori V_1 and V_2 glued along their boundary tori ∂V_1 and ∂V_2 and if K intersects each solid torus V_i in a trivial arc t_i for $i = 1$ and 2. The splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ is called a 1-*genus* 1-*bridge splitting* of (M, K), where $H_1 = V_1 \cap V_2 = \partial V_1 = \partial V_2$, the torus. We call also the torus H_1 a 1-*genus* 1-*bridge splitting*. We say $(1, 1)$-*knots* and $(1, 1)$-*splitting* for short.

We recall the definition of a $(2, 0)$-splitting. Let W be a handlebody, and K a knot in $\mathrm{int}\, W$. We say K is a *core* in W if there are a disk D and an annulus A such that D is properly embedded in W and intersects K transversely in a single point and that A is embedded in W with $K \subset \partial A$ and $A \cap \partial W = \partial A - K$.

2010 Mathematics Subject Classification. 57N10, 57M25

Key words and phrases. 1-genus 1-bridge knot, Heegaard splitting, unknotting tunnel

*The first and second authors are partially supported by Grant-in-Aid for Scientific Research, (No. 21540071 and 18540100), Ministry of Education, Science, Sports and Culture.

We say that the pair (M, K) admits a $(2,0)$-*splitting* if M is a union of two handlebodies of genus two, say W_1 and W_2, glued along ∂W_1 and ∂W_2 and if K forms a core in W_1. The closed surface $H_2 = \partial W_1 = \partial W_2 = W_1 \cap W_2$ gives the splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ and is called a $(2,0)$-*splitting surface* or *a* $(2,0)$-*splitting* for short. This is also called a $(2;1,0)$-*splitting surface* in [8]. It is easy to see that $\mathrm{cl}\,(W_1 - N(K))$ is a compression body, and $H_2 = \partial W_1 = \partial W_2$ gives a genus two Heegaard splitting of the exterior of K.

A $(1,1)$-knot admits a $(2,0)$-splitting naturally as follows. Let $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ be a $(1,1)$-splitting. We take a regular neighborhood $N(t_2)$ of the arc t_2 in V_2. Then $(M, K) = (V_1 \cup N(t_2), K) \cup (\mathrm{cl}\,(V_2 - N(t_2)), \emptyset)$ is a $(2,0)$-splitting. We may take a regular neighborhood of t_1 to obtain another $(2,0)$-splitting. Such $(2,0)$-splittings are characterized in the following manner. A $(2,0)$-splitting $(M, K) = (W_1, K) \cup_H (W_2, \emptyset)$ is *meridionally stabilized* if there is a disk D_i properly embedded in W_i for $i = 1$ and 2 such that ∂D_1 and ∂D_2 intersect each other transversely in a single point in $H = \partial W_1 = \partial W_2$ and that D_1 intersects K transversely in a single point. A $(2,0)$-splitting $(M, K) = (V_i \cup N(t_j), K) \cup (\mathrm{cl}\,(V_j - N(t_j)), \emptyset)$, which is derived from a $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup (V_2, t_2)$, is meridionally stabilized since we can take the disk D_1 to be a meridian disk of the arc t_j in $N(t_j)$, and the disk D_2 to be a canceling disk of the arc t_j. Conversely, we can obtain a $(1,1)$-splitting torus by compressing the meridionally stabilized $(2,0)$-splitting surface along D_1.

A torus knot is a $(1,1)$-knot. The result on unknotting tunnels of torus knots by Z. Boileau, M. Rost and H. Zieschang in [3] together with the results in [1], [2] and [11] implies that there is a torus knot which admits a $(2,0)$-splitting which is not derived from a $(1,1)$-splitting.

We consider the situation where a knot K in M admits both a $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ and a $(2,0)$-splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$. In most cases, under some technical conditions, we can place the splitting surfaces H_1 and H_2 so that they intersect each other in a non-empty collection of loops which are K-essential both in H_1 and H_2, where a loop of $H_1 \cap H_2$ is called K-*essential* in H_i if it does not bound a disk intersecting K in at most one point in H_i. This is proven by a similar argument introduced by H. Rubinstein and M. Scharlemann in [12] and developed by T. Kobayashi and O. Saeki in [10]. In this paper, we begin with the situation where H_1 and H_2 intersect each other in a non-empty collection of K-essential loops. Let ℓ denote the number of K-essential loops of $H_1 \cap H_2$. The number ℓ is said to be *minimum* if there is no isotopies of H_1 and H_2 in (M, K) so that they intersect each other in a non-empty collection of smaller number of loops which are K-essential both in H_1 and in H_2. We recall the result in the previous paper [4] in the case of $M = S^3$.

Theorem 1.1 ([4]). *Suppose M is the 3-sphere S^3, and ℓ is minimum and $\ell \neq 2$ (either $\ell \geq 3$ or $\ell = 1$). Then, at least one of the following conditions holds.*

(1) The $(2,0)$*-splitting* H_2 *is meridionally stabilized.*

(2) There is an arc γ *which forms a spine of* (W_1, K) *and is isotopic into the torus* H_1*. Moreover, we can take* γ *so that there is a canceling disk* C_i *of the arc* t_i *in* (V_i, t_i) *with* $\partial C_i \cap \gamma = \partial\gamma = \partial t_i$ *for* $i = 1$ *or* 2*.*

(3) The $(1,1)$*-splitting* H_1 *admits a satellite diagram of a longitudinal slope.*

We recall some terminologies for the conclusions (2) and (3) of the above theorem. An embedded arc γ in W_1 forms a *spine of* (W_1, K) if $\gamma \cap K = \partial\gamma$ and W_1 collapses to $K \cup \gamma$. We say that a $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ admits a *satellite diagram* if there is an essential simple loop l on the torus H_1 such that the arcs t_1 and t_2 have canceling disks which are disjoint from l. We call l the *slope* of the satellite diagram. We say that the slope of the satellite diagram is *longitudinal* if it is longitudinal on ∂V_1 or ∂V_2. If l is longitudinal on ∂V_1, then the boundary torus of the regular neighborhood of $(H_1 - N(l)) \cup C_2$ also gives a $(1,1)$-splitting, where C_2 is a canceling disk of t_2 with $C_2 \cap l = \emptyset$.

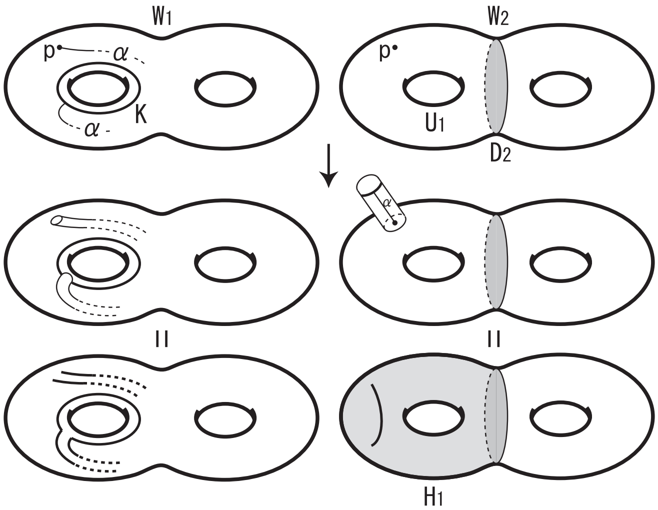

Figure 1

In this paper, we consider the case of the number of intersection loops $\ell = 2$.

Let X be a compact orientable 3-manifold, and T a compact 1-manifold properly embedded in X. For $i = 1$ and 2, let F_i be either a compact 2-submanifold of ∂X or a compact orientable 2-manifold which is properly embedded in X and is transverse to T. Suppose that $T \cap \partial F_i = \emptyset$ for $i = 1$ and 2. F_1 is said to be *T-compressible* in (X, T) if there is a disk D_1 embedded in X with $D_1 \cap F_1 = \partial D_1$ and $D_1 \cap T = \emptyset$ such that ∂D_1 does not bound a disk in $F_1 - T$. We call D_1 a

T*-compressing disk*. F_2 is said to be *meridionally compressible* in (X, T) if there is a disk D_2 embedded in X with $D_2 \cap F_2 = \partial D_2$ such that D_2 intersects T transversely in a single point and that ∂D_2 does not bound a disk which intersects T in a single point in F_2. We call D_2 a *meridionally compressing disk*.

A $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ is called *weakly* K*-reducible* if there is a t_i-compressing or meridionally compressing disk D_i of $H_1 = \partial V_i$ in (V_i, t_i) for $i = 1$ and 2 such that $\partial D_1 \cap \partial D_2 = \emptyset$. A $(1,1)$-splitting is called *strongly* K*-irreducible* if it is not weakly K-reducible. $(1,1)$-knots which admit a weakly K-reducible $(1,1)$-splitting are characterized in Lemma 3.2 in [7], which is recalled in Proposition 2.6.

A $(2,0)$-splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ is called *weakly* K*-reducible* if there is a K-compressing or meridionally compressing disk D_1 of $H_2 = \partial W_1$ in (W_1, K) and a compressing disk D_2 of $H_2 = \partial W_2$ in W_2 such that $\partial D_1 \cap \partial D_2 = \emptyset$. $(2,0)$-knots which admit a weakly K-reducible $(2,0)$-splitting are characterized in Proposition 2.14 in [5] which is recalled in Proposition 2.10. There we find that a meridionally stabilized $(2,0)$-splitting is weakly K-reducible.

To apply arguments by Kobayashi and Saeki in [10], to make the $(1,1)$-splitting H_1 and the $(2,0)$-splitting H_2 intersect in K-essential loops, we need the conditions that neither H_1 nor H_2 is weakly K-reducible and that M has a 2-fold branched cover with branch set K. See Section 1 in the previous paper [4] for detail. We also need the condition on a branched cover when we apply Proposition 2.12 (Proposition 3.4 in [9]).

In this paper, we will prove the following theorem.

Theorem 1.2. *Let* M *be the* 3*-sphere or a lens space* (*other than* $S^2 \times S^1$)*, and* K *a knot in* M*. Let* $(V_1, t_1) \cup_{H_1} (V_2, t_2)$ *and* $(W_1, K) \cup_{H_2} (W_2, \emptyset)$ *be a* $(1,1)$*-splitting and a* $(2,0)$*-splitting of* (M, K)*. Suppose that the surfaces* H_1 *and* H_2 *intersect each other in two loops which are* K*-essential both in* H_1 *and in* H_2*. Further, we assume that* M *has a* 2*-fold branched cover with branch set* K*. Then at least one of the six conditions* (*a*) $\sim$ (*f*) *below holds.*

- (*a*) *We can isotope* H_1 *and* H_2 *in* (M, K) *so that they intersect in one loop which is* K*-essential both in* H_1 *and in* H_2*.*
- (*b*) *The* $(2,0)$*-splitting* H_2 *is weakly* K*-reducible.*
- (*c*) *The knot* K *is a torus knot.*
- (*d*) *The knot* K *is a satellite knot.*
- (*e*) *The* $(1,1)$*-splitting* H_1 *admits a satellite diagram of a longitudinal slope.*
- (*f*) *There is an essential separating disk* D_2 *in* W_2*, and an arc* α *in* W_1 *such that* $\alpha \cap K$ *is one of the endpoints* $\partial\alpha$*, and* $\alpha \cap \partial W_1$ *is the other endpoint, say* p*, of* α *and that* D_2 *cuts off a solid tours* U_1 *from* W_2 *with* $p \in \partial U_1$ *and*

with the torus $\partial N(U_1 \cup \alpha)$ *isotopic to the* $(1,1)$*-splitting torus* H_1 *in* (M, K). *See Figure 1.*

We recall some terminologies for the conclusions (c) and (d) in the above theorem. We say that K is a *torus knot* if K can be isotoped into a torus which gives a genus one Heegaard splitting of M. We call K a *satellite knot* if the exterior $E(K) = \mathrm{cl}\,(M - N(K))$ contains an incompressible torus T which is not parallel to $\partial E(K)$. The torus T may not bound a solid torus in M.

Theorem 1.2 provides the case (3-1) of Theorem 1.3 in [4] so that the proof completes.

2. Preliminaries

Definition 2.1. Let X be an orientable 3-manifold, and T a compact 1-manifold properly embedded in X. Let F be a compact orientable 2-manifold properly embedded in X. Suppose that ∂F is disjoint from T and that T is transverse to F. We say that F is *T-∂-compressible* in (X, T) if there is a disk D embedded in X satisfying all of the following conditions:

(1) D is disjoint from T;

(2) $D \cap (F \cup \partial X) = \partial D$;

(3) $D \cap F$ is an essential arc properly embedded in $F - T$;

(4) $\partial D \cap \partial X$ is an essential arc in the surface obtained from $\partial X - T$ by cutting along ∂F.

We call such a disk D a *T-∂-compressing disk* of F. When there is not such a disk, we say that F is *T-∂-incompressible* in (X, T).

Remark. In the usual definition, the above condition (4) is omitted, but we add this in this paper as in [5] and [7]. Note that this definition is equivalent to the usual one when F is T-incompressible.

Lemma 2.2 (Lemma 2.10 in [7]). *Let V be a solid torus, and t a trivial arc properly embedded in V. Let F be a compact orientable 2-manifold properly embedded in V so that F is transverse to t and $\partial F \cap t = \emptyset$. Suppose that F is t-incompressible and t-∂-incompressible in (V, t). Then F is a union of finitely many surfaces of types* $(1) \sim (6)$ *below:*

(1) a 2-sphere disjoint from t;

(2) a 2-sphere intersecting t transversely in two points;

(3) a meridian disk of V disjoint from t;

(4) a meridian disk of V intersecting t transversely in a single point;

(5) a peripheral disk disjoint from t*;*

(6) a peripheral disk intersecting t *transversely in a single point.*

Lemma 2.3 (Lemma 3.10 in [5])**.** *Let* W *be a handlebody of genus two, and* K *a core loop in* W*. Let* F *be a compact orientable 2-manifold properly embedded in* W *so that* F *is transverse to* K*. Suppose that* F *is* K*-incompressible and* K*-*∂*-incompressible. Then* F *is a disjoint union of finitely many surfaces as below:*

(1) a 2-sphere disjoint from K*;*

(2) a 2-sphere which bounds a trivial 1-string tangle in (W, K)*;*

(3) an essential disk of W *disjoint from* K*;*

(4) an essential disk of W *intersecting* K *transversely in a single point;*

(5) a torus bounding a solid torus which forms a regular neighborhood of K *in* W*.*

Definition 2.4. A $(1,1)$-splitting $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ is called *K-reducible* if there are K-compressing disks D_1 and D_2 of H_1 in V_1 and V_2 respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$.

Definition 2.5. A knot K in M is called a *core knot* if its exterior is a solid torus.

Note that a knot in the 3-sphere is a core knot if and only if it is the trivial knot.

Proposition 2.6 (Lemmas 3.1 and 3.2 in [7])**.** *Let* M *be the* 3*-sphere or a lens space other than* $S^2 \times S^1$*, and* K *a knot in* M*. Let* $(M, K) = (V_1, t_1) \cup_{H_1} (V_2, t_2)$ *be a* $(1,1)$*-splitting. If it is weakly* K*-reducible, then one of the following occurs:*

(1) K *is a trivial knot;*

(2) K *is a core knot in a lens space;*

(3) K *is a 2-bridge knot in the* 3*-sphere;*

(4) K *is a connected sum of a core knot in a lens space and a 2-bridge knot in the* 3*-sphere.*

When the $(1,1)$*-splitting* H_1 *is* K*-reducible,* K *is trivial.*

Definition 2.7. A $(2,0)$-splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ is called *K-reducible* if there are a K-compressing disk D_1 of H_2 in (W_1, K) and an essential disk D_2 in W_2 such that $\partial D_1 = \partial D_2$ in H_2.

Definition 2.8. A knot K in M is called a *split knot* if its exterior $E(K)$ =cl$(M -$

$N(K))$ is reducible. A knot K is called *composite* if there is a 2-sphere S embedded in M such that S is separating in M, that S intersects K transversely in precisely two points and that the annulus $S \cap E(K)$ is incompressible and ∂-incompressible in $E(K)$.

Proposition 2.9 (Proposition 2.9 in [5]). *A* $(2,0)$*-splitting* $(M,K) = (W_1,K) \cup_{H_2} (W_2,\emptyset)$ *is* K*-reducible if and only if* K *is either a core knot or a split knot.*

Proposition 2.10 (Proposition 2.14 in [5]). *A* $(2,0)$*-splitting* $(M,K) = (W_1,K) \cup_{H_2} (W_2,\emptyset)$ *is weakly* K*-reducible if and only if one of the following occurs:*

(1) the $(2,0)$*-splitting* H_2 *is* K*-reducible;*

(2) the $(2,0)$*-splitting* H_2 *is meridionally stabilized; or*

(3) K *is a composite knot.*

Proposition 2.11 (Proposition 4.9 in [4]). *Suppose* (M,K) *has a* $(1,1)$*-splitting* H_1 *and a* $(2,0)$*-splitting* H_2*. Further, we suppose that* H_1 *admits a satellite diagram. Then one of the following holds:*

(1) the $(2,0)$*-splitting* H_2 *is* K*-reducible;*

(2) the knot K *is a torus knot;*

(3) the knot K *is a satellite knot;*

(4) the $(1,1)$*-splitting* H_1 *admits a satellite diagram of a longitudinal slope.*

In the proof of Theorem 1.2, we use the next proposition. The condition that M has a 2-fold branched cover with branch set K is necessary only when we apply this proposition.

Proposition 2.12 (Proposition 3.4 in [9]). *Let* M *be a closed orientable 3-manifold, and* L *a link in* M*. Assume that* M *has a 2-fold branched cover with branch set* L*. Let* H_i *be* (g_i,n_i)*-splitting of* (M,L) *for* $i=1$ *and* $i=2$*, and* W *a genus* g_2 *handlebody bounded by* H_2 *in* M*. Suppose that* H_1 *is contained in the interior of* W*, and that there is an* L*-compressing or meridionally compressing disk* D *of* H_2 *in* $(W, L\cap W)$ *with* $D\cap H_1=\emptyset$*. Then either* (i) $M=S^3$ *and* $L=\emptyset$ *or* L *is the trivial knot, or* (ii) *the splitting* H_2 *is weakly* L*-reducible.*

3. Separation of the proof into cases

We begin to prove Theorem 1.2. Let M be the 3-sphere or a lens space (other than $S^2\times S^1$), and K a knot in M. Let $(V_1,t_1)\cup_{H_1}(V_2,t_2)$ and $(W_1,K)\cup_{H_2}(W_2,\emptyset)$

be a $(1, 1)$-splitting and a $(2, 0)$-splitting of (M, K). According to the assumption of Theorem 1.2, we suppose that $H_1 \cap H_2$ consists of two loops, say l_1 and l_2, which are K-essential both in H_1 and in H_2.

Since the $(2, 0)$-splitting surface H_2 separates M, there are two patterns of intersection loops of $H_1 \cap H_2$ in H_1: (1) l_1 and l_2 together divide H_1 into a disk Q, an annulus A_1 and a torus with one hole H_1', or (2) l_1 and l_2 together divide H_1 into two annuli A_{11} and A_{12}. Since the $(1, 1)$-splitting torus H_1 also separates M, there are two patterns of intersection loops of $H_1 \cap H_2$ in H_2: (A) l_1 and l_2 are parallel separating essential loops, or (B) they are parallel non-separating essential loops. See Figure 2. In the following sections, we study each case.

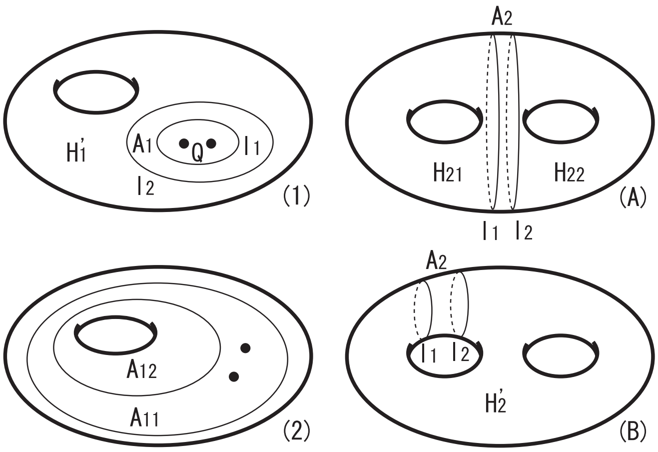

Figure 2

4. Case (1)(A)

In this section, we consider the case where $H_1 \cap H_2$ is of the pattern (1) in H_1, and of the pattern (A) in H_2 in Figure 2. In this case we show (a), (b), (c) or (d) of Theorem 1.2 holds. In Case (1), we assume, without loss of generality, that $l_1 = \partial Q$. The disk Q contains the two intersection points $K \cap H_1$ because the loop l_1 is K-essential in H_1. Then the torus with one hole H_1' and the annulus A_1 are disjoint from K. In particular, the loops l_1 and l_2 are parallel in $H_1 - K$. These loops are K-essential but inessential in H_1. Since the handlebody W_1 contains K as a core, and since Q intersects K, Q and H_1' are contained in W_1, and A_1 in W_2. In Case (A), for $i = 1$ and 2 the loop l_i bounds a torus with one hole, say H_{2i}, such that $H_{21} \cap H_{22} = \emptyset$. The complementary region is an annulus, say A_2, between l_1 and l_2. We can assume, without loss of generality, that A_2 is contained in the solid torus V_1, and $H_{21} \cup H_{22}$ in V_2.

By Lemma 2.3 (Lemma 3.10 in [5]), $Q \cup H_1'$ is K-compressible or K-∂-

compressible in (W_1, K). A_1 is compressible or ∂-compressible in the handlebody W_2.

Lemma 4.1. *If A_1 is compressible in W_2, then the $(2,0)$-splitting H_2 of (M,K) is weakly K-reducible.*

Proof. A compressing operation on A_1 yields a disk D_2 bounded by l_1 in W_2. By Lemma 2.3, the disk Q with two intersection points with K is K-compressible or K-∂-compressible in (W_1, K) if we ignore the once punctures torus H_1'. We perform a K-compressing or K-∂-compressing operation on Q, to obtain a K-compressing or meridionally compressing disk D_1 of H_2 in (W_1, K). We can slightly move D_1 near its boundary circle so that it is disjoint from l_1. Then the disks D_1 and D_2 together show that H_2 is weakly K-reducible. □

In the rest of this section, we assume that A_1 is incompressible in W_2. Then A_1 is ∂-compressible, and hence parallel to the annulus A_2 in W_2.

Lemma 4.2. *Suppose that A_1 is incompressible in W_2 and that W_1 contains either* (1) *a t_2-compressing disk of $H_{21} \cup H_{22}$,* (2) *a t_2-∂-compressing disk of $H_{21} \cup H_{22}$ incident to H_{21},* (3) *a K-compressing disk of $Q \cup H_1'$ in V_2 or* (4) *a K-∂-compressing disk of $Q \cup H_1'$ incident to Q. Then H_2 is weakly K-reducible.*

Proof. In Case (2), the t_2-∂-compressing disk is also a K-∂-compressing disk of Q by the unusual definition of a ∂-compressing disk. (See Definition 2.1.) In Case (4), the K-∂-compressing disk is incident to H_{21} rather than A_2 by the definition of a ∂-compressing disk. Hence, in Cases (2), (3) and (4), by performing a compressing or ∂-compressing operation on a copy of Q or H_1', we obtain a K-compressing or meridionally compressing disk D of $H_{21} \cup H_{22}$ in (W_1, K) such that $D \subset V_2 \cap W_1$ as in (1). We can isotope H_1 along the parallelism between A_1 and A_2 so that H_1 is contained in $\operatorname{int} W_1$ and is disjoint from D. Then Proposition 2.12 shows that H_2 is weakly K-reducible, or K is the trivial knot. In the latter case, H_2 is K-reducible by Proposition 2.9. □

Definition 4.3. We call a $(2,0)$-splitting $(M,K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ *semi-stabilized* if there is a K-compressing disk D_i of H_2 in $(W_i, K \cap W_i)$ for $i = 1$ and 2 such that ∂D_1 and ∂D_2 intersect each other transversely in precisely two points.

Proposition 4.4 (Theorem 7.2 in [5])**.** *If (M,K) admits a semi-stabilized strongly K-irreducible $(2,0)$-splitting, then one of the following occurs:*

(1) the knot K is a torus knot in M;

(2) the knot K is a satellite knot;

(3) the 3-manifold M admits a Seifert fibered structure over the 2-sphere with

three exceptional fibers, and K is an exceptional fiber; or

(4) the exterior of the knot K contains a non-separating torus.

When M is the 3-sphere or a lens space except for $S^2 \times S^1$, the conclusions (3) and (4) do not occur.

Lemma 4.5. *Suppose that A_1 is incompressible in W_2, and that $V_2 \cap W_1$ contains a t_2-∂-compressing disk of $H_{21} \cup H_{22}$ incident to H_{22} or a K-∂-compressing disk of $Q \cup H_1'$ incident to H_1'. Then one of the following conditions holds.*

(1) H_2 is weakly K-reducible.

(2) We can isotope H_1 in (M, K) so that $H_1 \cap H_2$ is a single loop which is K-essential both in H_1 and in H_2.

(3) H_2 is semi-stabilized, and K is a torus knot or a satellite knot.

Moreover, if $Q \cup H_1'$ is K-incompressible in (W_1, K), then (1) *or* (2) *holds.*

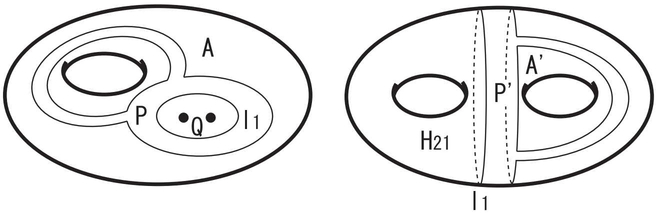

Figure 3

Proof. Let D be the ∂-compressing disk in the preliminary condition. Note that, when D is a K-∂-compressing disk of $Q \cup H_1'$, the arc $\partial D \cap \partial W_1$ is contained in H_{22} by Definition 2.1. We isotope H_1 along the ∂-compressing disk D slightly beyond the arc $\partial D \cap H_{22}$. Then, by the definition of a ∂-compressing disk, H_1' is deformed into an annulus A, each of the boundary loops ∂A is non-separating in H_{22}, and these loops cobound an annulus, say A', in H_{22}. The annulus A_1 is deformed into a disk with two holes, say P, in the handlebody W_2. Set $P' = \mathrm{cl}\,(H_2 - (H_{21} \cup A'))$. See Figure 3. Note that P is parallel to P' in W_2 since A_1 is parallel to A_2 in W_2 before the isotopy. By Lemma 2.3, $Q \cup A$ is K-compressible or K-∂-compressible in (W_1, K).

Case (*a*). We first assume that $Q \cup A$ is K-incompressible and has a K-∂-compressing disk, say R, in (W_1, K). (This holds if $Q \cup H_1'$ is K-incompressible before the isotopy.) If the arc $\partial R \cap H_2$ is contained in H_{21}, then R is there also before the isotopy, and H_2 is weakly K-reducible by Lemmas 4.1 and 4.2. If the arc $\partial R \cap H_2$ is contained in the annulus A', then A is parallel to A' in $W_1 - K$. We can isotope H_1 along the parallelism so that H_1 and H_2 intersect each other only in the loop l_1 which is K-essential both in H_1 and in H_2. Hence we may assume that the arc $\partial R \cap H_2$ is contained in the disk with two holes P'. If the disk R is incident to the annulus A, then by performing a K-∂-compressing operation on A along R, we obtain a disk, say R', disjoint from K. Since the arc $\partial R \cap H_2$ is an essential arc in P' and since it connects the two loops $\partial A'$, the boundary loop $\partial R'$ is parallel to $l_1 = \partial Q$. This implies that the disk Q is K-compressible in (W_1, K). This contradicts our assumption. Hence R is incident to Q. By ∂-compressing Q along R, we obtain a disk, say R_1, which intersects K transversely in a single point. We can isotope R_1 in (W_1, K) so that it is bounded by a component of $\partial A'$. By Lemma 2.2 (Lemma 2.10 in [7]), $H_{21} \cup A'$ is t_2-compressible or t_2-∂-compressible in (V_2, t_2). First we consider the former case. If $H_{21} \cup A'$ has a t_2-compressing disk in W_1, then by compressing a copy of $H_{21} \cup A'$, we obtain a K-compressing disk of $Q \cup A$, which contradicts our assumption. Hence $H_{21} \cup A'$ has a t_2-compressing disk in W_2. This disk together with R_1 shows that H_2 is weakly K-reducible.

We consider the latter case, where $H_{21} \cup A'$ has a t_2-∂-compressing disk, say C. If C is contained in W_1, then it is also a K-∂-compressing disk of $Q \cup A$. When C is incident to A', the annulus A is parallel to A' in (W_1, K), and we obtain the conclusion (2) of this lemma. When C is incident to H_{21}, it is a K-∂-compressing disk of H_{21} before the isotopy along D, and Lemmas 4.1 and 4.2 imply that H_2 is weakly K-reducible. Hence we may assume that C is contained in W_2. If C is incident to H_{21}, then this disk is extended to an essential disk with boundary loop in $H_{21} \cup P'$ because P is parallel to P' in W_2. This disk together with R_1 shows that H_2 is weakly K-reducible. If C is incident to the annulus A', then this disk is extended to an essential disk in W_2 such that its boundary loop intersects ∂R_1 transversely in a single point because P is parallel to P' in W_2. Then this disk together with R_1 shows that H_2 is meridionally stabilized, and hence is weakly K-reducible by Proposition 2.10 (Proposition 2.14 in [5]).

Case (*b*). We consider the case where $Q \cup A$ has a K-compressing disk E in (W_1, K). Note that E gives a K-compressing disk of $Q \cup H_1'$ before the isotopy. By Lemma 4.2, we can assume that E is contained in $W_1 \cap V_1$. Then a K-compressing operation on $Q \cup H_1'$ yields a K-compressing disk E' of the annulus A_2 in (W_1, K). After an adequate isotopy, we may assume that $\partial E' = l_1$ and E' is disjoint from $(\operatorname{int} Q) \cup H_1'$. After the isotopy along D, $H_{21} \cup A'$ is t_2-compressible or t_2-∂-compressible in (V_2, t_2) by Lemma 2.2. We first consider the case where $H_{21} \cup A'$ is t_2-compressible. If the compressing disk is in W_2, then it shows that H_2 is weakly K-reducible together with E'. If it is in W_1, then Lemma 4.2 shows that H_2 is weakly K-reducible.

Hence we can assume that $H_{21} \cup A'$ is t_2-incompressible in (V_2, t_2), and has a t_2-∂-compressing disk, say Z, in (V_2, t_2). First suppose that Z is incident to A'.

If Z is contained in W_1, then A is parallel to A' in $W_1 - K$, and the conclusion (2) holds. If Z is contained in W_2, then this disk gives an essential disk in W_2 such that its boundary loop is disjoint from $\partial E'$, since P is parallel to P' in W_2. This shows that H_2 is weakly K-reducible.

Thus we may assume that Z is incident to H_{21}. If Z is contained in W_1, then H_2 is weakly K-reducible by Lemma 4.2. Hence we may assume that Z is contained in W_2. The boundary loop ∂Z intersects P in an essential arc with both endpoints in l_1. Since P and P' are parallel in W_2, Z gives an essential disk in W_2 such that its boundary loop intersects $\partial E'$ transversely in two points. Thus H_2 is semi-stabilized, and the conclusion (3) holds by Proposition 4.4 and the note just after it. ☐

Lemma 4.6. *Suppose that A_1 is incompressible in W_2, and that $Q \cup H_1'$ is K-compressible in (W_1, K). Then one of the following conditions holds.*

(1) H_2 is weakly K-reducible.

(2) We can isotope H_1 in (M, K) so that $H_1 \cap H_2$ is a single loop which is K-essential both in H_1 and in H_2.

(3) H_2 is semi-stabilized, and K is a torus knot or a satellite knot.

Proof. By compressing $Q \cup H_1'$ we obtain a disk, say D, disjoint from K. An adequate isotopy moves D so that $\partial D = l_1$.

$H_{21} \cup H_{22}$ is t_2-compressible or t_2-∂-compressible in (V_2, t_2) by Lemma 2.2. In the former case, let R be a t_2-compressing disk of $H_{21} \cup H_{22}$. If R is contained in W_1, then H_2 is weakly K-reducible by Lemma 4.2. If R is contained in W_2, the disks D and R together show that H_2 is weakly K-reducible.

Hence we may assume that $H_{21} \cup H_{22}$ has a t_2-∂-compressing disk R' in (V_2, t_2). Since $H_1 \cap W_2 = A_1$ is an annulus disjoint from K, the disk R' is contained in W_1 rather than W_2 because of the definition of a t_2-∂-compressing disk. Thus R' is contained in $V_2 \cap W_1$, and we obtain the conclusion by Lemmas 4.2 and 4.5. ☐

Lemma 4.7. *Suppose that A_1 is incompressible in W_2 and that $Q \cup H_1'$ is K-incompressible in (W_1, K). Then either H_2 is weakly K-reducible, or we can isotope H_1 in (M, K) so that $H_1 \cap H_2$ is a single loop which is K-essential both in H_1 and in H_2.*

Proof. Because $Q \cup H_1'$ is K-incompressible in (W_1, K), it has a K-∂-compressing disk, say D. By the definition of a K-∂-compressing disk, the arc $\partial D \cap H_2$ is contained in H_{21} or H_{22} rather than in A_2. Then we obtain the conclusion by Lemmas 4.2 and 4.5. ☐

Thus, in Case (1)(A), we have Conclusion (a), (b), (c) or (d) of Theorem 1.2 by Lemmas 4.1, 4.6 and 4.7.

5. Case (1)(B)

In this section, we consider the case where the intersection loops $H_1 \cap H_2$ are of the pattern (1) in H_1 and of the pattern (B) in H_2 in Figure 2. In this case, we show (b), (c) or (d) of Theorem 1.2 holds. In Case (B), each of the loops l_1 and l_2 of $H_1 \cap H_2$ is non-separating in H_2, and they cobound an annulus, say A_2. The complementary region $H_2' = \mathrm{cl}\,(H_2 - A_2)$ is a torus with two holes. Let Q, A_1, H_1' be as in the previous section. In particular, $l_1 = \partial Q$ and $l_2 = \partial H_1'$. We may assume, without loss of generality, that A_2 is contained in V_1, and H_2' in V_2. See Figure 2.

By Lemma 2.3, $Q \cup H_1'$ is K-compressible or K-∂-compressible in (W_1, K). A_1 is compressible or ∂-compressible in the handlebody W_2. When A_1 is ∂-compressible, either A_1 is parallel to A_2 in W_2, or A_1 has a ∂-compressing disk that is also a t_2-∂-compressing disk of H_2' in (V_2, t_2).

Lemma 5.1. *Suppose that A_1 is incompressible and not parallel to A_2 in W_2. Then H_2 is weakly K-reducible.*

Proof. Since the annulus A_1 is incompressible and not parallel to A_2 in W_2, it has a ∂-compressing disk D such that the arc $\partial D \cap H_2$ is contained in H_2'. By ∂-compressing a copy of A_1 along D, we obtain a compressing disk D_2 of H_2' in W_2.

The annulus A_2 is t_1-compressible or t_1-∂-compressible in (V_1, t_1) by Lemma 2.2. First we consider the former case. Let R be a t_1-compressing disk of A_2. If R is contained in W_1, then R and D_2 together show that H_2 is weakly K-reducible. If R is contained in W_2, then by compressing a copy of A_2 along R we obtain a compressing disk of A_1 in W_2. This contradicts our assumption.

Hence we may assume that A_2 has a t_1-∂-compressing disk C in (V_1, t_1). This disk C is not contained in W_1 since the two boundary loops ∂A_2 are contained in distinct components of $H_1 \cap W_1 = Q \cup H_1'$. Then it is contained in W_2, and incident to A_1. Hence the annuli A_1 and A_2 are parallel in W_2. This again contradicts our assumption. □

In other cases, the arguments are similar to those in the previous section. The proofs of the next two lemmas are the same as those of Lemmas 4.1 and 4.2, and we omit them.

Lemma 5.2. *If A_1 is compressible in W_2, then H_2 is weakly K-reducible.*

The above two lemmas allow us to assume that A_1 is incompressible, and parallel to A_2 in W_2 in the rest of this section.

Lemma 5.3. *Suppose that A_1 is parallel to A_2 in W_2, and that W_1 contains either* (1) *a t_2-compressing disk of H_2',* (2) *a t_2-∂-compressing disk of H_2' incident* *to Q, (3) a K-compressing disk of $Q \cup H_1'$ in V_2 or (4) a K-∂-compressing disk of $Q \cup H_1'$ incident to Q. Then H_2 is weakly K-reducible.*

We call a $(2,0)$-splitting $(M, K) = (W_1, K) \cup_{H_2} (W_2, \emptyset)$ *K-stabilized* if there is a K-compressing disk of H_2 in $(W_i, K \cap W_i)$ for $i = 1$ and 2 such that ∂D_1 and ∂D_2 intersect each other transversely in a single point. A K-stabilized $(2,0)$-splitting is K-reducible. See, for example, Lemma 4.1 in [5].

Lemma 5.4. *Suppose that A_1 is parallel to A_2 in W_2. Assume that $V_2 \cap W_1$ contains a t_2-∂-compressing disk of H_2' incident to H_1' or a K-∂-compressing disk of $Q \cup H_1'$ incident to H_1'. Then one of the following conditions holds.*

(1) H_2 is weakly K-reducible.

(2) H_2 is semi-stabilized, and K is a torus knot or a satellite knot.

Moreover, if $Q \cup H_1'$ is K-incompressible in (W_1, K), then the conclusion (1) *holds.*

Proof. We isotope H_1 along the ∂-compressing disk D in the preliminary condition slightly beyond the arc $\partial D \cap H_2'$. Then, by the definition of a ∂-compressing disk, H_1' is deformed into an annulus A, each of the boundary loops ∂A is essential in H_2'. The annulus A_1 is deformed to a disk with two holes P. The annulus A_2 is deformed to a disk with two holes P_{21}. The torus with two holes H_2' is deformed to a 2-manifold H_2'', which is either a disk with two holes P_{22}, or a disjoint union of an annulus A' and a torus with one hole H_2^*. Note that one of the components of $\partial A'$ is $l_1 = \partial Q$. See Figure 4. Then P is parallel to P_{21} since A_1 is parallel to A_2 in W_2 before the isotopy. By Lemma 2.3, $Q \cup A$ is K-compressible or K-∂-compressible in (W_1, K).

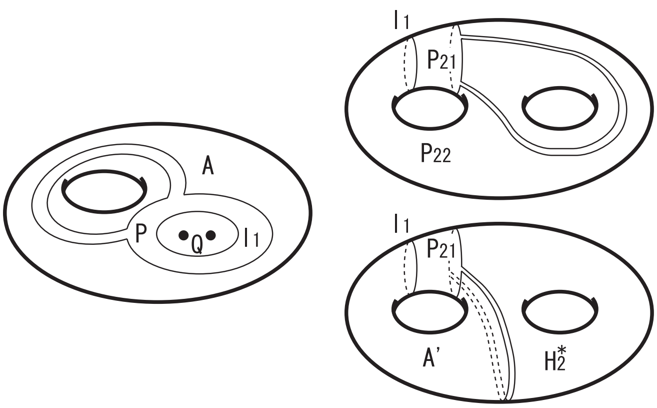

Figure 4

Case (*a*). First, we assume that $Q \cup A$ is K-incompressible and has a K-∂-compressing disk R in (W_1, K). (This holds if $Q \cup H_1'$ is K-incompressible before the isotopy.) The arc $\partial R \cap H_2$ is not contained in the annulus A' because of the definition of a ∂-compressing disk. The arc $\partial R \cap H_2$ is not contained in H_2^* since it contains only one component of ∂A. Thus the arc $\partial R \cap H_2$ is contained in a disk with two holes P_{21} or P_{22}.

If the arc $\partial R \cap H_2$ is contained in P_{22}, then by ∂-compressing $Q \cup A$ along R we obtain an essential disk R_1 in W_1 such that it intersects K in at most one point. Since P is parallel to P_{21} in W_2, we can isotope H_1 into $\operatorname{int} W_1$ so that it is disjoint from R_1. Then Proposition 2.12 shows that H_2 is weakly K-reducible.

Hence we may assume that the arc $\partial R \cap H_2$ is contained in P_{21}. If R is incident to A, then by performing a K-∂-compressing operation on A along R, we obtain a disk R_2, disjoint from K. Since the arc $\partial R \cap H_2$ is an essential arc in P_{21} and since it connects the two loops ∂A, the boundary loop ∂R_2 is parallel to $l_1 = \partial Q$. This implies that Q is K-compressible in (W_1, K), contradicting our assumption. Hence R is incident to Q. By compressing Q along R, we obtain two disks R_3 and R_4, each of which intersects K transversely in a single point. We can isotope R_3 and R_4 in (W_1, K) so that they are bounded by the two loops ∂A. When $H_2'' = A' \cup H_2^*$, one of the disks R_3 and R_4 is bounded by ∂H_2^*, and hence is separating in W_1. This contradicts that each of R_3 and R_4 intersects K transversely in a single point. Hence $H_2'' = P_{22}$. By Lemma 2.2, P_{22} is t_2-compressible or t_2-∂-compressible in (V_2, t_2). First we consider the former case. If the t_2-compressing disk of P_{22} is in W_1, then the same argument as in the third paragraph in this proof shows that H_2 is weakly K-reducible. If the t_2-compressing disk of P_{22} is in W_2, then this disk together with R_3 shows that H_2 is weakly K-reducible.

We consider the latter case, where P_{22} has a t_2-∂-compressing disk C. If C is contained in W_1, then it is also a K-∂-compressing disk of $Q \cup A$. We have considered this situation in the third paragraph of this proof. Hence we may assume that C is contained in W_2. The loop ∂C intersects one of the loops ∂R_3 and ∂R_4, say ∂R_3, in at most one point. Since P is parallel to P_{21} in W_2, and we can extend C to an essential disk in W_2 such that its boundary loop intersects ∂R_3 in at most one point. Hence H_2 is weakly K-reducible or meridionally stabilized. Also in the latter case, H_2 is weakly K-reducible by Proposition 2.10.

Case (*b*). We consider the case where $Q \cup A$ is K-compressible in (W_1, K). Then $Q \cup H_1'$ is K-compressible before the isotopy. If the compressing disk is in V_2, then H_2 is weakly K-reducible by Lemma 5.3. Hence we can assume that the K-compressing disk is in V_1, and a compressing operation on $Q \cup H_1'$ yields a K-compressing disk X of A_2 in $V_1 \cap W_1$. We can isotope so that $\partial X = l_1$. H_2'' is t_2-compressible or t_2-∂-compressible in (V_2, t_2). Suppose that H_2'' is t_2-compressible. If the t_2-compressing disk is in W_2, then this disk and X show that H_2 is weakly K-reducible. If it is in W_1, then H_2' has a K-compressing disk in (W_1, K) before the isotopy, and H_2 is weakly K-reducible by Lemma 5.3. Hence we can assume H_2'' is t_2-incompressible and has a t_2-∂-compressing disk Z in (V_2, t_2). First suppose that Z is contained in W_1. Then Z is not incident to the annulus A' because the two boundary loops of $\partial A'$ are contained in distinct components of $H_1 \cap W_1 = Q \cup A$ separately. Moreover, Z is not incident to the torus with one hole H_2^* because it contains only one component of ∂A. Hence $H_2'' = P_{22}$. By performing the K-∂-compressing operation on $Q \cup A$ along the disk Z, we obtain a disk Z_1 which intersects K in at most one point. Note that ∂Z_1 is essential in H_2 since Z is a t_2-∂-compressing disk of P_{22}. Along the parallelism of P and P_{21}, we can isotope H_1 so that P is pushed into $\operatorname{int} W_1$ and that $H_1 \cap Z_1 = \emptyset$. Then Proposition 2.12 shows that H_2 is weakly K-reducible. This is the conclusion (1).

Therefore, we may assume that the t_2-∂-compressing disk Z of H_2'' is contained in W_2. We can extend Z into an essential disk Z' in W_2 because P and P_{21} are parallel in W_2. Since ∂Z intersects the loop l_1 at most in two points, so does $\partial Z'$. Hence the disks Z' and X show that H_2 is either weakly K-reducible, K-stabilized or semi-stabilized. In the second case, H_2 is weakly K-reducible. In the last case, we have the conclusion (2) by Proposition 4.4 and the note just after it. □

Lemma 5.5. *Suppose that A_1 is parallel to A_2 in W_2, and that $Q \cup H_1'$ is K-compressible in (W_1, K). Then one of the following conditions holds.*

(1) The $(2,0)$-splitting H_2 of (M, K) is weakly K-reducible.

(2) H_2 is semi-stabilized, and K is a torus knot or a satellite knot.

Proof. Let D be a K-compressing disk of $Q \cup H_1'$ in (W_1, K). If D is in V_2, then Lemma 5.3 shows that H_2 is weakly K-reducible. Thus we may assume that D is in V_1. By compressing a copy of Q or H_1' along D, we obtain a disk D_1 which is disjoint from K. We can isotope D_1 in (W_1, K) so that $D_1 \cap (Q \cup H_1') = \partial D_1 \cap \partial Q = l_1$. Then D_1 forms a K-compressing disk of A_2.

H_2' is t_2-compressible or t_2-∂-compressible in (V_2, t_2) by Lemma 2.2. In the former case, let R be a t_2-compressing disk of H_2'. If R is contained in W_1, then Lemma 5.3 shows that H_2 is weakly K-reducible. If R is contained in W_2, the disks D_1 and R together show that H_2 is weakly K-reducible.

Hence we may assume that H_2' is t_2-incompressible and has a t_2-∂-compressing disk R' in (V_2, t_2). We first consider the case where R' is contained in W_2. Since $H_1 \cap W_2 = A_1$ is an annulus disjoint from K, the loop $\partial R'$ intersects each of the loop components of ∂A_1 transversely in a single point. Because A_1 is parallel to A_2 in W_2, we can extend R' to an essential disk in W_2 such that its boundary loop intersects the loop $l_1 = \partial D_1$ transversely in a single point. This disk and D_1 show that H_2 is K-stabilized, and we obtain the conclusion (1).

Hence we may assume that the disk R' is contained in W_1. Then we obtain the conclusion by Lemmas 5.3 and 5.4. □

Lemma 5.6. *Suppose that A_1 is parallel to A_2 in W_2, and that $Q \cup H_1'$ is K-incompressible in (W_1, K). Then the $(2,0)$-splitting H_2 of (M, K) is weakly K-reducible.*

Proof. Since $Q \cup H_1'$ is K-incompressible, it has a K-∂-compressing disk D in (W_1, K). By the definition of a K-∂-compressing disk, the arc $\partial D \cap H_2$ is contained in H_2' rather than A_2, since one component of ∂A_2 bounds Q and the other bounds H_1'. Then Lemmas 5.3 and 5.4 show that H_2 is weakly K-reducible. □

Thus, in Case (1)(B), we have Conclusion (b), (c) or (d) of Theorem 1.2 by Lemmas 5.1, 5.5 and 5.6.

6. Case (2)(A)

We consider in this section the case where the loops $H_1 \cap H_2$ are of the pattern (2) in H_1 and of the pattern (A) in H_2. (See Figure 2.) In this case we show (b) of Theorem 1.2 holds.

In Case (2), the loops l_1 and l_2 of $H_1 \cap H_2$ together separate the torus H_1 into two annuli A_{11} and A_{12}, where A_{1i} is contained in the handlebody W_i for $i = 1$ and 2. Note that the two intersection points $K \cap H_1$ are contained in A_{11} since the knot K is entirely contained in W_1. Let A_2, H_{21}, H_{22} be as in Section 4. We assume, without loss of generality, that A_2 is contained in V_1, and $H_{21} \cup H_{22}$ in V_2.

By Lemma 2.3, A_{11} is K-compressible or K-∂-compressible in (W_1, K). A_{12} is compressible or ∂-compressible in the handlebody W_2.

Lemma 6.1. *Suppose that A_{12} is compressible in W_2. Then H_2 is weakly K-reducible.*

Proof. Let D be a compressing disk of A_{12} in W_2. By compressing a copy of A_{12} along this disk D, we obtain two disks D_1 and D_2 bounded by l_1 and l_2 respectively. Suppose that D is contained in V_1. Then the disk D_1 is also contained in V_1, and we can isotope it slightly into $\operatorname{int} V_1$ so that $\partial D_1 \subset \operatorname{int} A_2$ and $D_1 \cap A_{12} = \emptyset$. The disk D_1 is separating in W_2 and separates the loops l_1 and l_2. This contradicts that the annulus A_{12} connects these loops and is disjoint from D_1. Hence the disks D, D_1 and D_2 are contained in V_2.

Suppose first that A_{11} is K-compressible in (W_1, K). Let R be a K-compressing disk of A_{11}. If ∂R is essential in A_{11} ignoring the intersection points with K, then, by performing a K-compressing operation on A_{11}, we obtain a disk R_1 bounded by l_1 or l_2 such that R_1 intersects K transversely in at most one point. Then the disks D_1 and R_1 together show that H_2 is weakly K-reducible.

Hence we may assume that ∂R is inessential in the annulus A_{11}. Suppose that R is contained in V_1. Then the disks R and D show that H_1 is K-reducible. This implies that K is the trivial knot by Proposition 2.6 (Lemma 3.1 in [7]), and hence H_2 is (weakly) K-reducible by Proposition 2.9 (Proposition 2.9 in [5]). Hence we may assume that the disk R is contained in V_2. By compressing A_{11} along R, we obtain an annulus A disjoint from K. Note that $\partial A = \partial A_{11} = l_1 \cup l_2$, and that K is entirely contained in the 3-manifold between A_2 and A in W_1. The annulus A is K-compressible or K-∂-compressible in (W_1, K) by Lemma 2.3. In the former case, by compressing A, we obtain two disks which are disjoint from K and bounded by the loops l_1 and l_2. Then the union of these disks and the annulus A_2 forms a 2-sphere, which bounds a 3-ball B in W_1 such that B entirely contains K. This contradicts that K is a core in W_1. In the latter case, let R' be a K-∂-compressing disk of A. Since the two loops of ∂A are contained in distinct components of $H_2 \cap V_2 = H_{21} \cup H_{22}$, the arc $\partial R' \cap H_2$ is contained in A_2. By performing a K-∂-compressing operation on the annulus A along this disk R', we obtain a peripheral disk which cuts off a 3-ball containing K from W_1. This again contradicts that K forms a core of W_1.

Hence we may assume that the annulus A_{11} is K-incompressible, and then it has a K-∂-compressing disk C in (W_1, K). Suppose first that C is contained in V_1. Then, by the definition of a K-∂-compressing disk, ∂C intersects the annulus A_2 in an essential arc, and C forms a t_1-∂-compressing disk of A_2 in (V_1, t_1). By performing a ∂-compressing operation on a copy of A_2 along C, we obtain a K-compressing disk of A_{11}. This contradicts our assumption. Hence we may assume that C is contained in V_2. Since the two boundary loops of ∂A_{11} are contained in distinct components of $H_2 \cap V_2 = H_{21} \cup H_{22}$, ∂C intersects the annulus A_{11} in an arc which is inessential on A_{11} ignoring the intersection points with K. By performing a ∂-compressing operation A_{11} along C, we obtain an annulus Z and a disk P. Note that one of the components of ∂Z is l_1 or l_2, say l_1, and hence the other component of ∂Z and ∂P are disjoint from the loop $l_1 = \partial D_1$. Moreover, the loops of ∂Z are essential and not parallel in H_2 because of the unusual definition of a K-∂-compressing disk. The disk P intersects K transversely in one or two points. When it intersects K in one point, it forms a meridionally compressing disk of H_2 in (W_1, K). Hence the disks D_1 and P together show that H_2 is weakly K-reducible. When P intersects K in two points, the annulus Z is disjoint form K. Then Z is K-compressible or K-∂-compressible in (W_1, K) by Lemma 2.3. In the former case, by compressing Z, we obtain a K-compressing disk of H_2 in (W_1, K) such that it is bounded by l_1. Hence this disk and D_1 together show that H_2 is weakly K-reducible. In the latter case, by performing a ∂-compressing operation on Z and isotoping the resulting disk slightly, we obtain a K-compressing disk of H_2 in (W_1, K) such that its boundary loop is disjoint from the loop l_1. Hence this disk and D_1 together show that H_2 is weakly K-reducible. □

Lemma 6.2. *Suppose that A_{12} is incompressible in W_2. Then H_2 is weakly K-reducible.*

Proof. Since A_{12} is incompressible in W_2, it is ∂-compressible. Then A_{12} is parallel to A_2 in W_2.

By Lemma 2.2, $H_{21} \cup H_{22}$ is t_2-compressible or t_2-∂-compressible in (V_2, t_2). We consider first the former case. Let D be a t_2-compressing disk of $H_{21} \cup H_{22}$. When D is in W_2, by compressing $H_{21} \cup H_{22}$, we obtain a compressing disk of A_{12}. This contradicts our assumption. When D is in W_1, we can isotope the torus H_1 into $\operatorname{int} W_1$ so that $H_1 \cap D = \emptyset$. Then Proposition 2.12 shows that H_2 is weakly K-reducible.

Hence we may assume that $H_{21} \cup H_{22}$ has a t_2-∂-compressing disk R. R cannot be in W_2, since A_{12} cannot contain the arc $\partial R \cap H_1$ by the definition of a ∂-compressing disk. Hence R is contained in W_1, and is a K-∂-compressing disk of the annulus A_{11} by the definition of a t_2-∂-compressing disk again. We assume, without loss of generality, that R is incident to H_{22} rather than H_{21}. We isotope the torus H_1 in (M, K) along the disk R slightly beyond the arc $\partial R \cap H_{22}$. Then the annulus A_{11} is deformed into an annulus, say A, and a disk, say R_1. The torus with one hole H_{22} is deformed into an annulus A'. The annuli A_{12} and A_2 are deformed into disks with two holes, say P_1 and P_2 respectively. See Figure 5. Note that P_1 is parallel to P_2 in W_2 since A_{12} is parallel to A_2 in W_2 before the isotopy. The disk R_1 intersects K transversely in one or two points. When it intersects K in one point, we can isotope the torus H_1 into $\operatorname{int} W_1$ along the parallelism between P_1 and P_2. Further, we can take a parallel copy of R_1 so that it is disjoint from H_1. Then Proposition 2.12 shows that H_2 is weakly K-reducible.

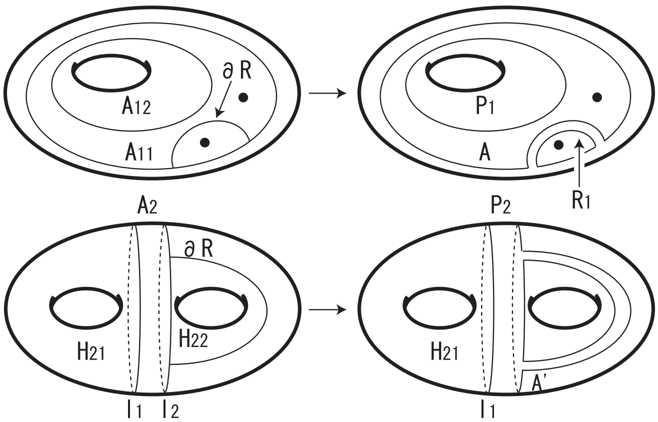

Figure 5

Hence we may assume that the disk R_1 intersects K in two points. Let R_2 be a disk bounded by the loop l_1 in W_1 such that it is obtained from the disk $A \cup A' \cup R_1$ by pushing its interior into $\text{int}\, V_2 \cap W_1$. Then this disk R_2 is bounded by the loop l_1, and intersects K transversely in two points. Moreover, R_2 divides the handlebody W_1 into two solid tori U_1 and U_2 where U_1 is bounded by the torus $H_{21} \cup R_2$. By Lemma 2.3, R_2 is K-compressible or K-∂-compressible in (W_1, K). We consider first the latter case. Let E be a K-∂-compressing disk of R_2. When E is contained in U_1, by performing a K-∂-compressing on R_2 along E, we obtain a disk E_1 intersecting K in a single point. We isotope E_1 slightly off of the disk R_2 so that ∂E_1 is in H_{21}. We can isotope the torus H_1 into $\text{int}\, W_1$ along the parallelism between P_1 and P_2 so that $H_1 \cap E_1 = \emptyset$. Then Proposition 2.12 shows that H_2 is weakly K-reducible. When E is contained in the other solid torus U_2, by performing a K-∂-compressing operation on R_2, we obtain a meridian disk, say E_2, intersecting K in a single point. We can form a knot K' taking a sum of the arc $K \cap U_2$ and an arc connecting the two points $K \cap R_2$ in R_2. Thus the disk R_1 intersects K' transversely in precisely two points, while the disk E_2 intersects K' transversely in a single point. This contradicts the fact that R_1 and E_2 represent the same homology class in $H_2(U_2; \partial U_2)$, and hence they have the same algebraic intersection number with $[K'] \in H_1(U_2)$.

Hence we may assume that R_2 has a K-compressing disk in (W_1, K). By performing a K-compressing operation on R_2, we obtain a disk, say G, bounded by the loop l_1. If G is contained in U_2, then it separates the intersection points $K \cap R_2$ from $K \cap R_1$, a contradiction. Hence G is contained in U_1. Then we move H_1 into $\text{int}\, W_1$ along the parallelism between P_1 and P_2 so that $H_1 \cap G = \emptyset$, to see that H_2 is weakly K-reducible by Proposition 2.12. □

Thus, in Case 2(A), we have the conclusion (b) of Theorem 1.2 by Lemmas 6.1 and 6.2.

7. Case (2)(B)

We consider in this section the case where the intersection loops $H_1 \cap H_2 = l_1 \cup l_2$ are of the pattern (2) in H_1 and of the pattern (B) in H_2 in Figure 2. In this case we show that one of the conclusions (a)–(f) of Theorem 1.2 holds, or we can isotope H_1 in (W_1, K) so that intersection of H_1 and H_2 is in case (1)(A) or (1)(B).

Let A_2, H_2' be as in Section 5, and A_{11}, A_{12} as in Section 6. See Figure 2. We may assume, without loss of generality, that A_2, H_2' are properly embedded in V_1, V_2 respectively. Note that A_{11}, A_{12} are properly embedded in W_1, W_2 respectively, and the two intersection points $K \cap H_1$ are contained in the annulus A_{11} since the knot K is entirely contained in W_1.

A_{12} is compressible or ∂-compressible in the handlebody W_2, so we have four cases below.

(i) A_{12} has a compressing disk in V_1.

(ii) A_{12} has a compressing disk in V_2.

(iii) A_{12} has a ∂-compressing disk in V_1.

(iv) A_{12} has a ∂-compressing disk in V_2.

By Lemma 2.3, A_{11} is K-compressible or K-∂-compressible in (W_1, K). Here we divide into seven cases as below.

(A) A_{11} has a K-compressing disk whose boundary loop is essential in A_{11}.

(B) In V_1, A_{11} has a K-compressing disk whose boundary loop is inessential in A_{11}.

(C) In V_2, A_{11} has a K-compressing disk whose boundary loop is inessential in A_{11}.

(D) In V_1, A_{11} has a K-∂-compressing disk.

(E) In V_2, A_{11} has a K-∂-compressing disk whose boundary loop intersects A_{11} in an essential arc.

(F) In V_2, A_{11} has a K-∂-compressing disk whose boundary loop intersects A_{11} in an inessential arc cutting off from A_{11} a disk which intersects K in a single point.

(G) In V_2, A_{11} has a K-∂-compressing disk whose boundary loop intersects A_{11} in an inessential arc cutting off from A_{11} a disk which intersects K in two points.

Hence we have $4 \times 7 = 28$ cases. By the next lemma, we do not need to consider the 10 cases (ii)(B), (ii)(C), (ii)(E), (ii)(F), (ii)(G), (iv)(B), (iv)(C), (iv)(E), (iv)(F) and (iv)(G).

Lemma 7.1. *At least one of the four conditions* (*i*), (*iii*), (*A*) *and* (*D*) *holds.*

Proof. By Lemma 2.2, A_2 is t_1-compressible or t_1-∂-compressible in (V_1, t_1). In the former case, let D be a t_1-compressing disk of A_2. If D is contained in W_1, then by compressing a copy of A_2 along D, we obtain a K-compressing disk D_1 of A_{11} such that ∂D_1 is essential in A_{11} ignoring the intersection points $K \cap A_{11}$. Thus the condition (A) holds. Hence we may assume that D is contained in W_2. By compressing a copy of A_2 along D, we obtain a compressing disk D_2 of A_{12} such that D_2 is contained in V_1. Thus the condition (i) holds.

In the latter case, let R be a t_1-∂-compressing disk of A_2. When R is contained in W_1, it is also a K-∂-compressing disk of A_{11} in (W_1, K) because of the definition of a t_1-∂-compressing disk. Thus the condition (D) holds. Hence we may assume that R is contained in W_2. Then R is also a ∂-compressing disk of A_{12}. Thus the condition (iii) holds. □

Lemma 7.2. *In Case* (*D*) *the condition* (*B*) *holds.*

Proof. In Case (D), there is a K-∂-compressing disk D of A_{11} in (W_1, K) such that D is contained in V_1. Then $\partial D \cap H_2$ is an essential arc in A_2 by the definition of a K-∂-compressing disk. Hence D is a t_1-∂-compressing disk of A_2 in (V_1, t_1). Note that the arc $\partial D \cap A_{11}$ is also essential in A_{11} ignoring the intersection points $K \cap A_{11}$. By performing a t_1-∂-compressing operation on A_2 along D, we obtain a K-compressing disk of A_{11} such that its boundary loop is inessential in A_{11} ignoring the intersection points $K \cap A_{11}$. Hence the condition (B) holds. □

We will show the present case of Theorem 1.2 in accordance with Table 1.

Table 1

	(A)	(B)	(C)	(D)	(E)	(F)	(G)
(i)	7.8	7.12	7.4	7.12	7.3	7.5	7.5
(ii)	7.8	—	—	7.9	—	—	—
(iii)	7.8	7.12	7.4	7.12	7.3	7.6	7.7
(iv)	7.8	—	—	7.10	—	—	—

Lemma 7.3. *In Case* (*E*) *we can isotope* H_1 *in* (M, K) *so that* $H_1 \cap H_2$ *consists of a single loop which is* K*-essential both in* H_1 *and in* H_2.

Proof. In Case (E), there is a K-∂-compressing disk D of A_{11} such that D is contained in V_2 and such that the arc $\partial D \cap A_{11}$ is essential in A_{11} ignoring the intersection points with K. We isotope H_1 along D, so that a band neighborhood of the arc $\partial D \cap A_{11}$ in A_{11} is isotoped into W_2. Then the annulus A_{11} is deformed into a disk Q intersecting K in two points. Note that the boundary loop ∂Q is essential in H_2 since the arc $\partial D \cap H_2'$ is essential in H_2' by the definition of a K-∂-compressing disk. □

Lemma 7.4. *In Case* (*C*), *the condition* (*E*) *holds.*

We have already considered Case (E) in Lemma 7.3.

Proof. In Case (C), there is a K-compressing disk D of A_{11} such that D is contained in V_2 and that ∂D bounds a disk D' in A_{11}. Then D' intersects K in two points, and a K-compressing operation on A_{11} along D yields an annulus A such that it is disjoint from K and that $\partial A = \partial A_{11}$. Since A_{11} is separating in W_1, so is A. Note that K is between the annuli A and A_2.

The annulus A is K-compressible or K-∂-compressible in (W_1, K) by Lemma 2.3. In the former case, by performing a K-compressing operation on A, we obtain two disks bounded by $H_1 \cap H_2 = l_1 \cup l_2$. The knot K is in the ball between these disks in W_1, which contradicts that K is a core of W_1. In the latter case, A has a K-∂-compressing disk R. We assume first that the arc $\partial R \cap H_2$ is contained in A_2. By performing a K-∂-compressing operation on A along R, we obtain a peripheral disk which cuts off a ball containing K from W_1. This is again a contradiction. Hence R is contained in V_2. We can isotope R so that it is disjoint from the copy of D in A. Then R gives a K-∂-compressing disk of A_{11}. Because the arc $\partial R \cap A$ is essential in A so is the arc $\partial R \cap A_{11}$ in A_{11}. Thus the condition (E) holds. □

Lemma 7.5. *In Cases* (*i*)(*F*) *and* (*i*)(*G*), H_2 *is weakly* K*-reducible.*

Proof. In Case (i), compressing A_{12} in W_2, we obtain two disks D_1 and D_2 ($\subset V_1$) bounded by the loops l_1 and l_2 respectively.

In Case (F), a K-∂-compressing operation on a copy of A_{11} along D yields a meridionally compressing disk Q of H_2 in (W_1, K). We can isotope Q slightly off of A_{11}. Then the disks D_1 and Q show that H_2 is weakly K-reducible.

In Case (G), there is a K-∂-compressing disk R of A_{11} in V_2 such that the arc $\partial R \cap A_{11}$ is inessential in A_{11} and cuts off a disk R' from A_{11} and that R' intersects K in two points. By the definition of a K-∂-compressing disk, the arc $\partial R \cap H_2$ is an essential arc in H_2'. By performing a K-∂-compressing operation on A_{11} along R, we obtain an annulus A which is disjoint from K. Note that one component of ∂A is l_1 or l_2, say l_1, and the other component is not parallel to l_1 in H_2. By Lemma 2.3, A is K-compressible or K-∂-compressible in (W_1, K). In the former case, by performing a K-compressing operation on A, we obtain a disk E_1 bounded by l_1. Note that E_1 is disjoint form K. Then the disks D_1 and E_1 show that H_2 is weakly K-reducible. In the latter case, by performing a K-∂-compressing operation on A, we obtain a disk E_2 such that it is disjoint form K and that ∂E_2 is essential in H_2. We can isotope E_2 slightly off of A, and hence off of l_1. Then the disks D_1 and E_2 show that H_2 is weakly K-reducible. □

Lemma 7.6. *In Case* (*iii*)(*F*), H_2 *is weakly* K*-reducible.*

Proof. In Case (iii), A_{12} is parallel to A_2 in W_2. In Case (F), a K-∂-compressing operation on A_{11} yields a meridionally compressing disk Q of H_2. We can isotope Q slightly off of A_{11}. We isotope H_1 in (M, K) along the parallelism between A_{12} and A_2, so that H_1 is contained in W_1 and that H_1 is disjoint from the disk Q. Then Proposition 2.12 shows that H_2 is weakly K-reducible. □

Lemma 7.7. *In Case* (*iii*)(*G*), *we can isotope* H_1 *in* (M, K) *so that* H_1 *and* H_2 *intersect each other transversely in two loops which are* K*-essential both in* H_1 *and* H_2 *and of the pattern* (1) *in* H_1 *in Figure 2.*

Proof. In Case (iii), A_{12} is parallel to A_2 in W_2.

In Case (G), we isotope H_1 along the K-∂-compressing disk of A_{11}. Then the annulus $A_{11} = H_1 \cap W_1$ is deformed into a disjoint union of an annulus A and a disk Q such that A is disjoint from K and Q intersects K in two points. Note that each loop of ∂Q and ∂A is essential in H_2. The annulus $A_{12} = H_1 \cap W_2$ is deformed into a disk with two holes P and also the annulus $A_2 = H_2 \cap V_1$ into a disk with two holes P'. Since A_{12} is parallel to A_2 in W_2, P is parallel to P' in W_2.

There is a ∂-compressing disk R of P in W_2 such that the arc $\partial R \cap H_2$ is contained in P' and connects the two boundary loops ∂A. We further isotope H_1 along R. Then P is deformed into an annulus, and A is deformed into a torus with one hole T. Note that ∂T is parallel to ∂Q in H_2 since the arc $\partial R \cap H_2$ is contained in P'. Thus we have isotoped H_1 so that H_1 and H_2 intersect each other transversely in two loops which are K-essential both in H_1 and in H_2 and of the pattern (1) in H_1 in Figure 2. □

Lemma 7.8. *In Case* (*A*)*, one of the two conditions below holds.*

(1) The $(2,0)$*-splitting* H_2 *is weakly* K*-reducible.*

(2) (*iii*)(*E*)*,* (*iii*)(*F*) *or* (*iii*)(*G*) *holds.*

We have already considered Cases (iii)(E), (iii)(F) and (iii)(G) in Lemmas 7.3, 7.6 and 7.7 respectively.

Proof. By performing a K-compressing operation on A_{11} along a K-compressing disk as in the condition (A), we obtain a disk D_1 in W_1, which is bounded by l_1 or l_2, say l_1, and intersects K transversely in at most one point.

In Cases (i) and (ii), by performing a compressing operation on A_{12}, we obtain a disk D_2 which is bounded by l_2. Hence the disks D_1 and D_2 show that H_2 is weakly K-reducible.

In Case (iv), by performing a ∂-compressing operation on A_{12}, we obtain a disk D_2' whose boundary loop $\partial D_2'$ is essential in H_2 and is disjoint from l_1 after an adequate small isotopy. Thus the disks D_1 and D_2' show that H_2 is weakly K-reducible.

In Case (iii), the annulus A_{12} is parallel to the annulus A_2 in W_2. By Lemma 2.2, H_2' is t_2-compressible or t_2-∂-compressible in (V_2, t_2). In the former case, let R be a t_2-compressing disk of H_2'. If R is contained in W_2, then the disks D_1 and R show that H_2 is weakly K-reducible. Therefore we may assume that R is contained in W_1. We can isotope H_1 in (M, K) along the parallelism between the annuli A_{12} and A_2 so that H_1 is contained in W_1 and so that it is disjoint from R. Then Proposition 2.12 shows that H_2 is weakly K-reducible. We consider the latter case, where H_2' has a t_2-∂-compressing disk R' in (V_2, t_2). If R' is contained in W_2, it is also a ∂-compressing disk of A_{12} in W_2 by the definition of a ∂-compressing disk. We have already considered this case in the third paragraph in this proof. Hence we can assume that R' is contained in W_1. Then R' is also a K-∂-compressing disk of A_{11} in (W_1, K) by the definition of a ∂-compressing disk. Thus one of the conditions (E), (F) and (G) holds. □

Lemma 7.9. *In Case (ii)(B), and hence, also in Case (ii)(D), H_2 is weakly K-reducible.*

Proof. In Case (ii), there is a compressing disk D_2 of A_{12} in $W_2 \cap V_2$. In Case (B), there is a K-compressing disk D_1 of A_{11} in $W_1 \cap V_1$. Then these disks D_1 and D_2 show that H_1 is K-reducible. Then H_2 is weakly K-reducible as shown in the third paragraph in the proof of Lemma 6.1. We have this lemma via Lemma 7.2. □

Lemma 7.10. *In Case (iv)(D), the conclusion (f) of Theorem 1.2 holds.*

Proof. In Case (iv), there is a ∂-compressing disk D of A_{12} in W_2 such that D is contained in V_2. Let $P = N(\partial A_{12} \cup (A_{12} \cap \partial D))$ be a neighborhood of the union of the two boundary loops ∂A_{12} and the arc $A_{12} \cup \partial D$ in A_{12}. Note that P is the disk with two holes. We can isotope H_1 in (M, K) along D so that P is isotoped into H_2'. After this isotopy, H_1 intersects $\operatorname{int} W_2$ in an open disk. Let D_2 be the closure of this open disk. Then ∂D_2 separates H_2 into the once punctured torus $A_2 \cup (H_2' \cap P)$ and the complementary once punctured torus, and D_2 cuts W_2 into two solid tori, one of which, say U_1, contains A_2.

In Case (D), there is a K-∂-compressing disk R of A_{11} in (W_1, K) such that R is contained in V_1. The arc $\beta = A_{11} \cap \partial R$ is essential in A_{11} ignoring the intersection points $K \cap A_{11}$ since the arc $A_2 \cap \partial R$ is essential in A_2 by the definition of a K-∂-compressing disk. Set $B = N(\beta)$, the band neighborhood of the arc β in A_{11}. We can isotope H_1 in (M, K) along R so that B is isotoped into A_2 and that $B \cap P \subset \partial A_2$. After this isotopy, H_1 intersects $\operatorname{int} W_1$ in an open disk. Let R_1 be the closure of this open disk. Since ∂R_1 is inessential on $\partial W_1 = H_2$, R_1 is a ∂-parallel disk in W_1 ignoring K, and cuts off a 3-ball X from W_1 such that X intersects K in a single arc t which is trivial in X. (See Lemma 3.2 in [5].)

After these isotopies, H_1 intersects H_2 in the torus with two holes $P \cup B$. The solid torus V_1 is the union $U_1 \cup X$. Hence $t = t_1$. We take an arc α in the 3-ball X so that an endpoint of α in $\operatorname{int} t$, that the other endpoint of α is in the disk $X \cap H_2$ and so that X collapses to $t \cup \alpha$. See Figure 1. Thus we obtain the conclusion (f) of Theorem 1.2. □

We need the next lemma to consider Case (iii)(B).

Lemma 7.11. *Suppose that the handlebody W_1 contains a separating disk D such that D cuts off from W_1 a solid torus U_1 disjoint from the knot and that the complementary 3-manifold $cl(M - U_1)$ is also a solid torus. Then the $(2, 0)$-splitting H_2 is meridionally stabilized, and hence H_2 is weakly K-reducible (see Proposition 2.10).*

Proof. The disk D cuts off another solid torus U_2 from W_1. Note that K forms a core of U_2 (Lemma 3.3 in [5]). There is a meridian disk Q of U_2 which intersects K transversely in a single point. We can take Q so that ∂Q intersects the disk D in a single arc. Let $N(Q)$ be a small regular neighborhood of Q in U_2. The solid torus $U_1' = U_1 \cup N(Q)$ intersects K in a trivial arc s_1. The 3-ball $\mathrm{cl}(U_2 - N(Q))$ forms a regular neighborhood of the complementary arc $s_2 = \mathrm{cl}\,(K - s_1)$ in the complementary solid torus $U_2' = \mathrm{cl}\,(M - U_1')$. The exterior of s_2 in U_2' is homeomorphic to W_2. Hence we can see that the arc s_2 is trivial in U_2', applying Theorem 1 in [6]. Therefore $(M, K) = (U_1', s_1) \cup (U_2', s_2)$ is a $(1, 1)$-splitting. Moreover, we can take a meridian disk D_1 of s_2 in $\mathrm{cl}\,(U_2 - N(Q))$ and a canceling disk D_2 of s_2 in $\mathrm{cl}\,(U_2' - \mathrm{cl}\,(U_2 - N(Q))) = W_2$ so that ∂D_1 and ∂D_2 intersects transversely in a single point. These disks D_1 and D_2 show that H_2 is meridionally stabilized. □

Lemma 7.12. *In Cases* (*i*)(*B*) *and* (*iii*)(*B*)*, one of the following four conditions holds.*

(1) The $(2, 0)$*-splitting* H_2 *is weakly* K*-reducible.*

(2) The $(1, 1)$*-splitting* H_1 *has a satellite diagram.*

(3) One of the conditions (*E*)*,* (*F*) *and* (*G*) *holds.*

(4) The conditions (*iv*) *and* (*D*) *hold.*

We have already considered the cases (i)(E), (i)(F), (i)(G), (iii)(E), (iii)(F), (iii)(G) and (iv)(D) in Lemmas 7.3, 7.5, 7.6, 7.7 and 7.10. In the case of the conclusion (2), we have the conclusion (b), (c), (d) or (e) of Theorem 1.2 by Proposition 2.11.

Proof. In Case (i), there is a compressing disk D of A_{12} in V_1. By compressing a copy of A_{12} along D, we obtain disks D_1 and D_2 bounded by the intersection loops $H_1 \cap H_2 = l_1 \cup l_2$. Note that these disks D_1 and D_2 are contained in $W_2 \cap V_1$, and form compressing disks of A_2.

In Case (iii), the annulus A_{12} is parallel to the annulus A_2 in W_2.

In Case (B), there is a K-compressing disk R of A_{11} in V_1 such that ∂R bounds in A_{11} a disk R' which intersects K in precisely two points. By compressing a copy of A_{11} along R, we obtain an annulus A which is disjoint from K.

The surface H_2' is t_2-compressible or t_2-∂-compressible in (V_2, t_2) by Lemma 2.2. Suppose first that H_2' has a t_2-compressing or t_2-∂-compressing disk P in W_1. When P is a t_2-compressing disk, D_1 and P show that H_2 is weakly K-reducible in Case (i), and in Case (iii) we isotope H_1 along the parallelism between A_{12} and A_2 so that H_1 is contained in $\mathrm{int}\, W_1$ and that H_1 is disjoint from P, to see that H_2 is weakly K-reducible by Proposition 2.12. This is the conclusion (1) of this lemma. When P is a t_2-∂-compressing disk, it is also a K-∂-compressing disk of

A_{11} by the definition of a t_2-∂-compressing disk. Thus one of the conditions (E), (F) and (G) holds. This is the conclusion (3) of this lemma.

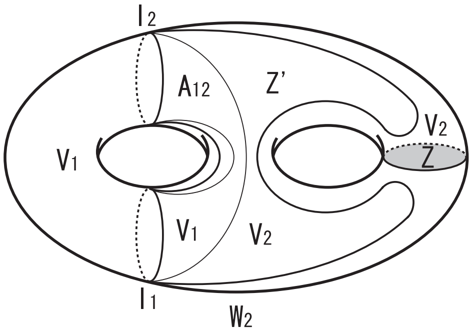

Figure 6

Hence we may assume that H_2' has a t_2-compressing or t_2-∂-compressing disk Z in W_2. When Z is a t_2-∂-compressing disk of H_2', it is also a ∂-compressing disk of A_{12}. By performing the ∂-compressing operation on a copy of A_{12} along Z, we obtain a compressing disk of H_2' in $W_2 \cap V_2$. Thus we may assume that Z is a t_2-compressing disk of H_2'. If ∂Z is parallel to a component of $\partial H_2'$, then $Z\,(\subset V_2)$ and the K-compressing disk $R\,(\subset V_1)$ of A_{11} show that H_1 is K-reducible. Hence H_2 is weakly K-reducible as shown in the third paragraph in the proof of Lemma 6.1. So we may assume that ∂Z is not parallel to a component of $\partial H_2'$. By compressing a copy of H_2' along R_2, we obtain an annulus Z' in $W_2 \cap V_2$ such that $\partial Z' = \partial H_2'$. We isotope $\operatorname{int} Z'$ slightly into $\operatorname{int}(V_2 \cap W_2)$ so that $Z' \cap H_2' = \partial Z' = \partial H_2'$. A typical example is described in Figure 6. This annulus Z' is t_2-compressible or t_2-∂-compressible in (V_2, t_2) by Lemma 2.2. In the former case, by performing a t_2-compressing on Z', we obtain a disk which is disjoint from K and bounded by a loop of $\partial H_2'$. This disk and R show that H_1 is K-reducible. This implies the conclusion (1) again. In the latter case, let Q be a t_2-∂-compressing disk of Z'. If the arc $\partial Q \cap H_1$ is contained in A_{11}, then by performing the t_2-∂-compressing operation on Z' along Q, we obtain a disk $Q'\,(\subset V_2)$ such that $\partial Q'$ bounds a disk in A_{11}, which intersects K in two points. Note that Q' is disjoint from K. The disks R and Q' show that H_1 has a satellite diagram on A_{11}. (In fact, for $i = 1$ and 2, we can take a canceling disk C_i of t_i in (V_i, t_i) so that C_1 is disjoint from R and C_2 is disjoint from Q.) This is the conclusion (2) of this lemma. Then we may assume that the arc $\partial Q \cap H_1$ is contained in A_{12}. This implies that the annulus Z' is parallel to A_{12} in W_2. Since Z' is obtained by a compression on H_2', it has a ∂-compressing disk G in (V_2, t_2) such that it is contained in $W_2 \cap V_2$ and $\partial G \cap H_2$ is an essential arc in H_2'. (There is an arc connecting the two loops l_1 and l_2 in H_2' such that it is disjoint from ∂Z.) Therefore A_{12} also has a ∂-compressing disk in $W_2 \cap V_2$. Thus the condition (iv) holds.

The annulus A, which was obtained from A_{11} by K-compressing along R, is K-compressible or K-∂-compressible in (W_1, K) by Lemma 2.3. In the former case, by performing the K-compressing operation on A, we obtain a K-compressing disk bounded by l_1. Then this disk and Z show that H_2 is weakly K-reducible. This is the conclusion (1). In the latter case, let C be a K-∂-compressing disk of A. First suppose that the arc $\partial C \cap H_2$ is contained in A_2. We can take C to be disjoint from the copy of R in A. Then C forms a K-∂-compressing disk of the annulus A_{11} such that the arc $A_{11} \cap \partial C$ is essential in A_{11}. Thus the condition (D) holds, and we obtain the conclusion (4) of this lemma. Hence we may assume that the arc $\partial C \cap H_2$ is contained in H_2'. In Case (i), ∂-compressing A along C, we obtain an essential disk disjoint from K in W_1. An adequate small isotopy moves this disk so that it is disjoint from l_1. Hence, this disk together with D_1 shows that H_2 is weakly K-reducible. We consider Case (iii). In this case, we will move H_1 ignoring K so that we can use Lemma 7.11. Recall that R is a K-compressing disk of A_{11} such that ∂R bounds a disk R' on A_{11}. In W_1 the 2-sphere $R \cup R'$ bounds a 3-ball, and hence A_{11} and A are isotopic in W_1 fixing their boundary loops $\partial A_{11} = \partial A$ ignoring K. Hence H_1 is isotopic to the torus $H = A \cup A_{12}$ in M, ignoring K. Since we are in Case (iii), we can isotope H along the parallelism between the annuli A_{12} and A_2 so that A_{12} is isotoped onto A_2. Recall that the K-∂-compressing disk C of A intersects H_2 in an essential arc in H_2'. (See Figure 7 for a typical example.) We isotope H along C so that $H \cap H_2$ is a torus with one hole H_0 and that $H \cap \operatorname{int} W_1$ is an open disk the closure of which is an essential separating disk C' bounded by the loop ∂H_0. Then the solid torus V_1 is isotopic to the solid torus U_1 bounded by the torus $H = H_0 \cup C'$ in W_1, ignoring K. The complementary 3-manifold $\operatorname{cl}(M - U_1)$ is isotopic to the solid torus V_2. Then Lemma 7.11 shows that H_2 is meridionally stabilized, and hence H_2 is weakly K-reducible by Proposition 2.10. □

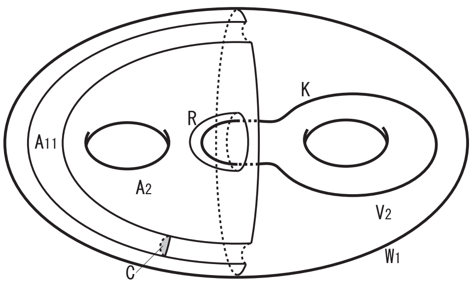

Figure 7

Remark 7.13. In Case (i)(B), we can delete the conclusion (2) of the above lemma. In fact, we can see the annuli Z' and A_{12} are parallel in W_2 as below. The torus $A_{12} \cup Z'$ bounds in W_2 a 3-manifold that is homeomorphic to an exterior E of a (possibly trivial) knot in S^3, because a handlebody is irreducible. Since the loop l_1 bounds the disk D_1 in W_2, it is of meridional slope on the boundary torus of the knot exterior E. Because l_1 is a meridian of the solid torus V_1 and M is not homeomorphic to $S^2 \times S^1$, l_1 is not a meridian of the solid torus V_2. Since E is cut off from V_2 by Z', it is not homeomorphic to the exterior of a non-trivial knot exterior, but is the exterior of the trivial knot in S^3 with l_1 being a meridian of the knot. Hence E is a solid torus with l_1 being a longitude, and hence A_{12} and Z' are parallel in W_2.

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