Saitama Math. J.
Vol. 29 (2012), 65–77

Normal Affine Surfaces with Non-Positive Euler Characteristic

Hideo Kojima*

(*Received 3 September, 2012; Revised 6 January, 2013; Accepted 6 January, 2013*)

Abstract

We prove that a normal affine surface with non-positive Euler characteristic has a structure of $\mathbb{C}$ or $\mathbb{C}^*$-fibration over a smooth curve.

1. Introduction

Throughout the present paper, we work over the complex number field $\mathbb{C}$. Let $e(T)$ denote the topological Euler characteristic of a topological space T.

It is well-known from the classification theory of algebraic surfaces that, for a smooth projective surface X, if $e(X) < 0$ (resp. $e(X) \leq 0$) then $\kappa(X) = -\infty$ (resp. $\kappa(X) \leq 1$), where $\kappa(X)$ denotes the Kodaira dimension of X. Several mathematicians have studied the topological Euler characteristics of open algebraic surfaces. We recall some results. It follows from the log Miyaoka–Yau inequality in [11] that every normal affine surface S with only quotient singular points and with $\overline{\kappa}(S \setminus \mathrm{Sing}S) = 2$ has positive Euler characteristic (see [20] and [5]). Gurjar–Parameswaran [6] and Veys [26] studied the pairs (X, D) of smooth projective surfaces X and connected curves D on X with $e(X \setminus \mathrm{Supp}D) \leq 0$. In particular, Gurjar–Parameswaran [6] proved that, for every smooth affine surface S with $e(S) \leq 0$, there exists a $\mathbb{C}$ or $\mathbb{C}^*$-fibration $\varphi : S \to T$ onto a smooth curve T, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. These results on open algebraic surfaces with non-positive Euler characteristic are very useful. For example, they have been applied for the study of topologically contractible curves on $\mathbb{Q}$-homology planes (see [20], [5], [7], [10], [27], [23], [1], etc.).

In this paper, by using the structure theorems on open algebraic surfaces (see, e.g., [17]) and the generalized log Miyaoka–Yau inequality in [15] and [21, 2.5 (ii)], we study the normal affine surfaces with non-positive Euler characteristic and attempt to generalize some results in [6]. In Section 3, we study structure of normal affine surfaces with non-positive Euler characteristic and prove the

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2010 Mathematics Subject Classification. Primary 14J26; Secondary 14R05.

Key words and phrases. affine surface, logarithmic Kodaira dimension

*The author is supported by Grant-in-Aid for Scientific Research (No. 23740008 and 24340006) from JSPS.

following results.

Theorem 1.1. *Let S be a normal affine surface and S_0 its smooth part. Then the following assertions hold true.*

(1) *If $\overline{\kappa}(S_0) = 2$, then $e(S) > 0$.*

(2) *If $\overline{\kappa}(S_0) = 0$ or 1, then $e(S) \geq 0$.*

(3) *If $\overline{\kappa}(S_0) \geq 0$ and $e(S) = 0$, then S is smooth and there exists a $\mathbb{C}^*$-fibration $\varphi : S \to T$ onto a smooth curve T.*

(4) *If $\overline{\kappa}(S_0) = e(S) = 0$, then S is isomorphic to either $\mathbb{C}^* \times \mathbb{C}^*$ or $H[-1, 0, -1]$ (for the definition of $H[-1, 0, -1]$,* see [2, 8.5] *or* [12, Example 4.2]*).*

(5) *If $e(S) \leq 0$ and $\overline{\kappa}(S_0) = -\infty$, then S is affine ruled, i.e., there exists a $\mathbb{C}$-fibration $\varphi : S \to T$ onto a smooth curve T.*

Theorem 1.2. *Let S be a normal affine surface with $e(S) \leq 0$ and S_0 its smooth part. Then $\overline{\kappa}(S_0) \geq 0$ if and only if $\overline{P}_2(S_0) > 0$.*

In Section 4, we study reduced curves on a normal complete rational surface of Picard number one (see [24] for the definition of the Picard number of a normal complete surface). By using Theorem 1.1 and the structure theorems on open algebraic surfaces, we prove the following result.

Theorem 1.3. *Let X be a normal complete rational surface of Picard number one and B a non-empty reduced algebraic curve on X. If $e(X \setminus B) \leq 0$, then every irreducible component of B is a rational curve.*

When X in Theorem 1.3 is smooth (i.e., $X = \mathbb{P}^2$), Theorem 1.3 was conjectured by Veys (cf. [25]) and was proved by de Jong and Steenbrink [8]. See [6] and [13] for other proofs.

2. Preliminaries

For $\mathbb{Q}$-divisors A and B, $A \equiv B$ means A and B are numerically equivalent. We denote by K_X the canonical divisor of an algebraic variety X. For a connected smooth quasi-projective variety S, we denote by $\overline{P}_n(S)$ ($n \geq 1$) (resp. $\overline{\kappa}(S)$) the logarithmic n-genus of S (resp. the logarithmic Kodaira dimension of S). For the definitions, see [17]. By a $(-n)$-curve, we mean a smooth projective rational curve with self-intersection number $-n$. A reduced effective divisor is called an SNC-divisor if it has only simple normal crossings.

We recall some basic notions in the theory of peeling. For more details, see [17, Chapter 2]. Let (X, B) be a pair of a smooth projective surface X and an SNC-divisor B on X. We call such a pair an *SNC-pair*. A connected curve T

consisting of irreducible components of B (a connected curve in B, for short) is called a *twig* if each irreducible component of T is rational, the dual graph of T is a linear chain and T meets $B-T$ in a single point at one of the end components of T, the other end of T is called the *tip* of T. A connected curve R (resp. F) in B is called a *rational rod* (resp. a *rational fork*) if R (resp. F) is a connected component of B and consists only of rational curves and if the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of the minimal resolution of a non-cyclic quotient singular point). A connected curve E in B is said to be *admissible* if Supp$E$ contains no (-1)-curves and the intersection matrix of E is negative definite. An admissible rational twig T in B is said to be *maximal* if it cannot be extended to an admissible rational twig with more irreducible components of B.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all maximal admissible rational twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of B as a sum of effective $\mathbb{Q}$-divisors $B = B^\# + \mathrm{Bk}\, B$ such that the following two conditions (i) and (ii) are satisfied:

(i) $\mathrm{Supp}(\mathrm{Bk}\, B) = (\cup_\lambda T_\lambda) \cup (\cup_\mu R_\mu) \cup (\cup_\nu F_\nu)$;

(ii) $(B^\# + K_X) \cdot Z = 0$ for every irreducible component Z of $\mathrm{Supp}(\mathrm{Bk}\, B)$.

Let $\pi : X \to \overline{X}$ be the contraction of $\mathrm{Supp}(\mathrm{Bk}\, B)$ to quotient singular points and put $\overline{B} := \pi_*(B)$. It then follows from the condition (ii) that $\pi^*(\overline{B} + K_{\overline{X}}) \equiv B^\# + K_X$.

Definition 2.1. An SNC-pair (X, B) is said to be *almost minimal* if, for every irreducible curve C on X, either $(B^\# + K_X) \cdot C \geq 0$ or $(B^\# + K_X) \cdot C < 0$ and the intersection matrix of $C + \mathrm{Bk}\, B$ is not negative definite.

Lemma 2.2. *Let (X, B) be an SNC-pair. Then there exists a birational morphism $\mu : X \to V$ onto a smooth projective surface V such that the following four conditions* (1) – (4) *are satisfied*:

(1) $D := \mu_*(B)$ *is an SNC-divisor.*

(2) $\mu_*(\mathrm{Bk}\, B) \leq \mathrm{Bk}\, D$ *and* $\mu_*(B^\# + K_X) \geq D^\# + K_V$.

(3) $\overline{P}_n(X \setminus \mathrm{Supp}B) = \overline{P}_n(V \setminus \mathrm{Supp}D)$ *for every integer* $n \geq 1$. *In particular,* $\overline{\kappa}(X \setminus \mathrm{Supp}B) = \overline{\kappa}(V \setminus \mathrm{Supp}D)$.

(4) *The pair* (V, D) *is almost minimal.*

Proof. See [17, Theorem 2.3.11.1 (p. 107)]. □

We call the pair (V, D) as in Lemma 2.2 an *almost minimal model* of (X, B).

3. Proofs of Theorems 1.1 and 1.2

In order to prove Theorems 1.1 and 1.2, we construct an almost minimal model of a normal affine surface.

Now, let S be a normal affine surface and $\mathrm{Sing}S = \{P_1, \ldots, P_q\}$ the set of all singular points on S. Set $S_0 := S \setminus \{P_1, \ldots, P_q\}$. Let $\epsilon : \tilde{S} \to S$ be a minimal good resolution of S, i.e., $\tilde{S}$ is a smooth surface, the reduced exceptional divisor $\tilde{E}$ is an SNC-divisor, $E_i \cdot (\tilde{E} - E_i) \geq 2$ for any (-1)-curve $E_i \subset \mathrm{Supp}\tilde{E}$ and the equality holds if and only if E_i meets a unique irreducible component of $\tilde{E} - E_i$. We put $\tilde{E}_i := \epsilon^{-1}(P_i)$ for $i = 1, \ldots, q$. Let (X, Δ) be an SNC-pair such that $X \setminus \mathrm{Supp}\Delta \cong \tilde{S}$ and put $B := \Delta + \tilde{E}$. Then B is an SNC-divisor. We may assume that, for any (-1)-curve $\Delta' \subset \mathrm{Supp}\Delta$, $\Delta' \cdot (\Delta - \Delta') \geq 2$ and the equality holds if and only if Δ' meets a unique irreducible component of $\Delta - \Delta'$. We assume further that P_i $(1 \leq i \leq r)$ is not a quotient singular point and P_j $(r + 1 \leq j \leq q)$ is a quotient singular point. With the same notations as in Section 2, we have the following:

(a) $B^{\#} = \Delta^{\#} + \sum_{i=1}^{q} \tilde{E}_i^{\#}$;

(b) $\lfloor \tilde{E}_i^{\#} \rfloor \neq 0$ for $i = 1, \ldots, r$ and $\lfloor \tilde{E}_i^{\#} \rfloor = 0$ for $i = r + 1, \ldots, q$;

(c) $\tilde{E}_i^{\#} = 0$ if and only if P_i is a rational double point.

Suppose that (X, B) is not almost minimal. Then there exists an irreducible curve C on X such that $C \cdot (B^{\#} + K_X) < 0$ and the intersection matrix of $C + \mathrm{Bk}\, B$ is negative definite. We consider the following cases separately.

Case 1: C is not an irreducible component of B. Since $C^2 < 0$ and $C \cdot K_X \leq C \cdot (B^{\#} + K_X) < 0$, C is a (-1)-curve. Since $S = \epsilon(X \setminus \mathrm{Supp}\Delta)$ contains no complete curves, $C \cdot \Delta > 0$. Let $Z_1, \ldots, Z_n$ be all the irreducible components of B meeting C. Then we infer from [17, Chapter 2, 3.6 (pp. 95–97)] that:

(i) $Z_i \subset \mathrm{Supp}(\mathrm{Bk}\, B)$ for every $i = 1, \ldots, n$;

(ii) $Z_i \cdot C = 1$ for every $i = 1, \ldots, n$;

(iii) $(Z_i)^2 = -2$ at most one index i;

(iv) $Z_i \cdot Z_j = 0$ if $i \neq j$;

(v) $n \leq 2$.

Suppose that $n = 1$. Then $Z_1 \subset \mathrm{Supp}\Delta$ and so $C \cdot \tilde{E} = 0$. Here we note that Δ is a big divisor since S is affine and $\mathrm{Supp}\Delta = X \setminus \tilde{S}$. So Δ is neither an admissible rational rod nor an admissible rational fork. Since the coefficient of Z_1 in $D^{\#} < 1$, we see that Z_1 is a component of an admissible maximal rational twig in Δ. We know that $\epsilon_*(C|_{\tilde{S}}) \cong \mathbb{C}$. In particular, $e(\epsilon_*(C|_{\tilde{S}})) = 1$.

Suppose that $n = 2$. Since $C \cdot \Delta > 0$, we may assume that $Z_1 \subset \mathrm{Supp}\Delta$. Then Z_1 is a component of an admissible maximal rational twig in Δ (see the preceding paragraph). Let A_i $(i = 1, 2)$ be the connected component of $\mathrm{Supp}(\mathrm{Bk}\, B)$ containing Z_i. By [17, Lemma 2.3.7.1 (p. 97)], $A_1 \neq A_2$. Moreover, we infer from [17, Lemma 2.3.7.1] and its proof (see also [17, 2.3.8 (p. 101)]) that A_2 is an admissible rational rod or fork. So $A_2 = \tilde{E}_i$ for some i, $r+1 \leq i \leq q$. We see that $\epsilon_*(C|_{\tilde{S}})$ is a topologically contractible curve on S. In particular, $e(\epsilon_*(C|_{\tilde{S}})) = 1$.

Case 2: C is an irreducible component of B. Since $C \cdot (B^{\#} + K_X) < 0$, C is not a component of $\mathrm{Supp}(\mathrm{Bk}\, B)$. So the coefficient of C in $B^{\#}$ equals one. If $g(C) > 0$, then we have

$$0 > C \cdot (B^{\#} + K_X) = C \cdot (C + K_X) + C \cdot (B^{\#} - C) \geq C \cdot (C + K_X) \geq 0,$$

which is a contradiction. Hence $g(C) = 0$. Let $Z_1, \ldots, Z_n$ be all the irreducible components of $B - C$ meeting C and let α_i $(1 \leq i \leq n)$ be the coefficient of Z_i in $B^{\#}$. If $\alpha_i < 1$, then Z_i is one of the terminal components of an admissible maximal rational twig A_i in B and Z_i is not a tip if A_i is reducible. We infer from [17, p. 89] that $\alpha_i = 1 - \frac{1}{m_i}$, where m_i is an integer ≥ 2. So $\frac{1}{2} \leq \alpha_i \leq 1$ for every $i = 1, \ldots, n$. Since $C \cdot (B^{\#} + K_X) < 0$, we have

$$2 = -C \cdot (C + K_X) > C \cdot (B^{\#} - C) = \sum_{i=1}^{n} \alpha_i C \cdot Z_i.$$

In particular, $n \leq 3$.

Suppose that $n = 3$. Since

$$2 > C \cdot (\alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3) \geq \frac{1}{2} C \cdot (Z_1 + Z_2 + Z_3),$$

we know that $C \cdot Z_i = 1$ and $\alpha_i < 1$ for $i = 1, 2, 3$. Hence Z_i is one of the terminal components of an admissible maximal rational twig A_i in B. Since the intersection matrix of $C + A_1 + A_2 + A_3$ is negative definite, it follows from [17, Lemma 2.3.4.1 (pp. 90–91)] and [17, Remark 2.3.4.3 (p. 93)] that $C + A_1 + A_2 + A_3$ is an admissible rational fork in B. This contradicts $C \not\subset \mathrm{Supp}(\mathrm{Bk}\, B)$.

Suppose that $n = 2$. Since

$$2 > C \cdot (\alpha_1 Z_1 + \alpha_2 Z_2) \geq \frac{1}{2} C \cdot (Z_1 + Z_2),$$

we may assume that $\alpha_2 < 1$. Then Z_2 is one of the terminal components of an admissible maximal rational twig A_2 in B and Z_2 is not a tip if A_2 is reducible. In particular, $C \cdot Z_2 = 1$. Since $2 - \frac{1}{2} > \alpha_1 C \cdot Z_1$, we know that $C \cdot Z_1 = 1$. Indeed, if $C \cdot Z_1 \geq 2$, then $\alpha_1 = 1$ and $2 - \frac{1}{2} > \alpha_1 C \cdot Z_1 \geq 2$, a contradiction.

Since, for any (-1)-curve $E \subset \mathrm{Supp}B$, $E \cdot (B - E) \geq 2$ and the equality holds then E meets a unique irreducible component of $B - E$, we know that $C^2 \leq -2$. Then C is a component of $\mathrm{Supp}(\mathrm{Bk}\, B)$, which is a contradiction.

Suppose that $n = 1$. Then $2 > \alpha_1 C \cdot Z_1$. If $C^2 = -1$, then $C \cdot Z_1 \geq 2$ because $C \cdot (B - C) \geq 2$. So $\alpha_1 = 1$, a contradiction. Assume that $C^2 \leq -2$. If $C \cdot Z_1 \geq 2$, then $\alpha_1 = 1$, a contradiction. If $C \cdot Z_1 = 1$, then C is a component of $\mathrm{Supp}(\mathrm{Bk}\, B)$, a contradiction.

Finally suppose that $n = 0$. Then C is an isolated component of B. Since Δ is a big divisor, C is a connected component of $\mathrm{Supp}\tilde{E}$. Since $\epsilon : \tilde{S} \to S$ is a minimal good resolution, C is not a (-1)-curve. Then $C^2 \leq -2$ and so C is a component of $\mathrm{Supp}(\mathrm{Bk}\, B)$, a contradiction.

As seen from the arguments as in Cases 1 and 2, we know that C is a (-1)-curve and $C \not\subset \mathrm{Supp}B$.

Now, let $f_1 : X \to X'$ be the composite of the contraction of C and contractions of all subsequently (smoothly) contractible components of $\mathrm{Supp}(\mathrm{Bk}\, B)$. Namely, f_1 is an operation (C) which is explained below (see before Lemma 3.1). Let Δ' be the connected component of $\mathrm{Supp}((f_1)_*(B))$ containing $(f_1)_*(\Delta)$. Further, let $f_2 : X' \to X_1$ be a successive contractions of (-1)-curves in $\mathrm{Supp}(\Delta')$ such that $(f_2)_*(\Delta')$ is an SNC-divisor and that, for any (-1)-curve $E' \subset \mathrm{Supp}((f_2)_*(\Delta'))$, either $E' \cdot ((f_2)_*(\Delta') - E') \geq 3$ or $E' \cdot ((f_2)_*(\Delta') - E') = 2$ and E' meets a unique irreducible component of $(f_2)_*(\Delta') - E'$. Namely, f_2 is an operation (A) which is explained below (see before Lemma 3.1). Let $f = f_2 \circ f_1$, $B_1 = f_*(B)$, $\tilde{E}_1 = f_*(\tilde{E})$, $\tilde{E}_{1,i} = f_*(\tilde{E}_i)$ for $i = 1, \ldots, q$ and $\Delta_1 = (f_2)_*(\Delta')$.

Claim.

(1) B_1 is an SNC-divisor.

(2) Each connected component of $f(\mathrm{Supp}(\mathrm{Bk}\, B))$ is an admissible rational twig, rod or fork.

(3) $f_*(\mathrm{Bk}\, B) \leq \mathrm{Bk}\,(B_1)$ and $f_*(B^\# + K_X) \geq B_1^\# + K_{X_1}$.

(4) $C \cdot \tilde{E} = 0$ or 1. If $C \cdot \tilde{E} = C \cdot \tilde{E}_i = 1$ for some i, then $i \geq r + 1$. In particular, $\tilde{E}_1, \ldots, \tilde{E}_r$ are not changed by f.

(5) Let $\overline{\epsilon}_1 : X_1 \to \overline{X}_1$ be the contraction of $\mathrm{Supp}(\mathrm{Bk}\,(B_1)) \cup (\cup_{i=1}^r \tilde{E}_{1,i})$ and let $S_1 = \overline{X}_1 \setminus \overline{\epsilon}_1(\Delta_1)$. Then S_1 is an affine open subset of S and $S \setminus S_1$ is a topologically contractible curve and contains at most one singular point of S, which is a quotient singular point.

Proof. The assertions (1)–(3) follow from [17, Lemma 2.3.7.1 (p. 97)]. The assertion (4) follows from the argument as above. As for the assertion (5), we infer from the argument as above that $S \setminus S_1$ contains at most one singular point of

S, which is a quotient singular point. Since S has only rational singular points on $S \setminus S_1$, we infer from [3, Theorem 2] that S_1 is an affine open subset of S. □

For an SNC-pair (X, B), its almost minimal model (V, D) is obtained by a composite of the following operations (cf. [17, Chapter 2, 3.11]):

(A) Contract all possible superfluous exceptional components of B, where a component E of B is called a *superfluous exceptional component* if E is a (-1)-curve, $E \cdot (B - E) \leq 2$ and the equality holds then E meets just two components of $B - E$;

(B) If there are no superfluous exceptional components in B, then construct $B^{\#}$ for the divisor B;

(C) Find a (-1)-curve C such that $C \not\subset \mathrm{Supp}B$, $C \cdot (B^{\#} + K_X) < 0$ and the intersection matrix of $C + \mathrm{Bk}\, B$ is negative definite. If there exists none then we are done. If there exists one, consider the contraction $\sigma : X \to X_1$ of C and all possible (smoothly) contractible components of $\mathrm{Supp}(\mathrm{Bk}\, B)$, and let $B_1 = \sigma_*(B)$;

(D) Repeat the operations (A), (B) and (C) all over again.

We note that, under our assumption that $\epsilon : \tilde{S} \to S$ is a minimal good resolution, $\tilde{E} = \tilde{E}_1 + \cdots + \tilde{E}_q = \epsilon^{-1}(\mathrm{Sing}S)$ contains no superfluous exceptional components.

By virtue of the above argument, we obtain the following lemma.

Lemma 3.1. *With the same notations and assumptions as above, let $\mu : X \to V$ be a birational morphism such that (V, D) $(D = \mu_*(B))$ is an almost minimal model of (X, B). Let $\overline{D}$ be the connected component of D containing $\mu_*(\Delta)$. Then the following assertions hold true.*

(1) *The divisor $D - \overline{D}$ has negative definite intersection matrix. In fact, $D - \overline{D}$ is contained in the image of $\tilde{E}_1 + \cdots + \tilde{E}_q$ via μ.*

(2) *Let $\overline{\epsilon} : V \to \overline{V}$ be the contraction of* $\mathrm{Supp}(D - \overline{D})$, *which exists by* (1), *and set $\overline{S} := \overline{V} \setminus \mathrm{Supp}(\overline{\epsilon}_*(\overline{D}))$. Then $\overline{S}$ is an affine open subset of S and $S \setminus \overline{S}$ is either an emptyset or a disjoint union of topologically contractible curves.*

(3) *Each irreducible component of $S \setminus \overline{S}$ contains at most one singular point of S, which is a quotient singular point.*

(4) *The surface $V \setminus \mathrm{Supp}D$ is a Zariski open subset of $S_0 = X \setminus \mathrm{Supp}B$ and $e(V \setminus \mathrm{Supp}D) \leq e(S_0)$.*

Proof. We recall that an almost minimal model of the SNC-pair (X, B) is obtained by a composite of the operations (A)–(C). The birational morphism μ can be decomposed as follows:

$$\mu = g_{A,n} \circ g_{C,n} \circ g_{A,n-1} \circ g_{C,n-1} \circ \cdots g_{A,1} \circ g_{C,1},$$

where $g_{A,i}$ $(i = 1, \dots, n)$ is either the identity map or an operation (A) and $g_{C,i}$ $(i = 1, \dots, n)$ is an operation (C). Here we note that the SNC-divisor $B = \Delta + \tilde{E}$ contains no superfluous exceptional components (see the second paragraph of this section). By the construction of f before Claim as above, we know that $f = g_{A,1} \circ g_{C,1}$. So μ is a composite of birational morphisms which are explained before Claim as above. Hence all the assertions follow from Claim as above. □

In Lemma 3.1, we call the surface $\overline{S}$ an *almost minimal model* of S.

Now we prove Theorems 1.1 and 1.2. We use the same notations as above.

Lemma 3.2. *With the same notations and assumptions as above, assume further that* $\overline{\kappa}(S_0) \geq 0$. *Let* (V, D) *be an almost minimal model of* (X, B) (*cf.* *Lemma* 3.1). *For each connected component of* D *that is also a connected component of* $\mathrm{Supp}(\mathrm{Bk}\, D)$ (*hence it is contractible to a quotient singular point* P), *denote by* G_P *the local fundamental group of the respective singular point* P. *Then*

$$\frac{1}{3}(K_V + D^{\#})^2 \leq e(V \setminus \mathrm{Supp} D) + \sum_P \frac{1}{|G_P|}.$$

Proof. By (3) of Lemma 2.2, we have $\overline{\kappa}(V \setminus \mathrm{Supp} D) = \overline{\kappa}(S_0) \geq 0$. The assertion then follows from [21, Corollary 2.5 (ii)]. □

Proof of Theorem 1.1. Let $D^{(1)}, \dots, D^{(\ell)}$ be the set of all admissible rational rods and forks of D and let $\mu'(D^{(i)})$ $(i = 1, \dots, \ell)$ be the proper transform of $D^{(i)}$ by μ. As seen from the construction of an almost minimal model of (X, B), we know that every $\mu'(D^{(i)})$ $(i = 1, \dots, \ell)$ is an admissible rational rod or fork in B, i.e., $\mu'(D^{(i)}) = \tilde{E}_j$ for some j, $r + 1 \leq j \leq q$. Set $Q_i := \overline{\epsilon}(D^{(i)})$ for $i = 1, \dots, \ell$ (see Lemma 3.1 for the definition of $\overline{\epsilon}$). Suppose that $\overline{\kappa}(S_0) \geq 0$. We infer from Lemma 3.2 that

$$\frac{1}{3}(K_V + D^{\#})^2 \leq e(V \setminus \mathrm{Supp} D) + \sum_{i=1}^{\ell} \frac{1}{|G_{Q_i}|},$$

where G_{Q_i} $(i = 1, \dots, \ell)$ is the local fundamental group of Q_i. Since $\ell \leq q$ and $e(V \setminus \mathrm{Supp} D) \leq e(S_0) = e(S) - q$, we have

$$\begin{aligned} (0 \leq) \quad \frac{1}{3}(K_V + D^{\#})^2 &\leq e(V \setminus \mathrm{Supp} D) + \sum_{i=1}^{\ell} \frac{1}{|G_{Q_i}|} \\ &\leq e(S) - q + \frac{\ell}{2} \\ &\leq e(S) - \frac{q}{2}. \end{aligned}$$

Therefore, $e(S) \geq 0$. If $\overline{\kappa}(S_0) = 2$, then $(K_V + D^{\#})^2 > 0$ (cf. [9] and [17, Chapter 2, Section 6]) and hence $e(S) > \frac{q}{2} \geq 0$. We consider the case $\overline{\kappa}(S_0) = 0$ or 1. Then $(K_V + D^{\#})^2 = 0$ (cf. [9] and [17, Chapter 2, Section 6]). If $e(S) = 0$, then $q = 0$ and so S is smooth. If $\overline{\kappa}(S) = 1$, then we infer from [17, Theorem 2.6.1.5 (p. 175)] (see also [9]) that $S = S_0$ has a structure of $\mathbb{C}^*$-fibration. Assume that $\overline{\kappa}(S) = 0$. We infer from [12] that S is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $H[-1, 0, -1]$. As seen from the constructions of $\mathbb{C}^* \times \mathbb{C}^*$ or $H[-1, 0, -1]$ in [2, Section 8], we know that S has a structure of $\mathbb{C}^*$-fibration. The assertions (1)–(4) are thus verified.

We consider the case where $e(S) \leq 0$ and $\overline{\kappa}(S_0) = -\infty$ and prove the assertion (5). If S is smooth, then S is affine ruled (cf. [17, Theorem 3.1.3.2 (p. 194)]). We assume further that $q > 0$. Since S is affine and $V \setminus \mathrm{Supp}D \subset S_0$, we know that if $V \setminus \mathrm{Supp}D$ is affine ruled then so is S. Suppose that $V \setminus \mathrm{Supp}D$ is not affine ruled. By [17, Theorem 2.5.1.2 (p. 143)], which is originally proved in [19], the surface $V \setminus \mathrm{Supp}D$ is a Platonic $\mathbb{C}^*$-fiber space (for the definition, see [17, Chapter 2, Section 5] or [19]). So, $e(V \setminus \mathrm{Supp}D) = 0$. On the other hand, by Lemma 3.1, we have $0 = e(V \setminus \mathrm{Supp}D) \leq e(S_0) = e(S) - q \leq -q < 0$. This is a contradiction. This proves the assertion (5). □

Proof of Theorem 1.2. Assume that $e(S) \leq 0$ and $\overline{\kappa}(S_0) \geq 0$. Then S is smooth, $e(S) = 0$ and $\overline{\kappa}(S) \leq 1$ by (1)–(3) of Theorem 1.1. If $\overline{\kappa}(S) = 0$, then S is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $H[-1, 0, -1]$ (cf. Proof of Theorem 1.1). So $\overline{P}_2(S) > 0$ by [12]. Assume that $\overline{\kappa}(S) = 1$. Since S is smooth affine surface with $e(S) = 0$, it follows from [14, Lemma 3.1 and Remark 3.2] that $\overline{P}_2(S) > 0$. This proves Theorem 1.2. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Let X be a normal complete rational surface of Picard number one and B a non-empty reduced algebraic curve on X. Set $\mathrm{Sing}X = \{P_1, \ldots, P_n\}$ and assume that $P_1, \ldots, P_k \in B$ and $P_{k+1}, \ldots, P_n \notin B$. Let $\mu : \tilde{X} \to X$ be a minimal good resolution of X, $\tilde{E}_i = \mu^{-1}(P_i)$ for $1 \leq i \leq n$ and $\tilde{B}$ the proper transform B on $\tilde{X}$. Then $\mu^{-1}(B) = \tilde{B} + \sum_{i=1}^{k} \tilde{E}_i$ as a reduced divisor. Let $B = \sum_{i=1}^{r} B_i$ be the decomposition of B into irreducible components. We use the intersection theory given in [24]. Since $\rho(X) = 1$, for every two Weil divisors L_1 and L_2 on X, L_1 is numerically equivalent to αL_2 for some rational number α. So, B_j $(j = 2, \ldots, r)$ is numerically equivalent to $a_j B_1$ for some rational number a_j. For $j = 2, \ldots, r$, we note that $B_j \not\equiv 0$, that $B_j \cdot B_1 \geq 0$ and that $B_j \cdot B_1 > 0 \Longleftrightarrow B_j \cap B_1 \neq \emptyset$ (cf. the intersection theory given in [24, Section 1]). If $(B_1)^2 \leq 0$, then there exist no divisors Δ such that $\mathrm{Supp}\Delta \subset \mathrm{Supp}(\tilde{B} + \sum_{i=1}^{n} \tilde{E}_i)$ and $\Delta^2 > 0$. This is impossible because the irreducible components of $\tilde{B} + \sum_{i=1}^{n} \tilde{E}_i$ generates $\mathrm{Pic}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$. So $(B_1)^2 > 0$ and $a_j > 0$ for $j = 2, \ldots, r$. Hence B is connected. Moreover, $S := X \setminus B$ contains no complete algebraic curves.

Let $\nu : V \to \tilde{X}$ be a composite of blowing-ups of points on $\mathrm{Supp}\tilde{B}$ including its infinitely near points such that $D = \nu^*(\tilde{B} + \sum_{i=1}^{n} \tilde{E}_i)_{\mathrm{red}}$ becomes an SNC-divisor.

Lemma 4.1. *With the same notations as above, the surface S is affine.*

Proof. Since $\rho(X) = 1$, the assertion is clear. The assertion can be verified also by using [22, Corollary 2.6]. □

Lemma 4.2. *Assume that $\overline{\kappa}(S \setminus \mathrm{Sing}S) = -\infty$. Then every irreducible component of D is rational.*

Proof. The assumption implies that $|D + K_V| = \emptyset$. Since V is a rational surface, it follows from [17, Lemma 2.2.2.2 (p. 73)] that every irreducible component of D is a rational curve. □

By Lemma 4.2, Theorem 1.3 is verified when $\overline{\kappa}(S \setminus \mathrm{Sing}S) = -\infty$. In particular, if $e(S) < 0$, then $\overline{\kappa}(S \setminus \mathrm{Sing}S) = -\infty$ by Theorem 1.1, and so Theorem 1.3 holds true.

From now on, we assume that $e(S) = 0$ and $\overline{\kappa}(S \setminus \mathrm{Sing}S) \geq 0$. Then Theorem 1.1 implies that S is smooth and $\overline{\kappa}(S) = 0$ or 1.

Lemma 4.3. *With the same notations and assumptions as above, assume further that $\overline{\kappa}(S) = 0$. Then every irreducible component of D is rational.*

Proof. Since $\overline{\kappa}(S) = e(S) = 0$, S is isomorphic to either $\mathbb{C}^* \times \mathbb{C}^*$ or the surface $H[-1, 0, -1]$ by (4) of Theorem 1.1. The assertion can be verified easily by considering the constructions of $\mathbb{C}^* \times \mathbb{C}^*$ and $H[-1, 0, -1]$ in [2, Section 8]. □

Finally we assume further that $\overline{\kappa}(S) = 1$. By virtue of [17, Theorem 2.6.1.5 (p. 175)], there exists a $\mathbb{P}^1$-fibration $\Phi : V \to \mathbb{P}^1$ onto $\mathbb{P}^1$ such that $\Phi|_S$ gives rise to a $\mathbb{C}^*$-fibration on S.

Suppose to the contrary that B contains an irrational curve, say B_1. Let D_1 be the proper transform of B_1 on V. Since Φ is a $\mathbb{P}^1$-fibration, D_1 is not a fiber component of Φ, i.e., D_1 is a horizontal component. Moreover, $\Phi|_{D_1} : D_1 \to \mathbb{P}^1$ is a morphism of degree two because $FD = 2$ for every fiber F of Φ and D_1 is irrational. So every component of $D - D_1$ is a fiber component of Φ. All the exceptional curves with respect to $\mu \circ \nu : V \to X$ are contained in $\mathrm{Supp}(D - D_1)$. Hence we know that Φ induces a fibration on X whose general fiber is isomorphic to $\mathbb{P}^1$. However, this contradicts $\rho(X) = 1$. Therefore every irrational curve in $\mathrm{Supp}D$ can be contracted to a point by $\mu \circ \nu$.

The proof of Theorem 1.3 is thus completed.

5. Remarks on topologically contractible curves on $\mathbb{Q}$-homology planes

A $\mathbb{Q}$-homology plane is, by definition, a normal surface with Betti numbers of the affine plane $\mathbb{C}^2$. It is well-known that every $\mathbb{Q}$-homology plane is affine and birationally ruled (cf. [22, Theorem 1.1 (1)]). In this section, we give a following result on topologically contractible curves on $\mathbb{Q}$-homology planes.

Proposition 5.1. *Let S be a $\mathbb{Q}$-homology plane and C a topologically contractible algebraic curve on S. If $S \setminus C$ is not smooth (i.e., $\mathrm{Sing}S \setminus (C \cap \mathrm{Sing}S) \neq \emptyset$), then there exists a $\mathbb{C}$-fibration $\varphi : S \to \mathbb{A}^1$ onto the affine line $\mathbb{A}^1$ such that C is the reduced part of some fiber of φ. In particular, $\overline{\kappa}(S \setminus \mathrm{Sing}S) = \overline{\kappa}(S \setminus (C \cup \mathrm{Sing}S)) = -\infty$ and S has only cyclic quotient singularities.*

Proof. Set $S' := S \setminus C$. Then S' is a normal affine surface by [22, Corollaries 2.6 and 3.2 (iv)], $e(S') = 0$ and $\mathrm{Sing}S' \neq \emptyset$. It then follows from Theorem 1.1 that S' is affine ruled and so we have a $\mathbb{C}$-fibration φ' on S'. Since S is affine, this $\mathbb{C}$-fibration φ' can be extended to a $\mathbb{C}$-fibration $\varphi : S \to T$ on S onto a smooth curve T and C is contained in a fiber F_0 of φ. Since S is affine ruled, it has only cyclic quotient singular points by [16, Theorem 1]. It then follows from [18, Theorem 2.8] (or [22, Proposition 3.3]) that $T \cong \mathbb{A}^1$ and every fiber of φ is irreducible. So C is the reduced part of F_0. □

As an easy consequence of Proposition 5.1, we obtain the following corollary.

Corollary 5.2. (cf. [1, Theorem 5.1], [4, Sublemma in the proof of Theorem 3.6]) *Let $S = \mathbb{A}^2/G$, where G is a non-abelian, finite, small subgroup of $\mathrm{GL}(2, \mathbb{C})$. Then $S \setminus \mathrm{Sing}S$ contains no topologically contractible algebraic curves.*

Proof. Since S has a non-cyclic quotient singular point, it is not affine ruled by [16, Theorem 1]. So the assertion follows from Proposition 5.1. □

Acknowledgements

The author would like to express his gratitude to the referee for pointing out errors and for giving useful comments and suggestions which improved the first version of this paper.

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Mathematics Division, Department of Information Engineering,
Faculty of Engineering, Niigata University,
8050 Ikarashininocho, Nishi-ku,
Niigata City 950–2181, JAPAN
e-mail: kojima@ie.niigata-u.ac.jp