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Maximal operators associated to the wave equation for radial data in $\mathbb{R}^{3+1}$

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Abstract

We consider $L^4_x L^\infty_t$ boundedness for a solution to the wave equation with radial data in $\mathbb{R}^{3+1}$. We derive Hardy type inequalities and Morrey type inequality. We also use the rearrangement of functions to solve the problem.

1. Introduction

Let u be the solution to the following Cauchy problem for the wave equation

$$\partial_{tt}u(t,x) = \Delta u(t,x), \qquad u(0,x) = f(x),\ \partial_t u(0,x) = g(x). \tag{1.1}$$

We write the solution of this problem by using the Fourier transform with respect to x

$$u(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(\cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi))d\xi.$$

Moreover we define the closely related operator $e^{\pm it\sqrt{-\Delta}}$ by

$$e^{\pm it\sqrt{-\Delta}}f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi \pm t|\xi|)}\hat{f}(\xi)d\xi. \tag{1.2}$$

We set the critical order for indices: For $n \in \mathbb{N}$

$$q^* = q^*(n) := \frac{2(n+1)}{n-1}, \qquad s^* = s^*(n) := \frac{n}{n+1} = \frac{n}{2} - \frac{n}{q^*}. \tag{1.3}$$

We note that $H^{s^*} \hookrightarrow L^{q^*}$. Rogers and Villarroya [9] have shown that, for any $n \in \mathbb{N}$, the inequality

$$\|e^{\pm it\sqrt{-\Delta}}f\|_{L^q_x L^\infty_t} \lesssim \|f\|_{H^s(\mathbb{R}^n)} \tag{1.4}$$

holds for $q \in [q^*, \infty]$ and $s > s^*$, and does not hold for $q < q^*$ nor $s < s^*$, where

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$L^q_x L^\infty_t = L^q_x(\mathbb{R}^n; L^\infty_t(\mathbb{R}))$ and

$$\|e^{\pm it\sqrt{-\Delta}} f\|_{L^q_x L^\infty_t} = \|\sup_{t\in\mathbb{R}} |e^{\pm it\sqrt{-\Delta}} f|\|_{L^q_x(\mathbb{R}^n)}.$$

Rogers and Villarroya's proof of (1.4) used the embedding $H^s_q(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}), s > 1/q$ with respect to the variable t, and so it was not possible for them to obtain the endpoint regularity $s = s^*$. For the proof for the negative result, they set $t = |x|$ to obtain

$$\|e^{\pm it\sqrt{-\Delta}} f\|_{L^q_x L^\infty_t} \geq \|e^{\pm i|x|\sqrt{-\Delta}} f\|_{L^q_x(\mathbb{R}^n)},$$

and they showed that the right hand side is not able to be bounded by $\|f\|_{H^s}$ for $s < s^*$ for some non-radial function f. It seems that the critical case $s = s^*$ still remains open. Here we give one remark for the case $n = 1$. We have $q^*(1) = \infty$ and $s^*(1) = 1/2$, and the solution u of (1.1) with $g = 0$ can be written as

$$u(t,x) = \frac{f(x+t) + f(x-t)}{2}, \quad t, x \in \mathbb{R}.$$

Alternatively, we can also write this $e^{\pm it\sqrt{-\partial_{xx}}} f = f(x \pm t)$. Obviously $\|f(x \pm t)\|_{L^\infty_x L^\infty_t} = \|f(x)\|_{L^\infty_x}$ is not bounded by $\|f\|_{H^{1/2}}$. This corresponds to the failure of the Sobolev embedding theorem. Therefore we are interested in only cases where $n \geq 2$. We also remark that these $L^q_x L^\infty_t$ bound problems have been intensively studied with respect to solutions for Schrödinger equations. This is closely related to the problem of showing pointwise convergence of $e^{it\Delta} f$ to f as t tends to zero. See the introduction in [10] and the references within.

We will add a contribution to the wave equation case when $n = 3$. We consider the solution with $f = 0$ and radial functions g of (1.1) in a three dimensional case. Note that $q^*(3) = 4$ and $s^*(3) = 3/4$. The best available result is

$$\|u\|_{L^4_x(\mathbb{R}^3; L^\infty_t)} \lesssim \|g\|_{H^{-1/4+\varepsilon}(\mathbb{R}^3)}$$

for $\varepsilon > 0$. We note $-1/4 = s^*(3) - 1$. We state our result.

Theorem 1. *Let $n = 3$ and $f = 0$. Then the solution of* (1.1) *satisfies*

$$\|u\|_{L^4_x(\mathbb{R}^3; L^\infty_t)} \lesssim \|g\|_{L^{12/7}(\mathbb{R}^3)} \tag{1.5}$$

for any radial data $g \in L^{\frac{12}{7}}(\mathbb{R}^3)$.

We remark that we have the embedding $L^{\frac{12}{7}}(\mathbb{R}^3) \hookrightarrow H^{-\frac{1}{4}}(\mathbb{R}^3)$. For the proof of Theorem 1, which is contained in Section 4, we use the following explicit representation for the solution u with radial data g,

$$u(t,x) = \frac{t}{2\pi} \int_{y\in\mathbb{R}^3, |y|<1} \frac{g(x+ty)}{\sqrt{1-|y|^2}}\, dy = \frac{1}{2r} \int_{|r-t|}^{r+t} \lambda \overset{o}{g}(\lambda) d\lambda, \tag{1.6}$$

where $r = |x|$ and for a radial function g, we set $\overset{o}{g}(r) = g(x)$. We will use Theorem 3 to estimate this. To deal with taking L_t^∞ norm on the solution first, we apply some results on the rearrangement of functions. We will introduce the definition and some properties which we require for the rearrangement of functions in Section 3.

Moreover if the radial function g satisfies a certain monotonicity property, it is easy to deal with taking L_t^∞ norm on the solution u. We have the following result.

Theorem 2. *Let $n = 3$ and $f = 0$. Then the solution of* (1.1) *satisfies*

$$\|u\|_{L_x^4(\mathbb{R}^3;L_t^\infty)} \lesssim \|g\|_{\dot{H}^{-1/4}(\mathbb{R}^3)} \tag{1.7}$$

for any radial data $g \in C^\infty(\mathbb{R}^3)$ satisfying

$$r|\overset{o}{g}(r)| \geq \lambda|\overset{o}{g}(\lambda)| \qquad (0 < r < \lambda), \tag{1.8}$$

where $\overset{o}{g}(|x|) = g(x)$.

Here we remark that we have the following decay estimate from the monotone decreasing property (1.8),

$$|\overset{o}{g}(\lambda)| \lesssim \frac{1}{\lambda}$$

as $\lambda \to \infty$. One may wonder whether this is too strong to deal with all $H^{-1/4}$ functions. We have the Sobolev embedding $L^{12/7}(\mathbb{R}^3) \hookrightarrow H^{-1/4}(\mathbb{R}^3)$ and we estimate the typical function $\langle x\rangle^{-p}$ in $L^{12/7}(\mathbb{R}^3)$ norm as

$$\int_{\mathbb{R}^3} \frac{1}{\langle x\rangle^{12p/7}} dx \sim \int_0^\infty \frac{1}{\langle r\rangle^{\frac{12p}{7}-2}} dr$$

to conclude $\langle x\rangle^{-p} \in L^{12/7}(\mathbb{R}^3)$ if and only if $p > 7/4 (> 1)$. In this sense, we may say that the condition (1.8) is not so strong.

Before closing this section, we give some notation. We use the bracket

$$\langle x\rangle = 1 + |x|, \quad x \in \mathbb{R}^n. \tag{1.9}$$

We write the Fourier transform of function,

$$\hat{f}(\xi) = (\mathcal{F}_{\mathbb{R}^n} f(x))(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

We sometimes also use the Fourier transform for functions on $(0, \infty)$,

$$(\mathcal{F}_{\mathbb{R}_+} f(x))(\xi) = \int_0^\infty e^{-ix\xi} f(x)dx.$$

For any $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$ and its norm is

$$\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}, \qquad \|f\|_{L^\infty(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |f(x)|.$$

We sometimes use the Lebesgue space on the half line $\mathbb{R}_+ = \{x \ge 0\}$

$$\|f\|_{L^p(\mathbb{R}_+)} = (\int_0^\infty |f(x)|^p dx)^{1/p}, \qquad \|f\|_{L^\infty(\mathbb{R}_+)} = \sup_{x\ge 0} |f(x)|.$$

Let $\mathcal{S}$ be the Schwartz space. For any $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^n)$ denotes homogeneous Sobolev space on $\mathbb{R}^n$ defined as the space $\mathcal{S}'$ of classes of distributions modulo polynomials for which the following (semi-)norm is finite,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \||\xi|^s \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

For $0 < \gamma \le 1$, $C^{0,\gamma}$ denotes Hölder space on $\mathbb{R}^n$ and its norm is

$$\|f\|_{C^{0,\gamma}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\ne y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

See [2, 8] for more general information on these function spaces.

2. Inequalities

In this section, we introduce two inequalities which we use in the proof of Theorem 1 and Theorem 2 respectively. First we introduce the well-known inequality, see [11] for example.

Theorem 3. ([11] et al.) *Let $n \in \mathbb{N}, a, b \in \mathbb{R}$ and $1 \le q \le p \le \infty$. Then the inequality*

$$\|\frac{1}{|x|^a} \int_{|y|<|x|} \frac{1}{|y|^b} f(y)dy\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}. \tag{2.1}$$

holds if and only if

$$a - \frac{n}{p} = n - \frac{n}{q} - b > 0. \tag{2.2}$$

We give an alternative proof for this, that is using a dyadic decomposition of $\mathbb{R}^n$.

The second inequality is similar to Morrey's inequality which can be stated as follows. For $n < p \le \infty, \gamma = 1 - n/p$ and a function $f : \mathbb{R} \to \mathbb{C}$,

$$\|f\|_{C^{0,\gamma}(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,p}(\mathbb{R}^n)}. \tag{2.3}$$

Our result is

Theorem 4. *Let* $n \in \mathbb{N}, 2 \le p \le \infty, a, s \in \mathbb{R}$ *satisfy*

$$0 < a - \frac{n}{p} = s - \frac{n}{2} < 1. \tag{2.4}$$

Then

$$\|\frac{f(x) - f(0)}{|x|^a}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)} \tag{2.5}$$

holds.

Proof of Theorem 3. The desired inequality (2.1) is equivalent to

$$\int_{\mathbb{R}^3} \frac{1}{|x|^a} \int_{|y|<|x|} \frac{1}{|y|^b} |f(y)| dy |\phi(x)| dx \lesssim \|f\|_{L^q} \|\phi\|_{L^{p'}}, \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We decompose the left hand side,

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{1}{|x|^a} \int_{|y|<|x|} \frac{1}{|y|^b} |f(y)| dy |\phi(x)| dx \\ &\sim \sum_{-\infty<j<k<\infty} 2^{-ak-bj} \left(\int_{|x|\sim 2^k} |\phi(x)| dx \right) \left(\int_{|y|\sim 2^j} |f(y)| dy \right). \end{aligned}$$

By using Hölder inequality with $1 \le p, q \le \infty$,

(2.7)
$$\int_{|x|\sim 2^k} |\phi(x)| dx \lesssim 2^{nk/p} \|\phi\|_{L^{p'}(|x|\sim 2^k)}, \qquad \int_{|x|\sim 2^j} |f(y)| dy \lesssim 2^{nj/q'} \|f\|_{L^q(|x|\sim 2^j)},$$

and so

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{1}{|x|^a} \int_{|y|<|x|} \frac{1}{|y|^b} |f(y)| dy |\phi(x)| dx \\ &\lesssim \sum_{j<k} 2^{(\frac{n}{p} - a)k + (\frac{n}{q'} - b)j} \|\phi\|_{L^{p'}(|x|\sim 2^k)} \|f\|_{L^q(|x|\sim 2^j)}. \end{aligned} \tag{2.8}$$

We now set $\sigma = a - \frac{n}{p} = \frac{n}{q'} - b$. We remark that $\sigma > 0$. We set also $c_k = \|\phi\|_{L^{p'}(|x|\sim 2^k)}$ and $d_j = \|f\|_{L^q(|x|\sim 2^j)}$ for simplicity. By Young's inequality

$$\sum_{j<k} 2^{\sigma(j-k)} c_k d_j \lesssim \sum_{j<k} 2^{\sigma(j-k)} ((Ac_k)^{p'} + (A^{-1} d_j)^p) \lesssim A^{p'} \sum_{j\in\mathbb{Z}} c_k^{p'} + A^{-p} \sum_{k\in\mathbb{Z}} d_j^p$$

$$= A^{p'}\|c_k\|^{p'}_{l^{p'}_k(\mathbb{Z})} + A^{-p}\|d_j\|^{p}_{l^{p}_j(\mathbb{Z})} \le A^{p'}\|c_k\|^{p'}_{l^{p'}_k(\mathbb{Z})} + A^{-p}\|d_j\|^{p}_{l^{q}_j(\mathbb{Z})},$$

where at the last inequality we used the condition $p \ge q$. The desired estimate (2.6) follows by setting

$$A = \|c_k\|^{\frac{-1}{p}}\|d_j\|^{\frac{1}{p'}}. \tag{2.9}$$

For the necessity of the condition (2.2), we put the rescaled function $f_\lambda(x) = f(\lambda x)$ into (2.1) to obtain

$$\lambda^{b+a-(n/p)-n}\|\frac{1}{|x|^a}\int_{|y|<|x|}\frac{1}{|y|^b}f(y)dy\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{-n/q}\|f\|_{L^q(\mathbb{R}^n)}.$$

Therefore the equality in (2.2) is needed. If we put a non zero function $f \in C_0^\infty$ into (2.1), the right hand side is finite, however the left hand side is infinite for $a \le n/p$ and $p < \infty$. In the case $p = \infty$, we can take $f(x) = \langle x\rangle^{-n/q}(\log\langle x\rangle)^{-(n/q)-\varepsilon} \in L^q(\mathbb{R}^n)$, but $|x|^{-b}f \notin L^1(\mathbb{R}^n)$ for a sufficiently small $\varepsilon > 0$. □

We sometimes call the argument in the lines from (2.8) to (2.9) the Schur test. We also use this argument in the proof Theorem 4.

Proof of Theorem 4. We take $I = \mathcal{F}^{-1}\mathcal{F}$ as

$$f(x) - f(0) = \int_{\mathbb{R}^n}(e^{ix\xi}-1)\widehat{f}(\xi)d\xi = \sum_{j\in\mathbb{Z}}\int_{|\xi|\sim 2^j}(e^{ix\xi}-1)\widehat{f}(\xi)d\xi,$$

and for any $j \in \mathbb{Z}$ we estimate $|e^{ix\xi}-1| \le \min\{2, |x||\xi|\} \le \min\{2, 2^{j+k}\}$ for $|x| \sim 2^k$. We will prove (2.5) in the following duality form:

$$\Big|\int_{\mathbb{R}^n}\frac{f(x)-f(0)}{|x|^a}\phi(x)dx\Big| \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)}\|\phi\|_{L^{p'}(\mathbb{R}^n)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We estimate the left hand side by using Hölder's inequality as in (2.7),

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\frac{f(x)-f(0)}{|x|^a}\phi(x)dx\Big| \\
&\lesssim \sum_{j,k\in\mathbb{Z}} 2^{-ak}\min\{1, 2^{j+k}\}\left(\int_{|\xi|\sim 2^j}|\widehat{f}(\xi)|d\xi\right)\left(\int_{|x|\sim 2^k}|\phi(x)|dx\right) \\
&\lesssim \sum_{j,k\in\mathbb{Z}} 2^{(-a+\frac{n}{p})k+\frac{nj}{2}}\min\{1, 2^{j+k}\}\|\widehat{f}\|_{L^2(|\xi|\sim 2^j)}\|\phi\|_{L^{p'}(|x|\sim 2^k)}.
\end{aligned} \tag{2.10}$$

Here we write the Sobolev norm as $\|f\|_{\dot{H}^s(\mathbb{R}^n)} \sim \|c_j\|_{l^2}$ with $c_j = 2^{sj}\|\widehat{f}\|_{L^2(|\xi|\sim 2^j)}$.

We also denote $d_k = \|\phi\|_{L^{p'}(|x|\sim 2^k)}$ for simplicity. Thus it suffices to estimate the following

$$\sum_{j,k\in\mathbb{Z}} 2^{(-a+\frac{n}{p})k+(-s+\frac{n}{2})j} \min\{1, 2^{j+k}\} c_j d_k = \sum_{j,k\in\mathbb{Z}} \min\{2^{-\sigma(k+j)}, 2^{(1-\sigma)(k+j)}\} c_j d_k$$

where we have set $\sigma = a - \frac{n}{p} = s - \frac{n}{2}$ and used the relation $-\sigma < 0 < 1-\sigma$ from the condition (2.4). Then we use the inclusion $\|c_j\|_{l^p} \leq \|c_j\|_{l^2}$ for $p \geq 2$ on the way for the Schur test. □

3. The rearrangement of functions

In this section, we introduce the definition of the equimeasurable decreasing rearrangement of functions and some useful results. We consider functions on measure space (M, m), but in later parts of this section we restrict the measure space to be the Lebesgue measure on $\mathbb{R}$, that is $(M, m) = (\mathbb{R}, m)$.

Definition 5. *For any measurable function f on (M, m), the distribution function $\lambda_f : (0,\infty) \to [0,\infty]$, and the rearrangement $f^* : (0,\infty) \to (0,\infty)$ are defined respectively by*

$$\lambda_f(s) = m(\{x \in M : |f(x)| > s\}), \quad s > 0,$$

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}, \quad t > 0.$$

We set out the properties on those functions without proofs: $\lambda_f(s)$ is nonincreasing and right continuous with s. $f^*(t)$ is nonincreasing and right continuous with t. The following inequality is useful:

$$\int_E |f(x)g(x)|dx \leq \int_0^{m(E)} f^*(t)g^*(t)dt, \tag{3.1}$$

where $E \subset M$, see [1], [6] for instance.

We call the following type of function a "simple function":

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}, \tag{3.2}$$

where $c_1 > \cdots > c_N > 0$ and measurable sets $E_j \subset E, j = 1, 2, \ldots, N$ satisfy

$$E_j \cap E_k = \varnothing, \text{ if } j \neq k, \quad \bigcup_{j=1}^{N} E_j = E,$$

and χ_E is the characteristic function of E. For any measurable function $f \geq 0$

on $E \subset M$, there exists a sequence of simple functions $\{f_n\}$ such that

$$0 \le f_n(x) \le f_{n+1}(x) \le \cdots \le f(x), \qquad \lim_{n\to\infty} f_n(x) = f(x) \tag{3.3}$$

pointwisely $x \in E$, and these functions satisfy

$$0 \le f_n^*(t) \le f_{n+1}^*(t) \le \cdots \le f^*(t), \qquad \lim_{n\to\infty} f_n^*(t) = f^*(t) \tag{3.4}$$

pointwisely $t \in \mathbb{R}$, see [1].

From here we restrict the measurable space to be the Lebesgue measure on $\mathbb{R}$. The follwing estimate can be derived in the same spirit of the proof for (3.1):

Lemma 6. *For any nonnegative, nondecreasing function f and any $0 < p < \infty$,*

$$\|fg^*\|_{L^p(\mathbb{R}_+)} \le \|fg\|_{L^p(\mathbb{R}_+)}. \tag{3.5}$$

Proof. See [1]. We show this for the readers' convenience. If we suppose that f and g are simple functions $f = \sum_{j=1}^N c_j\chi_{E_j}, g = \sum_{i=1}^L d_i\chi_{F_i}$ and all $m(E_j), j = 1, 2, \ldots, N$ and $m(F_i), i = 1, 2, \ldots, L$ are irrational numbers, the estimate can be reduced to the following discretized estimate: For any $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$,

$$\sum_{j=1}^n a_j b_j \le \sum_{j=1}^n a_j b_{\sigma(j)} \tag{3.6}$$

for any bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. This is true. Indeed, we have $a_{j_0}b_{j_0} + a_j b_k \le a_{j_0}b_k + a_j b_{j_0}$ for the minimum number j_0 satisfying $\sigma(j_0) \ne j_0$. So (3.6) follows from the required number of times of this exchanging positions for a_j and a_{j_0} as a pair of b_j and b_{j_0} respectively. From (3.4), we know $f_n g_n^* \nearrow fg^*$ pointwisely. We use Beppo-Levi's theorem and the dominated convergence theorem respectively for the following two equalities to obtain

$$\|fg^*\|_{L^p(\mathbb{R}_+)} = \lim_{n\to\infty} \|f_n g_n^*\|_{L^p(\mathbb{R}_+)} \le \lim_{n\to\infty} \|f_n g_n\|_{L^p(\mathbb{R}_+)} = \|fg\|_{L^p(\mathbb{R}_+)}. \tag{3.7}$$

□

The following lemma is the key lemma for the norm of $L_x^4 L_t^\infty$ in Section 4.

Lemma 7. *For any $r > 0$,*

$$\sup_{t>0} \int_t^{t+r} f(s)ds \le \int_0^r f^*(s)ds. \tag{3.8}$$

Proof. This follows from (3.1) with $g = \chi_{[t,t+r]}(s)$ and so $g^*(s) = \chi_{[0,r]}(s)$. □

4. Bound in $L^4_{x\in\mathbb{R}^3}L^\infty_t$

Now we are going to prove Theorem 1, that is, for $r = |x|$ and the radial function $g(x) = \overset{o}{g}(r)$, we will show that

$$\|\sup_{t>0}\frac{1}{2r}\int_{|r-t|}^{r+t}\lambda\overset{o}{g}(\lambda)d\lambda\|_{L^4(\mathbb{R}^3)} \lesssim \|g\|_{L^{12/7}(\mathbb{R}^3)}. \tag{4.1}$$

Proof of Theorem 1. We may assume $g \geq 0$. For each r,

$$\begin{aligned}\sup_{t>0}\frac{1}{2r}\int_{|r-t|}^{r+t}\lambda\overset{o}{g}(\lambda)d\lambda &\leq \sup_{0<t<r}\frac{1}{2r}\int_{r-t}^{r+t}\lambda\overset{o}{g}(\lambda)d\lambda + \sup_{t>r}\frac{1}{2r}\int_{t-r}^{r+t}\lambda\overset{o}{g}(\lambda)d\lambda \\ &= \frac{1}{2r}\int_0^{2r}\lambda\overset{o}{g}(\lambda)d\lambda + \sup_{t>0}\frac{1}{2r}\int_t^{t+2r}\lambda\overset{o}{g}(\lambda)d\lambda. \end{aligned}\tag{4.2}$$

We take the L^4 norm and apply Theorem 3 to the first term

$$\|\frac{1}{2r}\int_0^{2r}\lambda\overset{o}{g}(\lambda)d\lambda\|_{L^4(\mathbb{R}^3)} \sim \|\frac{1}{|x|}\int_{|y|<|x|}\frac{1}{|y|}g(y)dy\|_{L^4(\mathbb{R}^3)} \lesssim \|g\|_{L^{12/7}(\mathbb{R}^3)}.$$

For the second term, we remove $\sup_t$ by using Lemma 7 and

$$\begin{aligned}&\|\sup_{t>0}\frac{1}{2r}\int_t^{t+2r}\lambda\overset{o}{g}(\lambda)d\lambda\|_{L^4(\mathbb{R}^3)} \\ &\leq \|\frac{1}{2r}\int_0^{2r}(\lambda\overset{o}{g})^*(s)ds\|_{L^4(\mathbb{R}^3)} \sim \|\frac{1}{\sqrt{r}}\int_0^r(\lambda\overset{o}{g})^*(s)ds\|_{L^4(\mathbb{R}_+)} \\ &\lesssim \|r^{\frac{1}{6}}(r\overset{o}{g})^*\|_{L^{\frac{12}{7}}(\mathbb{R}_+)} \leq \|r^{\frac{7}{6}}\overset{o}{g}\|_{L^{\frac{12}{7}}(\mathbb{R}_+)} \sim \|g\|_{L^{12/7}(\mathbb{R}^3)},\end{aligned}$$

where we used Theorem 3 in the following form

$$\|\frac{1}{\sqrt{r}}\int_0^r\frac{1}{s^{\frac{1}{6}}}f(s)ds\|_{L^4(\mathbb{R}_+)} \lesssim \|f\|_{L^{\frac{12}{7}}(\mathbb{R}_+)}$$

and we used Lemma 6 with $f(r) = r^{\frac{1}{6}}$. □

From here we consider the case restricted by $t = |x|$ for the operator (1.2). This setting was studied in [9] to give the counter example for (1.4) in the supercritical case $s < s^*$. Here we consider the critical case $s = s^*$ for radial functions.

Lemma 8. *Let $n = 3$. Then*

$$\|e^{\pm i|x|\sqrt{-\Delta}}f\|_{L^4_x(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^{3/4}(\mathbb{R}^3)} \tag{4.3}$$

for any radial data $f \in \dot{H}^{3/4}(\mathbb{R}^3)$.

We remark that Theorem 2 is a corollary of this lemma since the supremum in (4.2) is attained at $t = |x|$ from (1.8), and the form of the solution of free wave equation gives

$$\|\sup_{t>0} \frac{1}{2r} \int_{|r-t|}^{r+t} \lambda \overset{o}{g}(\lambda) d\lambda \|_{L^4(\mathbb{R}^3)} = \| \frac{1}{2r} \int_0^{2r} \lambda \overset{o}{g}(\lambda) d\lambda \|_{L^4(\mathbb{R}^3)}$$
$$\sim \| \frac{\sin(|x|\sqrt{-\Delta})}{\sqrt{-\Delta}} g \|_{L^4(\mathbb{R}^3)} \lesssim \|g\|_{\dot{H}^{-1/4}(\mathbb{R}^3)}.$$

Proof of Lemma 8. We calculate (1.2) with a radial function f in $\mathbb{R}^3$ by using the polar coordinate $|x| = r, |\xi| = \rho$ as

$$\begin{aligned}(e^{it\sqrt{-\Delta}} f)(t,x) &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(x\cdot\xi + t|\xi|)} \hat{f}(\xi) d\xi \\ &= C \int_0^{\pi} \int_0^{\infty} e^{i\rho(r\cos\theta + t)} \rho^2 \hat{f}(\rho) \sin\theta d\rho d\theta \\ &= C \int_{-1}^{1} \int_0^{\infty} e^{i\rho(ry+t)} \rho^2 \hat{f}(\rho) d\rho dy,\end{aligned}$$

where θ is the angle between x and ξ, and we have changed $y = \cos\theta$. If we set the function g (on $\mathbb{R}$) as

$$g(z) := \int_0^{\infty} e^{i\rho z} \rho^2 \hat{f}(\rho) d\rho, \tag{4.4}$$

then we have one formula as

$$(e^{it\sqrt{-\Delta}} f)(t,x) = C \int_{-1}^{1} g(ry+t) dy = \frac{C}{r} \int_{t-r}^{t+r} g(z) dz.$$

From here we set $t = r$. We apply Theorem 4 with $(n,p,a,s) = (1, 4, \frac{1}{2}, \frac{3}{4})$ as

$$\|\frac{1}{r} \int_0^r g(z) dz \|_{L^4(\mathbb{R}^3)} \sim \| \frac{1}{\sqrt{r}} \int_0^r g(z) dz \|_{L^4(\mathbb{R}_+)} \lesssim \|g\|_{\dot{H}^{-\frac{1}{4}}(\mathbb{R})}. \tag{4.5}$$

We can rewrite (4.4) as

$$\begin{aligned} g(z) &= \int_0^{\infty} e^{i\rho z} \rho^2 (\mathcal{F}_{\mathbb{R}^3} f)(\rho) d\rho \\ &= \int_{-\infty}^{\infty} e^{i\rho z} \rho^2 H(\rho) (\mathcal{F}_{\mathbb{R}^3} f)(\rho) d\rho \\ &= \mathcal{F}_{\mathbb{R}}^{-1} (\rho^2 H(\rho) (\mathcal{F}_{\mathbb{R}^3} f)(\rho)),\end{aligned}$$

where H is the Heaviside function. From this we can continue the calculation for

(4.5),

$$\begin{aligned}\|g\|^2_{\dot{H}^{-\frac{1}{4}}(\mathbb{R})} &= \||\rho|^{-\frac{1}{4}}\mathcal{F}_{\mathbb{R}}g\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\rho|^{-\frac{1}{2}}|\mathcal{F}_{\mathbb{R}}g|^2 d\rho \\ &= \int_{\mathbb{R}} \rho^{\frac{7}{2}} H(\rho)|(\mathcal{F}_{\mathbb{R}^3}f)(\rho)|^2 d\rho \sim \int_{\mathbb{R}^3} |\xi|^{\frac{3}{2}}|(\mathcal{F}_{\mathbb{R}^3}f)(\xi)|^2 d\xi = \|f\|^2_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^3)}\end{aligned}$$

to conclude the desired estimate. □

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