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On the global existence of generalized rotational hypersurfaces with prescribed mean curvature in Euclidean spaces, II

Takeyuki Nagasawa†

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Abstract

It was shown that there exist generalized rotational hypersurfaces of Types I and II, for which the mean curvature is any prescribed continuous function, by Kenmotsu and the present author [4]. In this paper, the existence of generalized rotational hypersurfaces of all types with the prescribed continuous mean curvature is proven.

1. Introduction

Let H be a continuous function on $\mathbb{R}$. Our purpose is to construct generalized rotational hypersurfaces with mean curvature H. A generalized rotational hypersurface M in the n-dimensional Euclidean space $\mathbb{R}^n$ where $n \geqq 3$ is defined by a compact Lie group G and its representation to $\mathbb{R}^n$, i.e., M is invariant under an isometric transformation group $(G, \mathbb{R}^n)$ with codimension two principal orbit type. Such transformation groups $(G, \mathbb{R}^n)$ have already been known and been classified in 5 types by Hsiang [2]:
Type I $\quad (G, \mathbb{R}^n) = (O(n-1), \mathbb{R}^n)$.
Type II $\quad (G, \mathbb{R}^n) = (O(\ell+1) \times O(m+1), \mathbb{R}^{\ell+m+2})$.
Type III $\quad (G, \mathbb{R}^n) = (SO(3), \mathbb{R}^5)$, $(SU(3), \mathbb{R}^8)$, $(Sp(3), \mathbb{R}^{14})$, $(F_4, \mathbb{R}^{26})$.
Type IV $\quad (G, \mathbb{R}^n) = (SO(5), \mathbb{R}^{10})$, $(U(5), \mathbb{R}^{20})$, $(U(1) \times Spin(10), \mathbb{R}^{32})$, $(SO(2) \times SO(m), \mathbb{R}^{2m})$, $(S(U(2) \times U(m)), \mathbb{R}^{4m})$, $(Sp(2) \times Sp(m), \mathbb{R}^{8m}))$.
Type V $\quad (G, \mathbb{R}^n) = (SO(4), \mathbb{R}^8)$, $(G_2, \mathbb{R}^{14})$.

The hypersurfaces of Type I with prescribed mean curvature were constructed by Kenmotsu [3] when $n = 3$ and H is a continuous function, and by Dorfmeister-Kenmotsu [1] when $n \geqq 4$ and H is an analytic function. In the previous paper [4] we have shown the global existence of hypersurfaces of Types I and II with

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mean curvature H which is merely continuous. We shall show similar results for all types here.

Let $(x(s), y(s))$ be the generating curve with archlength parameter s:

$$(x'(s))^2 + (y'(s))^2 = 1. \tag{1.1}$$

The equation of a generalized rotational hypersurface with mean curvature H can be found in [2] for each Type. Introducing notation

$$\boldsymbol{x}(s) = {}^t(x(s), y(s)), \quad \boldsymbol{x}'(s) = {}^t(x'(s), y'(s)), \quad \boldsymbol{x}''(s)^{\perp} = {}^t(-y''(s), x''(s)),$$

and

$$\boldsymbol{e}(\phi) = {}^t(\cos\phi, \sin\phi), \quad \boldsymbol{e}(\phi)^{\perp} = {}^t(-\sin\phi, \cos\phi),$$

we can describe the equation uniformly.

Fact 1.1. *There exist a finite set J, angles $\phi_j \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and natural numbers $n_j \in \mathbb{N}$ for $j \in J$ such that the equation can be described by*

$$\boldsymbol{x}''(s)^{\perp} \cdot \boldsymbol{x}'(s) + \sum_{j\in J} \frac{n_j \boldsymbol{e}(\phi_j) \cdot \boldsymbol{x}'(s)}{\boldsymbol{e}(\phi_j)^{\perp} \cdot \boldsymbol{x}(s)} = (n-1)H(s), \quad \|\boldsymbol{x}'(s)\|^2 = 1, \tag{1.2}$$

where

$$\sum_{j\in J} n_j = n - 2$$

is the number of principal curvatures except the curvature of generating curve. For each type, the set J, angles ϕ_j, and natural numbers n_j for $j \in J$ are given as follows:

Type I $J = \{0\}$, $\phi_0 = 0$, $n_0 = n - 2$.

Type II $J = \{0, 1\}$, $\phi_1 = \frac{\pi}{2}$, $\phi_0 = 0$, $n_0 = m$, $n_1 = \ell$.

Type III $J = \{-1, 0, 1\}$ $\phi_1 = \frac{\pi}{3}$, $\phi_j = j\phi_1$. *And* $n_j \equiv 1$, 2, 4, *or* 8 *for* $(SO(3), \mathbb{R}^5)$, $(SU(3), \mathbb{R}^8)$, $(Sp(3), \mathbb{R}^{14})$, *or* $(F_4, \mathbb{R}^{26})$ *respectively.*

Type IV $J = \{-1, 0, 1, 2\}$, $\phi_1 = \frac{\pi}{4}$, $\phi_j = j\phi_1$. *And* $n_{\pm 1} = \ell$, $n_0 = n_2 = k$, *where* $(k, \ell) = (2, 2)$, $(5, 4)$, $(9, 6)$, $(m-2, 1)$, $(2m-3, 2)$, $(4m-5, 4)$ *for* $(SO(5), \mathbb{R}^{10})$, $(U(5), \mathbb{R}^{20})$, $(U(1) \times Sp(10), \mathbb{R}^{32})$, $(SO(2) \times SO(m), \mathbb{R}^{2m})$, $(S(U(2) \times U(m)), \mathbb{R}^{4m})$, $(Sp(2) \times Sp(m), \mathbb{R}^{8m}))$ *respectively.*

Type V $J = \{-2, -1, 0, 1, 2, 3\}$, $\phi_1 = \frac{\pi}{6}$, $\phi_j = j\phi_1$. *And* $n_j \equiv 1$, *or* 2 *for* $(SO(4), \mathbb{R}^8)$, *or* $(G_2, \mathbb{R}^{14})$ *respectively.*

This is by direct calculations. For example the equation of Type II, which we have already investigated in [4], is

$$x''(s)y'(s) - y''(s)x'(s) - \frac{\ell y'(s)}{x(s)} + \frac{m x'(s)}{y(s)} = (n-1)H(s),$$

and (1.1). Each term in the left-hand side of the above equation is

$$x''(s)y'(s) - y''(s)x'(s) = \boldsymbol{x}''(s)^\perp \cdot \boldsymbol{x}'(s),$$

$$-\frac{\ell y'(s)}{x(s)} = \frac{\ell\, {}^t(0,1) \cdot {}^t(x'(s), y'(s))}{{}^t(-1,0) \cdot {}^t(x(s), y(s))} = \frac{n_1 \boldsymbol{e}(\phi_1) \cdot \boldsymbol{x}'(s)}{\boldsymbol{e}(\phi_1)^\perp \cdot \boldsymbol{x}(s)},$$

$$\frac{m x'(s)}{y(s)} = \frac{m\, {}^t(1,0) \cdot {}^t(x'(s), y'(s))}{{}^t(0,1) \cdot {}^t(x(s), y(s))} = \frac{n_0 \boldsymbol{e}(\phi_0) \cdot \boldsymbol{x}'}{\boldsymbol{e}(\phi_0)^\perp \cdot \boldsymbol{x}(s)},$$

and

$$\sum_{j \in J} n_j = \ell + m = n - 2.$$

Our main result is:

Theorem 1.1. *Let H be a continuous function on $\mathbb{R}$. Put*

$$S = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, \boldsymbol{e}(\phi_j)^\perp \cdot \boldsymbol{x} = 0 \text{ for some } j \in J\}.$$

For any $\boldsymbol{x}_0 \notin S$ and $s_0 \in \mathbb{R}$, there exists a solution $\boldsymbol{x}(s)$ to (1.2) on $\mathbb{R}$ satisfying $\boldsymbol{x}(s_0) = \boldsymbol{x}_0$.

Our equation is singular on the set S. Since $\boldsymbol{x}(s_0) \notin S$, then it is easy to construct a solution near $s = s_0$, and we can extend the solution as long as $\boldsymbol{x}(s) \notin S$. To extend the solution, a problem happen when $\boldsymbol{x}(s)$ approaches to S as $s \to s_*$ for some $s_* \in \mathbb{R}$. It is not trivial that the solution can be extended beyound $s = s_*$. We must study asymptotic behavior of $\boldsymbol{x}'(s)$ as $s \to s_*$, in particular the existence of $\lim_{s \to s_*} \boldsymbol{x}'(s)$, say $\boldsymbol{x}'_*$. Furthermore we must construct solutions beyond s_* with $\boldsymbol{x}'(s_*) = \boldsymbol{x}'_*$.

By formal blow-up argument we can evaluate the limit $\boldsymbol{x}'_*$. For simplicity we assume $s_* = 0$. Let the generating curve be in the sector

(1.3)
$$S_i = \begin{cases} \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, \boldsymbol{e}(0)^\perp \cdot \boldsymbol{x} > 0\} = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} & \text{for Type I,} \\ \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, \boldsymbol{e}(\phi_{i+1})^\perp \cdot \boldsymbol{x} < 0 < \boldsymbol{e}(\phi_i)^\perp \cdot \boldsymbol{x}\} & \text{for Types II–V} \end{cases}$$

for $i = 0$ (Type I), or for some $i \in J \setminus \{\max J\}$ (Types II–V). There are three cases: (i) $\lim_{s \to 0} \boldsymbol{x}(s) = \boldsymbol{x}_* \neq \boldsymbol{o}$, $\boldsymbol{e}(\phi_i)^\perp \cdot \boldsymbol{x}_* = 0$ (The condition $\boldsymbol{x}_* \neq \boldsymbol{o}$ can be removed for Type I by translation.); (i)′ $\lim_{s \to 0} \boldsymbol{x}(s) = \boldsymbol{x}_* \neq \boldsymbol{o}$, $\boldsymbol{e}(\phi_{i+1})^\perp \cdot \boldsymbol{x}_* = 0$; and (ii) $\lim_{s \to 0} \boldsymbol{x}(s) = \boldsymbol{o}$ for Types II–V. Since the argument for (i)′ is similar to that for (i), we consider (i) and (ii) only.

For the case (i), we assume that there exists the limit $\lim_{s \to +0} \boldsymbol{x}'(s) = \boldsymbol{e}(\theta_*)$ and that $\boldsymbol{x}''(s)$ is bounded. Put

$$\boldsymbol{x}_\lambda(s) = \lambda^{-1}(\boldsymbol{x}(\lambda s) - \boldsymbol{x}_*)$$

for $\lambda > 0$, and then it is easy to see

$$\boldsymbol{x}_\lambda''(s)^\perp \cdot \boldsymbol{x}_\lambda'(s) + \sum_{j\in J} \frac{\lambda n_j \boldsymbol{e}(\phi_j) \cdot \boldsymbol{x}_\lambda'(s)}{\boldsymbol{e}(\phi_j)^\perp \cdot (\lambda \boldsymbol{x}_\lambda(s) + \boldsymbol{x}_*)} = (n-1)\lambda H(\lambda s). \tag{1.4}$$

Since we have

$$\lim_{\lambda\to+0} \boldsymbol{x}_\lambda(s) = s\boldsymbol{x}'(0) = s\boldsymbol{e}(\theta_*), \quad \lim_{\lambda\to+0} \boldsymbol{x}_\lambda'(s) = \boldsymbol{x}'(0) = \boldsymbol{e}(\theta_*), \quad \lim_{\lambda\to+0} \boldsymbol{x}_\lambda''(s) = \boldsymbol{o},$$

we get

$$\boldsymbol{e}(\phi_i) \cdot \boldsymbol{e}(\theta_*) = 0$$

by passing $\lambda \to +0$ in (1.4). Consequently, $\theta_* = \phi_i + \frac{\pi}{2}$. We can obtain a similar result when $s \to -0$. Thus the generating curve touches perpendicularly the boundary of sector S_i.

For the case (ii), we assume the existence of $\lim_{s\to+0} \boldsymbol{x}'(s) = \boldsymbol{e}(\theta_*)$, $\theta_* \neq \phi_j$ for $j \in J$, and the boundedness of $\boldsymbol{x}''(s)$. Putting

$$\boldsymbol{x}_\lambda(s) = \lambda^{-1}\boldsymbol{x}(\lambda s),$$

we have

$$\boldsymbol{x}_\lambda''(s)^\perp \cdot \boldsymbol{x}_\lambda'(s) + \sum_{j\in J} \frac{n_j \boldsymbol{e}(\phi_j) \cdot \boldsymbol{x}_\lambda'(s)}{\boldsymbol{e}(\phi_j)^\perp \cdot \boldsymbol{x}_\lambda(s)} = (n-1)\lambda H(\lambda s),$$

and

$$\sum_{j\in J} n_j \cot(\theta_* - \phi_j) = 0$$

by $\lambda \to +0$. Put

$$A(\theta) = \sum_{j\in J} n_j \cot(\theta - \phi_j). \tag{1.5}$$

Since $A(\cdot)$ is monotone decreasing on each interval (ϕ_i, ϕ_{i+1}), and since

$$\lim_{\theta\to\phi_i+0} A(\theta) = \infty, \qquad \lim_{\theta\to\phi_{i+1}-0} A(\theta) = -\infty$$

there exists a unique θ_i on each interval (ϕ_i, ϕ_{i+1}) such that $A(\theta_i) = 0$. Thus the generating curve approaches to the origin with angle θ_i.

In the above argument we have assumed the existence of $\lim_{s\to0} \boldsymbol{x}'(s)$, the boundedness $\boldsymbol{x}''(s)$ and so on. In the following sections, we shall prove the asymptotic behavior as above without these assumptions, and shall show the existence of solutions of (1.2) with the initial value

$$\boldsymbol{e}(\phi_i)^{\perp} \cdot \boldsymbol{x}(0) = 0, \quad \boldsymbol{x}(0) \neq \boldsymbol{o}, \quad \boldsymbol{x}'(0) = \boldsymbol{e}\left(\phi_i + \frac{\pi}{2}\right) \tag{1.6}$$

for Types I–V, or

$$\boldsymbol{x}(0) = \boldsymbol{o}, \quad \boldsymbol{x}'(0) = \boldsymbol{e}(\theta_i) \tag{1.7}$$

for Types II–V.

Remark 1.1. *By direct calculations, we know the explicit values of* θ_i*:*
Type II $\theta_0 = \arctan\sqrt{\frac{n_0}{n_1}}$.
Type III $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$.
Type IV $\theta_{\pm 1} = -\frac{1}{2}\arctan\sqrt{\frac{k}{\ell}}$, $\theta_0 = \frac{1}{2}\arctan\sqrt{\frac{k}{\ell}}$.
Type V $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$.
See Fact 5.1 in Appendix.

We shall discuss the following two cases in §§ 3–4 respectively:
Case (i) The asymptotic behavior of $\boldsymbol{x}'(s)$ when $\lim_{s\to 0} \boldsymbol{x}(s) = \boldsymbol{x}_* \neq \boldsymbol{o}$, and the solvability of initial-value problem (1.2) and (1.6), as Propositions 3.1–3.2.
Case (ii) The asymptotic behavior of $\boldsymbol{x}'(s)$ when $\lim_{s\to 0} \boldsymbol{x}(s) = \boldsymbol{o}$, and the solvability of initial-value problem (1.2) and (1.7), as Propositions 4.1–4.2.

Theorem 1.1 follows from these Propositions. For all types in the case (i) and for Types II–III in the case (ii), the derivation of setting the problem is more complicate than [4], however the discussions is similar to those of [4]. Therefore, we shall mention the setting in detail, but various estimates briefly. For the Types IV–V in the case (ii), we need one more extra procedure than [4]. Though we can prove our results without the extra procedure for Types II–III as [4], this procedure is applicable not only for Types IV–V but also for all types. In this sense our proof is universal.

2. A transformation

Let define the matrix $R(\psi)$ and a vector $\boldsymbol{u} = {}^t(u, v)$ by

$$R(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}, \quad \boldsymbol{u} = {}^t(u, v) = R(\psi)\boldsymbol{x}.$$

It is easy to see that when $\boldsymbol{x}(s)$ satisfies (1.2), the new unknown functions $\boldsymbol{u}(s) = (u(s), v(s))$ satisfies

$$\boldsymbol{u}''(s)^{\perp} \cdot \boldsymbol{u}'(s) + \sum_{j\in J} \frac{n_j \boldsymbol{e}(\phi_j + \psi) \cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j + \psi)^{\perp} \cdot \boldsymbol{u}(s)} = (n-1)H(s), \quad \|\boldsymbol{u}'(s)\|^2 = 1. \tag{2.1}$$

This is a useful transformation for our purpose.

Assume that the generating curve $\boldsymbol{x}(s)$ is in the sector S_i defined by (1.3). We transform $\boldsymbol{x}$ to $\boldsymbol{u}$ with $\psi = -\phi_i$, then $\boldsymbol{u}(s)$ is in the sector S_0 in uv-plane.

3. Case (i)

First we show

Proposition 3.1. *Let the generating curve $\boldsymbol{x}(s)$ be in the sector S_i, and assume that*

$$\lim_{s\to 0} \boldsymbol{e}(\phi_i)^{\perp} \cdot \boldsymbol{x}(s) = 0, \quad \lim_{s\to 0} \boldsymbol{x}(s) = \boldsymbol{x}_0 \neq 0.$$

Then there exists the limit of $\boldsymbol{x}'(s)$ as $s \to 0$ and

$$\lim_{s\to 0} \boldsymbol{e}(\phi_i) \cdot \boldsymbol{x}'(s) = 0.$$

Proof. As stated in § 2, we transform $\boldsymbol{x}(s)$ to $\boldsymbol{u}(s)$ with $\psi = -\phi_i$. And then $\boldsymbol{u}(s) = (u(s), v(s))$ satisfies

$$u''v' - v''u' + \frac{n_i u'}{v} + \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}} = (n-1)H, \quad \|\boldsymbol{u}'\|^2 = 1. \tag{3.1}$$

The assumption on $\lim_{s\to 0} \boldsymbol{x}(s)$ is written as

$$\lim_{s\to 0} u(s) > 0, \quad \lim_{s\to 0} v(s) = +0 \tag{3.2}$$

in terms of $u(s)$ and $v(s)$. What we want to show is

$$\lim_{s\to 0} u'(s) = 0.$$

Multiplying both sides of the first equation of (3.1) by $v^{n_i} v'$, and using the second relation, we have

$$(v^{n_i} u')' = \left\{ (n-1)H - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}} \right\} v^{n_i} v'.$$

Taking (3.2) into account, we get

$$u'(s) = \frac{1}{v^{n_i}(s)} \int_0^s \left\{ (n-1)H(t) - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'(t)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(t)} \right\} v^{n_i}(t) v'(t)\, dt$$

by integration of the equation of $u(s)$.

Now we show $v'(s) \neq 0$ near $s = 0$. Assume that there exists a sequence $\{s_k\}$ such that $v'(s_k) = 0$, $\lim_{k\to\infty} s_k = 0$. Inserting $s = s_k$ into $(u')^2 + (v')^2 \equiv 1$ and $u'u'' + v'v'' \equiv 0$, we have

$$u'(s_k) = \pm 1, \quad u''(s_k) = 0.$$

From these it follows that

$$v''(s_k) = \mp \left(1 + \sum_{j\in J} n_j\right) H(s_k) + \frac{n_i}{v(s_k)} + \sum_{j\neq i} \frac{n_j \cos(\phi_j - \phi_i)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s_k)}$$

by evaluating the equation (3.1) at $s = s_k$. It holds that

$$\lim_{k\to\infty} \boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s_k) = -u(0)\sin(\phi_j - \phi_i) \neq 0 \quad \text{for} \quad j \neq i.$$

Therefore, we have

$$\lim_{k\to\infty} v''(s_k) = \text{a finite value} + \lim_{k\to\infty} \frac{n_i}{v(s_k)} = \infty.$$

Consequently, $v(s_k)$'s are always local minimum values for large k. This contradicts with $\lim_{s\to 0} v(s) = +0$.

Since $v'(s) \neq 0$ near $s = 0$, we can use L'Hospital theorem to obtain

$$\begin{aligned}
&\lim_{s\to 0} u'(s) \\
&= \lim_{s\to 0} \frac{\left\{\left(1 + \sum_{j\in J} n_j\right) H(s) - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i)\cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s)}\right\} v^{n_i}(s) v'(s)}{n_i v^{n_i - 1}(s) v'(s)} \\
&= \lim_{s\to 0} \left\{\left(1 + \sum_{j\in J} n_j\right) H(s) - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i)\cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s)}\right\} \frac{v(s)}{n_i} \\
&= 0.
\end{aligned}$$

Here we use

$$\lim_{s\to 0} \boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s) = -u(0)\sin(\phi_j - \phi_i) \neq 0 \quad \text{for} \quad j \neq i.$$

□

Next we prove the converse of Proposition 3.2 below, *i.e.*, the solvability of (1.2) and (1.6). The problem is equivalent to (3.1) and

$$u'(0) = 0. \tag{3.3}$$

Since $v'(0)^2 = 1$, the map $s \mapsto v$ is monotone near $s = 0$. Therefore, there exists the inverse function $s = s(v)$. Put

$$q = \frac{du}{dv} = \frac{u'}{v'}$$

as a function of v. We divide both sides of the first equation of (3.1) by $(v')^3$. Using $\|\boldsymbol{u}'(s)\| \equiv 1$, we get

$$\begin{aligned}
\frac{dq}{dv} + \frac{n_i q}{v} &= -\frac{n_i q^3}{v} \\
&+ \sum_{j \neq i} \frac{n_j \left(q^2+1\right)\{q\cos(\phi_j-\phi_i)+\sin(\phi_j-\phi_i)\}}{\left(u(0)+\int_0^v q(\eta)\,d\eta\right)\sin(\phi_j-\phi_i) - v\cos(\phi_j-\phi_i)} \\
&+ (n-1)\left(q^2+1\right)^{\frac{3}{2}}\tilde{H},
\end{aligned} \tag{3.4}$$

where

$$\tilde{H} = (\operatorname{sgn} v')\,H.$$

We multiply both sides of (3.4) by v^{n_i} and integrate from 0 to v. Since

$$\lim_{v\to 0} q(v) = \lim_{s\to 0}\frac{u'(s)}{v'(s)} = 0,$$

we obtain

$$q(v) = \frac{1}{v^{n_i}}\int_0^v \omega(q)(\eta)\,d\eta, \tag{3.5}$$

where

$$\begin{aligned}
\omega(q)(\eta) &= \omega_1(q)(\eta)+\omega_2(q)(\eta)+\omega_3(q)(\eta), \\
\omega_1(q)(\eta) &= -n_i q(\eta)^3 \eta^{n_i-1}, \\
\omega_2(q)(\eta) &= \sum_{j\neq i}\frac{\left(q^2+1\right)\{q(\eta)\cos(\phi_j-\phi_i)+\sin(\phi_j-\phi_i)\}\,\eta^{n_i}}{\left(u(0)+\int_0^\eta q(\zeta)\,d\zeta\right)\sin(\phi_j-\phi_i)-\eta\cos(\phi_j-\phi_i)}, \\
\omega_3(q)(\eta) &= (n-1)\left(q(\eta)^2+1\right)^{\frac{3}{2}}\tilde{H}(\eta)\eta^{n_i}.
\end{aligned}$$

Define the Banach space X_V and its bounded set $X_{V,M}$ by

$$X_V = \{f \in C(0,V] \mid \|f\| < \infty\}, \quad \|f\| = \sup_{v\in(0,V]}\left|\frac{f(v)}{v}\right|,$$

$$X_{V,M} = \{f \in X_V \mid \|f\| \leqq M\}.$$

Using the boundedness of H, we can show that if M is large and if V is small, the map

$$\Phi(q)(v) = \frac{1}{v^{n_i}} \int_0^v \omega(q)(\eta)\, d\eta$$

defined on $X_{V,M}$ into itself and it is contraction. Indeed we have

$$\left\| \frac{1}{v^{n_i}} \int_0^v \omega_1(q)(\eta)\, d\eta \right\| \leqq CM^3V^2,$$

$$\left\| \frac{1}{v^{n_i}} \int_0^v \omega_2(q)(\eta)\, d\eta \right\| \leqq C\left(1 + M^3V^2\right),$$

$$\left\| \frac{1}{v^{n_i}} \int_0^v \omega_3(q)(\eta)\, d\eta \right\| \leqq C\left(1 + M^3V^2\right)$$

for $q \in X_{M,V}$;

$$\left\| \frac{1}{v^{n_i}} \int_0^v (\omega_1(q_1)(\eta) - \omega_1(q_2)(\eta))\, d\eta \right\| \leqq CM^2V^2 \|q_1 - q_2\|,$$

$$\left\| \frac{1}{v^{n_i}} \int_0^v (\omega_2(q_1)(\eta) - \omega_2(q_2)(\eta))\, d\eta \right\| \leqq C\left(M^2V^3 + V + MV^2 + V^2\right) \|q_1 - q_2\|,$$

$$\left\| \frac{1}{v^{n_i}} \int_0^v (\omega_3(q_1)(\eta) - \omega_3(q_2)(\eta))\, d\eta \right\| \leqq C\left(M^2V^3 + MV^2\right) \|q_1 - q_2\|$$

for $q_1 \in X_{M,V}$ and $q_2 \in X_{M,V}$. Since these estimates can be obtained in the same way as [4], we omit details. Hence, there exists the unique fixed point of Φ in $X_{M,V}$, which solves (3.5). If H is continuous, then it solves (3.4) satisfying $q(0) = 0$. We can derive the solvability of original problem from this fact.

Proposition 3.2. *Let H be continuous. Then there exists a unique local solution $\boldsymbol{x}$ to* (1.2) *and* (1.6).

4. Case (ii)

Consider the equation of Types II–V.

Proposition 4.1. *Let the generating curve $\boldsymbol{x}(s)$ be in the sector S_i, and assume that*

$$\lim_{s \to \pm 0} \boldsymbol{x}(s) = \boldsymbol{o}.$$

Then there exists the limit of $\boldsymbol{x}'(s)$ as $s \to \pm 0$ and

$$\lim_{s\to\pm 0} \boldsymbol{x}'(s) = \pm\boldsymbol{e}(\theta_i).$$

Here θ_i *is the unique angle satisfying*

$$\sum_{j\in J} n_j \cot(\theta_i - \phi_j) = 0, \quad \phi_i < \theta_i < \phi_{i+1}.$$

This proposition is proved by a series of Lemmas. In what follows, $\boldsymbol{x}(s)$ satisfies the assumption of Proposition 4.1. For simplicity we consider only the case $s \to +0$, and assume that $\boldsymbol{x}(s)$ is defined on $(0, s_0]$. As in the previous section, we transform $\boldsymbol{x}$ to $\boldsymbol{u}$ with $\psi = -\phi_i$. Then it holds that

$$(v^{n_i} u')' = \left\{ (n-1)H - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i)\cdot \boldsymbol{u}'}{\boldsymbol{e}(\phi_j - \phi_i)^\perp \cdot \boldsymbol{u}} \right\} v^{n_i} v'.$$

We integrate this from $s \in (0, s_0)$ to s_0, and get

$$v^{n_i}(s) u'(s) = \int_{s_0}^{s} \left\{ (n-1)H(t) - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i)\cdot \boldsymbol{u}'(t)}{\boldsymbol{e}(\phi_j - \phi_i)^\perp \cdot \boldsymbol{u}(t)} \right\} v^{n_i}(t) v'(t)\, dt + v^{n_i}(s_0) u'(s_0).$$

Since the left-hand side tends to 0 as $s \to +0$, so does the right-hand side. Hence, we get

(4.1)

$$u'(s) = \frac{1}{v^{n_i}(s)} \left[\int_{s_0}^{s} \left\{ (n-1)H(t) - \sum_{j\neq i} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i)\cdot \boldsymbol{u}'(t)}{\boldsymbol{e}(\phi_j - \phi_i)^\perp \cdot \boldsymbol{u}(t)} \right\} v^{n_i}(t) v'(t)\, dt + v^{n_i}(s_0) u'(s_0) \right],$$

and want to apply L'Hospital theorem to this.

Lemma 4.1. *If the limit* $\boldsymbol{u}'(s)$ *as* $s \to +0$ *exists, then it holds that*

$$\lim_{s\to +0} \boldsymbol{u}'(s) = \boldsymbol{e}(\theta_i - \phi_i).$$

Proof. Since $\boldsymbol{u}'(s)$ is a unit vector, so is its limit, say $\boldsymbol{e}(\psi_*)$. Under the assumption we can apply L'Hospital theorem to (4.1), and get

$$\begin{aligned}\cos\psi_* &= \frac{1}{n_i}\lim_{s\to+0}\left\{(n-1)H(s)-\sum_{j\neq i}\frac{n_j\boldsymbol{e}(\phi_j-\phi_i)\cdot\boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}(s)}\right\}v(s)\\ &= -\frac{1}{n_i}\lim_{s\to+0}\sum_{j\neq i}\frac{n_j\boldsymbol{e}(\phi_j-\phi_i)\cdot\boldsymbol{e}(\psi_*)v(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}(s)}\\ &= -\frac{1}{n_i}\lim_{s\to+0}\sum_{j\neq i}\frac{n_j\cos(\phi_j-\phi_i-\psi_*)v(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}(s)}.\end{aligned}\tag{4.2}$$

Since $\boldsymbol{u}\in S_0$ in uv-plane, it holds that $\psi_*\in[0,\phi_{i+1}-\phi_i]$.

We will show $\psi_*\in(0,\phi_{i+1}-\phi_i)$. Assume $\psi_*=0$, and then

$$\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{e}(\psi_*)=\sin(\phi_j-\phi_i)\neq 0$$

for $j\neq i$. By use of L'Hopital's theorem again we have

$$\lim_{s\to+0}\frac{v(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}(s)}=\lim_{s\to+0}\frac{v'(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}'(s)}=\frac{\sin\psi_*}{\sin(\phi_j-\phi_i)}=0.$$

Hence, from (4.2) it follows that

$$\cos\psi_*=-\frac{1}{n_i}\times 0=0.$$

This contradicts with $\psi_*=0$. We can show $\psi_*\neq\phi_{i+1}-\phi_i$ in a similar argument to $R(-\phi_{i+1})\boldsymbol{x}$.

We have already known $\psi_*\in(0,\phi_{i+1}-\phi_i)$, and therefore

$$\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{e}(\psi_*)=\sin(\phi_j-\phi_i-\psi_*)\neq 0$$

for $j\neq i$. We obtain

$$\begin{aligned}\cos\psi_* &= -\frac{1}{n_i}\lim_{s\to+0}\sum_{j\neq i}\frac{n_j\cos(\phi_j-\phi_i-\psi_*)v(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}(s)}\\ &= -\frac{1}{n_i}\lim_{s\to+0}\sum_{j\neq i}\frac{n_j\cos(\phi_j-\phi_i-\psi_*)v'(s)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{u}'(s)}\\ &= -\frac{\sin\psi_*}{n_i}\sum_{j\neq i}\frac{n_j\cos(\phi_j-\phi_i-\psi_*)}{\boldsymbol{e}(\phi_j-\phi_i)^\perp\cdot\boldsymbol{e}(\psi_*)}\\ &= -\frac{\sin\psi_*}{n_i}\sum_{j\neq i}\frac{n_j\cos(\phi_j-\phi_i-\psi_*)}{-\sin(\phi_j-\phi_i-\psi_*)}\\ &= -\frac{\sin\psi_*}{n_i}\sum_{j\neq i}n_j\cot(\psi_*+\phi_i-\phi_j)\end{aligned}$$

by L'Hospital theorem. This shows

$$\sum_{j\in J} n_j \cot(\psi_* + \phi_i - \phi_j) = 0,$$

and therefore $\psi_* = \theta_i - \phi_i$. □

Lemma 4.2. *On a neighborhood of $s \to +0$, we have*

$$u'(s) > 0, \quad v'(s) > 0, \quad 0 < \frac{v'(s)}{u'(s)} < \tan(\phi_{i+1} - \phi_i).$$

Proof. Since $\lim_{s\to+0} v(s) = 0$, and since $v(s) > 0$, $v'(s) \neq 0$ for small $s > 0$, we have

$$v'(s) = \boldsymbol{e}(0)^\perp \cdot \boldsymbol{u}'(s) > 0.$$

Applying a similar argument to $R(-\phi_{i+1})\boldsymbol{x}$, we have

$$-\boldsymbol{e}(\phi_{i+1} - \phi_i)^\perp \cdot \boldsymbol{u}'(s) > 0,$$

i.e.,

$$u'(s)\sin(\phi_{i+1} - \phi_i) - v'(s)\cos(\phi_{i+1} - \phi_i) > 0.$$

Because of $0 < \phi_{i+1} - \phi_i \leqq \frac{\pi}{2}$, we have $\sin(\phi_{i+1} - \phi_i) > 0$, $\cos(\phi_{i+1} - \phi_i) \geqq 0$. Therefore, we obtain

$$u'(s) > \frac{v'(s)\cos(\phi_{i+1} - \phi_i)}{\sin(\phi_{i+1} - \phi_i)} \geqq 0, \quad 0 < \frac{v'(s)}{u'(s)} < \tan(\phi_{i+1} - \phi_i)$$

□

Corollary 4.1. *It holds that*

$$0 \leqq \liminf_{s\to+0} \frac{v'(s)}{u'(s)} \leqq \liminf_{s\to+0} \frac{v(s)}{u(s)} \leqq \limsup_{s\to+0} \frac{v(s)}{u(s)} \leqq \limsup_{s\to+0} \frac{v'(s)}{u'(s)} \leqq \tan(\phi_{i+1} - \phi_i).$$

Proof. It is by virtue of previous lemma and L'Hospital theorem for limit superior and limit inferior. □

Lemma 4.3. *There exists the limit of $\dfrac{\boldsymbol{u}(s)}{\|\boldsymbol{u}(s)\|}$ as $s \to +0$.*

Proof. Put

$$w(s) = \frac{v(s)}{u(s)}, \quad z(s) = \frac{v'(s)}{u'(s)}, \quad \liminf_{s\to b-0} w(s) = \underline{L}, \quad \limsup_{s\to b-0} w(s) = \bar{L},$$

and

$$L = \tan(\theta_i - \phi_i).$$

Assume $\underline{L} \neq \bar{L}$ and $\underline{L} < L$. And, then, taking into consideration of the shape of the geretating curve, there exists sequences $\{s_j\}$ and $\{\tilde{s}_j\}$ such that

$$s_j > \tilde{s}_j > s_{j+1} > \tilde{s}_{j+1}, \quad \lim_{j\to\infty} s_j = +0, \quad \lim_{j\to\infty} \tilde{s}_j = +0,$$

$$\lim_{j\to\infty} w(s_j) = \underline{L}, \quad \lim_{j\to\infty} w(\tilde{s}_j) = \bar{L},$$

the generating curve is tangent to the line $v = L_j u$ at $s = s_j$, and $\lim_{j\to\infty} L_j = \underline{L}$. The last property implies

$$z(s_j) = L_j \to \underline{L} \quad \text{as} \quad j \to \infty.$$

Put

$$B_\varepsilon = \{(w, z) \in \mathbb{R}^2 \mid (w - \underline{L})^2 + (z - \underline{L})^2 < \varepsilon^2\}.$$

If $\varepsilon > 0$ is sufficiently small, then we may assume that

$$(w(s_j), z(s_j)) \in B_\varepsilon, \quad (w(\tilde{s}_j), z(\tilde{s}_j)) \in B_\varepsilon^c.$$

Hence, there exists $\{\hat{s}_j\}$ such that

$$s_j > \hat{s}_j > \tilde{s}_j, \quad (w(s), z(s)) \in \bar{B}_\varepsilon \quad \text{for} \quad s \in (\hat{s}_j, s_j], \quad (w(\hat{s}_j), z(\hat{s}_j)) \in \partial B_\varepsilon.$$

Now we consider the behavior of $(w(s), z(s))$ on the interval $I_j = [\hat{s}_j, s_j]$. Then

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}(w(s) - L)^2 &= (w(s) - L)w'(s) \\ &= (w(s) - L)\frac{v'(s)u(s) - v(s)u'(s)}{u(s)^2} \\ &= \frac{u'(s)}{u(s)}(w(s) - L)(z(s) - w(s)). \end{aligned}$$

When $s \in I_j$,

$$|w(s) - L| \leqq C,$$

$$|z(s) - w(s)| = |z(s) - \underline{L} - (w(s) - \underline{L})| \leqq 2\varepsilon.$$

Therefore,

$$\left|\frac{u'(s)}{u(s)} - \frac{v'(s)}{v(s)}\right| = |w(s) - z(s)| \left|\frac{u'(s)}{v(s)}\right| \leqq 2\varepsilon \left|\frac{u'(s)}{v(s)}\right|,$$

which implies

$$\frac{u'(s)}{u(s)} = \frac{v'(s) + O(\varepsilon)u'(s)}{v(s)}.$$

Consequently,

$$\left|\frac{1}{2}\frac{d}{ds}(w(s) - L)^2\right| = \left|\frac{v'(s) + O(\varepsilon)y'(s)}{v(s)}\right| O(\varepsilon) = \frac{O(\varepsilon)}{v(s)}.$$

Here we use $|u'(s)| \leqq 1$, $|v'(s)| \leqq 1$. On the other hand

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}(z(s) - L)^2 &= (z(s) - L)z'(s) \\
&= (z(s) - L)\frac{v''(s)u'(s) - v'(s)u''(s)}{(u'(s))^2} \\
&= -\frac{z(s) - L}{(u'(s))^2}\boldsymbol{u}''(s)^{\perp} \cdot \boldsymbol{u}'(s) \\
&= -\frac{z(s) - L}{(u'(s))^2}\left\{-\sum_{j\in J}\frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s)} + (n-1)H(s)\right\}.
\end{aligned}$$

Define $\hat{\theta}$ and $\check{\theta}$ by $w(s) = \tan(\hat{\theta}(s) - \phi_i)$, and $z(s) = \tan(\check{\theta}(s) - \phi_i)$. Then

$$\boldsymbol{u}(s) = \frac{u(s)}{\cos(\hat{\theta}(s) - \phi_i)}\boldsymbol{e}(\hat{\theta}(s) - \phi_i), \quad \boldsymbol{u}'(s) = \frac{u'(s)}{\cos(\check{\theta}(s) - \phi_i)}\boldsymbol{e}(\hat{\theta}(s) - \phi_i),$$

$$\frac{\boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s)} = \frac{u'(s)w(s)\cos(\check{\theta}(s) - \phi_i)\cos(\check{\theta}(s) - \phi_j)}{v(s)\cos(\hat{\theta}(s) - \phi_i)\sin(\hat{\theta}(s) - \phi_j)}.$$

Define $\underline{\theta}$ by $\underline{L} = \tan(\underline{\theta} - \phi_i)$. When $(w(s), z(s)) \in B_{\varepsilon}$, we have

$$\hat{\theta}(s) = \underline{\theta} + O(\varepsilon), \quad \check{\theta}(s) = \underline{\theta} + O(\varepsilon).$$

Hence,

$$\begin{aligned}
\sum_{j\in J}\frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^{\perp} \cdot \boldsymbol{u}(s)} &= \frac{u'(s)\cos(\check{\theta}(s) - \phi_i)}{u(s)\cos(\hat{\theta}(s) - \phi_i)}\sum_{j\in J} n_j\left(\cot(\underline{\theta} - \phi_j) + O(\varepsilon)\right) \\
&= \frac{u'(s)\cos(\check{\theta}(s) - \phi_i)}{u(s)\cos(\hat{\theta}(s) - \phi_i)}A(\underline{\theta}) + O(\varepsilon).
\end{aligned}$$

Our assumption $\underline{L} < L$ implies $\underline{\theta} < \theta_i$, and therefore $A(\underline{\theta}) > A(\theta_i) = 0$. There exists $\lambda \in [0, 1)$ such that

$$0 < w(s) < \frac{(1+\lambda)L}{2}, \quad 0 < z(s) < \frac{(1+\lambda)L}{2}, \quad 0 < u'(s) \leqq 0, \quad v(s) > 0,$$

$$\cos(\hat{\theta}(s) - \phi_i)) = \cos(\underline{\theta} - \phi_i) + O(\varepsilon), \quad \cos(\check{\theta}(s) - \phi_i)) = \cos(\underline{\theta} - \phi_i) + O(\varepsilon)$$

hold on I_j for large j. Hence, there exists $\delta > 0$ independent of ε such that

$$\frac{z(s) - L}{(u'(s))^2} \sum_{j \in J} \frac{n_j \boldsymbol{e}(\phi_j - \phi_i) \cdot \boldsymbol{u}'(s)}{\boldsymbol{e}(\phi_j - \phi_i)^\perp \cdot \boldsymbol{u}(s)} \leqq -\frac{\delta}{v(s)}.$$

On the interval I_j,

$$\frac{1}{(x'(s))^2} = \frac{\underline{L}^2(1 + o(1))}{(y'(s))^2} = \frac{\underline{L}^2(1 + o(1))}{1 - (x'(s))^2}.$$

If there exists a sequence $\{\bar{s}_k\} \subset \bigcup_j I_j$ such that

$$\lim_{k \to \infty} \bar{s}_k = +0, \quad \lim_{k \to \infty} u'(\bar{s}_k) = 0,$$

then as $k \to \infty$

$$\infty \leftarrow \frac{1}{(u'(\bar{s}_k))^2} = \frac{\underline{L}^2(1 + o(1))}{1 - (u'(\bar{s}_k))^2} \to \underline{L}^2 < L^2.$$

This is contradiction, and therefore we may assume

$$\inf \left\{ (u'(s))^2 \;\middle|\; s \in \bigcup_j I_j \right\} > 0.$$

Hence,

$$\left| \frac{(n-1)H(s)(z(s) - L)}{(u'(s))^2} \right| \leqq C.$$

Consquently,

$$\frac{1}{2} \frac{d}{ds} \left\{ (z(s) - L)^2 + (w(s) - L)^2 \right\} \leqq -\frac{1}{v(s)} \left(\delta + O(\varepsilon) \right) + C$$

on I_j. If j is sufficiently large, then $v(s) > 0$ is sufficiently small. Taking ε small, we have

$$\frac{1}{2} \frac{d}{ds} \left\{ (z(s) - L)^2 + (w(s) - L)^2 \right\} \leqq -\frac{\delta}{2v(s)} < 0$$

on I_j for large j. Hence,

$$\begin{aligned} (w(\hat{s}_j) - L)^2 + (z(\hat{s}_j) - L)^2 &\geqq (z(s_j) - L)^2 + (w(s_j) - L)^2 \\ &= 2(L_j - L)^2. \end{aligned}$$

Taking a suitable subsequence, we have $(w(\hat{s}_j), z(\hat{s}_j)) \to (\hat{w}, \hat{z})$, where

$$(\hat{w}, \hat{z}) \in \partial B_\varepsilon \cap \left\{ (w, z) \in \mathbb{R}^2 \;\middle|\; (w - L)^2 + (z - L)^2 \geqq 2(\underline{L} - L)^2 \right\}.$$

This shows that

$$\hat{w} < \underline{L} \quad \text{or} \quad \hat{z} < \underline{L}.$$

This is contradition. Indeed, if $\hat{w} < \underline{L}$, then

$$\liminf_{s \to +0} w(s) = \underline{L} > \hat{w} = \lim_{j \to \infty} w(\hat{s}_j) \geqq \liminf_{s \to +0} w(s).$$

If $\hat{z} < \underline{L}$, then

$$\liminf_{s \to +0} z(s) = \underline{L} > \hat{z} = \lim_{j \to \infty} z(\hat{s}_j) \geqq \liminf_{s \to +0} z(s).$$

Now we go back to the 5th line of the proof, and $\underline{L} = \bar{L}$ or $L \leqq \underline{L}$ has been proved.

Similarly we have $\underline{L} = \bar{L}$ or $\bar{L} \leqq L$.

Combining these, we finally get $\underline{L} = \bar{L}$, proving Lemma 4.3. □

Put

$$A(\alpha, \beta) = \sum_{j \in J} \frac{n_j \cos(\alpha - \phi_j)}{\sin(\beta - \phi_j)}.$$

Here we assume $\beta \neq \phi_j$ for all $j \in J$. It is easy to see

$$\frac{\partial A}{\partial \alpha} = -\sum_{j \in J} \frac{n_j \sin(\alpha - \phi_j)}{\sin(\beta - \phi_j)}.$$

When $\phi_i < \alpha < \phi_{i+1}$ and $\phi_i < \beta < \phi_{i+1}$, it holds that

$$\operatorname{sgn} \sin(\alpha - \phi_j) = \operatorname{sgn} \sin(\beta - \phi_j).$$

Therefore, we have

$$\frac{\partial A}{\partial \alpha} < 0.$$

Lemma 4.4. *There exists the limit of $\boldsymbol{u}'(s)$ as $s \to +0$ and*

$$\lim_{s \to +0} \boldsymbol{u}'(s) = \lim_{s \to +0} \frac{\boldsymbol{u}(s)}{\|\boldsymbol{u}(s)\|} = \boldsymbol{e}(\theta_i - \phi_i).$$

Proof. It is enough to show

$$\lim_{s \to +0} \frac{v'(s)}{u'(s)} = \lim_{s \to +0} \frac{v(s)}{u(s)} = \tan(\theta_i - \phi_i).$$

If $\dfrac{v'(s)}{u'(s)}$ is monotone near $s=+0$, then there exists $\displaystyle\lim_{s\to+0}\frac{v'(s)}{u'(s)}$.

Otherwise we put

$$\liminf_{s\to+0}\frac{v'(s)}{u'(s)}=\tan(\underline{\theta}'-\phi_i),\quad \limsup_{s\to+0}\frac{v'(s)}{u'(s)}=\tan(\bar{\theta}'-\phi_i).$$

There exists a sequence $\{s_k\}$ such that $v'(s)u'(s)$ takes a minimum value at $s=s_k$ and

$$s_k\to 0,\quad \frac{v'(s_k)}{u'(s_k)}\to\tan(\underline{\theta}'-\phi_i),\quad \left(\frac{v'}{u'}\right)'(s_k)=0$$

as $k\to\infty$. From the third relation it follows that $\boldsymbol{u}''(s_k)^{\perp}\cdot\boldsymbol{u}'(s_k)=0$. By using the equation (2.1) with $\psi=-\phi_i$ and $s=s_k$, we have

$$\|\boldsymbol{u}(s_k)\|(n-1)H(s_k)=\sum_{j\in J}\frac{n_j\boldsymbol{e}(\phi_j-\phi_i)\cdot\boldsymbol{u}'(s_k)}{\boldsymbol{e}(\phi_j-\phi_i)^{\perp}\cdot\frac{\boldsymbol{u}(s_k)}{\|\boldsymbol{u}(s_k)\|}}\to A(\underline{\theta}',\bar{\theta})$$

as $k\to\infty$. Because of the boundedness of H, it is clear that

$$\|\boldsymbol{u}(s_k)\|(n-1)H(s_k)\to 0.$$

Hence, we have $A(\underline{\theta}',\bar{\theta})=0$. Using a sequence of s where $v'(s)u'(s)$ takes a maximum value, we get $A(\bar{\theta}',\bar{\theta})=0$. Combining these, we know $A(\underline{\theta}',\bar{\theta})=A(\bar{\theta}',\bar{\theta})$. Since $\dfrac{\partial A}{\partial\alpha}<0$, we obtain $\underline{\theta}'=\bar{\theta}'$.

Consequently, in any cases, there exists $\displaystyle\lim_{s\to+0}\frac{v'(s)}{u'(s)}$. From Lemma 4.1 it follows that the limit value is $\tan(\theta_i-\phi_i)$. Finally we know $\displaystyle\lim_{s\to+0}\frac{v(s)}{u(s)}=\tan(\theta_i-\phi_i)$ by Corollary 4.1, proving Lemma 4.4. □

Thus we complete the proof of Proposition 4.1.

Next we prove the converse of Proposition 4.1, *i.e.*, the solvability of (1.2) and (1.7). We define the function q as before. For the case (ii),

$$\lim_{v\to0}q(v)=\cot(\theta_i-\phi_i).$$

Hence, we introduce new unknown functions r and ρ by

$$r(v)=q(v)-\cot(\theta_i-\phi_i),\quad \rho(v)=\frac{1}{v}\int_0^v r(\eta)\,d\eta.$$

Then our problem is equivalent to

$$\text{(4.3)}\qquad\begin{cases} v\dfrac{dr}{dv}(v)+(n-2)r(v)=-\gamma(n-2)\rho(v)+\displaystyle\sum_{k=1}^{5}F_k(r,\rho)(v),\\ r(0)=\rho(0)=0,\end{cases}$$

where

$$
\begin{aligned}
\gamma &= \sharp J - 1,\\
F_1(r,\rho) &= F_1(r) = -(n-2)r^2\left\{r + 2\cot(\theta_i-\phi_i)\right\}\sin^2(\theta_i-\phi_i),\\
F_2(r,\rho) &= -\gamma(n-2)r\rho\left\{r + 2\cot(\theta_i-\phi_i)\right\}\sin^2(\theta_i-\phi_i),\\
F_3(r,\rho) &= -r\rho\left\{r^2 + 2r\cot(\theta_i-\phi_i) + \operatorname{cosec}^2(\theta_i-\phi_i)\right\}\\
&\quad\times\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}},\\
F_4(r,\rho) &= -\rho^2\left\{r^2 + 2r\cot(\theta_i-\phi_i) + \operatorname{cosec}^2(\theta_i-\phi_i)\right\}\\
&\quad\times\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\sin^2(\phi_j-\phi_i)\cos(\theta_i-\phi_j)}{\sin^2(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}
\end{aligned}
$$

and

$$F_5(r,\rho)(v) = F_5(r)(v) = (n-1)\left[\left\{\cot(\theta_i-\phi_i) + r(v)\right\}^2 + 1\right]^{\frac{3}{2}}\tilde{H}(v)v.$$

The derivation of (4.3) is elementary but needs lengthy calculations, so we perform it in Appendix.

Multiplying both sides of the first equation in (4.3) by v^{n-3}, and integrating from 0 to v, we have

$$r(v) = -\frac{\gamma(n-2)}{v^{n-2}}\int_0^v \rho(\eta)\eta^{n-3}d\eta + \frac{1}{v^{n-2}}\int_0^v\sum_{k=1}^{5}\psi_k(r,\rho)(\eta)\,d\eta,$$

where

$$\psi_k(r,\rho)(\eta) = F_k(r,\rho)(\eta)\eta^{n-3}.$$

Since the function ρ is defined by r, we can define the map Ψ by

$$\Psi(r)(v) = -\frac{\gamma(n-2)}{v^{n-2}}\int_0^v \rho(\eta)\eta^{n-3}d\eta + \frac{1}{v^{n-2}}\int_0^v\sum_{k=1}^{5}\psi_k(r,\rho)(\eta)\,d\eta.$$

Taking M large, and V small, we can show this is a contraction map from $X_{V,M}$ into itself for Types II–III. This fact can be proved in the same way as [4]. Indeed, the principal part of Ψ is the map

$$\bar{\Psi}\,:\,r \mapsto -\frac{\gamma(n-2)}{v^{n-2}}\int_0^v \rho(\eta)\eta^{n-3}d\eta. \tag{4.4}$$

Because

$$\left| -\frac{\gamma(n-2)}{v^{n-1}} \int_0^v (\rho_1(\eta) - \rho_2(\eta))\eta^{n-3} d\eta \right| \leqq \frac{\gamma(n-2)\|\rho_1 - \rho_2\|}{v^{n-1}} \int_0^v \eta^{n-2} d\eta$$
$$\leqq \frac{\gamma(n-2)}{2(n-1)} \|r_1 - r_2\|,$$

$\bar{\Psi}$ is contractive when $\gamma \leqq 2$, which is fulfilled for Types II–III. Since Ψ is a small perturbation of $\bar{\Psi}$, it is also contractive for Types II–III. Testing linear functions $r_i = c_i v$, we find that the map $\bar{\Psi}$ is expansive for Types IV–V. This suggests that Ψ is not contractive for these types, and therefore we must deal with our problem more carefully.

Since

$$\frac{d\rho}{dv} = -\frac{1}{v^2} \int_0^v r \, d\eta + \frac{r}{v} = -\frac{\rho}{v} + \frac{r}{v},$$

we have

$$v\frac{d}{dv} \begin{pmatrix} r \\ \rho \end{pmatrix} + \begin{pmatrix} n-2 & \gamma(n-2) \\ -1 & 1 \end{pmatrix} \begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{5} F_k(r, \rho) \\ 0 \end{pmatrix}.$$

Eigenvalues of the matrix in the left-hand side are

$$\lambda_\pm = \frac{n - 1 \pm \sqrt{n^2 - 2(2\gamma + 3)n + 8\gamma + 9}}{2}.$$

Since

$$(2\gamma + 3)^2 - (8\gamma + 9) = 4\gamma(\gamma + 1) > 0,$$

we know $\lambda_+ \neq \lambda_-$. Therefore, there exists a non-singular matrix P such that

$$P^{-1} \begin{pmatrix} (n-2) & \gamma(n-2) \\ -1 & 1 \end{pmatrix} P = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}.$$

Put $P = (p_{ij})$, $P^{-1} = (p^{ij})$. These are matrices with constant entries. Define $\hat{r}$ and $\hat{\rho}$ by

$$\begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} = P^{-1} \begin{pmatrix} r \\ \rho \end{pmatrix}.$$

Then the equation can be rewritten as

$$v\frac{d}{dv} \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} + \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} p^{11} \sum_{k=1}^{5} \hat{F}_k(\hat{r}, \hat{\rho}) \\ p^{21} \sum_{k=1}^{5} \hat{F}_k(\hat{r}, \hat{\rho}) \end{pmatrix},$$

$$\hat{r}(0) = \hat{\rho}(0) = 0,$$

where

$$\hat{F}_k(\hat{r}, \hat{\rho}) = F_k(p_{11}\hat{r} + p_{12}\hat{\rho}, p_{21}\hat{r} + p_{22}\hat{\rho})(= F_k(r, \rho)).$$

Hence, we have

$$\hat{r}(v) = \frac{p^{11}}{v^{\lambda_+}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta)\eta^{\lambda_+ - 1} d\eta,$$
$$\hat{\rho}(v) = \frac{p^{21}}{v^{\lambda_-}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta)\eta^{\lambda_- - 1} d\eta.$$

Let $X_V \times X_V$ be a Banach space with norm

$$\|(\hat{r}, \hat{\rho})\|_{X_V \times X_V} = \|\hat{r}\|_{X_V} + \|\hat{\rho}\|_{X_V}.$$

Define the map $\hat{\Psi}$ by

$$\hat{\Psi}(\hat{r}, \hat{\rho})(v) = \left(\frac{p^{11}}{v^{\lambda_+}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta)\eta^{\lambda_+ - 1} d\eta, \frac{p^{21}}{v^{\lambda_-}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta)\eta^{\lambda_- - 1} d\eta \right).$$

We will show that if M is large, and if V is small, then $\hat{\Psi}$ is a contraction map from $X_{V,M} \times X_{V,M}$ into itself.

When $n^2 - 2(2\gamma + 3)n + 8\gamma + 9 < 0$,

$$\Re\lambda_\pm = \frac{n-1}{2} > 0.$$

If $n^2 - 2(2\gamma + 3)n + 8\gamma + 9 \geqq 0$, then

$$0 \leqq n^2 - 2(2\gamma + 3)n + 8\gamma + 9 = (n-1)^2 - 4(\gamma + 1)(n-2) < (n-1)^2,$$

and hence

$$\Re\lambda_\pm = \frac{n - 1 \pm \sqrt{n^2 - 2(2\gamma + 3)n + 8\gamma + 9}}{2} > 0.$$

Therefore, in any cases, the integral $\int_0^v \eta^{\Re\lambda_\pm + p} d\eta$ converges for $p > -1$, and

$$\int_0^v \eta^{\Re\lambda_\pm + p} d\eta = \frac{v^{\Re\lambda_\pm + p + 1}}{\Re\lambda_\pm + p + 1}.$$

Since both P and P^{-1} are constant matrices, it holds that

$$\|(r, \rho)\| \leqq C\|(\hat{r}, \hat{\rho})\|, \quad \|(\hat{r}, \hat{\rho})\| \leqq C\|(r, \rho)\|.$$

Let $(\hat{r},\hat{\rho}) \in X_{V,M} \times X_{V,M}$, and then we have

$$\begin{aligned}|\hat{F}_1(\hat{r},\hat{\rho})(\eta)| = |F_1(r,\rho)(\eta)| &\leqq C\|r\|^2\eta^2\left(\|r\|\eta+1\right)\\ &\leqq C\|(\hat{r},\hat{\rho})\|^2\eta^2\left(\|(\hat{r},\hat{\rho})\|\eta+1\right) \leqq CM^2\eta^2\left(M\eta+1\right).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\left|\frac{1}{v^{\lambda_+ +1}}\int_0^v \hat{F}_1(\hat{r},\hat{\rho})(\eta)\eta^{\lambda_+ -1}d\eta\right| &\leqq \frac{C}{v^{\Re\lambda_+ +1}}\int_0^v M^2\eta^2\left(M\eta+1\right)\eta^{\Re\lambda_+ -1}d\eta\\ &\leqq \frac{C}{v^{\Re\lambda_+ +1}}\left(M^3v^{\Re\lambda_+ +3}+M^2v^{\Re\lambda_+ +2}\right)\\ &\leqq C\left(M^3V^2+M^2V\right).\end{aligned}$$

We can estimate $\frac{1}{v^{\lambda_+ +1}}\int_0^v \hat{F}_k(\hat{r},\hat{\rho})(\eta)\eta^{\Re\lambda_+ -1}d\eta$ for $k=2,3,4,5$ in a similar manner, and can obtain

$$\left|\frac{p^{11}}{v^{\Re\lambda_+ +1}}\int_0^v\sum_{k=1}^{5}\hat{F}_k(\hat{r},\hat{\rho})(\eta)\eta^{\lambda_+ -1}d\eta\right| \leqq C\left(M^3V^2+M^2V+M^4V^3+M^3V^3+1\right).$$

We can derive

$$\left|\frac{p^{21}}{v^{\Re\lambda_- +1}}\int_0^v\sum_{k=1}^{5}\hat{F}_k(\hat{r},\hat{\rho})(\eta)\eta^{\lambda_- -1}d\eta\right| \leqq C\left(M^3V^2+M^2V+M^4V^3+M^3V^3+1\right)$$

in the same way. From these it follows that

$$\|\hat{\Psi}(\hat{r},\hat{\rho})\| \leqq C\left(M^3V^2+M^2V+M^4V^3+M^3V^3+1\right).$$

Consequently, $\hat{\Psi}$ is a map from $X_{V,M}\times X_{V,M}$ into itself provided M is large and V is small.

Using

$$\|(r_1,\rho_1)-(r_2,\rho_2)\| \leqq C\|(\hat{r}_1,\hat{\rho}_1)-(\hat{r}_2,\hat{\rho}_2)\|,$$

$$\|(\hat{r}_1,\hat{\rho}_1)-(\hat{r}_2,\hat{\rho}_2)\| \leqq C\|(r_1,\rho_1)-(r_2,\rho_2)\|,$$

we can get

$$\begin{aligned}&\|\hat{\Psi}(\hat{r}_1,\hat{\rho}_1)-\hat{\Psi}(\hat{r}_2,\rho_2)\|\\ &\quad\leqq C\left(M^2V^2+MV+M^4V^4+M^2V^3+V\right)\|(\hat{r}_1,\hat{\rho}_1)-(\hat{r}_2,\hat{\rho}_2)\|.\end{aligned}$$

Indeed, from

$$\begin{aligned}&\left|\hat{F}_1(\hat{r}_1,\hat{\rho}_1)-\hat{F}_1(\hat{r}_2,\hat{\rho}_2)\right| = |F_1(r_1)-F_1(r_2)|\\ &\quad= \left|-(n-2)(r_1-r_2)\left\{r_1^2+r_1r_2+r_2+2(r_1+r_2)\cot(\theta_i-\phi_i)\right\}\sin^2(\theta_i-\phi_i)\right|\\ &\quad\leqq C\|r_1-r_2\|\eta\left(M^2\eta^2+M\eta\right)\\ &\quad\leqq C\|\hat{r}_1-\hat{r}_2\|\eta\left(M^2\eta^2+M\eta\right)\end{aligned}$$

it follows that

$$\begin{aligned}
&\left|\frac{p^{11}}{v^{\lambda_++1}}\int_0^v \left(\hat{F}_1(r_1,\rho_1)(\eta)-\hat{F}_1(r_2,\rho_2)(\eta)\right)\eta^{\lambda_+-1}d\eta\right| \\
&\quad\leqq \frac{C\|\hat{r}_1-\hat{r}_2\|}{v^{\Re\lambda_++1}}\int_0^v \left(M^2\eta^3+M\eta^2\right)\eta^{\Re\lambda_+-1}d\eta \\
&\quad\leqq \frac{C\|\hat{r}_1-\hat{r}_2\|}{v^{\Re\lambda_++1}}\left(M^2v^{\Re\lambda_++3}+Mv^{\Re\lambda_++2}\right) \\
&\quad\leqq C\left(M^2V^2+MV\right)\|\hat{r}_1-\hat{r}_2\|.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\left|\frac{p^{11}}{v^{\lambda_++1}}\int_0^v \left(\hat{F}_2(r_1,\rho_1)(\eta)-\hat{F}_2(r_2,\rho_2)(\eta)\right)\eta^{\lambda_+-1}d\eta\right| \\
&\quad\leqq C\left(M^2V^2+MV\right)\|\hat{r}_1-\hat{r}_2\|,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{p^{11}}{v^{\lambda_++1}}\int_0^v \left(\hat{F}_k(r_1,\rho_1)(\eta)-\hat{F}_k(r_2,\rho_2)(\eta)\right)\eta^{\lambda_+-1}d\eta\right| \\
&\quad\leqq C\left(M^4V^4+MV\right)\left(\|\hat{r}_1-\hat{r}_2\|+\|\hat{\rho}_1-\hat{\rho}_2\|\right)\quad\text{for}\quad k=3,\,4,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{p^{11}}{v^{\lambda_++1}}\int_0^v \left(\hat{F}_5(r_1,\rho_1)(\eta)-\hat{F}_5(r_2,\rho_2)(\eta)\right)\eta^{\lambda_+-1}d\eta\right| \\
&\quad\leqq C\left(M^2V^3+V\right)\|\hat{r}_1-\hat{r}_2\|
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
&\left\|\frac{p^{11}}{v^{\lambda_+}}\int_0^v \sum_{k=1}^5 \left(\hat{F}_k(\hat{r}_1,\hat{\rho}_1)(\eta)-\hat{F}_k(\hat{r}_2,\hat{\rho}_2)(\eta)\right)\eta^{\lambda_+-1}d\eta\right\|_{X_V} \\
&\quad\leqq C\left(M^2V^2+MV+M^4V^4+M^2V^3+V\right)\|(\hat{r}_1,\hat{\rho}_1)-(\hat{r}_2,\hat{\rho}_2)\|
\end{aligned}$$

We can derive

$$\begin{aligned}
&\left\|\frac{p^{21}}{v^{\lambda_-}}\int_0^v \sum_{k=1}^5 \left(\hat{F}_k(\hat{r}_1,\hat{\rho}_1)(\eta)-\hat{F}_k(\hat{r}_2,\hat{\rho}_2)(\eta)\right)\eta^{\lambda_--1}d\eta\right\|_{X_V} \\
&\quad\leqq C\left(M^2V^2+MV+M^4V^4+M^2V^3+V\right)\|(\hat{r}_1,\hat{\rho}_1)-(\hat{r}_2,\hat{\rho}_2)\|
\end{aligned}$$

in the same way.

Consequently, the map $\hat{\Psi}$ is contraction if V is sufficiently small. The unique fixed point is a local solution to (4.3).

Because eigenvalues $\lambda_\pm$ and matrix P are not necessarily real, our solution might not be real-valued. Therefore, we must conform that our r and ρ are real-valued. Putting $r_I = \Im r$ and $\rho_I = \Im \rho$, we want to show $r_I = \rho_I \equiv 0$. It is easy to see that r_I satisfies

$$v\frac{d}{dv}r_I + (n-2)r_I + \gamma(n-2)\rho_I = \sum_{k=1}^{5} \Im F_k(r,\rho).$$

Multiplying both sides by $2r_I$, we have

$$\frac{d}{dv}\left[v\left\{r_I^2 + \gamma(n-2)\rho_I^2\right\}\right] + (2n-5)r_I^2 + \gamma(n-2)\rho_I^2 = 2r_I\sum_{k=1}^{5} \Im \hat{F}_k(\hat{r},\hat{\rho}).$$

Lemma 4.5. *Assume that V is sufficiently small. There exists a positive constant C depending on M and V such that*

$$|\Im \hat{F}_k(\hat{r},\hat{\rho})| \leqq Cv\left(|r_I| + |\rho_I|\right)$$

We shall give the proof of this lemma later. Using the lemma, $n-2>0$, $2n-5>0$, and $\gamma>0$, we have

$$v\left\{r_I^2 + \gamma(n-2)\rho_I^2\right\} \leqq C\int_0^v \eta\left\{r_I^2 + \gamma(n-2)\rho_I^2\right\} d\eta.$$

From Gronwall's lemma it follows that

$$r_I \equiv 0, \quad \rho_I \equiv 0.$$

Proof of Lemme 4.5 for $\Im\hat{F}_1, \cdots, \Im\hat{F}_4$. Put $\Re r = r_R$, $\Re\rho = \rho_R$. Since r_R, r_I, ρ_R, $\rho_I \in X_{V,M}$, these modulus are dominated by Cv. Therefore, we have

$$\begin{aligned}
|\Im \hat{F}_1(\hat{r},\hat{\rho})| = |\Im F_1(r)| &\leqq C\left\{\left|\Im\left(r^3\right)\right| + \left|\Im\left(r^2\right)\right|\right\}\\
&\leqq C\left(\left|r_R^2 r_I\right| + \left|r_I^3\right| + |r_R r_I|\right)\\
&\leqq C\left(v^2+v\right)|r_I|\\
&\leqq Cv|r_I|,
\end{aligned}$$

$$\begin{aligned}
|\Im \hat{F}_2(\hat{r},\hat{\rho})| = |\Im F_2(r,\rho)| &\leqq C\left\{\left|\Im\left(r^2\rho\right)\right| + |\Im(r\rho)|\right\}\\
&\leqq C\left(r^2|r_I| + |r_R\rho_R||r_I| + |\rho_R||r_I| + |r_R||\rho_I|\right)\\
&\leqq C\left(v^2+v\right)\left(|r_I| + |\rho_I|\right)\\
&\leqq Cv\left(|r_I| + |\rho_I|\right).
\end{aligned}$$

$\Im\hat{F}_3$ is estimated as follows:

$$\begin{aligned}
&|\Im \hat{F}_3(\hat{r},\hat{\rho})| = |\Im F_3(r,\rho)| \\
&\quad \leqq \left|\Im\left[r\rho\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \\
&\qquad \times\left|\Re\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right| \\
&\quad + \left|\Re\left[r\rho\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \\
&\qquad \times\left|\Im\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right|.
\end{aligned}$$

We can estimate each term as follows:

$$\begin{aligned}
&\left|\Im\left[r\rho\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \\
&\quad \leqq C\left(\left|\Im\left(r^3\rho\right)\right|+\left|\Im\left(r^2\rho\right)\right|+|\Im(r\rho)|\right) \\
&\quad \leqq Cv\left(|r_{\scriptscriptstyle I}|+|\rho_{\scriptscriptstyle I}|\right),
\end{aligned}$$

$$\left|\Re\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right| \leqq C,$$

$$\left|\Im\left[r\rho\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \leqq Cv^2 \leqq Cv,$$

$$\begin{aligned}
&\left|\Im\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right| \\
&\quad \leqq C\sum_{j\in J}\frac{\left|\Im\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}\right|}{\left|\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}\right|^2} \\
&\quad \leqq C|\rho_{\scriptscriptstyle I}|.
\end{aligned}$$

Similarly it holds for $\Im\hat{F}_4$ that

$$\begin{aligned}
&|\Im \hat{F}_4(\hat{r},\hat{\rho})| = |\Im F_4(r,\rho)| \\
&\quad \leqq \left|\Im\left[\rho^2\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \\
&\qquad \times\left|\Re\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\sin^2(\phi_j-\phi_i)\cos(\theta_i-\phi_j)}{\sin^2(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right| \\
&\quad + \left|\Re\left[\rho^2\left\{r^2+2r\cot(\theta_i-\phi_i)+\operatorname{cosec}^2(\theta_i-\phi_i)\right\}\right]\right| \\
&\qquad \times\left|\Im\left[\sum_{j\in J}\frac{n_j\sin^2(\theta_i-\phi_i)\sin^2(\phi_j-\phi_i)\cos(\theta_i-\phi_j)}{\sin^2(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}\right]\right| \\
&\quad \leqq C\left(\left|\Im\left(\rho^2r^2\right)\right|+\left|\Im\left(r\rho^2\right)\right|+\left|\Im\left(\rho^2\right)\right|\right)+Cv^2|\rho_{\scriptscriptstyle I}| \\
&\quad \leqq Cv\left(|r_{\scriptscriptstyle I}|+|\rho_{\scriptscriptstyle I}|\right).
\end{aligned}$$

□

We need the following lemma for the estimate of $\Im \hat{F}_5$.

Lemma 4.6. *Let B_R be the closed disc in $\mathbb{C}$ with center O and with radius R. Assume that the function f is analytic on B_R, and that $f|_{B_R \cap \mathbb{R}}$ is real-valued. Then there exists $C > 0$ such that*

$$|\Im f(r)| \leqq C|r_{_I}|$$

holds for $r \in B_{R/2}$.

Proof. We have

$$f(r) = \sum_{k=0}^{\infty} c_k r^k \quad \text{for} \quad r \in B_R, \quad \text{and} \quad \sum_{k=0} |c_k| R^k < \infty.$$

Since $f|_{B_R \cap \mathbb{R}}$ is real-valued, so are c_k's. Therefore, we get

$$\Im f(r) = \sum_{k=1}^{\infty} c_k \Im \left(r^k\right).$$

Furthermore we have

$$\begin{aligned} \left|\Im \left(r^k\right)\right| &= \left|\Im \left(r_{_R} + \sqrt{-1} r_{_I}\right)^k\right| \\ &\leqq \sum_{\ell=1}^{k} {}_kC_\ell \left|r_{_R}^{k-\ell} r_{_I}^{\ell}\right| \\ &\leqq \sum_{\ell=0}^{k} {}_kC_\ell |r|^{k-1} |r_{_I}| \\ &= \frac{(2|r|)^k}{R} |r_{_I}|. \end{aligned}$$

Consequently,

$$\left|\sum_{k=1}^{\infty} c_k \Im \left(r^k\right)\right| \leqq \frac{|r_{_I}|}{R} \sum_{k=1}^{\infty} c_k (2|r|)^k,$$

and the right-hand side converges for $r \in B_{R/2}$. □

Proof of Lemme 4.5 for $\Im \hat{F}_5$.

The function $f(r) = \left[\{\cot(\theta_i - \phi_i) + r\}^2 + 1\right]^{\frac{3}{2}}$ satisfies the assumption of Lemma 4.6. Taking V small, we may assume $|r_{_I}(v)| \leqq \dfrac{R}{2}$. Hence, we have

$$\left|\Im \left[\{\cot(\theta_i - \phi_i) + r(v)\}^2 + 1\right]^{\frac{3}{2}}\right| \leqq C|r_{_I}|.$$

Consequently, it holds that

$$|\Im\hat{F}_5(\hat{r},\hat{\rho})| = |\Im F_5(r)| \leqq Cv \left| \Im \left[\{\cot(\theta_i-\phi_i) + r(v)\}^2 + 1 \right]^{\frac{3}{2}} \right| \leqq Cv|r_{_I}|.$$

□

Now we complete the proof of existence of a real-valued local solution to (4.3), which implies the following proposition.

Proposition 4.2. *Let H be continuous. Then there exists a unique local solution $\boldsymbol{x}$ to* (1.2) *and* (1.7).

5. Appendix

5.1 The values of θ_i

Fact 5.1. *If $A(\theta_i) = 0$, $\phi_i < \theta_i < \phi_{i+1}$, then*

Type II $\theta_0 = \arctan\sqrt{\frac{n_0}{n_1}}$,

Type III $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$,

Type IV $\theta_{\pm 1} = -\frac{1}{2}\arctan\sqrt{\frac{k}{\ell}}$, $\theta_0 = \frac{1}{2}\arctan\sqrt{\frac{k}{\ell}}$,

Type V $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$.

Proof. Type II: Since $J = \{0, 1\}$, $\phi_j = \frac{j}{2}\pi$, we have

$$0 = \sum_{j\in J} n_j \cot(\theta_0 - \phi_j) = n_0 \cot\theta_0 + n_1 \cot\left(\theta_0 - \frac{\pi}{2}\right) = n_0 \cot\theta_0 - n_1 \tan\theta_0.$$

Combining this with $\theta_0 \in (0, \frac{\pi}{2})$, we get the assertion.

We prepare the following for Types III–V. When $\pm j \in J$, we have $\phi_{-j} = -\phi_j$, $n_{-j} = n_j$. Therefore, it holds that

$$\cot(\theta_i - \phi_{-j}) + \cot(\theta_i - \phi_j) = \frac{\sin 2\theta_i}{\sin^2\theta_i \cos^2\phi_j - \cos^2\theta_i \sin^2\phi_j}.$$

Furthermore $-\max J \notin J$, $n_0 = n_{\max J}$, $\phi_0 = 0$ and $\phi_{\max J} = \frac{\pi}{2}$ for Types IV and V. For these cases

$$\cot(\theta_i - \phi_0) + \cot(\theta_i - \phi_{\max J}) = 2\cot 2\theta_i.$$

Using these relation we have

$$\begin{aligned} 0 &= \sum_{j\in J} n_j \cot(\theta_i - \phi_j) = n_0 \sum_{j=-1}^{1} \cot\left(\theta_i - \frac{j}{3}\pi\right) \\ &= n_0 \left[\left\{ \cot\left(\theta_i + \frac{\pi}{3}\right) + \cot\left(\theta_i - \frac{\pi}{3}\right) \right\} + \cot\theta_i \right] \\ &= \frac{3n_0 \cos\theta_i\,(2\sin\theta_i - 1)\,(2\sin\theta_i + 1)}{\sin\theta_i\,\left(\sin^2\theta_i - 3\cos^2\theta_i\right)} \end{aligned}$$

for Type III;

$$\begin{aligned} 0 =& \sum_{j\in J} n_j \cot(\theta_i - \phi_j) \\ =& \ \ell \left\{ \cot\left(\theta_i + \frac{\pi}{4}\right) + \cot\left(\theta_i - \frac{\pi}{4}\right)\right\} + k\left(\cot\theta_i - \tan\theta_i\right) \\ =& -2\ell\tan 2\theta_i + 2k\cot 2\theta_i \end{aligned}$$

for Type IV;

$$\begin{aligned} 0 =& \sum_{j\in J} n_j \cot(\theta_i - \phi_j) = n_0 \sum_{j=-2}^{3} \cot\left(\theta_i - \frac{j}{6}\pi\right) \\ =& \ n_0 \left[\left\{ \cot\left(\theta_i + \frac{\pi}{3}\right) + \cot\left(\theta_i - \frac{\pi}{3}\right)\right\} + \left\{ \cot\left(\theta_i + \frac{\pi}{6}\right) + \cot\left(\theta_i - \frac{\pi}{6}\right)\right\} \right. \\ & \left. + \left\{ \cot\theta_i + \cot\left(\theta_i - \frac{\pi}{2}\right)\right\}\right] \\ =& \ \frac{6n_0 \cos 2\theta_i \left(1 - 2\sin 2\theta_i\right)\left(1 + 2\sin 2\theta_i\right)}{\sin 2\theta_i \left(\sin^2\theta_i - 3\cos^2\theta_i\right)\left(3\sin^2\theta_i - \cos^2\theta_i\right)} \end{aligned}$$

for Type V. Taking $\theta_i \in (\phi_i, \phi_{i+1})$ into consideration, we have the assertion. □

Remark 5.1. *The result $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$ for Types III and V is by virtue of symmetry $n_j \equiv n_0$.*

5.2 The derivation of (4.3)

We insert $r = q - \cot(\theta_i - \phi_i)$ into (3.4) with $u(0) = 0$. It is trivial that

$$\frac{dq}{dv} = \frac{dr}{dv},$$

$$(n-1)\left(q^2+1\right)^{\frac{3}{2}} \tilde{H} = (n-1)\left[\left\{\cot(\theta_i - \phi_i) + r\right\}^2 + 1\right]^{\frac{3}{2}} \tilde{H} = \frac{F_5(r)}{v}.$$

Summation of remainder terms of (3.4) are

$$\begin{aligned} & -\frac{n_i\left(q^2+1\right)q}{v} + \sum_{j\neq i} \frac{n_j\left(q^2+1\right)\left\{q\cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i)\right\}}{\sin(\phi_j - \phi_i)\displaystyle\int_0^v q(\eta)\,d\eta - v\cos(\phi_j - \phi_i)} \\ &= \left(q^2+1\right)\sum_{j\in J} \frac{n_j\left\{q\cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i)\right\}}{\sin(\phi_j - \phi_i)\displaystyle\int_0^v q(\eta)\,d\eta - v\cos(\phi_j - \phi_i)} \\ &= \left[\left\{\cot(\theta_i - \phi_i) + r\right\}^2 + 1\right] \\ &\quad \times \sum_{j\in J} \frac{n_j\left[\left\{\cot(\theta_i - \phi_i) + r\right\}\cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i)\right]}{\sin(\phi_j - \phi_i)\left\{v\cot(\theta_i - \phi_i) + \displaystyle\int_0^v r(\eta)\,d\eta\right\} - v\cos(\phi_j - \phi_i)} \end{aligned}$$

$$= -\left\{\cot^2(\theta_i - \phi_i) + 2r\cot(\theta_i - \phi_i) + r^2 + 1\right\} \times \sum_{j\in J} \frac{n_j\{\cos(\theta_i - \phi_j) + r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}}{v\{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}}.$$

Using $\sum_{j\in J} n_j \cot(\theta_i - \phi_j) = 0$, we have

$$\begin{aligned}
&-\left\{\cot^2(\theta_i - \phi_i) + 2r\cot(\theta_i - \phi_i) + r^2 + 1\right\} \\
&\qquad \times \sum_{j\in J} \frac{n_j\{\cos(\theta_i - \phi_j) + r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}}{v\{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}} \\
&= -\left\{r^2 + 2r\cot(\theta_i - \phi_i) + \operatorname{cosec}^2(\theta_i - \phi_i)\right\} \\
&\qquad \times \sum_{j\in J} \frac{n_j}{v}\left\{\frac{\cos(\theta_i - \phi_j) + r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_i)}{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)} - \frac{\cos(\theta_i - \phi_j)}{\sin(\theta_i - \phi_j)}\right\} \\
&= -\left\{r^2 + 2r\cot(\theta_i - \phi_i) + \operatorname{cosec}^2(\theta_i - \phi_i)\right\} \\
&\qquad \times \sum_{j\in J} \frac{n_j\sin(\theta_i - \phi_i)\{r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_j) + \rho\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)\}}{v\sin(\theta_i - \phi_j)\{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}}.
\end{aligned}$$

Extracting the linear parts with respect to r and ρ, we get

$$\begin{aligned}
&-\left\{r^2 + 2r\cot(\theta_i - \phi_i) + \operatorname{cosec}^2(\theta_i - \phi_i)\right\} \\
&\qquad \times \sum_{j\in J} \frac{n_j\sin(\theta_i - \phi_i)\{r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_j) + \rho\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)\}}{v\sin(\theta_i - \phi_j)\{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)\}} \\
&= -\left\{r^2 + 2r\cot(\theta_i - \phi_i) + \operatorname{cosec}^2(\theta_i - \phi_i)\right\} \\
&\qquad \times \sum_{j\in J} \frac{n_j\sin(\theta_i - \phi_i)}{v}\left[\frac{r\cos(\phi_j - \phi_i)}{\sin(\theta_i - \phi_j)} + \frac{\rho\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)}{\sin^2(\theta_i - \phi_j)}\right. \\
&\qquad\quad + \frac{\{r\cos(\phi_j - \phi_i)\sin(\theta_i - \phi_j) + \rho\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)\}}{\sin(\theta_i - \phi_j)} \\
&\qquad\qquad \left.\times\left\{\frac{1}{\sin(\theta_i - \phi_j) - \rho\sin(\phi_j - \phi_i)\sin(\theta_i - \phi_i)} - \frac{1}{\sin(\theta_i - \phi_j)}\right\}\right] \\
&= -r\sum_{j\in J} \frac{n_j\cos(\phi_j - \phi_i)}{v\sin(\theta_i - \phi_i)\sin(\theta_i - \phi_j)} \\
&\qquad - r^2\{r + 2\cot(\theta_i - \phi_i)\}\sum_{j\in J} \frac{n_j\sin(\theta_i - \phi_i)\cos(\phi_j - \phi_i)}{v\sin(\theta_i - \phi_j)} \\
&\qquad - \sum_{j\in J} \frac{n_j\rho\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)}{v\sin(\theta_i - \phi_i)\sin^2(\theta_i - \phi_j)} \\
&\qquad - r\{r + 2\cot(\theta_i - \phi_i)\}\sum_{j\in J} \frac{n_j\rho\sin(\theta_i - \phi_i)\sin(\phi_j - \phi_i)\cos(\theta_i - \phi_j)}{v\sin^2(\theta_i - \phi_j)}
\end{aligned}$$

$$
\begin{aligned}
&- \left\{r^2 + 2r\cot(\theta_i-\phi_i) + \operatorname{cosec}^2(\theta_i-\phi_i)\right\} \\
&\quad\times\sum_{j\in J} \frac{n_j r\rho\sin^2(\theta_i-\phi_i)\cos(\phi_j-\phi_i)\sin(\phi_j-\phi_i)}{v\sin(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}} \\
&- \left\{r^2 + 2r\cot(\theta_i-\phi_i) + \operatorname{cosec}^2(\theta_i-\phi_i)\right\} \\
&\quad\times\sum_{j\in J} \frac{n_j \rho^2\sin^2(\theta_i-\phi_i)\sin^2(\phi_j-\phi_i)\cos(\theta_i-\phi_j)}{v\sin^2(\theta_i-\phi_j)\left\{\sin(\theta_i-\phi_j)-\rho\sin(\phi_j-\phi_i)\sin(\theta_i-\phi_i)\right\}}.
\end{aligned}
$$

We use $\sum_{j\in J} n_j\cot(\theta_i-\phi_j) = 0$ again, and then the coefficients of linear terms are simplified as follows:

$$
\begin{aligned}
\sum_{j\in J}\frac{n_j\cos(\phi_j-\phi_i)}{\sin(\theta_i-\phi_i)\sin(\theta_i-\phi_j)} &= \sum_{j\in J}\frac{n_j\cos\left\{(\phi_j-\theta_i)+(\theta_i-\phi_i)\right\}}{\sin(\theta_i-\phi_i)\sin(\theta_i-\phi_j)} \\
&= \sum_{j\in J}\frac{n_j\left\{\cos(\phi_j-\theta_i)\cos(\theta_i-\phi_i) - \sin(\phi_j-\theta_i)\sin(\theta_i-\phi_i)\right\}}{\sin(\theta_i-\phi_i)\sin(\theta_i-\phi_j)} \\
&= \cot(\theta_i-\phi_i)\sum_{j\in J} n_j\cot(\theta_i-\phi_j) + \sum_{j\in J} n_j = \sum_{j\in J} n_j = n-2,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j\in J}\frac{n_j\sin(\phi_j-\phi_i)\cos(\theta_i-\phi_j)}{\sin(\theta_i-\phi_i)\sin^2(\theta_i-\phi_j)} \\
&= \sum_{j\in J}\frac{n_j\sin\left\{(\phi_j-\theta_i)+(\theta_i-\phi_i)\right\}\cos(\theta_i-\phi_j)}{\sin(\theta_i-\phi_i)\sin^2(\theta_i-\phi_j)} \\
&= \sum_{j\in J}\frac{n_j\left\{\sin(\phi_j-\theta_i)\cos(\theta_i-\phi_i) + \cos(\phi_j-\theta_i)\sin(\theta_i-\phi_i)\right\}\cos(\theta_i-\phi_j)}{\sin(\theta_i-\phi_i)\sin^2(\theta_i-\phi_j)} \\
&= -\cot(\theta_i-\phi_i)\sum_{j\in J} n_j\cot(\theta_i-\phi_j) + \sum_{j\in J} n_j\cot^2(\theta_i-\phi_j) \\
&= \sum_{j\in J} n_j\cot^2(\theta_i-\phi_j).
\end{aligned}
$$

Consequently, we get (4.3) if

$$
\sum_{j\in J} n_j\cot^2(\theta_i-\phi_j) = \gamma(n-2),
$$

which we shall prove in the remainder. To do this, we use Fact 5.1.

Type II: Since $J=\{0,1\}$, $\gamma = \sharp J - 1 = 1$, $\phi_j = \frac{j}{2}\pi$, and $\tan^2\theta_0 = \frac{n_0}{n_1}$, we get

$$
\sum_{j\in J} n_j\cot^2(\theta_0-\phi_j) = n_0\cot^2\theta_0 + n_1\cot^2\left(\theta_0 - \frac{\pi}{2}\right)
$$

$$
\begin{aligned}
&= n_0 \cot^2 \theta_0 + n_1 \tan^2 \theta_0 \\
&= n_0 \cdot \frac{n_1}{n_0} + n_1 \cdot \frac{n_0}{n_1} = n_1 + n_0 \\
&= \sum_{j \in J} n_j = \gamma(n-2).
\end{aligned}
$$

As in the proof of Fact 5.1, we prepare the following for Types III–V. When $\pm j \in J$, we have $\phi_{-j} = -\phi_j$, $n_{-j} = n_j$. Therefore, it holds that

$$
\cot^2(\theta_i - \phi_{-j}) + \cot^2(\theta_i - \phi_j) = \frac{2\left(\sin^2\theta_i \cos^2\theta_i + \sin^2\phi_j \cos^2\phi_j\right)}{\left(\sin^2\theta_i \cos^2\phi_j - \cos^2\theta_i \sin^2\phi_j\right)^2}.
$$

Furthermore $-\max J \notin J$, $n_0 = n_{\max J}$, $\phi_0 = 0$ and $\phi_{\max J} = \frac{\pi}{2}$ for Types IV and V. For these cases

$$
\cot^2(\theta_i - \phi_0) + \cot^2(\theta_i - \phi_{\max J}) = \frac{2(2 - \sin^2 2\theta_i)}{\sin^2 2\theta_i}
$$

Type III: Since $J = \{-1, 0, 1\}$, $\gamma = \sharp J - 1 = 2$, $\phi_j = \frac{j}{3}\pi$, $n_j \equiv n_0$, and $\sin^2\theta_i = \frac{1}{4}$, $\cos^2\theta_i = \frac{3}{4}$ for all $i \in J$, we have

$$
\begin{aligned}
&\sum_{j \in J} n_j \cot^2(\theta_i - \phi_j) = n_0 \left\{\cot^2(\theta_i - \phi_{-1} + \cot^2(\theta_i - \phi_1) + \cot^2\theta_i\right\} \\
&\quad = n_0 \left\{ \frac{2\left(\sin^2\theta_i \cos^2\theta_i + \sin^2\phi_1 \cos^2\phi_1\right)}{\left(\sin^2\theta_i \cos^2\phi_1 - \cos^2\theta_i \sin^2\phi_1\right)^2} + \frac{\cos^2\theta_i}{\sin^2\theta_i} \right\} \\
&\quad = n_0 \left\{ \frac{2\left(\frac{1}{4}\cdot\frac{3}{4} + \frac{3}{4}\cdot\frac{1}{4}\right)}{\left(\frac{1}{4}\cdot\frac{1}{4} - \frac{3}{4}\cdot\frac{3}{4}\right)^2} + \frac{\frac{3}{4}}{\frac{1}{4}} \right\} \\
&\quad = n_0 \left(\frac{\frac{3}{4}}{\frac{1}{4}} + 3 \right) = 6n_0 = 2\sum_{j \in J} n_j = \gamma(n-2).
\end{aligned}
$$

Type IV: Since $J = \{-1, 0, 1, 2\}$, $\gamma = \sharp J - 1 = 3$, $\phi_j = \frac{j}{4}\pi$, $n_{-1} = n_1 = \ell$, $n_0 = n_2 = k$, and

$$
\sin^2 2\theta_i = \frac{k}{k+\ell}, \quad \cos^2 2\theta_i = \frac{\ell}{k+\ell}
$$

for all $i \in J$, we have

$$
\begin{aligned}
&\sum_{j \in J} n_j \cot^2(\theta_i - \phi_j) \\
&\quad = \ell \left\{ \cot^2\left(\theta_i + \frac{\pi}{4}\right) + \cot^2\left(\theta_i - \frac{\pi}{4}\right) \right\} + k \left\{ \cot^2\theta_i + \cot^2\left(\theta_i - \frac{\pi}{2}\right) \right\} \\
&\quad = \frac{2\ell\left(\sin^2\theta_i \cos^2\theta_i + \sin^2\frac{\pi}{4}\cos^2\frac{\pi}{4}\right)}{\left(\sin^2\theta_i \cos^2\frac{\pi}{4} - \cos^2\theta_i \sin^2\frac{\pi}{4}\right)^2} + \frac{2k\left(2 - \sin^2 2\theta_i\right)}{\sin^2 2\theta_i}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\ell\left(\sin^2\theta_i\cos^2\theta_i+\frac14\right)}{\frac14\left(\sin^2\theta_i-\cos^2\theta_i\right)^2}+\frac{2k\left(1+\cos^2 2\theta_i\right)}{\sin^2 2\theta_i}\\
&= \frac{2\ell\left(1+\sin^2 2\theta_i\right)}{\cos^2 2\theta_i}+\frac{2k\left(1+\cos^2 2\theta_i\right)}{\sin^2 2\theta_i}\\
&= 2(k+\ell+k)+2(k+\ell+\ell)=6(k+\ell)\\
&= 3\sum_{j\in J} n_j=\gamma(n-2).
\end{aligned}
$$

Type V: Since $J=\{-2,-1,0,1,2,3\}$, and $n_j\equiv n_0$, we have

$$
\begin{aligned}
&\sum_{j\in J} n_j\cot^2(\theta_i-\phi_j)\\
&\quad= n_0\left[\sum_{j=1}^{2}\left\{\cot^2(\theta_i-\phi_{-j})+\cot^2(\theta_i-\phi_j)\right\}+\cot^2\theta_i+\cot^2(\theta_i-\phi_3)\right]\\
&\quad= n_0\left\{\sum_{j=1}^{2}\frac{2\left(\sin^2\theta_i\cos^2\theta_i+\sin^2\phi_j\cos^2\phi_j\right)}{\left(\sin^2\theta_i\cos^2\phi_j-\cos^2\theta_i\sin^2\phi_j\right)^2}+\frac{2\left(2-\sin^2 2\theta_i\right)}{\sin^2 2\theta_i}\right\}.
\end{aligned}
$$

When $i=-2$, 1, it holds that

$$\sin^2\theta_i=\cos^2\theta_i=\frac12,\quad \sin^2 2\theta_i=1.$$

Using $\gamma=\sharp J-1=5$ and $\phi_j=\frac{j}{6}\pi$, we get

$$
\begin{aligned}
\sum_{j\in J} n_j\cot^2(\theta_i-\phi_i) &= n_0\left\{\sum_{j=1}^{2}\frac{2\left(\frac14+\sin^2\phi_j\cos^2\phi_j\right)}{\frac14\left(\cos^2\phi_j-\sin^2\phi_j\right)^2}+2(2-1)\right\}\\
&= n_0\left\{\sum_{j=1}^{2}\frac{2\left(2+\sin^2 2\phi_j\right)}{\cos^2 2\phi_j}+2\right\}\\
&= n_0\left\{\frac{2\left(1+\sin^2\frac{\pi}{3}\right)}{\cos^2\frac{\pi}{3}}+\frac{2\left(1+\sin^2\frac23\pi\right)}{\cos^2\frac23\pi}+2\right\}\\
&= n_0\left\{\frac{2\left(1+\frac34\right)}{\frac14}+\frac{2\left(1+\frac34\right)}{\frac14}+2\right\}\\
&= 30n_0=5\sum_{j\in J} n_j=\gamma(n-2).
\end{aligned}
$$

When $i=-1$, 0, 2, it holds that

$$\sin^2 2\theta_i=\frac14,\quad \sin^2\theta_i=\frac{2-\sqrt3}{4},\quad \cos^2\theta_i=\frac{2+\sqrt3}{4}.$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{j\in J} n_j \cot^2(\theta_i - \phi_j) \\
&= n_0 \left[\sum_{j=1}^{2} \frac{2\left(\frac{2-\sqrt{3}}{4}\cdot\frac{2+\sqrt{3}}{4} + \sin^2\phi_j\cos^2\phi_j\right)}{\frac{1}{16}\left\{\left(2-\sqrt{3}\right)\cos^2\phi_j - \left(2+\sqrt{3}\right)\sin^2\phi_j\right\}^2} + \frac{2\left(2-\frac{1}{4}\right)}{\frac{1}{4}} \right] \\
&= n_0 \left\{ \sum_{j=1}^{2} \frac{2\left(1+4\sin^2 2\phi_j\right)}{\left(2\cos 2\phi_j - \sqrt{3}\right)^2} + \frac{2\cdot\frac{7}{4}}{\frac{1}{4}} \right\} \\
&= n_0 \left\{ \frac{2\left(1+4\sin^2\frac{\pi}{3}\right)}{\left(2\cos\frac{\pi}{3} - \sqrt{3}\right)^2} + \frac{2\left(1+4\sin^2\frac{2}{3}\pi\right)}{\left(2\cos\frac{2}{3}\pi - \sqrt{3}\right)^2} + 14 \right\} \\
&= n_0 \left\{ \frac{2(1+3)}{\left(1-\sqrt{3}\right)^2} + \frac{2(1+3)}{\left(-1-\sqrt{3}\right)^2} + 14 \right\} \\
&= n_0 \left(\frac{4}{2-\sqrt{3}} + \frac{4}{2+\sqrt{3}} + 14 \right) \\
&= n_0 \left[4\left\{ \left(2+\sqrt{3}\right) + \left(2-\sqrt{3}\right) \right\} + 14 \right] \\
&= 30 n_0 = 5 \sum_{j\in J} n_j = \gamma(n-2).
\end{aligned}$$

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Takeyuki Nagasawa
Department of Mathematics
Graduate School of Science and Engineering
Saitama University
255 Shimo-Okubo
Sakura-ku, Saitama 338-8570, Japan
e-mail: tnagasaw@rimath.saitama-u.ac.jp