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A note on modular representations of elementary abelian p-groups

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Abstract

Let k be an algebraically closed field of positive characteristic p, let $\mathbb{G}_a$ denote the additive group of k, and let $\mathbb{Z}/p\mathbb{Z}$ denote the cyclic group of order p. Given a modular representation $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$, we ask whether or not ρ can be extended, by an arbitrary group embedding $\iota : (\mathbb{Z}/p\mathbb{Z})^r \to \mathbb{G}_a$, to a representation $\varphi : \mathbb{G}_a \to GL(n, k)$, i.e., $\rho = \varphi \circ \iota$. We consider some classes of modular representations of elementary abelian p-groups, and give some partial positive answers to the above problem. Besides, we classify up to equivalence four-dimensional modular representations $\rho : (\mathbb{Z}/2\mathbb{Z})^r \to GL(4, k)$ in characteristic two.

0. Introduction

Let k be an algebraically closed field of positive characteristic p and let $\mathbb{G}_a$ denote the additive group of k. A map $\varphi : \mathbb{G}_a \to GL(n, k)$ is said to be a *representation* of $\mathbb{G}_a$ if φ is a homomorphism of algebraic groups over k. An *elementary abelian p-group of rank r* is a finite abelian group which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$, where $\mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p.

In this article, we consider the following problem:

Given a modular representation $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$, we ask whether or not ρ can be extended, by an arbitrary injective group homomorphism $\iota : (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \mathbb{G}_a$, to a representation $\varphi : \mathbb{G}_a \to GL(n, k)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathbb{G}_a & \xrightarrow{\ \varphi\ } & GL(n,k) \\ {\scriptstyle\iota}\uparrow & \nearrow_{\rho} & \\ (\mathbb{Z}/p\mathbb{Z})^r & & \end{array}$$

We remark that there exists a one-to-one correspondence between the set of all injective group homomorphisms $\iota : (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \mathbb{G}_a$ and the set of all elements

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$(\alpha_1, \ldots, \alpha_r) \in k^r$ such that $\alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{F}_p$.

Let $\mathbb{A}^n_k$ denote the affine space in dimension n over k and let $E_r := (\mathbb{Z}/p\mathbb{Z})^r$. If the above problem is affirmative, any linear action of E_r on $\mathbb{A}^n_k$ can be extended to a linear action of $\mathbb{G}_a$ on $\mathbb{A}^n_k$, and then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{A}^n_k & \xleftarrow{\mathrm{id}_{\mathbb{A}^n_k}} & \mathbb{A}^n_k \\ \downarrow & & \downarrow \\ \mathbb{A}^n_k/\mathbb{G}_a & \longleftarrow & \mathbb{A}^n_k/E_r \end{array}$$

However, we still do not know whether the quotient $\mathbb{A}^n_k/\mathbb{G}_a$ is an affine algebraic variety over k. We are in progress for solving this quotient problem (see [3, 4, 5, 6]). In this article, we consider the extension problem in order to study modular representations of elementary abelian p-groups through $\mathbb{A}^1_k$-fibrations on the affine space $\mathbb{A}^n_k$.

In the following, we state our theorems and corollaries in this article:

We say that matrices $X_1, \ldots, X_r$ of $\mathrm{Mat}(n, k)$ are *p-pyramidic* if $X_1, \ldots, X_r$ satisfy

$$\prod_{i=1}^{r} X_i^{l_i} = O_n \qquad \text{for all } l_1, \ldots, l_r \geq 0 \text{ with } l_1 + \cdots + l_r \geq p.$$

For $1 \leq i \leq r$, an element e_i of $(\mathbb{Z}/p\mathbb{Z})^r$ is defined as the i-th component of e_i is 1 and the other components of e_i are zeros.

A modular representation $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$ is said to be *p-pyramidic* if r matrices $\rho(e_1) - I_n, \ldots, \rho(e_r) - I_n$ are p-pyramidic.

The following theorem gives a partial positive answer to the extension problem.

Theorem 1 *Let $r \geq 1$ and let $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$ be a modular representation. Assume that one of the following conditions* (1), (2) *and* (3) *holds true:*

(1) $r = 1$.

(2) ρ *is p-pyramidic.*

(3) $1 \leq n \leq p$.

Then, for any injective group homomorphism $\iota : (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \mathbb{G}_a$, there exists a representation $\varphi : \mathbb{G}_a \to GL(n, k)$ satisfying $\rho = \varphi \circ \iota$.

Let $1 \leq j \leq r$. We say that matrices $X_1, \ldots, X_r$ of $\mathrm{Mat}(n, k)$ are of *j-mutually annihilating* if $X_1, \ldots, X_r$ satisfy $X_{i_1} \cdots X_{i_j} = O_n$ for all distinct j

integers $i_1, \dots, i_j$ within $1 \le i_1, \dots, i_j \le r$.

Let $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n,k)$ be a modular representation. For $2 \le j \le r$, we say that ρ is of *j-mutually annihilating* if the matrices $\rho(e_1) - I_n, \dots, \rho(e_r) - I_n$ are of j-mutually annihilating.

In particular when $p = 2$, we have the following partial positive answer to the problem.

Theorem 2 *Let $p = 2$ and let $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n,k)$ be a modular representation. Assume that one of the following conditions* (1) *and* (2) *holds true:*

(1) $2 \le r \le 3$.

(2) $r \ge 4$, *and ρ is of* 3*-mutually annihilating.*

Then, for any injective group homomorphism $\iota : (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \mathbb{G}_a$, there exists a representation $\varphi : \mathbb{G}_a \to GL(n,k)$ such that $\rho = \varphi \circ \iota$.

If the dimension n of a modular representation $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n,k)$ is in the range $1 \le n \le 4$, we have the following partial positive answer to the problem.

Corollary 3 *Let $r \ge 1$ and let $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n,k)$ be a modular representation. Assume that one of the following conditions* (1) *and* (2) *holds true:*

(1) $1 \le n \le 3$.

(2) $p = 2$ *and* $n = 4$.

Then, for any injective group homomorphism $\iota : (\mathbb{Z}/p\mathbb{Z})^r \hookrightarrow \mathbb{G}_a$, there exists a representation $\varphi : \mathbb{G}_a \to GL(n,k)$ such that $\rho = \varphi \circ \iota$.

We know the following concerning modular representations of elementary abelian p-groups: There are exactly p inequivalent indecomposable modular representations of $\mathbb{Z}/p\mathbb{Z}$. Bašev [1] classifies indecomposable modular representations of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ over an algebraically closed field of characteristic two. Campbell, Shank and Wehlau [2] give parametrizations of modular representations of elementary abelian p-groups whose representation spaces are in dimensions two and three.

In the following Corollary 4, with assuming $p = 2$, we describe, up to equivalence, four-dimensional modular representations of elementary abelian p-groups. We define subsets $\mathcal{A}_{2,2}$, $\mathcal{A}_{3,1}$, $\mathcal{H}_\mu$ $(\mu \in k)$ of $GL(4,k)$ as follows:

$$\mathcal{A}_{2,2} := \left\{ \begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \alpha,\, \beta,\, \gamma,\, \delta \in k \right\},$$

$$\mathcal{A}_{3,1} := \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ \alpha,\, \beta,\, \gamma \in k \right\},$$

$$\mathcal{H}_{\mu} := \left\{ \begin{pmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & \mu\beta \\ 0 & 0 & 1 & \mu\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ \alpha,\, \beta,\, \gamma \in k \right\}.$$

Let $(U_i)_{i=1}^r$ be a sequence taken from one of the subsets $\mathcal{A}_{2,2}$, $\mathcal{A}_{3,1}$, $\mathcal{H}_\mu$ ($\mu \in k$). Then we can define a modular representation $\sigma : (\mathbb{Z}/p\mathbb{Z})^r \to GL(4,k)$ as $\sigma(n_1, \ldots, n_r) := U_1^{n_1} \cdots U_r^{n_r}$.

Corollary 4 *Assume* $p = 2$. *Let* $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(4,k)$ *be a modular representation of* $(\mathbb{Z}/p\mathbb{Z})^r$. *Then there exists a modular representation* $\sigma : (\mathbb{Z}/p\mathbb{Z})^r \to GL(4,k)$ *satisfying the following conditions* (1) *and* (2)*:*

(1) σ *is equivalent to* ρ.

(2) *The set* $\{\sigma(e_i) \mid 1 \leq i \leq r\}$ *is included in one of the subsets* $\mathcal{A}_{2,2}$, $\mathcal{A}_{3,1}$, $\mathcal{H}_\mu$ ($\mu \in k$),

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Notations and definitions. For any field $\mathbb{F}$, we denote by $\mathbb{F}[x_1, \ldots, x_r]$ a polynomial ring in r variables over $\mathbb{F}$. Let $\mathbb{F}_p$ denote the finite field consisting of p elements.

For a commutative ring R with unity, we denote by $\mathrm{Mat}(n, R)$ the ring of all $n \times n$ matrices whose entries belong to R, and write O_n (and I_n) for the zero element (resp. unity). For any $A \in \mathrm{Mat}(n, R)$, we denote by $\det(A)$ the determinant of A. We denote by $GL(n, R)$ the group of all invertible matrices of $\mathrm{Mat}(n, R)$.

Let G be a group. Two representations $\rho_1 : G \to GL(n, R)$ and $\rho_2 : G \to GL(n, R)$ of G are *equivalent* if there exists a regular matrix $P \in GL(n, R)$ such that $P^{-1}\rho_1(g)P = \rho_2(g)$ for all $g \in G$.

Let $k[T]$ be a polynomial ring in one variable over k. We say that a polynomial $f(T) \in k[T]$ is a *p-polynomial* if $f(T)$ has the form $f(T) = \sum_{i=0}^{s} a_i T^{p^i}$ for some $a_0, \ldots, a_s \in k$.

1. A correspondence between $\mathfrak{N}(A)$ and $\mathfrak{U}(A)$

Let $\Bbbk$ be a field of positive characteristic p and let A be a not-necessarily commutative $\Bbbk$-algebra with unity. We denote by O the zero element of A under addition and denote by I the unity of A under multiplication. Let $\mathfrak{N}(A)$ be the set of all p-nilpotent elements of A, and let $\mathfrak{U}(A)$ be the set of all p-unipotent elements of A, i.e.,

$$\begin{cases} \mathfrak{N}(A) & := & \{N \in A \mid N^p = O\}, \\ \mathfrak{U}(A) & := & \{U \in A \mid U^p = I\}. \end{cases}$$

1.1 The truncated exponential of p-nilpotent elements

We can define a map $\mathrm{Exp} : \mathfrak{N}(A) \to \mathfrak{U}(A)$ as

$$\mathrm{Exp}(N) := \sum_{i=0}^{p-1} \frac{N^i}{i!}.$$

We know the following lemma:

Lemma 5 *Let N_1, N_2 be elements of $\mathfrak{N}(A)$ satisfying both conditions $N_1N_2 = N_2N_1$ and $N_1^i N_2^j = O$ for all $i, j \geq 0$ with $i + j \geq p$. Then we have* $\mathrm{Exp}(N_1 + N_2) = \mathrm{Exp}(N_1)\,\mathrm{Exp}(N_2)$.

1.2 The truncated logarithm of p-unipotent elements

We can define a map $\mathrm{Log} : \mathfrak{U}(A) \to \mathfrak{N}(A)$ as

$$\mathrm{Log}(U) := \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i}(U - I)^i.$$

Lemma 6 *The truncated logarithm* Log *is injective.*

Proof. Choose arbitrary $U_1, U_2 \in \mathfrak{U}(A)$ and assume that $\mathrm{Log}(U_1) = \mathrm{Log}(U_2)$. Let $N_1 := U_1 - I$ and $N_2 := U_2 - I$. We have

$$\sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} N_1^i = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} N_2^i.$$

Calculating the $(p - 1)$th power of both sides of the above equality, we have $N_1^{p-1} = N_2^{p-1}$, which implies

$$\sum_{i=1}^{p-2} \frac{(-1)^{i-1}}{i} N_1^i = \sum_{i=1}^{p-2} \frac{(-1)^{i-1}}{i} N_2^i.$$

Calculating $(p-2)$th power of both sides of the above equality, we have $N_1^{p-2} = N_2^{p-2}$, which implies

$$\sum_{i=1}^{p-3} \frac{(-1)^{i-1}}{i} N_1^i = \sum_{i=1}^{p-3} \frac{(-1)^{i-1}}{i} N_2^i.$$

We can repeat the above arguments in finitely many steps until we have $N_1 = N_2$.

Q.E.D.

1.3 A correspondence between $\mathfrak{N}(A)$ and $\mathfrak{U}(A)$

We shall use the following lemma on proving Lemma 8.

Lemma 7 *Let p be a prime number. Then the following assertions* (1) *and* (2) *hold true:*

(1) *For all $0 \leq j' \leq p-2$, we have $\sum_{\ell=j'}^{p-2} \binom{\ell}{j'} \equiv (-1)^{j'+1} \pmod{p}$.*

(2) *Assume $p \geq 3$. For all $1 \leq n \leq p-2$, we have $\sum_{j=1}^{p-1} j^n \equiv 0 \pmod{p}$.*

Proof. (1) In the polynomial ring $\mathbb{F}_p[x]$, compare the coefficients of $x^{j'}$ $(0 \leq j' \leq p-2)$ of the both sides of the equality

$$\sum_{\ell=0}^{p-2} (x+1)^\ell = \sum_{j=1}^{p-1} \binom{p-1}{j} x^{j-1}.$$

(2) Let $\mathbb{F}_p^*$ denote the set of all invertible elements of the field $\mathbb{F}_p$. Since $\mathbb{F}_p^*$ is a cyclic group of order $p-1$, there exists an element $\zeta \in \mathbb{F}_p^*$ such that $\mathbb{F}_p^* = \{\zeta^i \mid 1 \leq i \leq p-1\}$. Since $1 \leq n \leq p-2$, we have $\zeta^n \neq 1$, and thereby have

$$\sum_{j=1}^{p-1} j^n = \sum_{i=1}^{p-1} \zeta^{in} = \frac{\zeta^{pn} - \zeta^n}{\zeta^n - 1} = 0 \in \mathbb{F}_p.$$

Q.E.D.

The following lemma states that there exists a one-to-one correspondence between $\mathfrak{N}(A)$ and $\mathfrak{U}(A)$.

Lemma 8 *We have* $\mathrm{Log} \circ \mathrm{Exp} = \mathrm{id}_{\mathfrak{N}(A)}$ *and* $\mathrm{Exp} \circ \mathrm{Log} = \mathrm{id}_{\mathfrak{U}(A)}$.

Proof. We first prove $\mathrm{Log} \circ \mathrm{Exp} = \mathrm{id}_{\mathfrak{N}(A)}$. Choose an arbitrary element N of $\mathfrak{N}(A)$.

$$\begin{aligned}&(\mathrm{Log} \circ \mathrm{Exp})(N) \\ &= \sum_{\ell=1}^{p-1} \frac{(-1)^{\ell-1}}{\ell} (\mathrm{Exp}(N) - I)^\ell = \sum_{\ell=1}^{p-1} \sum_{j=0}^{\ell} \frac{(-1)^{j+1}}{\ell} \binom{\ell}{j} \mathrm{Exp}(jN)\end{aligned}$$

$$= \sum_{\ell=1}^{p-1} \frac{-1}{\ell} I + \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{\ell} \binom{\ell}{j} \operatorname{Exp}(jN).$$

Let

$$\alpha := \sum_{\ell=1}^{p-1} \frac{-1}{\ell} I = -\sum_{\ell=1}^{p-1} \ell^{p-2} I \qquad \text{and} \qquad \beta := \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{\ell} \binom{\ell}{j} \operatorname{Exp}(jN).$$

So, we have

$$(\operatorname{Log} \circ \operatorname{Exp})(N) = \alpha + \beta.$$

We can express α as

$$\alpha = \begin{cases} I & \text{if} \quad p = 2, \\ O & \text{if} \quad p \geq 3. \end{cases}$$

We can express β as

$$\begin{aligned}
\beta &= \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{\ell} \binom{\ell}{j} \operatorname{Exp}(jN) = \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \frac{(-1)^{j+1}}{j} \binom{\ell-1}{j-1} \operatorname{Exp}(jN) \\
&= \sum_{j=1}^{p-1} \sum_{\ell=j}^{p-1} \frac{(-1)^{j+1}}{j} \binom{\ell-1}{j-1} \operatorname{Exp}(jN) \overset{\text{(a)}}{=} \sum_{j=1}^{p-1} \frac{-1}{j} \operatorname{Exp}(jN) \\
&= \sum_{j=1}^{p-1} \sum_{m=0}^{p-1} (-j^{m-1}) \frac{N^m}{m!} = -\sum_{m=0}^{p-1} \left(\sum_{j=1}^{p-1} j^{m-1} \right) \frac{N^m}{m!} \\
&\overset{\text{(b)}}{=} \begin{cases} -\sum_{j=1}^{p-1} \frac{1}{j} I - (p-1) \frac{N}{1!} & \text{if} \quad p = 2, \\ -\sum_{j=1}^{p-1} \frac{1}{j} I - (p-1) \frac{N}{1!} - \sum_{m=2}^{p-1} \left(\sum_{j=1}^{p-1} j^{m-1} \right) \frac{N^m}{m!} & \text{if} \quad p \geq 3 \end{cases} \\
&= \alpha + N,
\end{aligned}$$

where we use assertions (1) (and (2)) of Lemma 7 for proving the above equalities (a) (resp. (b)). Thus we have $(\operatorname{Log} \circ \operatorname{Exp})(N) = N$.

We next prove $\operatorname{Exp} \circ \operatorname{Log} = \mathrm{id}_{\mathfrak{U}(A)}$. Choose an arbitrary element U of $\mathfrak{U}(A)$. Let $U' := (\operatorname{Exp} \circ \operatorname{Log})(U)$. Then we have $\operatorname{Log}(U') = \operatorname{Log}(U)$. Since Log is injective, we know that $U' = U$. Q.E.D.

2. A proof of Theorem 1

2.1 Lemmas

Let $\Bbbk$ be as above, i.e., $\Bbbk$ is a field of positive characteristic p. We define a polynomial matrix $F_r(x_1,\ldots,x_r)$ of $\mathrm{Mat}(r,\Bbbk[x_1,\ldots,x_r])$ as

$$F_r(x_1,\ldots,x_r) := \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_1^p & x_2^p & \cdots & x_r^p \\ \vdots & & \ddots & \vdots \\ x_1^{p^{r-1}} & x_2^{p^{r-1}} & \cdots & x_r^{p^{r-1}} \end{pmatrix}.$$

Let ζ be a generator of the cyclic group $\mathbb{F}_p^*$ of order $p-1$.

For any $\ell \geq 1$, we define a polynomial $g_\ell(x_1,\ldots,x_\ell) \in \Bbbk[x_1,\ldots,x_\ell]$ as

$$\begin{aligned} &g_\ell(x_1,\ldots,x_\ell) \\ &:= x_\ell \cdot \left(\prod_{\substack{1\leq i_1\leq p-1\\ 1\leq j_1\leq \ell-1}} (x_\ell - \zeta^{i_1}x_{j_1}) \right) \cdot \left(\prod_{\substack{1\leq i_1,i_2\leq p-1\\ 1\leq j_1<j_2\leq \ell-1}} (x_\ell - \zeta^{i_1}x_{j_1} - \zeta^{i_2}x_{j_2}) \right) \\ &\qquad \cdots\cdots \left(\prod_{\substack{1\leq i_1,\ldots,i_{\ell-1}\leq p-1\\ 1\leq j_1<\cdots<j_{\ell-1}\leq \ell-1}} (x_\ell - \zeta^{i_1}x_{j_1} - \cdots - \zeta^{i_{\ell-1}}x_{j_{\ell-1}}) \right). \end{aligned}$$

Clearly, $g_1(x_1) = x_1$.

Lemma 9 *We have*

$$\det(F_r(x_1,\ldots,x_r)) = \prod_{\ell=1}^{r} g_\ell(x_1,\ldots,x_\ell).$$

In particular if $\alpha_1,\ldots,\alpha_r$ are linearly independent over $\mathbb{F}_p$, then $F_r(\alpha_1,\ldots,\alpha_r)$ is a regular matrix.

Proof. We can express $\det(F_r)$ and g_r as

$$\begin{cases} \det(F_r) &= \det(F_{r-1}) \cdot x_r^{p^{r-1}} + (\text{ terms of lower degree in } x_r \text{ }), \\ g_r &= x_r^{p^{r-1}} + (\text{ terms of lower degree in } x_r \text{ }). \end{cases}$$

Since g_r divides $\det(F_r)$ in $\Bbbk[x_1,\ldots,x_r]$, we have $\det(F_r) = \det(F_{r-1}) \cdot g_r$, which implies the desired expression. Q.E.D.

For a matrix $A \in \mathrm{Mat}(r,\Bbbk)$, we define a submatrix $A_{i_1,i_2,\ldots,i_\ell}^{j_1,j_2,\ldots,j_\ell}$ $(1 \leq i_1 < i_2 < \cdots < i_\ell \leq r$, $1 \leq j_1 < j_2 < \cdots < j_\ell \leq r)$ of A as

$$A_{i_1,i_2,\ldots,i_\ell}^{j_1,j_2,\ldots,j_\ell} := \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_\ell} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_\ell} \\ \vdots & & \ddots & \vdots \\ a_{i_\ell,j_1} & a_{i_\ell,j_2} & \cdots & a_{i_\ell,j_\ell} \end{pmatrix}.$$

Let $\Gamma := \{(\mu, \nu) \mid 1 \le \mu < \nu \le r\}$ be an ordered set whose ordering $\preceq$ is given as follows: For $\gamma_1, \gamma_2 \in \Gamma$, we write $\gamma_1 \preceq \gamma_2$ if the first non-zero component of $\gamma_2 - \gamma_1$ is positive or $\gamma_1 = \gamma_2$. The number of elements of Γ is $r' := (r(r-1))/2$.

For any $A = (a_{i,j}) \in \mathrm{Mat}(r, \Bbbk)$, we define a matrix $\widetilde{A} := (\widetilde{a}_{\gamma,\delta})_{\gamma\in\Gamma,\delta\in\Gamma} \in \mathrm{Mat}(r', \Bbbk)$ as

$$\widetilde{a}_{\gamma,\delta} := \det(A_\gamma^\delta).$$

Lemma 10 *If A is a regular matrix of* $\mathrm{Mat}(r, \Bbbk)$*, then $\widetilde{A}$ is a regular matrix of* $\mathrm{Mat}(r', \Bbbk)$.

Proof. For any $\gamma = (\gamma_1, \gamma_2) \in \Gamma$, we let $|\gamma| := \gamma_1 + \gamma_2$. We define a matrix $B = (b_{\gamma,\delta})_{(\gamma,\delta)\in\Gamma\times\Gamma} \in \mathrm{Mat}(r', \Bbbk)$ as follows: $b_{\gamma,\delta} := (-1)^{|\gamma|+|\delta|} \det(A_{s-\delta}^{s-\gamma})$, where s is a sequence defined by $s := (1, 2, \ldots, r)$, and for any $(\mu, \nu) \in \Gamma$, $s - (\mu, \nu)$ is a subsequence of s obtained from s by deleting μ and ν, i.e., $s - (\mu, \nu) := (1, \ldots, \widehat{\mu}, \ldots, \widehat{\nu}, \ldots, r)$. Clearly, $\widetilde{A} \cdot B = \det(A) \cdot I_{r'}$. Q.E.D.

Now, we prove Theorem 1.

(1) Let $\alpha_1 := \iota(e_1)$. Clearly, $\alpha_1 \neq 0$. Let $M_1 := \rho(e_1) \in \mathrm{Mat}(n, k)$. Clearly, $M_1^p = I_n$. So, let $N_1 := \alpha_1^{-1} \cdot \mathrm{Log}(M_1)$. We can define a map $\varphi : \mathbb{G}_a \to GL(n, k)$ as

$$\varphi(t) := \mathrm{Exp}(tN_1).$$

Clearly, φ is a representation of $\mathbb{G}_a$ and $\rho(e_1) = \varphi(\alpha_1)$, which implies $\rho = \varphi \circ \iota$.

(2) Let $\alpha_i := \iota(e_i)$ and let $M_i := \rho(e_i)$ for $1 \le i \le r$. Since $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$ is a modular representation, we have the following (i) and (ii):

(i) $M_i^p = I_n$ for all $1 \le i \le r$.

(ii) $M_i M_j = M_j M_i$ for all $1 \le i, j \le r$.

Let $\mathfrak{N}$ be the set of all p-nilpotent matrices of $\mathrm{Mat}(n, k)$ and let $\mathfrak{U}$ be the set of all p-unipotent matrices of $\mathrm{Mat}(n, k)$. Let $\mathrm{Exp} : \mathfrak{N} \to \mathfrak{U}$ be the truncated exponential map and let $\mathrm{Log} : \mathfrak{U} \to \mathfrak{N}$ be the truncated logarithmic map. So, we have the following (iii) and (iv):

(iii) $\mathrm{Log}(M_i) \in \mathfrak{N}$ for all $1 \le i \le r$.

(iv) $\mathrm{Log}(M_i)\,\mathrm{Log}(M_j) = \mathrm{Log}(M_j)\,\mathrm{Log}(M_i)$ for all $1 \le i, j \le r$.

There exist matrices $N_1, \ldots, N_r \in \mathrm{Mat}(n, k)$ satisfying

$$\mathrm{Log}(M_i) = \sum_{\lambda=1}^{r} \alpha_i^{p^{\lambda-1}} N_\lambda \qquad \text{for all } 1 \le i \le r,$$

since $\det(\alpha_i^{p^{\lambda-1}})_{1\le i,\lambda\le r} \neq 0$. Thus we have the following (v) and (vi):

(v) $N_i^p = O_n$ for all $1 \leq i \leq r$.

(vi) $N_i N_j = N_j N_i$ for all $1 \leq i, j \leq r$.

Now, we can define a map $\varphi : \mathbb{G}_a \to GL(n, k)$ as

$$\varphi(t) := \mathrm{Exp}\left(\sum_{\lambda=1}^{r} t^{p^{\lambda-1}} N_\lambda \right).$$

Since ρ is p-pyramidic, φ is a representation of $\mathbb{G}_a$. Clearly, $\rho(e_i) = \varphi(\alpha_i)$ for all $1 \leq i \leq r$, which implies $\rho = \varphi \circ \iota$.

(3) It is enough to show that ρ is p-pyramidic. Let $X_i := \rho(e_i) - I_n$ $(1 \leq i \leq r)$. Since $X_i X_j = X_j X_i$ for all $1 \leq i, j \leq r$, there exists a regular matrix $P \in GL(n, k)$ such that $P^{-1} X_i P$'s $(1 \leq i \leq r)$ are upper triangular matrices. Since $X_i^p = O_n$, the all diagonal entries of $P^{-1} X_i P$ are zeros. Since $1 \leq n \leq p$, we have

$$\prod_{i=1}^{r} (P^{-1} X_i P)^{\ell_i} = O_n \qquad \text{for all } \ell_1, \ldots, \ell_r \geq 0 \text{ with } \ell_1 + \cdots + \ell_r \geq p.$$

This completes the proof of Theorem 1.

3. A proof of Theorem 2

3.1 A proof of assertion (2) of Theorem 2

Let $M_i := \rho(e_i)$ for $1 \leq i \leq r$. We can solve the following equations $(*)$ for $N_\lambda \in \mathrm{Mat}(n, k)$ $(1 \leq \lambda \leq r)$ and $N_{\mu,\nu} \in \mathrm{Mat}(n, k)$ $(1 \leq \mu < \nu \leq r)$:

$$(*) \begin{cases} M_i - I_n \\ \quad = \displaystyle\sum_{\lambda=1}^{r} \alpha_i^{p^{\lambda-1}} N_\lambda + \sum_{1 \leq \mu < \nu \leq r} \alpha_i^{p^{\mu-1}+p^{\nu-1}} N_{\mu,\nu} \qquad (\, 1 \leq i \leq r \,), \\ (M_i - I_n)(M_j - I_n) \\ \quad = \displaystyle\sum_{1 \leq \mu < \nu \leq r} (\alpha_i^{p^{\mu-1}} \alpha_j^{p^{\nu-1}} + \alpha_i^{p^{\nu-1}} \alpha_j^{p^{\mu-1}}) N_{\mu,\nu} \qquad (\, (i,j) \in \Gamma \,). \end{cases}$$

Let $A := F_r(\alpha_1, \ldots, \alpha_r) \in \mathrm{Mat}(r, k)$. Recall that A is a regular matrix (see Lemma 9) and that $\widetilde{A}$ is also a regular matrix (see Lemma 10). It follows that

$$\begin{aligned} (N_{\mu,\nu})_{(\mu,\nu)\in\Gamma} &= ((M_i - I_n)(M_j - I_n))_{(i,j)\in\Gamma} \cdot \widetilde{A}^{-1}, \\ (N_\lambda)_{1\leq\lambda\leq r} &= \left(M_i - I_n - \sum_{1 \leq \mu < \nu \leq r} \alpha_i^{p^{\mu-1}+p^{\nu-1}} N_{\mu,\nu} \right)_{1 \leq i \leq r} \cdot A^{-1}. \end{aligned}$$

Since $(M_i - I_n)^2 = O_n$ for all $1 \leq i \leq r$ and $(M_i - I_n)(M_j - I_n) = (M_j - I_n)(M_i - I_n)$ for all $1 \leq i < j \leq r$, we have

$$N_{\mu,\nu}\, N_{\mu',\nu'} = O_n \qquad (\,(\mu,\nu),(\mu',\nu') \in \Gamma\,),$$
$$N_\lambda^2 = O_n \qquad (\,1 \leq \lambda \leq r\,),$$
$$N_\mu N_\nu = N_\nu N_\mu \qquad (\,(\mu,\nu) \in \Gamma\,).$$

Since ρ is of 3-mutually annihilating, we have

$$N_\lambda N_{\mu,\nu} = O_n \qquad (\,1 \leq \lambda \leq r,\ (\mu,\nu) \in \Gamma\,).$$

By the first equation of $(*)$, we have

$$(M_i - I_n)(M_j - I_n) = \sum_{1 \leq \mu < \nu \leq r} (\alpha_i^{p^{\mu-1}} \alpha_j^{p^{\nu-1}} + \alpha_i^{p^{\nu-1}} \alpha_j^{p^{\mu-1}}) N_\mu N_\nu \quad (\,(i,j) \in \Gamma\,).$$

The second equality of $(*)$ implies

$$N_\mu N_\nu = N_{\mu,\nu} \qquad (\,1 \leq \mu < \nu \leq r\,).$$

Now, the first equation of $(*)$ implies that

$$M_i = \prod_{\lambda=1}^{r} (I_n + \alpha^{p^{\lambda-1}} N_\lambda) \qquad (\,1 \leq i \leq r\,).$$

Let $\varphi : \mathbb{G}_a \to GL(n,k)$ be the map defined by

$$\varphi(t) = \prod_{\lambda=1}^{r} (I_n + t^{p^{\lambda-1}} N_\lambda).$$

Clearly, φ is a representation. So, $\rho(e_i) = \varphi(\alpha_i)$ for all $1 \leq i \leq r$, which implies $\rho = \varphi \circ \iota$. Q.E.D.

3.2 A proof of assertion (1) of Theorem 2

3.2.1 $r = 2$

We first consider the case $r = 2$. Let $M_i := \rho(e_i)$ for $i = 1, 2$. Let $A := F_2(\alpha_1, \alpha_2)$. We can solve the following equations $(*)$ for $N_1, N_2, N_{1,2} \in \mathrm{Mat}(n,k)$:

$$(*) \begin{cases} M_i - I_n = \alpha_i N_1 + \alpha_i^p N_2 + \alpha_i^{p+1} N_{1,2} & (\,1 \leq i \leq 2\,), \\ (M_1 - I_n)(M_2 - I_n) = \det(A) N_{1,2}. \end{cases}$$

Clearly, we have

$$N_1^2 = N_2^2 = O_n, \quad N_1N_2 = N_2N_1, \quad N_1N_{1,2} = O_n, \quad N_2N_{1,2} = O_n.$$

Calculate $(M_1 - I_n)(M_2 - I_n)$ by using the first equation of $(*)$. Thus $N_{1,2} = N_1N_2$. Hence we have $M_i = (I_n + \alpha_i N_1)(I_n + \alpha_i^p N_2)$ for all $1 \le i \le 2$. Now, we can define a representation $\varphi : \mathbb{G}_a \to GL(n,k)$ as

$$\varphi(t) := (I_n + tN_1)(I_n + t^pN_2).$$

Clearly, $\rho(e_i) = \varphi(\alpha_i)$ for all $i = 1, 2$, which implies $\rho = \varphi \circ \iota$.

3.2.2 $r = 3$

We next consider the case $r = 3$. Let $M_i = \rho(e_i)$ for $1 \le i \le 3$, let $A := F_3(\alpha_1, \alpha_2, \alpha_3)$ and let $\widetilde{A}$ be as above. So,

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^p & \alpha_2^p & \alpha_3^p \\ \alpha_1^{p^2} & \alpha_2^{p^2} & \alpha_3^{p^2} \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} \alpha_1\alpha_2^p + \alpha_1^p\alpha_2 & \alpha_1\alpha_3^p + \alpha_1^p\alpha_3 & \alpha_2\alpha_3^p + \alpha_2^p\alpha_3 \\ \alpha_1\alpha_2^{p^2} + \alpha_1^{p^2}\alpha_2 & \alpha_1\alpha_3^{p^2} + \alpha_1^{p^2}\alpha_3 & \alpha_2\alpha_3^{p^2} + \alpha_2^{p^2}\alpha_3 \\ \alpha_1^p\alpha_2^{p^2} + \alpha_1^{p^2}\alpha_2^p & \alpha_1^p\alpha_3^{p^2} + \alpha_1^{p^2}\alpha_3^p & \alpha_2^p\alpha_3^{p^2} + \alpha_2^{p^2}\alpha_3^p \end{pmatrix}.$$

We can solve the following equations $(*)$ for $N_1, N_2, N_3, N_{1,2}, N_{1,3}, N_{2,3}, N_{1,2,3} \in \mathrm{Mat}(n,k)$:

$$(*) \begin{cases} M_i - I_n \\ \quad = \alpha_i N_1 + \alpha_i^p N_2 + \alpha_i^{p^2} N_3 \\ \qquad + \alpha_i^{p+1} N_{1,2} + \alpha_i^{p^2+1} N_{1,3} + \alpha_i^{p^2+p} N_{2,3} + \alpha_i^{p^2+p+1} N_{1,2,3} \\ \hfill (1 \le i \le 3), \\ \\ (M_i - I_n)(M_j - I_n) \\ \quad = (\alpha_i\alpha_j^p + \alpha_i^p\alpha_j)N_{1,2} + (\alpha_i\alpha_j^{p^2} + \alpha_i^{p^2}\alpha_j)N_{1,3} + (\alpha_i^p\alpha_j^{p^2} + \alpha_i^{p^2}\alpha_j^p)N_{2,3} \\ \qquad + \begin{pmatrix} \alpha_i\alpha_j^{p^2+p} + \alpha_i^p\alpha_j^{p^2+1} + \alpha_i^{p^2}\alpha_j^{p+1} \\ + \alpha_i^{p+1}\alpha_j^{p^2} + \alpha_i^{p^2+1}\alpha_j^p + \alpha_i^{p^2+p}\alpha_j \end{pmatrix} N_{1,2,3} \\ \hfill (1 \le i < j \le 3), \\ \\ (M_1 - I_n)(M_2 - I_n)(M_3 - I_n) = \det(A)N_{1,2,3}. \end{cases}$$

In fact, letting $M_{i,j} := (M_i - I_n)(M_j - I_n)$ for $1 \le i, j \le 3$ and $M_{1,2,3} := (M_1 - I_n)(M_2 - I_n)(M_3 - I_n)$, we have, from the bottom to the top of the above equations $(*)$,

$$N_{1,2,3} = \frac{1}{\det(A)} M_{1,2,3},$$

$$(N_{1,2},\, N_{1,3},\, N_{2,3}) = (M_{1,2},\, M_{1,3},\, M_{2,3}) \cdot \widetilde{A}^{-1} + (b_{1,2},\, b_{1,3},\, b_{2,3}) \cdot M_{1,2,3} \quad \text{for some } b_{1,2},\, b_{1,3},\, b_{2,3} \in k,$$

$$\begin{aligned}(N_1,\, N_2,\, N_3) = {} & (M_1 - I_n,\, M_2 - I_n,\, M_3 - I_n) \cdot A^{-1} \\ & +(M_{1,2},\, M_{1,3},\, M_{2,3}) \cdot C + (d_{1,2},\, d_{1,3},\, d_{2,3}) \cdot M_{1,2,3} \\ & \text{for some } C \in \mathrm{Mat}(3,k) \text{ and } d_{1,2},\, d_{1,3},\, d_{2,3} \in k.\end{aligned}$$

Clearly, we have $N_i^2 = O_n$ for all $1 \le i \le 3$ and $N_i N_j = N_j N_i$ for all $1 \le i, j \le 3$. For $1 \le i, j \le 3$, let $A_{i,j}$ be the determinant of the submatrix formed by deleting the i-th row and the j-th column of A. So,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} A_{3,3} & A_{3,2} & A_{3,1} \\ A_{2,3} & A_{2,2} & A_{2,1} \\ A_{1,3} & A_{1,2} & A_{1,1} \end{pmatrix}, \qquad \widetilde{A}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \alpha_3^{p^2} & \alpha_3^p & \alpha_3 \\ \alpha_2^{p^2} & \alpha_2^p & \alpha_2 \\ \alpha_1^{p^2} & \alpha_1^p & \alpha_1 \end{pmatrix}.$$

Let $A = (a_{i,j})_{1 \le i,j \le 3}$. For all $1 \le i \le 3$ and $1 \le j < \ell \le 3$ and $m \in \{1, 2, 3\} \backslash \{j, \ell\}$, we have

$$\begin{aligned} N_i N_{j,\ell} &= \left(\frac{1}{\det(A)} (A_{i,1}(M_1 - I_n) + A_{i,2}(M_2 - I_n) + A_{i,3}(M_3 - I_n)) \right) \\ &\quad \cdot \left(\frac{1}{\det(A)} (a_{m,3} M_{1,2} + a_{m,2} M_{1,3} + a_{m,1} M_{2,3}) \right) \\ &= \frac{1}{\det(A)^2} (A_{i,1}\, a_{m,1} + A_{i,2}\, a_{m,2} + A_{i,3}\, a_{m,3}) M_{1,2,3} \\ &= \begin{cases} O_n & \text{if} \quad i \neq m, \\ \dfrac{1}{\det(A)} M_{1,2,3} & \text{if} \quad i = m. \end{cases} \end{aligned}$$

Calculate $(M_1 - I_n)(M_2 - I_n)(M_3 - I_n)$ by using the first equation of $(*)$. We have

$$N_1 N_2 N_3 = \frac{1}{\det(A)} M_{1,2,3}.$$

So, the third equality of $(*)$ implies

$$N_1N_2N_3 = N_{1,2,3}.$$

Expand $(M_i - I_n)(M_j - I_n)$ by using the first equation of $(*)$. The second equality of $(*)$ can imply

$$N_1N_2 = N_{1,2}, \qquad N_1N_3 = N_{1,3}, \qquad N_2N_3 = N_{2,3}.$$

Hence we have

$$M_i = (I_n + \alpha_i N_1)(I_n + \alpha_i^p N_2)(I_n + \alpha_i^{p^2} N_3) \qquad (1 \le i \le 3).$$

Let $\varphi : \mathbb{G}_a \to GL(n,k)$ be the map defined by

$$\varphi(t) = \prod_{\lambda=1}^{3} (I_n + t^{p^{\lambda-1}} N_\lambda).$$

Clearly, φ is a representation. Thus, $\rho(e_i) = \varphi(\alpha_i)$ for all $1 \le i \le 3$, which implies $\rho = \varphi \circ \iota$.

4. Proofs of Corollaries 3 and 4

4.1 Lemmas

Let

$$\mathfrak{a}_{2,2} := \left\{ \begin{pmatrix} 0 & 0 & a_{1,3} & a_{1,4} \\ 0 & 0 & a_{2,3} & a_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4} \in k \right\}$$

be a subset of $\mathrm{Mat}(4,k)$.

Lemma 11 *Let $X = (x_{i,j})$ be an upper triangular matrix of $\mathrm{Mat}(4,k)$ satisfying $X^2 = O_n$ and $x_{2,3} \neq 0$. Then the following assertions (1) and (2) hold true:*

(1) $X \in \mathfrak{a}_{2,2}$.

(2) *For any upper triangular matrix $Y = (y_{i,j})$ of $\mathrm{Mat}(4,k)$ satisfying $Y^2 = O_n$ and $XY = YX$, we have $Y \in \mathfrak{a}_{2,2}$.*

Proof. (1) The proof is straightforward.

(2) If $y_{2,3} \neq 0$, then $Y \in \mathfrak{a}_{2,2}$ (by the above assertion (1)). If $y_{2,3} = 0$, then $y_{1,2} = y_{3,4} = 0$ (since $XY = YX$), which implies $Y \in \mathfrak{a}_{2,2}$. Q.E.D.

Let

$$\mathfrak{h}_4 := \left\{ \begin{pmatrix} 0 & h_{1,2} & h_{1,3} & h_{1,4} \\ 0 & 0 & 0 & h_{2,4} \\ 0 & 0 & 0 & h_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; h_{1,2}, h_{1,3}, h_{1,4}, h_{2,4}, h_{3,4} \in k \right\}$$

be a subset of $\mathrm{Mat}(4, k)$.

Lemma 12 *Let N_i $(1 \leq i \leq r)$ be upper triangular matrices of* $\mathrm{Mat}(4, k)$ *satisfying both conditions $N_i^2 = O_n$ for all $1 \leq i \leq r$ and $N_i N_j = N_j N_i$ for all $1 \leq i, j \leq r$. Then one of the following cases* (1) *and* (2) *can occur:*

(1) $N_i \in \mathfrak{a}_{2,2}$ *for all* $1 \leq i \leq r$.

(2) $N_i \in \mathfrak{h}_4$ *for all* $1 \leq i \leq r$.

Proof. Suppose that there exists at least one matrix N_j among N_i $(1 \leq i \leq r)$ such that N_j does not belongs to $\mathfrak{h}_4$. By Lemma 11, $N_j \in \mathfrak{a}_{2,2}$ and then the other $(r-1)$ matrices $N_1, \ldots, \widehat{N_j}, \ldots, N_r$ belong to $\mathfrak{a}_{2,2}$. Q.E.D.

Lemma 13 *Assume $r \geq 3$. Let N_i $(1 \leq i \leq r)$ be matrices of* $\mathrm{Mat}(4, k)$ *satisfying both conditions $N_i^2 = O_n$ for all $1 \leq i \leq r$ and $N_i N_j = N_j N_i$ for all $1 \leq i, j \leq r$. Then the matrices $N_1, \ldots, N_r$ are of* 3*-mutually annihilating.*

Proof. The proof is straightforward by the above Lemma 12. Q.E.D.

4.2 A proof of Corollary 3

If $p \geq 3$, the corollary follows from assertion (3) of Theorem 1. So, if $p = 2$ and $2 \leq r \leq 3$, the corollary follows from assertion (1) of Theorem 2. If $p = 2$ and $r \geq 4$, the corollary follows from assertion (2) of Theorem 2 and Lemma 13.

4.3 A proof of Corollary 4

Let $\rho : (\mathbb{Z}/p\mathbb{Z})^r \to GL(4, k)$ be a modular representation. Since k is algebraically closed, there exists an injective group homomorphism $\iota : (\mathbb{Z}/p\mathbb{Z})^r \to \mathbb{G}_a$. Let $\alpha_i := \iota(e_i)$ for $1 \leq i \leq r$. By Theorem 1, we can factor ρ as $\rho = \varphi \circ \iota$ for some representation $\varphi : \mathbb{G}_a \to GL(4, k)$. By [6, Theorem 2.1], there exists a regular matrix $P \in GL(4, k)$ such that the representation $\psi(t) := P^{-1}\varphi(t)P$ of $\mathbb{G}_a$ has one of the following forms $A_{2,2}(t)$, $A_{3,1}(t)$, $H_\mu(t)$:

$$A_{2,2}(t) := \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c, d \text{ are } p\text{-polynomials}),$$

$$A_{3,1}(t) := \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c \text{ are } p\text{-polynomials}),$$

$$H_\mu(t) := \begin{pmatrix} 1 & a & b & \mu ab + c \\ 0 & 1 & 0 & \mu b \\ 0 & 0 & 1 & \mu a \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a, b, c \text{ are } p\text{-polynomials,} \\ a, b \text{ are linearly independent} \\ \text{over } k, \text{ and } \mu \in k \end{pmatrix}.$$

If $\psi(t) = A_{2,2}(t)$, then $\psi(\alpha_i) \in \mathcal{A}_{2,2}$ for all $1 \le i \le r$. If $\psi(t) = A_{3,1}(t)$, then $\psi(\alpha_i) \in \mathcal{A}_{3,1}$ for all $1 \le i \le r$. If $\psi(t) = H_\mu(t)$, then $\psi(\alpha_i) \in \mathcal{H}_\mu$ for all $1 \le i \le r$. Now we define a modular representation $\sigma : (\mathbb{Z}/p\mathbb{Z})^r \to GL(n, k)$ as $\sigma(g) := P^{-1}\rho(g)P$. Clearly, σ satisfies the condition (1) of Corollary 4. And σ satisfies the condition (2) of Corollary 4 since $\sigma(e_i) = \psi(\alpha_i)$ for all $1 \le i \le r$. This completes the proof of Corollary 4.

References

[1] V. A. Bašev, *Representations of the group $Z_2 \times Z_2$ in a field of characteristic* 2, Dokl. Akad. Nauk SSSR **141** (1961), 1015–1018.

[2] H. E. A. Campbell, R. J. Shank, D. L. Wehlau, *Rings of invariants for modular representations of elementary abelian p-groups*, Transform. Groups **18** (2013), no.1, 1–22.

[3] A. Fauntleroy, *On Weitzenböck's theorem in positive characteristic*, Proc. Amer. Math. Soc. **64** (1977), no.2, 209–213.

[4] R. Tanimoto, *An algorithm for computing the kernel of a locally finite iterative higher derivation*, J. Pure Appl. Algebra **212** (2008), no.10, 2284–2297.

[5] R. Tanimoto, *Representations of $\mathbb{G}_a$ of codimension two*, Affine algebraic geometry, 279–284, World Sci. Publ., Hackensack, NJ, 2013.

[6] R. Tanimoto, *A note on the Weitzenböck problem in dimension four*, Communications in Algebra **46** (2018), no.2, 588–596.

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