Saitama Math. J.
Vol. 34 (2022), 19–46

The best constant of discrete sobolev inequality corresponding to a discrete bending problem of a string

Hiroyuki Yamagishi and Atsushi Nagai

(*Received 1 September, 2020; Revised 3 June, 2021; Accepted 3 June, 2021*)

Abstract

The discrete Sobolev inequality shows that the maximum of deviation of a string is estimated from above by a constant multiples of the potential energy. We have found the best constant and the vector, which attain the equality. In the background, there is five boundary value problems of the 2nd-order difference equation, which describes a discrete version of a string bending problem. The solution is expressed by using Green or pseudo Green matrices. The best constant and vector are given by investigating the matrices. Moreover, we show the positivity and the hierarchical structure of Green matrices.

1. Introduction

For $N = 2, 3, 4, \cdots$, we consider the following boundary value problems of 2nd-order difference equation:

DBVP(X; a)

$$\left\{\begin{array}{ll} -u(i-1)+(2+a)u(i)-u(i+1)=f(i) & (0 \le i \le N-1) \\ \left\{\begin{array}{lll} u(-1)=0, & u(N)=0 & (\mathrm{X})=(0,0) \\ u(-1)=0, & u(N-1)-u(N)=0 & (\mathrm{X})=(0,1) \\ u(-1)-u(0)=0, & u(N)=0 & (\mathrm{X})=(1,0) \\ u(-1)-u(0)=0, & u(N-1)-u(N)=0 & (\mathrm{X})=(1,1) \\ u(-1)=u(N-1), & u(0)=u(N) & (\mathrm{X})=(\mathrm{P}) \end{array}\right. \end{array}\right. .$$

The physical meaning of the above equation is as follows. A string is supported by uniformly distributed springs with spring constant a on a fixed ceiling. Let $f(i)$ be a given function which means the load at i $(0 \le i \le N-1)$ and $u(i)$ be the bending displacement at i $(0 \le i \le N-1)$. (X) means a boundary condition in which $(0,0)$ is Clamped-Clamped, $(0,1)$ is Clamped-Free, $(1,0)$ is Free-Clamped, $(1,1)$ is Free-Free and (P) is Periodic boundary conditions. The Clamped and Free boundary conditions are called Dirichlet and Neumann boundary conditions in other words, respectively. We call DBVP(X; a) the discrete bending problem

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2010 Mathematics Subject Classification. Primary 46E39, Secondary 35J08
Key words and phrases. discrete Sobolev inequality, Chebyshev polynomial.

of a string.

We set hyperbolic functions $\mathrm{ch}(x) = \cosh(x)$ and $\mathrm{sh}(x) = \sinh(x)$ for short. For later convenience sake, we introduce constants x and y defined by

$$x = \frac{2+a}{2} = \cos(\sqrt{-1}y) = \mathrm{ch}(y) \qquad \Leftrightarrow \qquad a = 2(x-1) = 4\mathrm{sh}^2(y/2) \tag{1.1}$$
$$(1 < x < \infty, \quad 0 < a < \infty, \quad 0 < y < \infty).$$

Results are sometimes described by means of x or y in (1.1), instead of a. We note that the limit of $a \to 0$ is equivalent to $x \to 1$ and $y \to 0$.

Introducing vectors

$$\boldsymbol{u} = {}^t(u(0), \cdots, u(N-1)) \in \boldsymbol{C}^N, \qquad \boldsymbol{f} = {}^t(f(0), \cdots, f(N-1)) \in \boldsymbol{C}^N$$

and $N \times N$ identity matrix $\boldsymbol{I}$, we can rewrite DBVP(X; a) as

$$\begin{array}{l} \mathrm{DBVP}(\mathrm{X}; a) \\ (\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u} = \boldsymbol{f}, \end{array}$$

where $\boldsymbol{A} = \boldsymbol{A}(\mathrm{X})$ is a discrete Laplacian. $\boldsymbol{A} = \boldsymbol{A}(\mathrm{X})$ is an $N \times N$ matrix. Using $2 + a = 2x$ in (1.1), we have the concrete form of $\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}$ as

$$\boldsymbol{A}(m,n) + a\boldsymbol{I} = \begin{pmatrix} 2x-m & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-n \end{pmatrix}_{N\times N},$$

for $(m, n) = (0, 0), (0, 1), (1, 0), (1, 1)$ and

$$\boldsymbol{A}(\mathrm{P}) + a\boldsymbol{I} = \begin{cases} \begin{pmatrix} 2x & -2 \\ -2 & 2x \end{pmatrix} & (N = 2) \\ \begin{pmatrix} 2x & -1 & & & -1 \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ -1 & & & -1 & 2x \end{pmatrix}_{N\times N} & (N = 3, 4, 5, \cdots) \end{cases}.$$

5 kinds of $\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}$ are positive definite Hermitian matrices. Taking limit as $a \to 0$ (that is $x \to 1$) for $\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}$, we have the concrete form of $\boldsymbol{A}(\mathrm{X})$. $\boldsymbol{A}(0, 0)$, $\boldsymbol{A}(0, 1)$, $\boldsymbol{A}(1, 0)$ are positive definite Hermitian matrices. $\boldsymbol{A}(1, 1)$ and $\boldsymbol{A}(\mathrm{P})$ are non-negative definite and singular Hermitian matrices. In particular, $\boldsymbol{A}(\mathrm{P})$ is a cyclic matrix.

We introduce the maximum and minimum function defined by

$$\begin{cases} x \vee y = \max\{x, y\} \\ x \wedge y = \min\{x, y\} \end{cases} \quad \Leftrightarrow \quad \begin{cases} i \vee j = \dfrac{1}{2}(i + j + |i - j|) \\ i \wedge j = \dfrac{1}{2}(i + j - |i - j|) \end{cases}.$$

The following Lemma 1.1 states the uniqueness of the solution for DBVP(X; a).

Lemma 1.1. *For arbitrary* $\boldsymbol{f} \in \mathbf{C}^N$*,* DBVP(X; a) *has a unique solution* $\boldsymbol{u} = \boldsymbol{G}(a)\boldsymbol{f}$*. We call* $\boldsymbol{G}(a)$ *"Green matrix". Green matrix* $\boldsymbol{G}(a)$ *is expressed as*

$$\boldsymbol{G}(a) = (\boldsymbol{A} + a\boldsymbol{I})^{-1} = \Big(g(\mathrm{X}; a; i, j) \Big)_{0 \le i,j \le N-1},$$

where (i, j)*-th entries* $g(\mathrm{X}; a; i, j)$ *are given by*

$$g(0, 0; a; i, j) = \frac{U_{i\wedge j+1}(x)\, U_{N-i\vee j}(x)}{U_{N+1}(x)} = \frac{\mathrm{sh}((i \wedge j + 1)y)\, \mathrm{sh}((N - i \vee j)y)}{\mathrm{sh}((N + 1)y)\, \mathrm{sh}(y)} =$$
$$\frac{1}{2\mathrm{sh}((N + 1)y)\, \mathrm{sh}(y)} \Big[\mathrm{ch}((N + 1 - |i - j|)y) - \mathrm{ch}((N - 1 - i - j)y) \Big],$$

$$g(0, 1; a; i, j) = \frac{U_{i\wedge j+1}(x)\, (U_{N-i\vee j}(x) - U_{N-1-i\vee j}(x))}{U_{N+1}(x) - U_N(x)} =$$
$$\frac{\mathrm{sh}((i \wedge j + 1)y)\, \mathrm{ch}((N - 1/2 - i \vee j)y)}{\mathrm{ch}((N + 1/2)y)\, \mathrm{sh}(y)} =$$
$$\frac{1}{2\mathrm{ch}((N + 1/2)y)\, \mathrm{sh}(y)} \Big[\mathrm{sh}((N + 1/2 - |i - j|)y) - \mathrm{sh}((N - 3/2 - i - j)y) \Big],$$

$$g(1, 0; a; i, j) = g(0, 1; a; N-1-i, N-1-j) = \frac{(U_{i\wedge j+1}(x) - U_{i\wedge j}(x))\, U_{N-i\vee j}(x)}{U_{N+1}(x) - U_N(x)} =$$
$$\frac{\mathrm{ch}((i \wedge j + 1/2)y)\, \mathrm{sh}((N - i \vee j)y)}{\mathrm{ch}((N+1/2)y)\, \mathrm{sh}(y)} =$$
$$\frac{1}{2\mathrm{ch}((N + 1/2)y)\, \mathrm{sh}(y)} \Big[\mathrm{sh}((N + 1/2 - |i - j|)y) + \mathrm{sh}((N - 1/2 - i - j)y) \Big],$$

$$g(1, 1; a; i, j) = \frac{(U_{i\wedge j+1}(x) - U_{i\wedge j}(x))\, (U_{N-i\vee j}(x) - U_{N-1-i\vee j}(x))}{U_{N+1}(x) - 2U_N(x) + U_{N-1}(x)} =$$
$$\frac{\mathrm{ch}((i \wedge j + 1/2)y)\, \mathrm{ch}((N - 1/2 - i \vee j)y)}{\mathrm{sh}(Ny)\, \mathrm{sh}(y)} =$$
$$\frac{1}{2\mathrm{sh}(Ny)\, \mathrm{sh}(y)} \Big[\mathrm{ch}((N - |i - j|)y) + \mathrm{ch}((N - 1 - i - j)y) \Big],$$

$$g(\mathrm{P}; a; i, j) = \frac{U_{N-|i-j|}(x) + U_{|i-j|}(x)}{2(T_N(x) - 1)} = \frac{\mathrm{ch}((N/2 - |i - j|)y)}{2\mathrm{sh}(Ny/2)\, \mathrm{sh}(y)} =$$
$$\frac{1}{4\mathrm{sh}^2(Ny/2)\, \mathrm{sh}(y)} \Big[\mathrm{sh}((N - |i - j|)y) + \mathrm{sh}(|i - j|y) \Big].$$

In the above expressions, $T_N(x)$ *and* $U_N(x)$ *are Chebyshev polynomials of the first and second kinds defined by*

$$T_N(\cos(\theta)) = \cos(N\theta), \qquad U_N(\cos(\theta)) = \frac{\sin(N\theta)}{\sin(\theta)} \qquad (N = 0, 1, 2, \cdots). \quad (1.2)$$

We use constants x, y *in* (1.1) *instead of* a.

Lemma 1.1 shows that the element of Green matrix $g(\mathrm{X}; a; i, j)$ can be written by employing Chebyshev polynomials or hyperbolic functions in section 4. We call the former Chebyshev polynomial expression and the latter hyperbolic function expression in this paper. In particular, 4 kinds of Chebyshev polynomial expression $g(m, n; a; i, j)$ are given as

$$g(m, n; a; i, j) = \frac{\left(U_{i\wedge j+1}(x) - mU_{i\wedge j}(x)\right)\left(U_{N-i\vee j}(x) - nU_{N-1-i\vee j}(x)\right)}{U_{N+1}(x) - (m+n)U_N(x) + mnU_{N-1}(x)} \qquad (0 \le i, j \le N-1). \quad (1.3)$$

Next, we take the limit $a \to 0$. In the case of (X) $= (0, 0)$, $(0, 1)$, $(1, 0)$, all the eigenvalues of $\boldsymbol{A}$ are positive eigenvalues, as is shown later in Lemma 5.1. Hence, taking limit as $a \to 0$ for Lemma 1.1, we have the following lemma.

Lemma 1.2 ((X) $= (0, 0), (0, 1), (1, 0)$)**.** *For arbitrary* $\boldsymbol{f} \in \mathbf{C}^N$, DBVP(X; 0) *has a unique solution*

$$\boldsymbol{u} = \boldsymbol{G}(0)\boldsymbol{f},$$

where

$$\boldsymbol{G}(0) = \boldsymbol{A}^{-1} = \Big(g(\mathrm{X}; 0; i, j)\Big)_{0 \le i, j \le N-1}.$$

(i, j)*-th entries of* $g(\mathrm{X}; 0; i, j)$ *are given as*

$$g(0, 0; 0; i, j) = \frac{(i \wedge j + 1)(N - i \vee j)}{N+1},$$

$$g(0, 1; 0; i, j) = i \wedge j + 1,$$

$$g(1, 0; 0; i, j) = g(0, 1; 0; N-1-i, N-1-j) = N - i \vee j.$$

In the case of (X) $= (1, 1)$ and (P), $\boldsymbol{A}$ has an eigenvalue 0 whose corresponding eigenvector is $\mathbf{1} = {}^t(1, 1, \cdots, 1) \in \mathbf{C}^N$, as is shown later in Lemma 5.1. In order to guarantee the existence and uniqueness of the solution for DBVP(X; 0), we impose additional two condition as the following lemma.

Lemma 1.3 ((X)=(1,1), (P)). *For arbitrary $\boldsymbol{f} \in \mathbf{C}^N$ with the solvability condition ${}^t\mathbf{1}\boldsymbol{f} = 0$,* DBVP(X; 0) *with the orthogonality condition ${}^t\mathbf{1}\boldsymbol{u} = 0$ has a unique solution*

$$\boldsymbol{u} = \boldsymbol{G}_*\boldsymbol{f},$$

where

$$\boldsymbol{G}_* = \lim_{a\to 0}\left(\boldsymbol{G}(a) - a^{-1}\boldsymbol{E}_0\right) = \Big(g_*(\mathrm{X}; i, j)\Big)_{0\le i,j\le N-1}, \tag{1.4}$$

where $\boldsymbol{E}_0 = N^{-1}\mathbf{1}{}^t\mathbf{1}$ is a projection matrix to the eigenspace associated with the eigenvalue 0 [10, §2]. *(i,j)-th entries of $g_*(\mathrm{X}; i, j)$ are given as*

$$g_*(1,1;i,j) = b_2(2N;|i-j|) + b_2(2N;1+i+j) =$$
$$\frac{1}{6N}\Big[(N-1)(2N-1) - 3(i\vee j)(2N-1-i\vee j) + 3(i\wedge j)(i\wedge j+1)\Big],$$
$$g_*(\mathrm{P};i,j) = b_2(N;|i-j|) = \frac{1}{2N}|i-j|^2 - \frac{1}{2}|i-j| + \frac{N^2-1}{12N},$$

where $b_2(N;i)$ is the discrete Bernoulli polynomial [1] *given as*

$$b_2(N;i) = \frac{1}{2N}i^2 - \frac{1}{2}i + \frac{N^2-1}{12N} \qquad (0 \le i \le N-1). \tag{1.5}$$

The matrix $\boldsymbol{G}_*$ is a Moore-Penrose generalized inverse matrix of $\boldsymbol{A}$ [10, §3]. $\boldsymbol{G}_*$ satisfies the relations

$$\boldsymbol{A}\boldsymbol{G}_* = \boldsymbol{G}_*\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{E}_0, \qquad \boldsymbol{G}_*\boldsymbol{E}_0 = \boldsymbol{E}_0\boldsymbol{G}_* = \boldsymbol{O},$$

where $\boldsymbol{O}$ is the zero matrix.

Lemma 1.1, 1.2 and 1.3 show the uniqueness of the solution of DBVP(X; a) and DBVP(X; 0), respectively. Continuous versions of Lemma 1.1, 1.2 and 1.3 are given in [4, Lemma 2.1] and [5, Theorem 3.1].

The main purpose of this paper is to find the best constant of discrete Sobolev inequalities corresponding to 2nd-order difference equation which represents a string bending problem. The best constant and the best vector, which attains the equality in discrete Sobolev inequalities, are given by employing Chebyshev or discrete Bernoulli polynomials.

This paper is composed of five sections. In section 2, we state main results of this paper. In section 3, we show the determinant expressions of Chebyshev polynomials, which play important roles in the expressions of Green matrices. In section 4, we prove lemmas and theorems. Finally, in section 5, we consider the eigenvalue problem for the bending of a string. From spectral decompositions of Green or pseudo Green matrices, we obtain nontrivial and interesting equalities concerning trigonometric functions.

2. Main results

First, we state the positivity and hierarchical structure of Green matrices $\boldsymbol{G}(a)$ and $\boldsymbol{G}(0)$.

Theorem 2.1. *The elements $g(\mathrm{X};a;i,j)$ $(0 \le i,j \le N-1)$ of Green matrix $\boldsymbol{G}(a)$ satisfy the following inequalities:*

$$g(\mathrm{X};a;i,j) > 0 \qquad (0<a<\infty) \qquad (\mathrm{X}) = (0,0),(0,1),(1,0),(1,1),(\mathrm{P}), \tag{2.1}$$

$$0 < g(0,0;a;i,j) < \left\{ \begin{array}{l} g(0,1;a;i,j) \\ g(1,0;a;i,j) \end{array} \right\} < g(1,1;a;i,j) \qquad (0<a<\infty), \tag{2.2}$$

$$g(\mathrm{X};0;i,j) > 0 \qquad (\mathrm{X}) = (0,0),(0,1),(1,0), \tag{2.3}$$

$$0 < g(0,0;0;i,j) < \left\{ \begin{array}{l} g(0,1;0;i,j) \\ g(1,0;0;i,j) \end{array} \right\}. \tag{2.4}$$

(2.1) *and* (2.3) *show the positivity of Green matrices $\boldsymbol{G}(a)$ and $\boldsymbol{G}(0)$, respectively.* (2.2) *and* (2.4) *show the hierarchical structure of Green matrices $\boldsymbol{G}(a)$ and $\boldsymbol{G}(0)$.*

The hierarchical structure of Green matrices (2.2) shows that if boundary condition (X) becomes looser as $(0,0) \to (0,1)$ or $(1,0) \to (1,1)$, Green matrix gets larger. The continuous version is given in [3, Theorem 0.3]. It should be noted that $\boldsymbol{G}_*$ takes negative value owing to the orthogonality condition $\boldsymbol{G}_*\boldsymbol{E}_0 = \boldsymbol{O}$. Hence, we remove $\boldsymbol{G}_*$ from the hierarchical structure.

Next, we state the best constant of the discrete Sobolev inequality corresponding to Lemma 1.1 and 1.2. For $\boldsymbol{u},\boldsymbol{v} \in \mathbf{C}^N$, we introduce the inner products and norms as

$$(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{v}^*\boldsymbol{u}, \qquad \|\boldsymbol{u}\|^2 = (\boldsymbol{u},\boldsymbol{u}),$$
$$(\boldsymbol{u},\boldsymbol{v})_H = ((\boldsymbol{A}+a\boldsymbol{I})\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{v}^*(\boldsymbol{A}+a\boldsymbol{I})\boldsymbol{u}, \qquad \|\boldsymbol{u}\|_H^2 = (\boldsymbol{u},\boldsymbol{u})_H,$$

where $\boldsymbol{u}^* = {}^t\overline{\boldsymbol{u}}$. We prepare vector spaces

$$\mathbf{H}(\mathrm{X}) = \left\{ \boldsymbol{u} \in \mathbf{C}^N \;\middle|\; \left\{ \begin{array}{ll} \text{nothing} & (\mathrm{X}) = (0,0),(0,1),(1,0) \\ {}^t\mathbf{1}\boldsymbol{u} = 0 & (\mathrm{X}) = (1,1),(\mathrm{P}) \end{array} \right\} \right\}.$$

For $\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathrm{X})$, we introduce the inner product and norm as

$$(\boldsymbol{u},\,\boldsymbol{v})_A = (\boldsymbol{A}\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{v}^*\boldsymbol{A}\boldsymbol{u}, \qquad \|\boldsymbol{u}\|_A^2 = (\boldsymbol{u},\boldsymbol{u})_A.$$

$\|\boldsymbol{u}\|^2$, $\|\boldsymbol{u}\|_H^2$ and $\|\boldsymbol{u}\|_A^2$ are also expressed as

$$\|\boldsymbol{u}\|^2 = \sum_{i=0}^{N-1} |u(i)|^2,$$

$$\|\boldsymbol{u}\|_H^2 = \|\boldsymbol{u}\|_A^2 + a\|\boldsymbol{u}\|^2,$$

$$\|\boldsymbol{u}\|_A^2 = \begin{cases} |u(0)|^2 + \sum_{i=0}^{N-2} |u(i)-u(i+1)|^2 + |u(N-1)|^2 & (\mathrm{X}) = (0,0) \\ |u(0)|^2 + \sum_{i=0}^{N-2} |u(i)-u(i+1)|^2 & (\mathrm{X}) = (0,1) \\ \sum_{i=0}^{N-2} |u(i)-u(i+1)|^2 + |u(N-1)|^2 & (\mathrm{X}) = (1,0) \\ \sum_{i=0}^{N-2} |u(i)-u(i+1)|^2 & (\mathrm{X}) = (1,1) \\ \sum_{i=0}^{N-2} |u(i)-u(i+1)|^2 + |u(N-1)-u(0)|^2 & (\mathrm{X}) = (\mathrm{P}) \end{cases}.$$

We set Kronecker delta symbol $\delta(i) = 1 \ (i = 0), \quad 0 \ (i \neq 0)$. For any fixed $0 \le j \le N-1$, we introduce the delta vector

$$\boldsymbol{\delta}_j = {}^t(\delta(-j), \delta(1-j), \cdots, \delta(N-1-j)) \in \mathbf{C}^N.$$

We state the best constant of 2 kinds of the discrete Sobolev inequalities in two cases $a > 0$ and $a = 0$, corresponding to the bending problem of a string.

Theorem 2.2. *Let a be a positive constant. There exists a positive constant C such that for any $\boldsymbol{u} \in \mathbf{C}^N$ the discrete Sobolev inequality*

$$\left(\max_{0 \le j \le N-1} |\, u(j) \,| \right)^2 \le C \, \| \, \boldsymbol{u} \, \|_H^2$$

holds. Among such C, the best (least) constant $C_0(a)$ is

$$C_0(a) = \max_{0 \le j \le N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}(a) \boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}(a) \boldsymbol{\delta}_{j_0} = g(\mathrm{X}; a; j_0, j_0).$$

Here, $\boldsymbol{G}(a)$ is given in Lemma 1.1. The concrete forms of $C_0(a) = C_0(\mathrm{X}; a)$ are given as

$$C_0(0,0;a) = g(0,0;a;j_0,j_0) = \frac{U_{\lfloor \frac{N+1}{2} \rfloor}(x)\, U_{\lfloor \frac{N+2}{2} \rfloor}(x)}{U_{N+1}(x)} = \frac{\mathrm{sh}(\lfloor \frac{N+1}{2} \rfloor y)\, \mathrm{sh}(\lfloor \frac{N+2}{2} \rfloor y)}{\mathrm{sh}((N+1)y)\, \mathrm{sh}(y)},$$

$$j_0 = \begin{cases} \dfrac{N-2}{2}, \quad \dfrac{N}{2} & (N = 2, 4, 6, \cdots) \\ \dfrac{N-1}{2} & (N = 3, 5, 7, \cdots) \end{cases},$$

$$C_0(0,1;a) = g(0,1;a;j_0,j_0)\Big|_{j_0 = N-1} =$$

$$\frac{U_N(x)}{U_{N+1}(x)-U_N(x)}=\frac{\mathrm{sh}(Ny)}{2\,\mathrm{ch}((N+1/2)y)\,\mathrm{sh}(y/2)},$$

$$C_0(1,0;a)=g(1,0;a;j_0,j_0)\Big|_{j_0=0}=$$

$$\frac{U_N(x)}{U_{N+1}(x)-U_N(x)}=\frac{\mathrm{sh}(Ny)}{2\,\mathrm{ch}((N+1/2)y)\,\mathrm{sh}(y/2)},$$

$$C_0(1,1;a)=g(1,1;a;j_0,j_0)\Big|_{j_0=0,N-1}=$$

$$\frac{U_N(x)-U_{N-1}(x)}{U_{N+1}(x)-2U_N(x)+U_{N-1}(x)}=\frac{\mathrm{ch}((N-1/2)y)}{2\,\mathrm{sh}(Ny)\,\mathrm{sh}(y/2)},$$

$$C_0(\mathrm{P};a)=g(\mathrm{P};a;j_0,j_0)=\frac{U_N(x)}{2(T_N(x)-1)}=\frac{\mathrm{ch}(Ny/2)}{2\,\mathrm{sh}(Ny/2)\,\mathrm{sh}(y)},$$

for any $0\le j_0\le N-1$.

If we replaces C *by* $C_0(a)$ *in the above inequality, the equality holds if and only if the constant multiple of*

$$\boldsymbol{u}=\boldsymbol{G}(a)\boldsymbol{\delta}_{j_0}={}^t(\qquad g(\mathrm{X};a;i,j_0)\qquad)_{0\le i\le N-1}.$$

In the case of $(\mathrm{X})=(0,0)$, $\lfloor x\rfloor$ *is an integer part of a real number* x *as*

$$\lfloor x\rfloor=\sup\{n\in\mathbf{Z}\,|\,n\le x\,\}. \tag{2.5}$$

We use constants x,y *in* (1.1) *instead of* a.

The following theorem states the case $a=0$.

Theorem 2.3. *There exists a positive constant* C *such that for any* $\boldsymbol{u}\in\mathbf{H}(\mathrm{X})$ *the discrete Sobolev inequality*

$$\left(\max_{0\le j\le N-1}|\,u(j)\,|\right)^2\le C\,\|\,\boldsymbol{u}\,\|_A^2$$

holds. Among such C, *the best constant* C_0 *is given as*

$$C_0=\begin{cases}\displaystyle\max_{0\le j\le N-1}{}^t\boldsymbol{\delta}_j\boldsymbol{G}(0)\boldsymbol{\delta}_j={}^t\boldsymbol{\delta}_{j_0}\boldsymbol{G}(0)\boldsymbol{\delta}_{j_0}=g(\mathrm{X};0;j_0,j_0) & (\mathrm{X})=(0,0),(0,1),(1,0)\\ \displaystyle\max_{0\le j\le N-1}{}^t\boldsymbol{\delta}_j\boldsymbol{G}_*\boldsymbol{\delta}_j={}^t\boldsymbol{\delta}_{j_0}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}=g_*(\mathrm{X};j_0,j_0) & (\mathrm{X})=(1,1),(\mathrm{P})\end{cases}.$$

Here, $\boldsymbol{G}(0)$ *and* $\boldsymbol{G}_*$ *are given in Lemma* 1.2. *The concrete forms of* $C_0=C_0(\mathrm{X})$ *are given as*

$$C_0(0,0)=g(0,0;0;j_0,j_0)=\frac{1}{N+1}\left\lfloor\frac{N+1}{2}\right\rfloor\left\lfloor\frac{N+2}{2}\right\rfloor,$$

$$j_0 = \begin{cases} \dfrac{N-2}{2}, \quad \dfrac{N}{2} & (N = 2,4,6,\cdots) \\ \dfrac{N-1}{2} & (N = 3,5,7,\cdots) \end{cases},$$

$$C_0(0,1) = g(0,1;0;j_0,j_0)\Big|_{j_0=N-1} = N,$$

$$C_0(1,0) = g(1,0;0;j_0,j_0)\Big|_{j_0=0} = N,$$

$$C_0(1,1) = g_*(1,1;j_0,j_0)\Big|_{j_0=0,N-1} = b_2(2N;0)+b_2(2N;1) = \frac{1}{6N}(N-1)(2N-1),$$

$$C_0(\mathrm{P}) = g_*(\mathrm{P};j_0,j_0) = b_2(N;0) = \frac{N^2-1}{12N}, \qquad \textit{for any} \ \ 0 \le j_0 \le N-1.$$

If we replace C *by* C_0 *in the above inequality, the equality holds if and only if the constant multiple of*

$$\boldsymbol{u} = \begin{cases} \boldsymbol{G}(0)\boldsymbol{\delta}_{j_0} = {}^t(\ \ g(\mathrm{X};0;i,j_0)\ \ )_{0\le i\le N-1} & (\mathrm{X}) = (0,0),(0,1),(1,0) \\ \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} = {}^t(\ \ g_*(\mathrm{X};i,j_0)\ \ )_{0\le i\le N-1} & (\mathrm{X}) = (1,1),(\mathrm{P}) \end{cases}.$$

In the case of $(\mathrm{X}) = (0,0)$*, we use* (2.5).

The engineering meaning of the discrete Sobolev inequality is that the square of the maximum bending displacement of a string $u(i)$ is estimated from above by the constant multiple of its potential energy $\| \boldsymbol{u} \|_H^2$ or $\| \boldsymbol{u} \|_A^2$. In this paper, we have the best constants of discrete Sobolev inequalities which are obtained through the construction of Green matrix and the pseudo Green matrix. If we have the best constant of discrete Sobolev inequality, then we estimate the maximum of the bending of a string and have the shape of a string from the best vector.

We note that the main results concerning continuous and discrete cases are partially solved in our previous papers which are shown in Table 1. Although some results of this paper in the discrete cases of $a > 0$ and $a = 0$ are partially solved, as are shown in the above table, we also treat them for the sake of self-containedness.

Table 1 Concerning this paper and previous paper.

(X)	Continuous ($a > 0$)	Continuous ($a = 0$)	Discrete ($a > 0$)	Discrete ($a = 0$)
$(0,0)$	[4]	[5]	This paper	This paper
$(0,1)$	[4]	[5]	This paper	[5]
$(1,0)$	[4]	[5]	This paper	This paper
$(1,1)$	[4]	[5]	This paper	This paper
(P)	[4]	[5]	[7]	[1, 7]

3. Chebyshev polynomials

For $N = 0, 1, 2, \cdots$, we introduce Chebyshev polynomials $T_N(x)$ and $U_N(x)$. $T_N(x)$ is defined by the recurrence relation

$$\begin{cases} T_N(x) - 2xT_{N-1}(x) + T_{N-2}(x) = 0 \qquad (N = 2, 3, 4, \cdots) \\ T_0(x) = 1, \quad T_1(x) = x \end{cases} . \tag{3.1}$$

From this recurrence relation, we have

$$T_2(x)=2x^2-1, \quad T_3(x)=4x^3-3x, \quad T_4(x)=8x^4-8x^2+1, \quad T_5(x)=16x^5-20x^3+5x.$$

$U_N(x)$ is defined by the recurrence relation

$$\begin{cases} U_N(x) - 2xU_{N-1}(x) + U_{N-2}(x) = 0 \qquad (N = 2, 3, 4, \cdots) \\ U_0(x) = 0, \quad U_1(x) = 1 \end{cases} . \tag{3.2}$$

From this recurrence relation, we have

$$U_2(x) = 2x, \quad U_3(x) = 4x^2 - 1, \quad U_4(x) = 8x^3 - 4x, \quad U_5(x) = 16x^4 - 12x^2 + 1.$$

We note that

$$T_N(1) = 1, \qquad U_N(1) = N \qquad (N = 0, 1, 2, \cdots). \tag{3.3}$$

Chebyshev polynomials $T_N(x)$ and $U_N(x)$ are also defined by (1.2). From the definition of Chebyshev polynomials (1.2), we have

$$T_N(x)\Big|_{x=\frac{2+a}{2}=\cos(\sqrt{-1}y)=\mathrm{ch}(y)} = T_N(\cos(\sqrt{-1}y)) = \cos(\sqrt{-1}Ny) = \mathrm{ch}(Ny), \tag{3.4}$$

$$U_N(x)\Big|_{x=\frac{2+a}{2}=\cos(\sqrt{-1}y)=\mathrm{ch}(y)} = U_N(\cos(\sqrt{-1}y)) = \frac{\sin(\sqrt{-1}Ny)}{\sin(\sqrt{-1}y)} = \frac{\mathrm{sh}(Ny)}{\mathrm{sh}(y)}. \tag{3.5}$$

Taking a difference with respect to N, we have

$$(U_{N+1}(x) - U_N(x))\Big|_{x=\frac{2+a}{2}=\cos(\sqrt{-1}y)=\mathrm{ch}(y)} =$$
$$\frac{\mathrm{sh}((N+1)y) - \mathrm{sh}(Ny)}{\mathrm{sh}(y)} = \frac{2\mathrm{ch}((N+1/2)y)\mathrm{sh}(y/2)}{2\mathrm{sh}(y/2)\mathrm{ch}(y/2)} = \frac{\mathrm{ch}((N+1/2)y)}{\mathrm{ch}(y/2)}, \tag{3.6}$$

$$(U_{N+1}(x) - U_{N-1}(x))\Big|_{x=\frac{2+a}{2}=\cos(\sqrt{-1}y)=\mathrm{ch}(y)} =$$
$$\frac{\mathrm{sh}((N+1)y) - \mathrm{sh}((N-1)y)}{\mathrm{sh}(y)} = \frac{2\mathrm{ch}(Ny)\mathrm{sh}(y)}{\mathrm{sh}(y)} = 2\mathrm{ch}(Ny) = 2T_N(x) \tag{3.7}$$

by used (3.4) and (3.5). The following Lemma 3.1 states a determinant expression of Chebyshev polynomials.

Lemma 3.1. *$T_N(x)$ and $U_N(x)$, together with their difference, possess the following determinant expressions:*

$$U_N(x) = \begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x \end{vmatrix}_{(N-1)\times(N-1)}, \tag{3.8}$$

$$U_{N+1}(x) - U_N(x) = \begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{N\times N}, \tag{3.9}$$

$$U_{N+1}(x) - U_N(x) = \begin{vmatrix} 2x-1 & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x \end{vmatrix}_{N\times N}, \tag{3.10}$$

$$U_{N+1}(x) - 2U_N(x) + U_{N-1}(x) = \begin{vmatrix} 2x-1 & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{N\times N}, \tag{3.11}$$

$$2(T_N(x) - 1) = \begin{cases} \begin{vmatrix} 2x & -2 \\ -2 & 2x \end{vmatrix}_{2\times 2} & (N = 2) \\ \begin{vmatrix} 2x & -1 & & & -1 \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ -1 & & & -1 & 2x \end{vmatrix}_{N\times N} & (N = 3, 4, 5, \cdots) \end{cases}, \tag{3.12}$$

where $N = 3, 4, 5, \cdots$ in (3.8) and $N = 2, 3, 4, \cdots$ in (3.9) $\sim$ (3.12).

Proof of Lemma 3.1 We first prove (3.8). We introduce an $(N-1)\times(N-1)$ determinant defined by

$$V_N(x) = \begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x \end{vmatrix}_{(N-1)\times(N-1)} .$$

From the expansion with respect to the first row, we have

$$V_N(x) = 2xV_{N-1}(x) - V_{N-2}(x)$$

which is equivalent to recurrence relations of (3.1) and (3.2). Putting $N = 3, 4$, we have

$$V_3(x) = \begin{vmatrix} 2x & -1 \\ -1 & 2x \end{vmatrix} = 4x^2 - 1, \qquad V_4(x) = \begin{vmatrix} 2x & -1 & 0 \\ -1 & 2x & -1 \\ 0 & -1 & 2x \end{vmatrix} = 8x^3 - 4x.$$

From the uniqueness of difference equation, we obtain $V_N(x) = U_N(x)$.

Using the recurrence relation (3.2) and the determinant (3.8), (3.9) is obtained from

$$U_{N+1}(x) - U_N(x) = 2xU_N(x) - U_{N-1}(x) - U_N(x) = (2x-1)U_N(x) - U_{N-1}(x) =$$

$$(2x-1)\begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x \end{vmatrix}_{(N-1)\times(N-1)} +$$

$$\begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & 0 & -1 \end{vmatrix}_{(N-1)\times(N-1)} =$$

$$\begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{N\times N} .$$

(3.10) is shown in a similar way. Using the recurrence relation (3.2) and the determinant (3.8), (3.11) is obtained from

$$U_{N+1}(x) - 2U_N(x) + U_{N-1}(x) = (U_{N+1}(x) - U_N(x)) - (U_N(x) - U_{N-1}(x)) =$$
$$(2x-1)\,(U_N(x) - U_{N-1}(x)) - (U_{N-1}(x) - U_{N-2}(x)) =$$
$$(2x-1)\begin{vmatrix} 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{(N-1)\times(N-1)} +$$
$$\begin{vmatrix} -1 & 0 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{(N-1)\times(N-1)} =$$
$$\begin{vmatrix} 2x-1 & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x-1 \end{vmatrix}_{N\times N}.$$

We finally prove (3.12). In the case $N = 2, 3$, it is shown through simple calculations. We prove the case of $N \geq 4$. Using (3.7), the recurrence relation (3.2) and the determinant (3.8),

$$2(T_N(x) - 1) = U_{N+1}(x) - U_{N-1}(x) - 2 = 2xU_N(x) - 2U_{N-1}(x) - 2 =$$
$$2xU_N(x) - U_{N-1}(x) - 1 - 1 - U_{N-1}(x) =$$
$$2xU_N(x) +$$
$$\begin{vmatrix} -1 & 0 & & & -1 \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ & & & -1 & 2x \end{vmatrix}_{(N-1)\times(N-1)} + (-1)^N \begin{vmatrix} -1 & & & & -1 \\ 2x & -1 & & & \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \end{vmatrix}_{(N-1)\times(N-1)} =$$
$$\begin{vmatrix} 2x & -1 & & & -1 \\ -1 & 2x & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2x & -1 \\ -1 & & & -1 & 2x \end{vmatrix}_{N\times N}.$$

This completes the proof of Lemma 3.1. ■

4. Proof of lemmas and theorems

Proof of Lemma 1.1 We show the elements of $\boldsymbol{G}(a)$. We use the definition of the inverse matrix [2, p.61]

$$g(\mathrm{X}; a; i, j) = \frac{\Delta_{ji}(\mathrm{X})}{\Delta(\mathrm{X})} \qquad (0 \le i, j \le N-1),$$
$$\Delta_{ij}(\mathrm{X}) = (i, j) \text{ cofactor of } (\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}), \qquad \Delta(\mathrm{X}) = \det(\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}).$$

Since it loses essentially nothing, we illustrate through the case $N = 5$. First, we show $\Delta(\mathrm{X})$. From Lemma 3.1, we have

$$\begin{aligned} \Delta(m, n) &= \left\{ \begin{array}{ll} U_6(x) & (m, n) = (0, 0) \\ U_6(x) - U_5(x) & (m, n) = (0, 1), (1, 0) \\ U_6(x) - 2U_5(x) + U_4(x) & (m, n) = (1, 1) \end{array} \right\} \\ &= U_6(x) - (m+n)U_5(x) + mnU_4(x), \\ \Delta(\mathrm{P}) &= 2(T_5(x) - 1). \end{aligned}$$

Next, we show $\Delta_{ij}(\mathrm{X})$. Because $\boldsymbol{A}(\mathrm{X}) + a\boldsymbol{I}$ is a symmetric matrix, we consider $\Delta_{ij}(\mathrm{X})$ $(0 \le i \le j \le 4)$. For the case of $(\mathrm{X}) = (m, n)$, we have

$$\Delta_{00}(m, n) =$$
$$(-1)^{0+0} \begin{vmatrix} 2x & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x - n \end{vmatrix} = \left(U_1(x) - mU_0(x)\right)\left(U_5(x) - nU_4(x)\right),$$

$$\Delta_{01}(m, n) =$$
$$(-1)^{0+1} \begin{vmatrix} -1 & -1 & & \\ & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x - n \end{vmatrix} = \left(U_1(x) - mU_0(x)\right)\left(U_4(x) - nU_3(x)\right),$$

$$\Delta_{02}(m, n) =$$
$$(-1)^{0+2} \begin{vmatrix} -1 & 2x & & \\ & -1 & -1 & \\ & & 2x & -1 \\ & & -1 & 2x - n \end{vmatrix} = \left(U_1(x) - mU_0(x)\right)\left(U_3(x) - nU_2(x)\right),$$

$$\Delta_{03}(m,n) =$$
$$(-1)^{0+3}\begin{vmatrix} -1 & 2x & -1 & \\ & -1 & 2x & \\ & & -1 & -1 \\ & & & 2x-n \end{vmatrix} = \left(U_1(x) - mU_0(x)\right)\left(U_2(x) - nU_1(x)\right),$$

$$\Delta_{04}(m,n) =$$
$$(-1)^{0+4}\begin{vmatrix} -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x \\ & & & -1 \end{vmatrix} = \left(U_1(x) - mU_0(x)\right)\left(U_1(x) - nU_0(x)\right),$$

$$\Delta_{11}(m,n) =$$
$$(-1)^{1+1}\begin{vmatrix} 2x-m & & & \\ & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x-n \end{vmatrix} = \left(U_2(x) - mU_1(x)\right)\left(U_4(x) - nU_3(x)\right),$$

$$\Delta_{12}(m,n) =$$
$$(-1)^{1+2}\begin{vmatrix} 2x-m & -1 & & \\ & -1 & -1 & \\ & & 2x & -1 \\ & & -1 & 2x-n \end{vmatrix} = \left(U_2(x) - mU_1(x)\right)\left(U_3(x) - nU_2(x)\right),$$

$$\Delta_{13}(m,n) =$$
$$(-1)^{1+3}\begin{vmatrix} 2x-m & -1 & & \\ & -1 & 2x & \\ & & -1 & -1 \\ & & & 2x-n \end{vmatrix} = \left(U_2(x) - mU_1(x)\right)\left(U_2(x) - nU_1(x)\right),$$

$$\Delta_{14}(m,n) =$$
$$(-1)^{1+4}\begin{vmatrix} 2x-m & -1 & & \\ & -1 & 2x & -1 \\ & & -1 & 2x \\ & & & -1 \end{vmatrix} = \left(U_2(x) - mU_1(x)\right)\left(U_1(x) - nU_0(x)\right),$$

$$\Delta_{22}(m,n) =$$
$$(-1)^{2+2}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & & \\ & & 2x & -1 \\ & & -1 & 2x-n \end{vmatrix} = \left(U_3(x) - mU_2(x)\right)\left(U_3(x) - nU_2(x)\right),$$

$$\Delta_{23}(m,n) =$$
$$(-1)^{2+3}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & -1 & \\ & & -1 & -1 \\ & & & 2x-n \end{vmatrix} = \left(U_3(x) - mU_2(x)\right)\left(U_2(x) - nU_1(x)\right),$$

$$\Delta_{24}(m,n) =$$
$$(-1)^{2+4}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & -1 & \\ & & -1 & 2x \\ & & & -1 \end{vmatrix} = \left(U_3(x) - mU_2(x)\right)\left(U_1(x) - nU_0(x)\right),$$

$$\Delta_{33}(m,n) =$$
$$(-1)^{3+3}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & \\ & & & 2x-n \end{vmatrix} = \left(U_4(x) - mU_3(x)\right)\left(U_2(x) - nU_1(x)\right),$$

$$\Delta_{34}(m,n) =$$
$$(-1)^{3+4}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & & -1 \end{vmatrix} = \left(U_4(x) - mU_3(x)\right)\left(U_1(x) - nU_0(x)\right),$$

$$\Delta_{44}(m,n) =$$
$$(-1)^{4+4}\begin{vmatrix} 2x-m & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x \end{vmatrix} = \left(U_5(x) - mU_4(x)\right)\left(U_1(x) - nU_0(x)\right).$$

Hence we have

$$\boldsymbol{G}(a) = \left(\frac{\left(U_{i\wedge j+1}(x) - mU_{i\wedge j}(x)\right)\left(U_{5-i\vee j}(x) - nU_{4-i\vee j}(x)\right)}{U_6(x) - (m+n)U_5(x) + mnU_4(x)} \right)_{0\le i,j\le 4}.$$

For the case of (X) = (P), we have

$$\Delta_{00}(\mathrm{P}) = (-1)^{0+0}\begin{vmatrix} 2x & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x \end{vmatrix} = U_5(x) + U_0(x),$$

$$\Delta_{01}(\mathrm{P}) = (-1)^{0+1} \begin{vmatrix} -1 & -1 & & \\ & 2x & -1 & \\ & -1 & 2x & -1 \\ -1 & & -1 & 2x \end{vmatrix} = U_4(x) + U_1(x),$$

$$\Delta_{02}(\mathrm{P}) = (-1)^{0+2} \begin{vmatrix} -1 & 2x & & \\ & -1 & -1 & \\ & & 2x & -1 \\ -1 & & -1 & 2x \end{vmatrix} = U_3(x) + U_2(x),$$

$$\Delta_{03}(\mathrm{P}) = (-1)^{0+3} \begin{vmatrix} -1 & 2x & -1 & \\ & -1 & 2x & \\ & & -1 & -1 \\ -1 & & & 2x \end{vmatrix} = U_2(x) + U_3(x),$$

$$\Delta_{04}(\mathrm{P}) = (-1)^{0+4} \begin{vmatrix} -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x \\ -1 & & & -1 \end{vmatrix} = U_1(x) + U_4(x),$$

$$\Delta_{11}(\mathrm{P}) = (-1)^{1+1} \begin{vmatrix} 2x & & & -1 \\ & 2x & -1 & \\ & -1 & 2x & -1 \\ -1 & & -1 & 2x \end{vmatrix} = U_5(x) + U_0(x),$$

$$\Delta_{12}(\mathrm{P}) = (-1)^{1+2} \begin{vmatrix} 2x & -1 & & -1 \\ & -1 & -1 & \\ & & 2x & -1 \\ -1 & & -1 & 2x \end{vmatrix} = U_4(x) + U_1(x),$$

$$\Delta_{13}(\mathrm{P}) = (-1)^{1+3} \begin{vmatrix} 2x & -1 & & -1 \\ & -1 & 2x & \\ & & -1 & -1 \\ -1 & & & 2x \end{vmatrix} = U_3(x) + U_2(x),$$

$$\Delta_{14}(\mathrm{P}) = (-1)^{1+4} \begin{vmatrix} 2x & -1 & & \\ & -1 & 2x & -1 \\ & & -1 & 2x \\ -1 & & & -1 \end{vmatrix} = U_2(x) + U_3(x),$$

$$\Delta_{22}(\mathrm{P}) = (-1)^{2+2} \begin{vmatrix} 2x & -1 & & -1 \\ -1 & 2x & & \\ & & 2x & -1 \\ -1 & & -1 & 2x \end{vmatrix} = U_5(x) + U_0(x),$$

$$\Delta_{23}(\mathrm{P}) = (-1)^{2+3} \begin{vmatrix} 2x & -1 & & -1 \\ -1 & 2x & -1 & \\ & & -1 & -1 \\ -1 & & & 2x \end{vmatrix} = U_4(x) + U_1(x),$$

$$\Delta_{24}(\mathrm{P}) = (-1)^{2+4} \begin{vmatrix} 2x & -1 & & \\ -1 & 2x & -1 & \\ & & -1 & 2x \\ -1 & & & -1 \end{vmatrix} = U_3(x) + U_2(x),$$

$$\Delta_{33}(\mathrm{P}) = (-1)^{3+3} \begin{vmatrix} 2x & -1 & & -1 \\ -1 & 2x & -1 & \\ & -1 & 2x & \\ -1 & & & 2x \end{vmatrix} = U_5(x) + U_0(x),$$

$$\Delta_{34}(\mathrm{P}) = (-1)^{3+4} \begin{vmatrix} 2x & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ -1 & & & -1 \end{vmatrix} = U_4(x) + U_1(x),$$

$$\Delta_{44}(\mathrm{P}) = (-1)^{4+4} \begin{vmatrix} 2x & -1 & & \\ -1 & 2x & -1 & \\ & -1 & 2x & -1 \\ & & -1 & 2x \end{vmatrix} = U_5(x) + U_0(x).$$

Hence we have

$$\boldsymbol{G}(a) = \left(\quad \frac{U_{5-|i-j|}(x) + U_{|i-j|}(x)}{2(T_5(x) - 1)} \quad \right)_{0 \le i,j \le 4}.$$

Thus we have $g(\mathrm{X}; a; i, j)$ of Chebyshev polynomial expression. The hyperbolic function expression follows from Chebyshev polynomial expression and (3.4) $\sim$ (3.7). This proves Lemma 1.1. ■

Proof of Lemma 1.2 We show the elements of $\boldsymbol{G}(0)$ in the case of (X) =

$(0,0),(0,1),(1,0)$. Taking limit as $a\to 0$ for Chebyshev polynomial expression of $g(\mathrm{X};a;i,j)$ in Lemma 1.1 and using (3.3), we have $g(\mathrm{X};0;i,j)$. ■

Proof of Lemma 1.3 We show the elements of $\boldsymbol{G}_*$. First, we treat the case of (X) = (P). We calculate $\boldsymbol{G}_*$ which is based on (1.4). Using hyperbolic function expression of $g(\mathrm{P};a;i,j)$ in Lemma 1.1 and noting (1.1), we have

$$g(\mathrm{P};a;i,j)-\frac{1}{aN}=\frac{\mathrm{ch}((N/2-|i-j|)y)}{4\,\mathrm{sh}(Ny/2)\,\mathrm{sh}(y/2)\,\mathrm{ch}(y/2)}-\frac{1}{4N\mathrm{sh}^2(y/2)}=$$
$$\frac{Ny/2}{\mathrm{sh}(Ny/2)}\left(\frac{y/2}{\mathrm{sh}(y/2)}\right)^2\frac{1}{\mathrm{ch}(y/2)}\frac{2}{N^2y^3}h(y),$$
$$h(y)=N\,\mathrm{ch}((N/2-|i-j|)y)\,\mathrm{sh}(y/2)-\mathrm{sh}(Ny/2)\,\mathrm{ch}(y/2).$$

Using Taylor expansion of $h(y)$ as

$$h(y)=$$
$$N\left(1+\frac{1}{2!}\left(\frac{N}{2}-|i-j|\right)^2y^2+O(y^4)\right)\left(\frac{y}{2}+\frac{1}{3!}\left(\frac{y}{2}\right)^3+O(y^5)\right)-$$
$$\left(\frac{Ny}{2}+\frac{1}{3!}\left(\frac{Ny}{2}\right)^3+O(y^5)\right)\left(1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+O(y^4)\right)=$$
$$\left(\frac{N}{4}|i-j|^2-\frac{N^2}{4}|i-j|+\frac{1}{24}N(N^2-1)\right)y^3+O(y^5)\qquad(y\to 0),$$

we have

$$g_*(\mathrm{P};i,j)=\lim_{a\to 0}\left(g(\mathrm{P};a;i,j)-\frac{1}{aN}\right)=$$
$$\lim_{a\to 0}\left(\frac{\mathrm{ch}((N/2-|i-j|)y)}{2\mathrm{sh}(Ny/2)\,\mathrm{sh}(y)}-\frac{1}{4N\mathrm{sh}^2(y/2)}\right)=$$
$$\frac{1}{2N}|i-j|^2-\frac{1}{2}|i-j|+\frac{N^2-1}{12N}=b_2(N;|i-j|),$$

where $b_2(N;i)$ is given as (1.5). Next, we treat the case of (X) = (1, 1). We calculate $\boldsymbol{G}_*$ which is based on (1.4). Using hyperbolic function expression of $g(1,1;a;i,j)$ in Lemma 1.1 and noting (1.1), we have

$$g(1,1;a;i,j)-\frac{1}{aN}=$$
$$\frac{1}{2\mathrm{sh}(Ny)\,\mathrm{sh}(y)}\Big[\mathrm{ch}((N-|i-j|)y)+\mathrm{ch}((N-1-i-j)y)\Big]-\frac{1}{4N\mathrm{sh}^2(y/2)}=$$
$$\frac{\mathrm{ch}((N-|i-j|)y)}{2\mathrm{sh}(Ny)\,\mathrm{sh}(y)}-\frac{1}{8N\mathrm{sh}^2(y/2)}+\frac{\mathrm{ch}((N-1-i-j)y)}{2\mathrm{sh}(Ny)\,\mathrm{sh}(y)}-\frac{1}{8N\mathrm{sh}^2(y/2)}.$$

Using the relation

$$b_2(N;i) = \lim_{a\to 0}\left(\frac{\mathrm{ch}((N/2-i)y)}{2\mathrm{sh}(Ny/2)\,\mathrm{sh}(y)} - \frac{1}{4N\mathrm{sh}^2(y/2)}\right),$$

we have $g_*(1,1;i,j)$. This completes the proof of Lemma 1.2. ■

Proof of Theorem 2.1 We use the hyperbolic function expression of Green matrix $\boldsymbol{G}(a)$ in Lemma 1.1. The positivity (2.1) and (2.3) are obvious. We treat the hierarchical structure (2.2) and (2.4). We show the upper berth of (2.2) as

$$g(0,1;a;i,j) - g(0,0;a;i,j) = \frac{\mathrm{sh}((i\wedge j+1)y)}{\mathrm{sh}(y)\,\mathrm{sh}((N+1)y)\,\mathrm{ch}((N+1/2)y)}p(y),$$
$$g(1,1;a;i,j) - g(0,1;a;i,j) = \frac{\mathrm{ch}((N-1/2-i\vee j)y)}{\mathrm{sh}(y)\,\mathrm{sh}(Ny)\,\mathrm{ch}((N+1/2)y)}q(y),$$

where $p(y)$ and $q(y)$ are

$$p(y) =$$
$$\mathrm{sh}((N+1)y)\,\mathrm{ch}((N-1/2-i\vee j)y) - \mathrm{ch}((N+1/2)y)\,\mathrm{sh}((N-i\vee j)y) =$$
$$\frac{1}{2}\Big[\mathrm{sh}((3/2+i\vee j)y) + \mathrm{sh}((1/2+i\vee j)y)\Big] > 0,$$

$$q(y) =$$
$$\mathrm{ch}((N+1/2)y)\mathrm{ch}((i\wedge j+1/2)y) - \mathrm{sh}(Ny)\mathrm{sh}((i\wedge j+1)y) =$$
$$\frac{1}{2}\Big[\mathrm{ch}((N-i\wedge j)y) + \mathrm{ch}((N-1-i\wedge j)y)\Big] > 0.$$

We can show the lower berth of (2.2) in the same way. Taking limit as $a\to 0$ for (2.2), we have (2.4). This completes the proof of Theorem 2.1. ■

The basic idea for the best constant of the discrete Sobolev inequality can be seen in, for example, [1, 5, 6, 7, 8]. However, for the sake of self-containedness we give a proof. We prepare Lemma 4.1 and 4.2 which show that $\boldsymbol{G}$ and $\boldsymbol{G}_*$ are reproducing matrix for $\mathbf{C}^N$ with $(\cdot,\cdot)_H$ and $\mathbf{H}(\mathrm{X})$ with $(\cdot,\cdot)_A$, respectively. We set $\boldsymbol{G} = \boldsymbol{G}(a)$ for short.

Lemma 4.1. *For any $\boldsymbol{u}\in\mathbf{C}^N$ and fixed j $(0\le j\le N-1)$, we have the reproducing relations:*

$$u(j) = (\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_H. \tag{4.1}$$

In particular, putting $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_j$, we have

$${}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_j = \|\boldsymbol{G}\boldsymbol{\delta}_j\|_H^2. \tag{4.2}$$

Lemma 4.2. *For any* $\boldsymbol{u} \in \mathbf{H}(\mathrm{X})$ *and fixed* j $(0 \le j \le N-1)$*, we have the reproducing relations:*

$$u(j) = (\boldsymbol{u},\, \boldsymbol{G}_* \boldsymbol{\delta}_j)_A. \tag{4.3}$$

In particular, putting $\boldsymbol{u} = \boldsymbol{G}_* \boldsymbol{\delta}_j$*, we have*

$$^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = \|\boldsymbol{G}_* \boldsymbol{\delta}_j\|_A^2. \tag{4.4}$$

Proof of Lemma 4.1 Noting $\boldsymbol{G}^* = \boldsymbol{G}$, we have (4.1) as

$$(\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_H = ((\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j) = {}^t\boldsymbol{\delta}_j \boldsymbol{G}(\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} = u(j),$$

which completes the proof of Lemma 4.1. ■

Proof of Lemma 4.2 In the case of (X) = (0, 0), (0, 1), (1, 0), taking the limit as $a \to 0$ for Lemma 4.1, we have Lemma 4.2. In the case of (X) = (1, 1), (P), noting $\boldsymbol{G}_*^* = \boldsymbol{G}_*$ and $\boldsymbol{E}_0 \boldsymbol{u} = \boldsymbol{0}$, we have (4.3) as

$$(\boldsymbol{u},\, \boldsymbol{G}_*\boldsymbol{\delta}_j)_A = (\boldsymbol{A}\boldsymbol{u},\, \boldsymbol{G}_*\boldsymbol{\delta}_j) = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{A}\boldsymbol{u} = {}^t\boldsymbol{\delta}_j (\boldsymbol{I} - \boldsymbol{E}_0)\boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} = u(j),$$

which completes the proof of Lemma 4.2. ■

Proof of Theorem 2.2 Applying Schwarz inequality to (4.1) and using (4.2), we have

$$|u(j)|^2 \le \|\boldsymbol{u}\|_H^2 \|\boldsymbol{G}\boldsymbol{\delta}_j\|_H^2 = {}^t\boldsymbol{\delta}_j \boldsymbol{G}\boldsymbol{\delta}_j \|\boldsymbol{u}\|_H^2. \tag{4.5}$$

It should be noted that in performing Schwarz inequality in (4.5), the equality holds if and only if $\boldsymbol{u} = k\,\boldsymbol{G}\boldsymbol{\delta}_j$ $(k \neq 0,\ 0 \le j \le N-1)$. Taking the maximum with respect to j on both sides, we have the discrete Sobolev inequality

$$\left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le C_0(a) \|\boldsymbol{u}\|_H^2, \tag{4.6}$$

where

$$C_0(a) = \max_{0 \le j \le N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}\boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}\boldsymbol{\delta}_{j_0}. \tag{4.7}$$

The inequality (4.6) implies that $\|\boldsymbol{u}\|_H = 0$ holds if and only if $\boldsymbol{u} = \boldsymbol{0}$, which shows the positive definiteness of inner product $(\cdot, \cdot)_H$. If we take $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_{j_0}$ in (4.6), then we have

$$\left(\max_{0 \le j \le N-1} |{}^t\boldsymbol{\delta}_j \boldsymbol{G}\boldsymbol{\delta}_{j_0}| \right)^2 \le C_0(a) \|\boldsymbol{G}\boldsymbol{\delta}_{j_0}\|_H^2 = (C_0(a))^2.$$

Combining this with the trivial inequality

$$(C_0(a))^2 = ({}^t\boldsymbol{\delta}_{j_0}\boldsymbol{G}\boldsymbol{\delta}_{j_0})^2 \le \left(\max_{0\le j\le N-1} |{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_{j_0}|\right)^2,$$

we have

$$\left(\max_{0\le j\le N-1} |{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_{j_0}|\right)^2 = C_0(a)\,\|\boldsymbol{G}\boldsymbol{\delta}_{j_0}\|_H^2.$$

This shows that (4.7) is the best constant of (4.6) and the equality holds for $\boldsymbol{G}\boldsymbol{\delta}_{j_0}$.

Next, we search for j_0 which satisfies (4.7). We use the hyperbolic function expression of $g(\mathrm{X}; a; j, j)$ in Lemma 1.1. The maximum of

$$g(0,0;a;j,j) = \frac{\mathrm{ch}((N+1)y) - \mathrm{ch}((N-1-2j)y)}{2\mathrm{sh}((N+1)y)\mathrm{sh}(y)}$$

is attained at $j = (N-2)/2$ or $N/2$ if N is even and at $j = (N-1)/2$ if N is odd. The maximum of

$$g(0,1;a;j,j) = \frac{\mathrm{sh}((N+1/2)y) - \mathrm{sh}((N-3/2-2j)y)}{2\mathrm{ch}((N+1/2)y)\mathrm{sh}(y)}$$

is attained at $j = N-1$. The maximum of

$$g(1,0;a;j,j) = \frac{\mathrm{sh}((N+1/2)y) + \mathrm{sh}((N-1/2-2j)y)}{2\mathrm{ch}((N+1/2)y)\mathrm{sh}(y)}$$

is attained at $j = 0$. The maximum of

$$g(1,1;a;j,j) = \frac{\mathrm{ch}(Ny) + \mathrm{ch}((N-1-2j)y)}{2\mathrm{sh}(Ny)\,\mathrm{sh}(y)}$$

is attained at $j = 0$ or $N-1$. The maximum of $g(\mathrm{P}; a; j, j)$ is attained at any j_0 $(0 \le j_0 \le N-1)$, because $g(\mathrm{P}; a; j, j)$ is independent of j. This completes the proof of Theorem 2.2. ■

Proof of Theorem 2.3 In the case of $(\mathrm{X}) = (0,0), (0,1), (1,0)$, Theorem 2.3 follows from Theorem 2.2 by taking limit as $a \to 0$. Using (3.3), we have the concrete value of C_0.

We show the case of $(\mathrm{X}) = (1,1)$ and (P). Applying Schwarz inequality to (4.3) and using (4.4), we have

$$|u(j)|^2 \le \|\boldsymbol{u}\|_A^2 \|\boldsymbol{G}_*\boldsymbol{\delta}_j\|_A^2 = {}^t\boldsymbol{\delta}_j\boldsymbol{G}_*\boldsymbol{\delta}_j\|\boldsymbol{u}\|_A^2. \tag{4.8}$$

It should be noted that in performing Schwarz inequality in (4.8), the equality holds if and only if $\boldsymbol{u} = k\,\boldsymbol{G}_*\boldsymbol{\delta}_j$ $(k \ne 0,\ 0 \le j \le N-1)$. Taking the maximum with respect to j on both sides, we obtain the discrete Sobolev inequality

$$\left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le C_0 \|\boldsymbol{u}\|_A^2, \tag{4.9}$$

where

$$C_0 = \max_{0 \le j \le N-1} {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_* \boldsymbol{\delta}_{j_0}. \tag{4.10}$$

The inequality (4.9) implies that $\|\boldsymbol{u}\|_A = 0$ holds if and only if $\boldsymbol{u} = \boldsymbol{0}$, which shows the positive definiteness of inner product $(\cdot, \cdot)_A$. Here, j_0 is the same in Theorem 2.2 because of the definition of $\boldsymbol{G}_*$ shown in (1.4). If we take $\boldsymbol{u} = \boldsymbol{G}_* \boldsymbol{\delta}_{j_0}$ in (4.9), then we have

$$\left(\max_{0 \le j \le N-1} |{}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_{j_0}| \right)^2 \le C_0 \|\boldsymbol{G}_* \boldsymbol{\delta}_{j_0}\|_A^2 = C_0^2.$$

Combining this with the trivial inequality

$$C_0^2 = ({}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_* \boldsymbol{\delta}_{j_0})^2 \le \left(\max_{0 \le j \le N-1} |{}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_{j_0}| \right)^2,$$

we have

$$\left(\max_{0 \le j \le N-1} |{}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_{j_0}| \right)^2 = C_0 \, \|\boldsymbol{G}_* \boldsymbol{\delta}_{j_0}\|_A^2.$$

This shows that (4.10) is the best constant of (4.9) and the equality holds for $\boldsymbol{G}_* \boldsymbol{\delta}_{j_0}$. If we apply $j = j_0$ which is the same in Theorem 2.2 to $g(\mathrm{X}; 0; j, j)$ and $g_*(\mathrm{X}; j, j)$, then we have the concrete form of C_0. This completes the proof of Theorem 2.3. ∎

5. Eigenvalue problem of bending of a string

We consider the following discrete string bending problem,

DEVP(X)

$$\begin{cases} -u(i-1)+2u(i)-u(i+1)=\lambda u(i) & (0\le i\le N-1) \\ \begin{cases} u(-1)=0, & u(N)=0 & (\mathrm{X})=(0,0) \\ u(-1)=0, & u(N-1)-u(N)=0 & (\mathrm{X})=(0,1) \\ u(-1)-u(0)=0, & u(N)=0 & (\mathrm{X})=(1,0) \\ u(-1)-u(0)=0, & u(N-1)-u(N)=0 & (\mathrm{X})=(1,1) \\ u(-1)=u(N-1), & u(0)=u(N) & (\mathrm{X})=(\mathrm{P}) \end{cases} \end{cases}$$

or equivalently the following matrix eigenvalue problem,

$$\text{DEVP(X)}$$
$$\boldsymbol{A}\boldsymbol{u}=\lambda\boldsymbol{u}.$$

We state the eigenvalues and eigenvectors of $\boldsymbol{A}=\boldsymbol{A}(\mathrm{X})$ as the following Lemma 5.1.

Lemma 5.1. DEVP(X) *has the eigenvalues* $\lambda=\lambda_k$ $(0\le k\le N-1)$ *whose normalized orthogonal eigenvectors* $\boldsymbol{u}=\boldsymbol{\varphi}_k$ $(0\le k\le N-1)$*. The concrete forms of* λ_k *and* $\boldsymbol{\varphi}_k$ *are as follows:*

$(\mathrm{X})=(0,0)$

$$\lambda_k=4\sin^2\left(\frac{k+1}{2(N+1)}\pi\right), \qquad 0<\lambda_0<\lambda_1<\cdots<\lambda_{N-1}<4,$$

$$\boldsymbol{\varphi}_k=\sqrt{\frac{2}{N+1}}\,{}^t\left(\cdots,\sin\left(\frac{(i+1)(k+1)}{N+1}\pi\right),\cdots\right)_{0\le i\le N-1}.$$

$(\mathrm{X})=(0,1)$

$$\lambda_k=4\sin^2\left(\frac{k+1/2}{2(N+1/2)}\pi\right), \qquad 0<\lambda_0<\lambda_1<\cdots<\lambda_{N-1}<4,$$

$$\boldsymbol{\varphi}_k=\sqrt{\frac{2}{N+1/2}}\,{}^t\left(\cdots,\sin\left(\frac{(i+1)(k+1/2)}{N+1/2}\pi\right),\cdots\right)_{0\le i\le N-1}.$$

$(\mathrm{X})=(1,0)$

$$\lambda_k=4\sin^2\left(\frac{k+1/2}{2(N+1/2)}\pi\right), \qquad 0<\lambda_0<\lambda_1<\cdots<\lambda_{N-1}<4,$$

$$\boldsymbol{\varphi}_k=\sqrt{\frac{2}{N+1/2}}\,{}^t\left(\cdots,\cos\left(\frac{(i+1/2)(k+1/2)}{N+1/2}\pi\right),\cdots\right)_{0\le i\le N-1}.$$

(X) $=(1,1)$

$$\lambda_k = 4\sin^2\left(\frac{k}{2N}\pi\right), \qquad 0=\lambda_0<\lambda_1<\cdots<\lambda_{N-1}<4,$$

$$\boldsymbol{\varphi}_k = \begin{cases} \dfrac{1}{\sqrt{N}}\,{}^t(1,1,\cdots,1)_{0\le i\le N-1} & (k=0) \\ \sqrt{\dfrac{2}{N}}\,{}^t\left(\cdots,\cos\left(\dfrac{(i+1/2)k}{N}\pi\right),\cdots\right)_{0\le i\le N-1} & (1\le k\le N-1) \end{cases}.$$

(X) $=$ (P)

$$\lambda_k = 4\sin^2\left(\frac{k}{N}\pi\right),$$

$$N=2n+1+\varepsilon \qquad (n=1,2,3,\cdots,\ \varepsilon=0,1),$$

$$\lambda_0=0<\lambda_1=\lambda_{N-1}<\cdots<\lambda_n=\lambda_{N-n}\begin{cases} <4 & (\varepsilon=0) \\ <\lambda_{N/2}=4 & (\varepsilon=1) \end{cases},$$

$$\boldsymbol{\varphi}_k = \begin{cases} \dfrac{1}{\sqrt{N}}\,{}^t(1,1,\cdots,1)_{0\le i\le N-1} & (k=0) \\ \dfrac{1}{\sqrt{N}}\,{}^t\left(\cdots,\exp\left(\dfrac{\sqrt{-1}2ik}{N}\pi\right),\cdots\right)_{0\le i\le N-1} & (1\le k\le N-1) \end{cases}.$$

In the case of (X) $=(0,0),(0,1),(1,0),(1,1)$*, since* $\boldsymbol{A}$ *is a real symmetric matrix* ${}^t\boldsymbol{A}=\boldsymbol{A}$*, the eigenvalues* λ_k *are distinct and the eigenvectors* $\boldsymbol{\varphi}_k$ *satisfy* $\boldsymbol{\varphi}_k^*\boldsymbol{\varphi}_\ell=\delta(k-\ell)$*. In the case of* (X) $=$ (P)*, we have* $\lambda_k=\lambda_{N-k}\ \ (k=1,2,\cdots,n)$ *and the corresponding eigenspace is two-dimensional. In this case, the eigenvectors* $\boldsymbol{\varphi}_k$ *also satisfy* $\boldsymbol{\varphi}_k^*\boldsymbol{\varphi}_\ell=\delta(k-\ell)$*.*

The proof of the above lemma in the case of (X) $=(0,0)$ and (P) are given in [9] and [7, Lemma 2.1], respectively. Since the outline of the proof is basically similar, we omit the proof.

For 5 kinds of DEVP(X), we introduce the diagonal matrix $\widehat{\boldsymbol{A}}=\mathrm{diag}\{\lambda_0,\cdots,\lambda_{N-1}\}$, the unitary $N\times N$ matrix $\boldsymbol{W}=(\boldsymbol{\varphi}_0,\cdots,\boldsymbol{\varphi}_{N-1})$ and the orthogonal projection matrices $\boldsymbol{E}_k=\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^*$ $(0\le k\le N-1)$.

We have the relation

$$\boldsymbol{W}^*\boldsymbol{W}=\boldsymbol{W}\boldsymbol{W}^*=\boldsymbol{I}, \qquad \boldsymbol{E}_k\boldsymbol{E}_l=\delta(k-l)\boldsymbol{E}_k, \qquad \boldsymbol{E}_k^*=\boldsymbol{E}_k.$$

Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{I}$, $\boldsymbol{A}$ and $\boldsymbol{A}+a\boldsymbol{I}$ as

$$\boldsymbol{I}=\boldsymbol{W}\boldsymbol{W}^*=\sum_{k=0}^{N-1}\boldsymbol{\varphi}_k\boldsymbol{\varphi}_k^*=\sum_{k=0}^{N-1}\boldsymbol{E}_k,$$

$$\boldsymbol{A} = \boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^* = \sum_{k=0}^{N-1} \lambda_k \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \begin{cases} \displaystyle\sum_{k=0}^{N-1} \lambda_k \boldsymbol{E}_k & (\mathrm{X}) = (0,0),\ (0,1),\ (1,0) \\ \displaystyle\sum_{k=1}^{N-1} \lambda_k \boldsymbol{E}_k & (\mathrm{X}) = (1,1),\ (\mathrm{P}) \end{cases},$$

$$\boldsymbol{A} + a\boldsymbol{I} = \boldsymbol{W}(\widehat{\boldsymbol{A}} + a\boldsymbol{I})\boldsymbol{W}^* = \sum_{k=0}^{N-1} (\lambda_k + a)\boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{N-1} (\lambda_k + a)\boldsymbol{E}_k,$$

$$(\mathrm{X}) = (0,0),\ (0,1),\ (1,0),\ (1,1),\ (\mathrm{P}).$$

We note that $\lambda_0 = 0$ in the case of $(\mathrm{X}) = (1,1)$ and (P). Using the eigenvalues λ_k $(0 \le k \le N-1)$ and the eigenvectors $\boldsymbol{\varphi}_k$ $(0 \le k \le N-1)$, we have the spectral decomposition of $\boldsymbol{G}(a)$ and $\boldsymbol{G}_*$ [6, 8] as

$$\boldsymbol{G}(a) = \sum_{k=0}^{N-1} \frac{1}{\lambda_k + a}\boldsymbol{E}_k, \qquad \boldsymbol{G}_* = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}\boldsymbol{E}_k. \tag{5.1}$$

Combining the expression of $\boldsymbol{G}(a)$ and $\boldsymbol{G}_*$ given in Lemma 1.1 and 1.2 with the spectral decomposition of $\boldsymbol{G}(a)$ and $\boldsymbol{G}_*$ given in (5.1), we have the non-trivial identities of trigonometric functions shown as Proposition 5.1 and 5.2.

Proposition 5.1. *Combining* $\boldsymbol{G}(a)$ *in Lemma* 1.1 *with* $\boldsymbol{G}(a)$ *in* (5.1) *and putting* $a = \mathrm{sh}^2(y/2)$ *in* (1.1), *we have the following identities.*

$$g(0,0;a;i,j) = \frac{\mathrm{sh}((i \wedge j + 1)y)\,\mathrm{sh}((N - i \vee j)y)}{\mathrm{sh}((N+1)y)\,\mathrm{sh}(y)} =$$

$$\frac{1}{2(N+1)} \sum_{k=0}^{N-1} \frac{\sin\left(\frac{(i+1)(k+1)}{N+1}\pi\right)\sin\left(\frac{(j+1)(k+1)}{N+1}\pi\right)}{\sin^2\left(\frac{k+1}{2(N+1)}\pi\right) + \mathrm{sh}^2(y/2)},$$

$$g(0,1;a;i,j) = \frac{\mathrm{sh}((i \wedge j + 1)y)\,\mathrm{ch}((N - 1/2 - i \vee j)y)}{\mathrm{ch}((N+1/2)y)\,\mathrm{sh}(y)} =$$

$$\frac{1}{2(N+1/2)} \sum_{k=0}^{N-1} \frac{\sin\left(\frac{(i+1)(k+1/2)}{N+1/2}\pi\right)\sin\left(\frac{(j+1)(k+1/2)}{N+1/2}\pi\right)}{\sin^2\left(\frac{k+1/2}{2(N+1/2)}\pi\right) + \mathrm{sh}^2(y/2)},$$

$$g(1,0;a;i,j) = \frac{\mathrm{ch}((i \wedge j + 1/2)y)\,\mathrm{sh}((N - i \vee j)y)}{\mathrm{ch}((N+1/2)y)\,\mathrm{sh}(y)} =$$

$$\frac{1}{2(N+1/2)} \sum_{k=0}^{N-1} \frac{\cos\left(\frac{(i+1/2)(k+1/2)}{N+1/2}\pi\right)\cos\left(\frac{(j+1/2)(k+1/2)}{N+1/2}\pi\right)}{\sin^2\left(\frac{k+1/2}{2(N+1/2)}\pi\right) + \mathrm{sh}^2(y/2)},$$

$$g(1,1;a;i,j) = \frac{\mathrm{ch}((i \wedge j + 1/2)y)\,\mathrm{ch}((N - 1/2 - i \vee j)y)}{\mathrm{sh}(Ny)\,\mathrm{sh}(y)} =$$

$$\frac{1}{4N\mathrm{sh}^2(y/2)} + \frac{1}{2N}\sum_{k=1}^{N-1} \frac{\cos\left(\frac{(i+1/2)k}{N}\pi\right)\cos\left(\frac{(j+1/2)k}{N}\pi\right)}{\sin^2\left(\frac{k}{2N}\pi\right) + \mathrm{sh}^2(y/2)},$$

$$g(\mathrm{P};a;i,j) = \frac{\mathrm{ch}((N/2 - |i-j|)y)}{2\mathrm{sh}(Ny/2)\,\mathrm{sh}(y)} = \frac{1}{4N\mathrm{sh}^2(y/2)} + \frac{1}{4N}\sum_{k=1}^{N-1} \frac{\exp\left(\frac{\sqrt{-1}2(i-j)k}{N}\pi\right)}{\sin^2\left(\frac{k}{N}\pi\right) + \mathrm{sh}^2(y/2)}.$$

Proposition 5.2. *Combining* $\boldsymbol{G}_*$ *in Lemma* 1.2 *with* $\boldsymbol{G}_*$ *in* (5.1)*, we have the following identities.*

$$g_*(1,1;i,j) = b_2(2N;|i-j|) + b_2(2N;1+i+j) =$$

$$\frac{1}{2N}\sum_{k=1}^{N-1} \frac{\cos\left(\frac{(i+1/2)k}{N}\pi\right)\cos\left(\frac{(j+1/2)k}{N}\pi\right)}{\sin^2\left(\frac{k}{2N}\pi\right)},$$

$$g_*(\mathrm{P};i,j) = b_2(N;|i-j|) = \frac{1}{4N}\sum_{k=1}^{N-1} \frac{\exp\left(\frac{\sqrt{-1}2(i-j)k}{N}\pi\right)}{\sin^2\left(\frac{k}{N}\pi\right)}.$$

We note that $b_2(N;i)$ *is the discrete Bernoulli polynomial* (1.5)*.*

Acknowledgements

This research is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No.18K03340 and 18K03347.

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Hiroyuki Yamagishi
Tokyo Metropolitan College of Industrial Technology
1-10-40 Higashi-oi, Shinagawa, Tokyo 140-0011, Japan
e-mail: yamagisi@metro-cit.ac.jp

Atsushi Nagai
Department of Computer Sciences, College of Liberal Arts
Tsuda University, 2-1-1 Tsuda-machi, Kodaira, Tokyo 187-8577, Japan
e-mail: a.nagai@tsuda.ac.jp