Getzler's symbol calculus and the composition of pseudodifferential operators on contact Riemannian manifolds

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Abstract

Following Getzler's idea from the geometric viewpoint as to symbol calculus on a spin manifold, we introduce a new symbol calculus of H-pseudodifferential operators on a contact Riemannian manifold with contact distribution H, which will turn out to be an effective tool for understanding the contact Riemannian structure from the viewpoint of calculus. An explicit formula for the top grading part of the symbol of composition of H-differential operators is presented. For general H-pseudodifferential operators, we introduce a method of computing that of their composition.

0 Introduction

On a spin manifold, Getzler [7] introduced a new symbol calculus of pseudodifferential operators by unifying two kinds of ideas: that of Widom ([14], [15]) about symbol calculus on Riemannian manifold and that of Alvarez-Gaumé ([1]) who used the Clifford variables to propose a suitable filtration of symbol space. Getzler's symbol calculus simplifies the calculation of the principal part, or the top grading part (cf. [4]), of the composition of symbols ([7, Theorems 2.7 and 3.5]) and consequently provides a remarkably short proof of the Atiyah-Singer index theorem for the Dirac operator ([7, §3], [8], [1]).

In this paper, on a contact Riemannian manifold with contact distribution H, following Getzler's idea we will introduce a similar symbol calculus of H-pseudodifferential operators, which will turn out to be an effective tool for understanding the contact Riemannian structure from the viewpoint of calculus. The manifold possesses canonical Spin^c structure, the Clifford variables associated with which provide similarly a filtration of symbol space, so that Getzler's idea can be applicable. The first result in this paper is Theorem 3.5, which offers an explicit formula for the top grading part (cf. (2.13)) of the composition of polynomial symbols, that is, the symbols of H-differential operators. In the spin manifold case its counterpart is [7, Theorem 2.7], which was certified by using the Campbell-Hausdorff formula. To prove Theorem 3.5, we will employ not the CH formula but the formula (1.1), which gives an explicit expression of the connection

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coefficients of the hermitian Tanno connection. The CH formula is so daunting that Benameur-Heitsch [4], who applied Getzler's idea to the case of foliated spin manifold, used Atiyah-Bott-Patodi's formula [2, Proposition 3.7] for the proof of [4, Theorem 4.6] which is a foliation version of [7, Theorem 2.7]. Their idea led us to the use of (1.1). The author is actually uncertain whether the CH formula works for our case.

We have to be careful about an extension of Theorem 3.5 to general symbols. In the spin manifold case, the formula for polynomial symbols ([7, Theorem 2.7]) and Widom's formula [15, Proposition 3.6] lead almost automatically to the general formula ([7, Theorem 3.5]). Since Beals-Greiner's formula [3, Theorems 14.1 and 14.7] must be the counterpart of Widom's one in the contact Riemannian manifold case, it will be natural to expect that a general composition formula will be derived easily from Theorem 3.5 and Beals-Greiner's one. But the situation is not so simple. The author thinks it hard to present a concise general formula, but in fact we have a method of computing the top grading part of composition, that is, if two general ones are given concretely, then that of their composition can be computed exactly: refer to §4. Last the author want to state that he has a plan to apply the study in this paper to that of the Kohn-Rossi Laplacian (cf. [10]), the Toeplitz operator (cf. [9]), etc.

1 Preliminaries: contact Riemannian manifold and the canonical Spin^c structure

Let $M = (M, e^0,\ e_0, J, g)$ be a $(2n+1)$-dimensional contact Riemannian manifold. Here e^0 is a contact 1-form and e_0 is the unique vector field satisfying $e_0 \rfloor\, e^0 := e^0(e_0) = 1$, $e_0 \rfloor\, de^0 := de^0(e_0,\ ) = 0$, and (J, g) is a pair of $(1,1)$-tensor field and Riemannian metric satisfying $g(e_0, X) = e^0(X)$, $g(X, JY) = -de^0(X,Y) := -X(e^0(Y)) - Y(e^0(X)) + e^0([X,Y])$ and $J^2X = -X + e^0(X)e_0$. Referring to [10], [11] and [12], we will review briefly some basic properties of the hermitian Tanno connection and the canonical Spin^c structure on M, which are tools crucial for our study.

We set $H = \ker e^0$, $H_\pm = \{X \in \mathbb{C}H \mid JX = \pm\sqrt{-1}\,X\}$ ($\mathbb{C}H := H \otimes \mathbb{C}$). Without the assumption that J is integrable (i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$), we will equip M with the hermitian Tanno connection ${}^\sharp\nabla$ ([10]), which is known to be appropriate for the study of such a manifold and is characterized by the following conditions:

$$
\begin{gathered}
{}^\sharp\nabla e^0 = 0, \quad {}^\sharp\nabla g = 0, \quad {}^\sharp\nabla J = 0, \\
\pi_+ T({}^\sharp\nabla)(Z, W) = 0 \quad (Z \in H_+,\ W \in \mathbb{C}TM),
\end{gathered}
$$

where $T({}^\sharp\nabla)$ is the torsion tensor and $\pi_+ : \mathbb{C}TM = \mathbb{C}e_0 \oplus H_+ \oplus H_- \to H_+$ is the natural projection (cf. [10, Lemma 1.1], [12, §2]). Notice that it coincides with the Tanaka-Webster connection ([6, §1.2]) provided that J is integrable. Near each point $\mathbb{P} \in M$, we always take a local unitary frame $e^{\mathbb{C}}_{\bullet} = (e^{\mathbb{C}}_0, e^{\mathbb{C}}_1, \ldots, e^{\mathbb{C}}_n, e^{\mathbb{C}}_{\bar{1}}, \ldots, e^{\mathbb{C}}_{\bar{n}})$ of $\mathbb{C}TM$ ($e^{\mathbb{C}}_0 := e_0$, $e^{\mathbb{C}}_{\bar{\alpha}} = \overline{e^{\mathbb{C}}_\alpha} \in H_-$, $g(e^{\mathbb{C}}_\alpha, e^{\mathbb{C}}_{\bar{\beta}}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) which is ${}^\sharp\nabla$-parallel along all the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$. Its dual frame is denoted by $e_{\mathbb{C}}^{\bullet} = (e^0_{\mathbb{C}}, e^1_{\mathbb{C}}, \ldots, e^n_{\mathbb{C}}, e^{\bar{1}}_{\mathbb{C}}, \ldots, e^{\bar{n}}_{\mathbb{C}})$ (hence, $e^0_{\mathbb{C}} = e^0$). We assume that the Greek indices α, β, … vary from 1 to n, so that

$$
g = e^0_{\mathbb{C}} \otimes e^0_{\mathbb{C}} + \sum (e^\alpha_{\mathbb{C}} \otimes e^{\bar{\alpha}}_{\mathbb{C}} + e^{\bar{\alpha}}_{\mathbb{C}} \otimes e^\alpha_{\mathbb{C}})
$$

and the connection ${}^\sharp\nabla$ can be expressed as

$$ {}^\sharp\nabla e_0^{\mathbb{C}} = 0, \quad {}^\sharp\nabla e_\beta^{\mathbb{C}} = \sum e_\alpha^{\mathbb{C}} \cdot \omega({}^\sharp\nabla)^\alpha_\beta, $$
$$ {}^\sharp\nabla e_{\bar\beta}^{\mathbb{C}} = \sum e_{\bar\alpha}^{\mathbb{C}} \cdot \omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta}, \quad \omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta} = -\omega({}^\sharp\nabla)^\beta_\alpha. $$

The associated orthonormal frames $e_\bullet = (e_0, e_1, \cdots, e_{2n})$, $e^\bullet = (e^0, e^1, \cdots, e^{2n})$ with respect to the underlying Riemannian metric are given by

$$ e_\alpha = \frac{e_\alpha^{\mathbb{C}} + e_{\bar\alpha}^{\mathbb{C}}}{\sqrt{2}}, \quad e_{n+\alpha} = Je_\alpha, \quad e^\alpha = \frac{e^\alpha_{\mathbb{C}} + e^{\bar\alpha}_{\mathbb{C}}}{\sqrt{2}}, \quad e^{n+\alpha} = -Je^\alpha. $$

Certainly these frames are also ${}^\sharp\nabla$-parallel along the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$. We denote the ${}^\sharp\nabla$-exponential map from $\mathbb{P}$ by $\exp = \exp_{\mathbb{P}} : T_{\mathbb{P}}M \to M$, and, as coordinates near $\mathbb{P}$, we always adopt the associated ${}^\sharp\nabla$-normal coordinates $x = {}^t(x_0, x_1 \ldots, x_{2n})$ with $\partial/\partial x_j = e_j$ at $0 = \mathbb{P}$. Then [10, (2.2)] says that there is a Taylor expansion

$$ (1.1) \quad \omega({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x_j) = -\sum_{\ell=1}^\infty \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0), $$

where $F({}^\sharp\nabla)$ is the curvature 2-form of ${}^\sharp\nabla$. Further, if we set

$$ (1.2) \qquad e_\bullet = (\partial/\partial x_\bullet) \cdot v_\bullet, \quad e^\bullet = (dx_\bullet) \cdot v^\bullet \quad (\text{i.e., } e_j = \textstyle\sum v_{kj}\, \partial/\partial x_k, \text{ etc.}), $$

then the matrix-valued functions $v_\bullet$, $v^\bullet$ are also expanded explicitly: Let us set $z_0 = x_0$, $z_\alpha = (x_\alpha + ix_{n+\alpha})/\sqrt{2}$, $z_{\bar\alpha} = \overline{z_\alpha}$, $\partial/\partial z_0 = \partial/\partial x_0$, $\partial/\partial z_\alpha = (\partial/\partial x_\alpha - i\partial/\partial x_{n+\alpha})/\sqrt{2}$, $\partial/\partial z_{\bar\alpha} = \overline{\partial/\partial z_\alpha}$ and $(\partial/\partial z_\bullet) = (\partial/\partial z_0, \ldots, \partial/\partial z_n, \partial/\partial z_{\bar 1}, \ldots, \partial/\partial z_{\bar n})$. Then, in [10, (2.4)] we presented carefully the Taylor expansions of the complex ones $V_\bullet$, $V^\bullet$ defined by $e_\bullet^{\mathbb{C}} = (\partial/\partial z_\bullet) \cdot V_\bullet$, $e^\bullet_{\mathbb{C}} = (dz_\bullet) \cdot V^\bullet$. For later use, we will record here the beginnings of that of $V_\bullet$, instead of that of $v_\bullet$:

$$ (1.3) \qquad V_\bullet = \begin{pmatrix} 1 & \underset{(0,\beta)\text{-th entry}}{z_{\bar\beta}\frac{i}{2}} & \underset{(0,\bar\beta)\text{-th entry}}{z_\beta\frac{-i}{2}} \\ \underset{(\alpha,0)\text{-th entry}}{\sum z_{\bar\gamma}\frac{\widetilde{\mathcal{T}}_{\bar\alpha 0\bar\gamma}}{2}} & E_n & \underset{(\alpha,\bar\beta)\text{-th entry}}{z_0\frac{-\widetilde{\mathcal{T}}_{\bar\alpha 0\bar\beta}}{2} + \sum z_{\bar\gamma}\frac{-\widetilde{\mathcal{T}}_{\bar\alpha\bar\gamma\bar\beta}}{2}} \\ \underset{(\bar\alpha,0)\text{-th entry}}{\sum z_\gamma\frac{\widetilde{\mathcal{T}}_{\alpha 0\gamma}}{2}} & \underset{(\bar\alpha,\beta)\text{-th entry}}{z_0\frac{-\widetilde{\mathcal{T}}_{\alpha 0\beta}}{2} + \sum z_\gamma\frac{-\widetilde{\mathcal{T}}_{\alpha\gamma\beta}}{2}} & E_n \end{pmatrix} + O(|z|^2), $$

where we set $\widetilde{\mathcal{T}}_{\alpha\gamma\beta} = g(T({}^\sharp\nabla)(e_\gamma^{\mathbb{C}}, e_\beta^{\mathbb{C}}), e_\alpha^{\mathbb{C}})(0)$, etc.

Next, referring to [12, §2], let us recall that the hermitian vector bundle $(H, g|_H, J|_H)$ yields the canonical Spin^c structure over M with spinor bundle

$$ \mathcal{S}^c = \wedge_H^{0,*} T^*M := \{\omega \in \wedge^* \mathbb{C}T^*M \mid X \rfloor \omega = 0 \ (X \in \mathbb{R}e_0 \cup H_+)\} $$

accompanied with the Clifford action of $\mathbb{C}l(T^*M)$ given by

$$ (1.4) \qquad e^0_{\mathbb{C}} \diamond = (-1)^{*+1} i, \quad e^\alpha_{\mathbb{C}} \diamond = \sqrt{2}\, e^{\bar\alpha}_{\mathbb{C}} \wedge, \quad e^{\bar\alpha}_{\mathbb{C}} \diamond = -\sqrt{2}\, e^{\bar\alpha}_{\mathbb{C}} \vee, $$

where we set $e^{\bar\alpha}_{\mathbb{C}} \vee = e_{\bar\alpha}^{\mathbb{C}} \rfloor$. Obviously the spinor metric coincides with the canonical one on the right hand side, i.e., $g^{\mathcal{S}^c} = g^{\wedge_H^{0,*}}$, and [12, Proposition 2.4] says that so does the spinor connection, that is,

$$ \nabla^{\mathcal{S}^c} = {}^\sharp\nabla^{\wedge_H^{0,*}}, \quad \omega({}^\sharp\nabla^{\wedge_H^{0,*}}) := \sum \omega({}^\sharp\nabla)^\alpha_\beta \cdot e^{\bar\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} \vee. $$

Hence the curvature 2-form $F(\nabla^{\mathcal{S}^c}) = F({}^\sharp\nabla^{\wedge_H^{0,*}})$ is expressed as

$$ F({}^\sharp\nabla^{\wedge_H^{0,*}})(X,Y) = \sum F({}^\sharp\nabla)^\alpha_\beta(X,Y)\, e^{\bar\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} \vee. $$

2 Intrinsic symbol spaces $\mathcal{S}_H^\infty$, $\mathcal{SC}_H^\infty$ and H-pseudodifferential operators

Let us take a hermitian vector bundle (E, g^E) over M with connection ∇^E and set

$$F = \not{S}^c \otimes E = \wedge_H^{0,*} T^*M \otimes E$$

with canonically defined metric and connection (g^F, ∇^F). In this section, we will introduce two kinds of $\mathrm{End}(F)$-valued symbol spaces and associated H-pseudodifferential operators.

We set

$$\mathcal{F}_m^H = \mathcal{F}_m^H(M; \mathrm{End}(F)) = \{ f \in C^\infty(\pi^*\mathrm{End}(F) \backslash \{0\}) \mid f(\mathbb{P}, \lambda T) = \lambda^m f(\mathbb{P}, T) \}, \tag{2.1}$$

where $\pi : T^*M \to M$ is the projection and λT denotes the Heisenberg dilation

$$T^*M = \mathbb{R}e^0 \oplus H^* \ni T = (T^0, T^H) \mapsto \lambda T := (\lambda^2 T^0, \lambda T^H).$$

By using the ∇^F-parallel transport along the ${}^\sharp\nabla$-geodesic from $\mathbb{P}'$ to $\mathbb{P}$

$$\mathcal{T}_{\mathbb{P}'}^{\mathbb{P}} = \mathcal{T}_{\nabla^F}(\mathbb{P}, \mathbb{P}') : F_{\mathbb{P}'} \to F_{\mathbb{P}}, \tag{2.2}$$

the bundle F is trivialized on a neighborhood $U_{\mathbb{P}}$ of $\mathbb{P}$ as

$$F|_{U_{\mathbb{P}}} \;\cong\; U_{\mathbb{P}} \times F_{\mathbb{P}}, \quad u_{\mathbb{P}'} \leftrightarrow (\mathbb{P}', \mathcal{T}_{\mathbb{P}'}^{\mathbb{P}}(u_{\mathbb{P}'})). \tag{2.3}$$

Together with the trivialization

$$T^*U_{\mathbb{P}} \cong U_{\mathbb{P}} \times T_{\mathbb{P}}^*M = (U_{\mathbb{P}} \times \mathbb{R}^{2n+1}, (x, \sigma)), \quad e^\bullet(x) \cdot \sigma \leftrightarrow (x, e^\bullet(0) \cdot \sigma) = (x, \sigma), \tag{2.4}$$

it induces naturally a local expression of $q \in C^\infty(\pi^*\mathrm{End}(F))$, which we denote by

$$q(\mathbb{P}, e^\bullet; x, \sigma) \in \mathrm{End}(F)_{\mathbb{P}}.$$

The parallel transports for TM, T^*M, etc., are similarly defined and denoted also by $\mathcal{T}_{\mathbb{P}'}^{\mathbb{P}}$: therefore, $\mathcal{T}_{\exp(x)}^{\mathbb{P}}(e^\bullet(x)) = e^\bullet(0)$.

Let us define now one of the intrinsic symbol spaces, following the ideas due to Beals-Greiner ([3]) and Widom ([14], [15]), as

$$\begin{aligned} \mathcal{S}_H^m &= \mathcal{S}_H^m(M; \mathrm{End}(F)) \\ &= \Big\{ q \in C^\infty(\pi^*\mathrm{End}(F)) \;\Big|\; \text{there exist } q_k \in \mathcal{F}_k^H \ (k \le m) \text{ such that,} \\ &\qquad\qquad \text{for each } \mathbb{P}, \quad q \sim \sum_{k \le m} q_k \text{ at } \mathbb{P} \Big\}. \end{aligned} \tag{2.5}$$

Here "$q \sim \sum_{k \le m} q_k$ at $\mathbb{P}$" means that, for all multi-indices A, B and all $N > 0$, we have

$$\Big| \partial_x^A \partial_\sigma^B \Big(q - \sum_{k > m-N} q_k \Big)(\mathbb{P}, e^\bullet; 0, \sigma) \Big| \le c_{ABN} \, |\sigma|_H^{m - |B|_H - N} \quad (|\sigma|_H \ge 1) \tag{2.6}$$

$$\Big(|\sigma|_H := \{ |\sigma_0|^2 + \sum_{j \ge 1} |\sigma_j|^4 \}^{1/4}, \ |B|_H := 2B_0 + \sum_{j \ge 1} B_j = B_0 + |B| \Big),$$

where $c_{ABN} = c_{ABN}(\mathbb{P}) > 0$ are bounded functions. Further we set $\mathcal{S}_H^\infty = \bigcup_m \mathcal{S}_H^m$, $\mathcal{S}_H^{-\infty} = \bigcap_m \mathcal{S}_H^m$ as usual.

Lemma 2.1 *The symbol space $\mathcal{S}^m_H$ coincides with the one given by Beals-Greiner [3, §10].*

Proof. It will suffice to show that (2.6) holds not only for $(0,\sigma)$ but also for any (y,σ). At the point $\mathbb{P}(y) := \exp_{\mathbb{P}}(e_\bullet(0)\cdot y)$, let us check the relation between the frame $(\partial/\partial x_\bullet, \partial/\partial\sigma_\bullet)$ induced from the coordinates (x,σ) centered at $\mathbb{P}$, and the frame $(\partial/\partial w_\bullet, \partial/\partial\eta_\bullet)$ induced from the ones (w,η) centered at $\mathbb{P}(y)$. We have

$$\begin{aligned}&\mathbb{P}(x) = \mathbb{P}(y)(w) := \exp_{\mathbb{P}(y)}(e_\bullet(y)\cdot w):\ w = w(y,x) = O(|x-y|),\\ &e^\bullet(0)\cdot\eta = \mathcal{T}^{\mathbb{P}}_{\mathbb{P}(y)}\mathcal{T}^{\mathbb{P}(y)}_{\mathbb{P}(x)}\mathcal{T}^{\mathbb{P}(x)}_{\mathbb{P}}(e^\bullet(0)\cdot\sigma):\ \eta = \eta(y,x,\sigma) = a(y,x)\cdot\sigma,\ a(y,y) = E.\end{aligned} \tag{2.7}$$

Since ${}^\sharp\nabla e^0 = 0$, etc., obviously we have

$$a(y,x)^{-1} = {}^t a(y,x), \quad a(y,x)_{0k} = a(y,x)_{k0} = \begin{cases} 1 & (k=0),\\ 0 & (k\geq 1),\end{cases}$$

so that

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial w_j}{\partial x_i}\frac{\partial}{\partial w_j} + \sum_{j,k,\ell\geq 1}\frac{\partial a_{jk}}{\partial x_i}a_{\ell k}\eta_\ell\frac{\partial}{\partial\eta_j}, \quad \frac{\partial}{\partial\sigma_i} = \begin{cases}\dfrac{\partial}{\partial\eta_0} & (i=0),\\ \displaystyle\sum_{j\geq 1}a_{ji}\frac{\partial}{\partial\eta_j} & (i\geq 1).\end{cases}$$

Consequently, for example we have

$$\begin{aligned}\partial_{x_i}\Big(q - \sum_{k>m-N}q_k\Big)(e^\bullet(\mathbb{P});y,\sigma) &= \sum\frac{\partial w_j}{\partial x_i}\partial_{w_j}\Big(q - \sum_{k>m-N}q_k\Big)(e^\bullet(\mathbb{P}(y));0,\eta)\\ &+ \sum_{j,\ell\geq 1}\frac{\partial a_{j\ell}}{\partial x_i}\eta_\ell\,\partial_{\eta_j}\Big(q - \sum_{k>m-N}q_k\Big)(e^\bullet(\mathbb{P}(y));0,\eta).\end{aligned}$$

Here $(\mathbb{P},e^\bullet;y,\sigma)$, etc., are abbreviated to $(e^\bullet(\mathbb{P});y,\sigma)$, etc. We know it certainly satisfies (2.6). ∎

By further consideration we notice that the estimate at (2.7) is refined into

$$w = {}^t v^\bullet(y)(x-y) + O(|x-y|^2), \quad x = y + v_\bullet(y)w + O(|w|^2). \tag{2.8}$$

Next, let us introduce a class of pseudodifferential operators on M and associated intrinsic symbols. We adopt another trivialization

$$T^*U_{\mathbb{P}} \cong U_{\mathbb{P}}\times T^*_{\mathbb{P}} = (U_{\mathbb{P}}\times\mathbb{R}^{2n+1},(x,\xi)), \quad dx_\bullet(x)\cdot\xi \leftrightarrow (x, dx_\bullet(0)\cdot\xi) = (x,\xi),$$

which gives another local expression of $q\in C^\infty(\pi^*\mathrm{End}(F))$:

$$\begin{aligned}q(x,\xi) = q(\mathbb{P},dx_\bullet;x,\xi) = q(\mathbb{P},e^\bullet;x,\sigma(x,\xi)) \in F_{\mathbb{P}},\\ \text{hence, } q(\mathbb{P},\xi) := q(0,\xi) = q(\mathbb{P},e^\bullet;0,\sigma(0,\xi)) = q(\mathbb{P},e^\bullet;0,\xi).\end{aligned}$$

Referring to (1.2), we know that the transition rule between ξ and $\sigma = \sigma(x,\xi)$ is

$$(dx_\bullet)(x)\cdot\xi = e^\bullet(x)\cdot\sigma(x,\xi):\ \sigma(x,\xi) = v^\bullet(x)^{-1}\xi = {}^t v_\bullet(x)\xi. \tag{2.9}$$

For a symbol $p \in \mathcal{S}^m_H$ and a smooth bump function ϕ on $M \times M$ which is supported in a small neighborhood of the diagonal set and is equal to 1 on a still smaller one, we define the H-pseudodifferential operator

$$\theta(p) = \theta^\phi(p) : \Gamma(F) \to \Gamma(F)$$

as follows: For $u \in \Gamma(F)$, at each $\mathbb{P} \in M$ we set

$$\begin{aligned}
(\theta(p)u)(\mathbb{P}) &= \frac{1}{(2\pi)^{2n+1}} \int_{T_{\mathbb{P}}M \times T^*_{\mathbb{P}}M \ni (x,\xi)} e^{-i\langle x,\xi\rangle}\, p(0,\xi)\, \overline{u}_{\mathbb{P}}(x)\, dx d\xi \\
&= \frac{1}{(2\pi)^{2n+1}} \int_{T^*_{\mathbb{P}}M \ni \xi} p(0,\xi)\, \widehat{\overline{u}_{\mathbb{P}}}(\xi)\, d\xi, \\
\overline{u}_{\mathbb{P}} &:= \Big(T_{\mathbb{P}}M \ni x \mapsto \phi(\mathbb{P}, \exp(x))\, \mathcal{T}^{\mathbb{P}}_{\exp(x)}(u(\exp(x))) \in F_{\mathbb{P}}\Big),
\end{aligned}$$

where $\widehat{\overline{u}_{\mathbb{P}}}$ is the Fourier transform of $\overline{u}_{\mathbb{P}}$. The set of such operators is denoted by $Op\,\mathcal{S}^m_H$. By referring to [3, §10], it is certain that $Op\,\mathcal{S}^{-\infty}_H$ consists of operators with C^∞-kernels and, if we define the operator by using another bump function ψ, we have

$$\theta^\phi(p) = \theta^\psi(p) \pmod{Op\,\mathcal{S}^{-\infty}_H}.$$

For an operator $P : \Gamma(F) \to \Gamma(F)$, its intrinsic symbol (according to the idea of Widom [14], [15]) $\varsigma(P) \in C^\infty(T^*M, \pi^*\mathrm{End}(F))$ is now defined by

$$\varsigma(P)(\mathbb{P}, \xi)(u_{\mathbb{P}}) = P\Big(M \ni \exp(x) \mapsto e^{i\langle x,\xi\rangle}\, \phi(\mathbb{P}, \exp(x))\, \mathcal{T}^{\exp(x)}_{\mathbb{P}}(u_{\mathbb{P}})\Big)\Big|_{x=0}.$$

Then obviously we have

$$\begin{aligned}
&\varsigma(\theta(p)) = p \pmod{\mathcal{S}^{-\infty}_H}, \\
&\theta(\varsigma(\theta(p))) = \theta(p) \pmod{Op\,\mathcal{S}^{-\infty}_H}, \\
&\mathcal{S}^m_H \,/\, \mathcal{S}^{-\infty}_H \ \underset{\varsigma}{\overset{\theta}{\rightleftharpoons}}\ Op\,\mathcal{S}^m_H \,/\, Op\,\mathcal{S}^{-\infty}_H.
\end{aligned} \tag{2.10}$$

In a way similar to the idea of Getzler ([7]) (and Alvarez-Gaumé ([1])), let us define here another symbol space $\mathcal{SC}^m_H$. By (1.4), we know there are the identifications

$$\begin{aligned}
\mathrm{End}(\not{S}^c) = \mathrm{End}(\wedge^{0,*}_H T^*M) \ &\cong\ \mathbb{C}l(H^*) \ \cong\ \wedge^*\mathbb{C}H^* \ \subset \wedge^*\mathbb{C}T^*M, \\
&e^{j_1} \diamond e^{j_2} \diamond \cdots \underset{(j_1<j_2<\cdots)}{\leftrightarrow} e^{j_1} \wedge e^{j_2} \wedge \cdots
\end{aligned}$$

$$\begin{matrix} e^{\alpha}_{\mathbb{C}}\diamond \\ e^{\bar\alpha}_{\mathbb{C}}\diamond \end{matrix} \ = \ \begin{matrix} \sqrt{2}\cdot e^{\bar\alpha}_{\mathbb{C}}\wedge \\ -\sqrt{2}\cdot e^{\bar\alpha}_{\mathbb{C}}\vee \end{matrix} \quad \leftrightarrow \quad \begin{matrix} e^{\alpha}_{\mathbb{C}} \\ e^{\bar\alpha}_{\mathbb{C}} \end{matrix}$$

$$\mathrm{End}(F) = \mathrm{End}(\wedge^{0,*}_H T^*M \otimes E) \ \cong\ \wedge^*\mathbb{C}H^* \otimes \mathrm{End}(E).$$

For example, accordingly we have

$$F({}^\sharp\nabla^{\wedge^{0,*}_H})(X,Y) = \frac{1}{2}\sum F({}^\sharp\nabla)^{\bar\beta}_{\bar\alpha}(X,Y)\, e^{\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} =: \frac{1}{2}F({}^\sharp\nabla;\wedge)(X,Y), \tag{2.11}$$

$$\begin{aligned}
F(\nabla^F)(X,Y) &= F({}^\sharp\nabla^{\wedge^{0,*}_H})(X,Y) + F(\nabla^E)(X,Y) \\
&= \frac{1}{2}F({}^\sharp\nabla;\wedge)(X,Y) + F(\nabla^E)(X,Y).
\end{aligned} \tag{2.12}$$

We set

$$\mathcal{SC}_H^m = \mathcal{SC}_H^m(M; \mathrm{End}(F)) = \sum_{k=0}^{2n} \mathcal{S}_H^{m-k}(M; \wedge^k H^* \otimes \mathrm{End}(E))) \tag{2.13}$$
$$:= \sum_{k=0}^{2n} \mathcal{S}_H^{m-k} \cap C^\infty(T^*M,\ \pi^*(\wedge^k H^* \otimes \mathrm{End}(E))),$$

the element of which is said to have grading m (according to the naming in [4]). For $p \in \mathcal{SC}_H^\infty$, the H-pseudodifferential operator $\theta(p) \in Op\,\mathcal{SC}_H^\infty$ is defined by regarding p as an element of $\mathcal{S}_H^\infty$ via the canonical identification

$$\begin{array}{rcl} \mathcal{S}_H^\infty & \leftrightarrow & \mathcal{SC}_H^\infty \\ \sum \mathcal{S}_H^{m-k} & \hookleftarrow & \sum \mathcal{S}_H^{m-k}(M; \wedge^k H^* \otimes \mathrm{End}(E)) = \mathcal{SC}_H^m, \end{array}$$

so that (2.10) holds also for $\mathcal{SC}_H^m$, etc.

3 Formula for the composition of polynomial intrinsic symbols $\in \mathcal{PC}_H^\infty$

In this section, we concentrate on the space of polynomial intrinsic symbols, that is, the space of intrinsic symbols associated with H-differential operators,

$$\mathcal{PC}_H^m = \{p \in \mathcal{SC}_H^m \mid p(\mathbb{P}, \xi) \text{ is a polynomial in } \xi\},$$

and consider the composition

$$\mathcal{PC}_H^\infty \times \mathcal{PC}_H^\infty \to \mathcal{PC}_H^\infty, \quad (p, q) \ \mapsto\ p \circ q := \varsigma(\theta(p) \circ \theta(q)).$$

As was mentioned in Introduction, Getzler [7, Theorem 2.7] (and Block-Fox [5, Theorem 2.1]) derived an explicit expression of such a composition on spin manifold by means of the Campbell-Hausdorff formula, and so did Benameur-Heitsch [4, Theorem 4.6] on foliated spin manifold but by means of Atiyah-Bott-Patodi's formula [2, Proposition 3.7]. Stimulated by the latter method, the author tries to examine the composition in the contact Riemannian case by using the following formula (cf. (1.1)): Let $(u_1, u_2, \dots)$ be a local frame of F which is ∇^F-parallel along all the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$ and let us set $\nabla^F u_{i_2} = \sum \omega(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j)\, u_{i_1} \otimes dx_j$. Then, at $x = 0$, the connection coefficients are expanded as

$$\omega(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j) = -\sum_{\ell=1}^\infty \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0). \tag{3.1}$$

Let us start with calculating symbols of some H-differential operators.

Lemma 3.1 *For $X = X^0 + X^H = \sum X_j e_j \in \Gamma(TM = \mathbb{R}e_0 \oplus H)$, $\xi = \xi_0 + \xi_H = \sum \xi_j e^j(\mathbb{P}) \in T^*_{\mathbb{P}}M = \mathbb{R}e^0(\mathbb{P}) \oplus H^*_{\mathbb{P}}$, we have*

$$\varsigma(\nabla_X^F)(\mathbb{P}, \xi) = \langle iX_{\mathbb{P}}, \xi\rangle = \sum iX_j(\mathbb{P})\, \xi_j = \langle iX_{\mathbb{P}}^H, \xi_H\rangle + \langle iX_{\mathbb{P}}^0, \xi_0\rangle, \tag{3.2}$$

$$\theta(\langle iX, \xi\rangle) = \nabla_X^F. \tag{3.3}$$

Proof. We have $\phi(\mathbb{P}, x) = 1$ near $x = 0$, so that we may ignore the bump function ϕ. Since

$$\begin{aligned} X\langle \exp^{-1}(x), \xi\rangle\Big|_{x=0} &= \frac{d}{dt}\Big|_{t=0}\langle \exp^{-1}(\exp(tX_{\mathbb{P}})), \xi\rangle = \frac{d}{dt}\Big|_{t=0} t\langle X_{\mathbb{P}}, \xi\rangle = \langle X_{\mathbb{P}}, \xi\rangle, \\ \nabla^F_X(\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}}))\Big|_{x=0} &= 0 \quad (\mathcal{T}^x_{\mathbb{P}} := \mathcal{T}^{\exp(x)}_{\mathbb{P}}), \end{aligned} \tag{3.4}$$

we have

$$\begin{aligned} \varsigma(\nabla^F_X)(\mathbb{P}, \xi)(u_{\mathbb{P}}) &= \nabla^F_X\Big(e^{i\langle \exp^{-1}(x),\xi\rangle}\, \mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\Big)\Big|_{x=0} \\ &= X(i\langle \exp^{-1}(x), \xi\rangle)\Big|_{x=0} \cdot u_{\mathbb{P}} + \nabla^F_X(\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}}))\Big|_{x=0} = \langle iX_{\mathbb{P}}, \xi\rangle \cdot u_{\mathbb{P}}, \end{aligned}$$

that is, (3.2) is valid. Next, we have

$$\begin{aligned} (\theta(\langle iX, \xi\rangle)u)(\mathbb{P}) &= \sum X_j(\mathbb{P}) \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle Y,\xi\rangle}\, i\xi_j\, \overline{u}_{\mathbb{P}}(Y)\, dY\, d\xi \\ &= \sum X_j(\mathbb{P}) \frac{\partial}{\partial Y_j}\Big|_{Y=0} (\overline{u}_{\mathbb{P}}(Y)) = \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{T}^{\mathbb{P}}_{\exp(tX_{\mathbb{P}})}(u(\exp(tX_{\mathbb{P}}))) = \nabla^F_{X_{\mathbb{P}}} u. \end{aligned}$$

Namely, (3.3) is also valid. ∎

Proposition 3.2 *We have*

$$\begin{aligned} \varsigma(\nabla^F_X\nabla^F_Y)(\mathbb{P}, \xi) = &\underbrace{\langle iX^H_{\mathbb{P}}, \xi_H\rangle\, \langle iY^H_{\mathbb{P}}, \xi_H\rangle + \frac{1}{4}F(^\sharp\nabla; \wedge)(X^H_{\mathbb{P}}, Y^H_{\mathbb{P}})}_{\textit{grading } 2} \\ &+ \underbrace{\langle iX^0_{\mathbb{P}}, \xi_0\rangle\, \langle iY^0_{\mathbb{P}}, \xi_0\rangle}_{\textit{grading } 4} + \underbrace{\langle iX^H_{\mathbb{P}}, \xi_H\rangle\, \langle iY^0_{\mathbb{P}}, \xi_0\rangle + \langle iX^0_{\mathbb{P}}, \xi_0\rangle\, \langle iY^H_{\mathbb{P}}, \xi_H\rangle}_{\textit{grading } 3} \\ &+ \underbrace{\frac{1}{4}F(^\sharp\nabla; \wedge)(X^0_{\mathbb{P}}, Y^H_{\mathbb{P}}) + \frac{1}{4}F(^\sharp\nabla; \wedge)(X^H_{\mathbb{P}}, Y^0_{\mathbb{P}})}_{\textit{grading } 2} \\ &+ \underbrace{iX_{\mathbb{P}}(Y_x\langle \exp^{-1}(x), \xi_H\rangle)}_{\textit{grading } 1} + \underbrace{iX_{\mathbb{P}}(Y_x\langle \exp^{-1}(x), \xi_0\rangle)}_{\textit{grading } 2} + \underbrace{\frac{1}{2}F(\nabla^E)(X_{\mathbb{P}}, Y_{\mathbb{P}})}_{\textit{grading } 0} \end{aligned} \tag{3.5}$$

and

$$iX^H_{\mathbb{P}}(Y^H_x\langle \exp^{-1}(x), \xi_0\rangle) = \frac{\xi_0}{2i}\, de^0(X^H_{\mathbb{P}}, Y^H_{\mathbb{P}}) = \frac{\xi_0}{2i}\, de^0(X_{\mathbb{P}}, Y_{\mathbb{P}}). \tag{3.6}$$

Here note that $de^0 = i\sum e^\alpha_{\mathbb{C}} \wedge e^{\bar\alpha}_{\mathbb{C}} = \sum e^\alpha \wedge e^{n+\alpha}$.

Proof. We have

$$\begin{aligned} &\nabla^F_X\nabla^F_Y\Big(e^{i\langle \exp^{-1}(x),\xi\rangle}\, \mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\Big)\Big|_{x=0} \\ &= \Big(i(\nabla^F_X\nabla^F_Y\langle \exp^{-1}(x), \xi\rangle)\, e^{i\langle \exp^{-1}(x),\xi\rangle}\, \mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}}) \\ &\quad + i(\nabla^F_X\langle \exp^{-1}(x), \xi\rangle)\, i(\nabla^F_Y\langle \exp^{-1}(x), \xi\rangle)\, e^{i\langle \exp^{-1}(x),\xi\rangle}\, \mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}}) \end{aligned}$$

$$
\begin{aligned}
&+ i(\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle)\, e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_X\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\\
&+ i(\nabla^F_X\langle \exp^{-1}(x),\xi\rangle)\, e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\\
&+ e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_X\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\Big)\Big|_{x=0}\\
= {}& i\nabla^F_{X_{\mathbb{P}}}\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle\, u_{\mathbb{P}} + \langle iX,\xi\rangle\langle iY,\xi\rangle u_{\mathbb{P}} + \nabla^F_X\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\Big|_{x=0}
\end{aligned}
$$

and, by (3.1), (2.2), (2.3) and (2.12), we have

$$
\begin{aligned}
&\nabla^F_X\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}(u_{\mathbb{P}})\Big|_{x=0} = \nabla^F_X\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}\Big(\sum a_{i_2}(\mathbb{P})u_{i_2}(\mathbb{P})\Big)\Big|_{x=0}\\
&= \sum a_{i_2}(\mathbb{P})\nabla^F_X\Big(\sum_{i_1,j}\omega(\nabla^F)^{i_1}_{i_2}(\partial/\partial x_j)dx_j(Y)\,u_{i_1}(x)\Big)\Big|_{x=0}\\
&= \sum a_{i_2}(\mathbb{P})\nabla^F_X\Big(\sum_{i_1,j}\Big\{-\frac{1}{2}\sum x_{j_1}F(\nabla^F)^{i_1}_{i_2}(\partial/\partial x_j,\partial/\partial x_{j_1})(\mathbb{P})\\
&\qquad\qquad + \cdots\Big\}dx_j(Y)\,u_{i_1}(x)\Big)\Big|_{x=0}\\
&= \sum_{i_1.i_2,j,j_1} a_{i_2}(\mathbb{P})\Big\{-\frac{1}{2}X(x_{j_1})F(\nabla^F)^{i_1}_{i_2}(\partial/\partial x_j,\partial/\partial x_{j_1})(0)\Big\}dx_j(Y)\,u_{i_1}(0)\\
&= -\frac{1}{2}\sum_{i_1,i_2,j} a_{i_2}(\mathbb{P})F(\nabla^F)^{i_1}_{i_2}(Y,X)(0)\,u_{i_1}(\mathbb{P}) = -\frac{1}{2}F(\nabla^F)(Y,X)\,u_{\mathbb{P}}\\
&= \frac{1}{4}F(^\sharp\nabla;\wedge)(X,Y)u_{\mathbb{P}} + \frac{1}{2}F(\nabla^E)(X,Y)u_{\mathbb{P}}.
\end{aligned}
$$

They imply

$$
\begin{aligned}
&\varsigma(\nabla^F_X\nabla^F_Y)(\mathbb{P},\xi)\\
&= \langle iX,\xi\rangle\langle iY,\xi\rangle + \frac{1}{4}F(^\sharp\nabla;\wedge)(X,Y) + iX_{\mathbb{P}}(Y\langle\exp^{-1}(x),\xi\rangle) + \frac{1}{2}F(\nabla^E)(X,Y).
\end{aligned}
$$

Considering the gradings of the terms, we obtain (3.5). Next, (2.8) says

$$
\begin{aligned}
&X_{\mathbb{P}}(Y_x\langle\exp^{-1}(x),\xi\rangle) = \frac{d}{dt}\Big|_{t=0}\Big(Y_x\langle\exp^{-1}(x),\xi\rangle\Big|_{x=\exp(tX_{\mathbb{P}})}\Big)\\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\langle tX_{\mathbb{P}} + v_\bullet(tX_{\mathbb{P}})\cdot sY_x + \cdots,\xi\rangle = \frac{d}{dt}\Big|_{t=0}\langle v_\bullet(tX_{\mathbb{P}})\cdot Y_x,\xi\rangle\\
&= \langle\frac{d}{dt}\Big|_{t=0}v_\bullet(tX_{\mathbb{P}})\cdot Y_{\mathbb{P}} + \frac{d}{dt}\Big|_{t=0}Y_{\exp(tX_{\mathbb{P}})},\xi\rangle.
\end{aligned}
$$

Hence, by (1.3) we have

$$
\begin{aligned}
&iX^H_{\mathbb{P}}(Y^H_x\langle\exp^{-1}(x),\xi_0\rangle = \langle\frac{d}{dt}\Big|_{t=0}v_\bullet(tX^H_{\mathbb{P}})\cdot Y^H_{\mathbb{P}},\xi_0\rangle\\
&= \frac{i}{2}\sum\Big\{X_{n+\beta}(\mathbb{P})Y_\beta(\mathbb{P}) - X_\beta(\mathbb{P})Y_{n+\beta}(\mathbb{P})\Big\}\xi_0\\
&= \frac{\xi_0}{2i}\,de^0(X_{\mathbb{P}},Y_{\mathbb{P}}).
\end{aligned}
$$

Thus we obtain (3.6). ∎

Set

$$\mathscr{F}(^\sharp\nabla;\wedge) = F(^\sharp\nabla;\wedge) + \frac{2\xi_0}{\sqrt{-1}}\, de^0. \tag{3.7}$$

Then Proposition 3.2 yields

Corollary 3.3 *Denoting the grading of $\varsigma(\nabla^F_X)(\mathbb{P},\xi)$ by m_X, we have*

$$\begin{aligned}(\varsigma(\nabla^F_X)\circ\varsigma(\nabla^F_Y))(\mathbb{P},\xi) &= \langle iX_{\mathbb{P}},\xi\rangle\,\langle iY_{\mathbb{P}},\xi\rangle + \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X_{\mathbb{P}},Y_{\mathbb{P}})\\ &\quad + (\textit{terms of grading} < m_X+m_Y).\end{aligned} \tag{3.8}$$

This will suggest a formula for general polynomial symbols.

Definition 3.4 *For $p,q\in\mathcal{PC}^\infty_H$, we set*

$$\begin{aligned}&\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})p(\mathbb{P},\xi)\wedge q(\mathbb{P},\xi')\\ &= \sum_{i,j}\mathscr{F}(^\sharp\nabla;\wedge)(\partial/\partial x_i,\partial/\partial x_j)(\mathbb{P})\,\frac{\partial}{\partial\xi_i}p(\mathbb{P},\xi)\wedge\frac{\partial}{\partial\xi'_j}q(\mathbb{P},\xi')\\ &= \sum_{i,j}\Big\{\sum_{\alpha,\beta}F(^\sharp\nabla)^{\bar\beta}_{\bar\alpha}(\partial/\partial x_i,\partial/\partial x_j)(\mathbb{P})\,e^\alpha_{\mathbb{C}}(\mathbb{P})\wedge e^{\bar\beta}_{\mathbb{C}}(\mathbb{P})\wedge\\ &\qquad + \frac{2\xi_0}{\sqrt{-1}}\,de^0(\partial/\partial x_i,\partial/\partial x_j)(\mathbb{P})\Big\}\frac{\partial}{\partial\xi_i}p(\mathbb{P},\xi)\wedge\frac{\partial}{\partial\xi'_j}q(\mathbb{P},\xi'),\\ &e^{-\frac14\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}p(\mathbb{P},\xi)\wedge q(\mathbb{P},\xi')\Big|_{\xi'=\xi}\\ &= \sum_{k=0}^\infty\frac{1}{k!}\Big(-\frac14\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})\Big)^k p(\mathbb{P},\xi)\wedge q(\mathbb{P},\xi')\Big|_{\xi'=\xi}.\end{aligned}$$

Note that the summation in the last line is actually finite and the action of $\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})$ may lower the grading (cf. [13, Proposition 1.2(2)]).

The formula (3.8) then becomes

$$\begin{aligned}(\varsigma(\nabla^F_X)\circ\varsigma(\nabla^F_Y))(\mathbb{P},\xi) &= e^{-\frac14\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\varsigma(\nabla^F_X)(\mathbb{P},\xi)\wedge\varsigma(\nabla^F_Y)(\mathbb{P},\xi')\Big|_{\xi'=\xi}\\ &\quad + (\text{terms of grading} < m_X+m_Y)\end{aligned}$$

and the general one is given as follows.

Theorem 3.5 *There exists a series of bilinear differential operators*

$$a_k:\mathcal{PC}^\infty_H\times\mathcal{PC}^\infty_H\to\mathcal{PC}^\infty_H\quad(k=0,1,2,\ldots)$$

such that

$$\begin{aligned}&a_k(\mathcal{PC}^m_H,\mathcal{PC}^{m'}_H)\subset\mathcal{PC}^{m+m'-k}_H,\\ &p\circ q = \sum_{k=0}^\infty a_k(p,q),\\ &a_0(p,q)(\mathbb{P},\xi) = e^{-\frac14\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}p(\mathbb{P},\xi)\wedge q(\mathbb{P},\xi')\Big|_{\xi'=\xi}.\end{aligned} \tag{3.9}$$

It suffices to prove Theorem 3.5 at $\mathbb{P}$ for $p, q \in \mathcal{PC}_H^\infty$ with

$$p = g\langle iX, \xi\rangle^m, \quad q = h\langle iY, \xi\rangle^{m'}, \tag{3.10}$$

where $g, h \in \Gamma(\wedge^* H^* \otimes \mathrm{End}(E))$, $X = \sum_{j=0}^{2n} c_j\, \partial/\partial x_j$ and $Y = \sum_{j=0}^{2n} d_j\, \partial/\partial x_j$ (the coefficients c_j, d_j are constant) near $\mathbb{P}$ (cf. the comment in [4, §3]). In the following, X, Y, p, q are such ones if not specified. The grading of $\langle iX, \xi\rangle$ will be denoted by m_X and we will denote ∇^F simply by ∇.

Lemma 3.6 *We have*

$$\varsigma(g\nabla_X^m)(\mathbb{P}, \xi) = g(\mathbb{P})\langle iX_{\mathbb{P}}, \xi\rangle^m, \tag{3.11}$$

$$\theta(\varsigma(g\nabla_X^m)) = g\nabla_X^m. \tag{3.12}$$

Proof. As for (3.11): We have

$$X^m\langle Z, \xi\rangle\Big|_{Z=0} = X^m\langle \exp^{-1}(x), \xi\rangle\Big|_{x=0} = \begin{cases} \langle iX_{\mathbb{P}}, \xi\rangle & (m = 1), \\ 0 & (m \geq 2). \end{cases} \tag{3.13}$$

Indeed, (3.4) says it holds when $m = 1$, and we have

$$\begin{aligned} &X^2\langle Z, \xi\rangle\Big|_{Z=0} = X^2\langle \exp^{-1}(x), \xi\rangle\Big|_{x=0} = X_0\Big(X_x\langle \exp^{-1}(x), \xi\rangle\Big)\Big|_{x=0} \\ &= \frac{d}{dt}\Big|_{t=0}\Big(\frac{d}{dt_2}\Big|_{t_2=0}\langle \exp^{-1}(\exp((t+t_2)X_0)), \xi\rangle\Big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big(\frac{d}{dt_2}\Big|_{t_2=0}\langle (t+t_2)X_0, \xi\rangle\Big) = \frac{d}{dt}\Big|_{t=0}\Big(\langle X_0, \xi\rangle\Big) = 0, \end{aligned}$$

etc. Thus (3.13) was shown. Hence, ignoring the bump function ϕ, we have

$$\begin{aligned} \varsigma(\nabla_X^m)(\mathbb{P}, \xi)(u_{\mathbb{P}}) &= \nabla_X^m\Big(e^{i\langle \exp^{-1}(x), \xi\rangle}\, \mathcal{T}_{\mathbb{P}}^x(u_{\mathbb{P}})\Big)\Big|_{x=0} \\ &= \sum_{k=0}^m \binom{m}{k}\Big(\nabla_X^k e^{i\langle \exp^{-1}(x), \xi\rangle}\, \nabla_X^{m-k}\mathcal{T}_{\mathbb{P}}^x(u_{\mathbb{P}})\Big)\Big|_{x=0} \\ &= \sum_{k=0}^m \binom{m}{k}\langle iX, \xi\rangle^k\Big(\nabla_X^{m-k}\mathcal{T}_{\mathbb{P}}^x(u_{\mathbb{P}})\Big)\Big|_{x=0} = \langle iX, \xi\rangle^m u_{\mathbb{P}}. \end{aligned} \tag{3.14}$$

Since the coefficients of X are constant, we have $\nabla_X \mathcal{T}_{\mathbb{P}}^x(u_{\mathbb{P}}) = 0$, which implies the last equality above. Thus we obtain (3.11). As for (3.12), it suffices to show

$$(\theta(\varsigma(\nabla_X^m))u)(\mathbb{P}) = (X^m\, \overline{u}_{\mathbb{P}})(0) = (\nabla_X^m u)(\mathbb{P}).$$

First we have

$$\begin{aligned} &(\theta(\varsigma(\nabla_X^m))u)(\mathbb{P}) = \frac{1}{(2\pi)^{2n+1}}\int_{T_{\mathbb{P}}M \times T_{\mathbb{P}}^*M \ni (x,\xi)} e^{-i\langle x, \xi\rangle}\, \langle iX, \xi\rangle^m\, \overline{u}_{\mathbb{P}}(x)\, dxd\xi \\ &= \frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x, \xi\rangle}\, (\sum ic_j\, \xi_j)^m\, \overline{u}_{\mathbb{P}}(x)\, dxd\xi \\ &= \frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x, \xi\rangle}\, (\sum ic_j\, i^{-1}\partial_{x_j})^m\, \overline{u}_{\mathbb{P}}(x)\, dxd\xi \\ &= \frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x, \xi\rangle}\, X^m\, \overline{u}_{\mathbb{P}}(x)\, dxd\xi = (X^m\, \overline{u}_{\mathbb{P}})(0). \end{aligned}$$

The second equality is shown in a way similar to the last equality at (3.14). ∎

By Lemma 3.6, we have

$$
\begin{aligned}
p \circ q &= \varsigma(\theta(p) \circ \theta(q)) = \varsigma(\theta(g\langle iX, \xi\rangle^m) \circ \theta(h\langle iY, \xi\rangle^{m'})) \\
&= \varsigma(g\nabla_X^m \circ h\nabla_Y^{m'}) \equiv gh\,\varsigma(\nabla_X^m \nabla_Y^{m'}),
\end{aligned} \tag{3.15}
$$

where the last $\equiv$ means that the top grading parts of both sides coincide.

Proposition 3.7 *We have*

$$
\varsigma\Big(\nabla_X^m \nabla_Y^{m'}\Big) \equiv \underbrace{\sum_{k=0}^{\min(m,m')} k! \binom{m, m'}{k, k} \langle iX, \xi\rangle^{m-k} \langle iY, \xi\rangle^{m'-k} \Big(\frac{1}{4} \mathscr{F}(^\sharp\nabla; \wedge)(X, Y)\Big)^k}_{\text{grading} = mm_X + m'm_Y}, \tag{3.16}
$$

where we put $\binom{m,m'}{k,k'} = \binom{m}{k}\binom{m'}{k'}$.

Proof. It is obvious that

$$
\begin{aligned}
&\nabla_X^m \nabla_Y^{m'} \Big(e^{i\langle \exp^{-1}(x), \xi\rangle} u_j\Big)(0) \\
&= \sum_{m=j+k,\, m'=j'+k'} \binom{m, m'}{k, k'} \Big(X^j Y^{j'} e^{i\langle \exp^{-1}(x), \xi\rangle}\Big)(0) \Big(\nabla_X^k \nabla_Y^{k'} u_j\Big)(0).
\end{aligned} \tag{3.17}
$$

We want to show

$$
\Big(X^j Y^{j'} e^{i\langle \exp^{-1}(x), \xi\rangle}\Big)(0) \equiv \underbrace{\sum_{a=0}^{\min(j,j')} a! \binom{j, j'}{a, a} \langle iX, \xi\rangle^{j-a} \langle iY, \xi\rangle^{j'-a} (\frac{\xi_0}{2i} de^0(X, Y))^a}_{\text{grading} = jm_X + j'm_Y}, \tag{3.18}
$$

$$
\begin{aligned}
\Big(\nabla_X^k \nabla_Y^k u_j\Big)(0) &= k! \Big(\frac{1}{2} F(\nabla)(X, Y)\Big)^k u_j + (\text{terms with grading} < 2k) \\
&= \underbrace{\frac{k!}{4^k} F(^\sharp\nabla; \wedge)(X, Y)^k}_{\text{degree} = 2k} u_j + (\text{terms with grading} < 2k),
\end{aligned} \tag{3.19}
$$

(3.20) the grading of $\Big(\nabla_X^k \nabla_Y^{k'} u_j\Big)(0)$ is less than $k + k'$ (if $k \neq k'$).

As for (3.18): We have

$$
\begin{aligned}
&XYZ(e^{\langle i\exp^{-1}(x), \xi\rangle})\Big|_{x=0} = XY(Z\langle i\exp^{-1}(x), \xi\rangle e^{\langle i\exp^{-1}(x), \xi\rangle})\Big|_{x=0} \\
&= X\langle i\exp^{-1}(x), \xi\rangle \Big\{Y\langle i\exp^{-1}(x), \xi\rangle\, Z\langle i\exp^{-1}(x), \xi\rangle \\
&\quad + YZ\langle i\exp^{-1}(x), \xi\rangle\Big\}\Big|_{x=0} + \Big\{XY\langle i\exp^{-1}(x), \xi\rangle\, Z\langle i\exp^{-1}(x), \xi\rangle \\
&\quad + Y\langle i\exp^{-1}(x), \xi\rangle\, XZ\langle i\exp^{-1}(x), \xi\rangle + XYZ\langle i\exp^{-1}(x), \xi\rangle\Big\}\Big|_{x=0} \\
&\equiv \langle iX, \xi\rangle \cdot \langle iY, \xi\rangle \cdot \langle iZ, \xi\rangle + \langle iX, \xi\rangle \cdot YZ\langle i\exp^{-1}(x), \xi\rangle\Big|_{x=0} \\
&\quad + \langle iY, \xi\rangle \cdot XZ\langle i\exp^{-1}(x), \xi\rangle\Big|_{x=0} + \langle iZ, \xi\rangle \cdot XY\langle i\exp^{-1}(x), \xi\rangle\Big|_{x=0} \\
&= \langle iX, \xi\rangle \cdot \langle iY, \xi\rangle \cdot \langle iZ, \xi\rangle \\
&\quad + \langle iX, \xi\rangle \cdot \frac{\xi_0}{2i} de^0(Y, Z) + \langle iY, \xi\rangle \cdot \frac{\xi_0}{2i} de^0(X, Z) + \langle iZ, \xi\rangle \cdot \frac{\xi_0}{2i} de^0(X, Y)
\end{aligned}
$$

and, in general,

$$\begin{aligned}
&X^jY^{j'}(e^{\langle i\exp^{-1}(x),\xi\rangle})\Big|_{x=0}\\
&\equiv \sum_{a\le\min(j,j')} a!\binom{j,j'}{a,a}\langle iX,\xi\rangle^{j-a}\langle iY,\xi\rangle^{j'-a}(XY\langle i\exp^{-1}(x),\xi\rangle)^a\\
&= \sum_{a\le\min(j,j')} a!\binom{j,j'}{a,a}\langle iX,\xi\rangle^{j-a}\langle iY,\xi\rangle^{j'-a}(\frac{\xi_0}{2i}de^0(X,Y))^a.
\end{aligned}$$

As for (3.19), (3.20): We show them by induction. First, obviously we have

$$\begin{aligned}
&\Big(\nabla_X^k u_j\Big)(x) = \Big(\nabla_Y^k u_j\Big)(0) = 0 \quad (k>0),\\
&\Big(\nabla_X\nabla_Y u_j\Big)(0) = \frac{1}{2}F(\nabla)(X,Y)u_j \equiv \frac{1}{4}F({}^\sharp\nabla;\wedge)(X,Y)u_j.
\end{aligned}$$

Let us set $F=F(\nabla)=F(\nabla^F)$. Then, referring to (3.1), we have

$$\begin{aligned}
(3.21)\quad &\nabla_X^{k+1}\nabla_Y^{k'+1}u_{i_1}(x)\Big|_{x=0}\\
&= \nabla_X^{k+1}\nabla_Y^{k'}\Big(\sum\frac{\ell}{(\ell+1)!}\sum x_{j_1}\cdots x_{j_\ell}\frac{\partial^{\ell-1}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_\ell}}u_{i_2}(x)\Big)\Big|_{x=0}\\
&= \nabla_X\Big(\sum\frac{\ell}{(\ell+1)!}\sum\binom{k,k'}{\kappa,\kappa'}\\
&\qquad \nabla_X^{k-\kappa}\nabla_Y^{k'-\kappa'}(x_{j_1}\cdots x_{j_\ell})\frac{\partial^{\ell-1}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_\ell}}\nabla_X^{\kappa}\nabla_Y^{\kappa'}(u_{i_2}(x))\Big)\Big|_{x=0}\\
&= \Big(\sum\frac{\ell}{(\ell+1)!}\sum\binom{k,k'}{\kappa,\kappa'}\\
&\qquad \nabla_X^{k+1-\kappa}\nabla_Y^{k'-\kappa'}(x_{j_1}\cdots x_{j_\ell})\frac{\partial^{\ell-1}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_\ell}}\nabla_X^{\kappa}\nabla_Y^{\kappa'}(u_{i_2}(x))\Big)\Big|_{x=0}\\
&\quad + \Big(\sum\frac{\ell}{(\ell+1)!}\sum\binom{k,k'}{\kappa,\kappa'}\\
&\qquad \nabla_X^{k-\kappa}\nabla_Y^{k'-\kappa'}(x_{j_1}\cdots x_{j_\ell})\frac{\partial^{\ell-1}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_\ell}}\nabla_X^{\kappa+1}\nabla_Y^{\kappa'}(u_{i_2}(x))\Big)\Big|_{x=0}\\
&= \sum\frac{k+k'+1-\kappa-\kappa'}{(k+k'+2-\kappa-\kappa')!}\binom{k,k'}{\kappa,\kappa'}\nabla_X^{k+1-\kappa}\nabla_Y^{k'-\kappa'}(x_{j_1}\cdots x_{j_{k+k'+1-\kappa-\kappa'}})\\
&\qquad \frac{\partial^{k+k'-\kappa-\kappa'}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'+1-\kappa-\kappa'}}}\nabla_X^{\kappa}\nabla_Y^{\kappa'}(u_{i_2}(x))\Big|_{x=0}\\
&\quad +\sum\frac{k+k'-\kappa-\kappa'}{(k+k'+1-\kappa-\kappa')!}\binom{k,k'}{\kappa,\kappa'}\nabla_X^{k-\kappa}\nabla_Y^{k'-\kappa'}(x_{j_1}\cdots x_{j_{k+k'-\kappa-\kappa'}})\\
&\qquad \frac{\partial^{k+k'-\kappa-\kappa'-1}F_{i_1}^{i_2}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'-\kappa-\kappa'}}}\nabla_X^{\kappa+1}\nabla_Y^{\kappa'}(u_{i_2}(x))\Big|_{x=0}\\
&= \sum\frac{k+k'+1-\kappa-\kappa}{(k+k'+2-\kappa-\kappa)!}\binom{k,k'}{\kappa,\kappa}\nabla_X^{k+1-\kappa}\nabla_Y^{k'-\kappa}(x_{j_1}\cdots x_{j_{k+k'+1-\kappa-\kappa}})
\end{aligned}$$

$$\underbrace{\frac{\partial^{k+k'-\kappa-\kappa}F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'+1-\kappa-\kappa}}}\,\nabla^{\kappa}_X\nabla^{\kappa}_Y(u_{i_2}(x))\Big|_{x=0}}_{grading=2\kappa+2}$$

$$+\sum_{\kappa\neq\kappa'}\frac{k+k'+1-\kappa-\kappa'}{(k+k'+2-\kappa-\kappa')!}\binom{k,k'}{\kappa,\kappa'}\nabla^{k+1-\kappa}_X\nabla^{k'-\kappa'}_Y(x_{j_1}\cdots x_{j_{k+k'+1-\kappa-\kappa'}})$$
$$\underbrace{\frac{\partial^{k+k'-\kappa-\kappa'}F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'+1-\kappa-\kappa'}}}\,\nabla^{\kappa}_X\nabla^{\kappa'}_Y(u_{i_2}(x))\Big|_{x=0}}_{grading<\kappa+\kappa'+2}$$

$$+\sum\frac{k+k'-\kappa-\kappa-1}{(k+k'+1-\kappa-\kappa-1)!}\binom{k,k'}{\kappa,\kappa+1}\nabla^{k-\kappa}_X\nabla^{k'-\kappa-1}_Y(x_{j_1}\cdots x_{j_{k+k'-\kappa-\kappa-1}})$$
$$\underbrace{\frac{\partial^{k+k'-\kappa-\kappa-2}F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'-\kappa-\kappa-1}}}\,\nabla^{\kappa+1}_X\nabla^{\kappa+1}_Y(u_{i_2}(x))\Big|_{x=0}}_{grading=2\kappa+4}$$

$$+\sum_{\kappa+1\neq\kappa'}\frac{k+k'-\kappa-\kappa'}{(k+k'+1-\kappa-\kappa')!}\binom{k,k'}{\kappa,\kappa'}\nabla^{k-\kappa}_X\nabla^{k'-\kappa'}_Y(x_{j_1}\cdots x_{j_{k+k'-\kappa-\kappa'}})$$
$$\underbrace{\frac{\partial^{k+k'-\kappa-\kappa'-1}F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)}{\partial x_{j_2}\cdots\partial x_{j_{k+k'-\kappa-\kappa'}}}\,\nabla^{\kappa+1}_X\nabla^{\kappa'}_Y(u_{i_2}(x))\Big|_{x=0}}_{grading<\kappa+\kappa'+3}.$$

Hence, we know

$$\begin{aligned}
&\nabla^{k+1}_X\nabla^{k+1}_Y u_{i_1}(x)\Big|_{x=0}\\
&\equiv\frac{1}{2!}\sum\nabla_X(x_{j_1})F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)\,\nabla^k_X\nabla^k_Y(u_{i_2}(x))\Big|_{x=0}\\
&\quad+\frac{1}{2!}\binom{k}{k-1}\sum\nabla_X(x_{j_1})F^{i_2}_{i_1}(\partial/\partial x_{j_1},Y)\nabla^k_X\nabla^k_Y(u_{i_2}(x))\Big|_{x=0}\\
&=(k+1)\frac{1}{2}F^{i_2}_{i_1}(X,Y)\,\nabla^k_X\nabla^k_Y(u_{i_2}(x))\Big|_{x=0},
\end{aligned}$$

which inductively implies (3.19). Further, in the case $k\neq k'$, it is obvious that the grading of each line on the last side of (3.21) is less than $(k+1)+(k'+1)$. Notice that some terms in the third and fourth lines may not seem to be so at a glance but in fact such ones vanish. Thus we obtain (3.20). Now (3.17)-(3.20) yield

$$\begin{aligned}
&\varsigma\Big(\nabla^m_X\nabla^{m'}_Y\Big)\\
&\equiv\sum_{b\le\min(m,m')}^{m=j+b,m'=j'+b}\ \sum_{a\le\min(j,j')}a!\,\frac{b!}{4^b}\binom{m,m'}{b,b}\binom{j,j'}{a,a}\\
&\qquad\qquad\langle iX,\xi\rangle^{j-a}\langle iY,\xi\rangle^{j'-a}(\frac{\xi_0}{2i}de^0(X,Y))^a\,F(^\sharp\nabla;\wedge)(X,Y)^b\\
&=\sum_{b\le\min(m,m')}\ \sum_{a\le\min(m-b,m'-b)}a!\,\frac{b!}{4^b}\binom{m,m'}{b,b}\binom{m-b,m'-b}{a,a}
\end{aligned}$$

$$\langle iX,\xi\rangle^{m-a-b}\langle iY,\xi\rangle^{m'-a-b}(\frac{\xi_0}{2i}de^0(X,Y))^a\,F(^\sharp\nabla;\wedge)(X,Y)^b$$
$$=\sum_{k=0}^{\min(m,m')} k!\begin{pmatrix} m,m'\\ k,k\end{pmatrix}\langle iX,\xi\rangle^{m-k}\langle iY,\xi\rangle^{m'-k}\Big(\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X,Y)\Big)^k.$$

Namely, (3.16) was proved. ∎

Last, let us prove Theorem 3.5.

Proof of Theorem 3.5. We have

$$e^{-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}g\langle iX,\xi\rangle^m\wedge h\langle iY,\xi'\rangle^{m'}\Big|_{\xi'=\xi}$$
$$=g\wedge h\sum_{k=0}^{\infty}\frac{1}{k!}\Big(-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(e_i,e_j)\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi'_j}\Big)^k\langle iX,\xi\rangle^m\langle iY,\xi'\rangle^{m'}\Big|_{\xi'=\xi}$$
$$=g\wedge h\sum_{k=0}^{\infty}\frac{1}{k!}\frac{m!}{(m-k)!}\frac{m'!}{(m'-k)!}\Big(-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(e_{i_1},e_{j_1})iX_{i_1}iY_{j_1}\Big)\cdots$$
$$\cdots\Big(-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(e_{i_k},e_{j_k})iX_{i_k}iY_{j_k}\Big)\langle iX,\xi\rangle^{m-k}\langle iY,\xi'\rangle^{m'-k}\Big|_{\xi'=\xi}$$
$$=g\wedge h\sum_{k=0}^{\min(m,m')}k!\begin{pmatrix} m,m'\\ k,k\end{pmatrix}\langle iX,\xi\rangle^{m-k}\langle iY,\xi\rangle^{m'-k}\Big(\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X,Y)\Big)^k$$
$$\equiv g\wedge h\,\varsigma\Big(\nabla_X^m\nabla_Y^{m'}\Big),$$

which, together with (3.10) and (3.15), implies the formula (3.9). The other parts will be obvious. ∎

4 Beals-Greiner's formula and the composition of general intrinsic symbols $\in\mathcal{SC}_H^\infty$

As was mentioned in Introduction, in the spin manifold case (Getzler [7], Block-Fox [5], Benameur-Heitsch [4]) a composition formula for general symbols was derived almost automatically from the one for polynomial symbols and Widom's formula ([14], [15]). It will be thus natural to expect that, in the contact Riemannian manifold case, so can be a composition formula for intrinsic symbols $\in\mathcal{SC}_H^\infty$ from Theorem 3.5 and Beals-Greiner's formula ([3]) for symbols $\in\mathcal{S}_H^\infty$. But the situation is not so simple. To compute such a composition, it seems that, added to Beals-Greiner's one, not Theorem 3.5 but an extra computation is needed. In this section, referring only to their formula reviewed below we will give another proof of (3.9), by which we want to show what kind of extra computation is required.

With reference to [3, §9–§14], let us review Beals-Greiner's formula. Referring to (1.2), (1.3) and (2.9), we set

$$v_\bullet^0(x)=\begin{pmatrix} 1 & \underset{(0,\beta)\text{-th entry}}{\sqrt{2}\,\mathrm{Re}(z_{\bar\beta}\frac{i}{2})=\frac{x_{n+\beta}}{2}} & \underset{(0,n+\beta)\text{-th entry}}{\sqrt{2}\,\mathrm{Im}(z_\beta\frac{-i}{2})=-\frac{x_\beta}{2}}\\ \underset{(\alpha,0)\text{-th entry}}{0} & E_n & O\\ \underset{(n+\alpha,0)\text{-th entry}}{0} & O & E_n\end{pmatrix},$$

$$\sigma^0(x,\xi) = {}^t v^0_\bullet \cdot \xi, \quad \text{i.e., } \sigma^0_j(x,\xi) = \begin{cases} \xi_0 \quad (j=0), \\ \xi_\beta + \dfrac{x_{n+\beta}}{2}\xi_0 \quad (j=\beta), \\ \xi_{n+\beta} - \dfrac{x_\beta}{2}\xi_0 \quad (j=n+\beta). \end{cases}$$

Here $\sigma^0_j(x,\xi)$ are the symbols of certain operators written in the 0-coordinates (cf. [3, (11.28), (11.29)]). Next, referring to (2.1) and (2.5), we put

$$\mathcal{S}^H_m = \{h \in \mathcal{S}^m_H \mid h(\mathbb{P},\lambda T) = \lambda^m h(\mathbb{P},T) \ (|T| \geq 1)\}.$$

Notice that, for $f \in \mathcal{F}^H_m$ there exists a symbol $h \in \mathcal{S}^H_m$ with $h \sim f$ (at each $\mathbb{P}$), and, conversely, for $h \in \mathcal{S}^H_m$ there exists such a unique symbol $f \in \mathcal{F}^H_m$. For $h_j \in \mathcal{S}^H_{m_j}$ $(j=1,2)$, we set

$$(h_1 \# h_2)(\mathbb{P},\xi) = \frac{1}{(2\pi)^{2n+1}} \int_{T_\mathbb{P}M \times T^*_\mathbb{P}M \ni (x,\eta)} e^{-i\langle x,\eta\rangle}\, h_1(\mathbb{P},\xi+\eta)\, h_2(\mathbb{P},\sigma^0(x,\xi))\, dx d\eta.$$

In fact this is a kind of oscillatory integral (cf. [3, (12.17)–(12.19)]) as in the case of classical symbol calculus and we know $h_1 \# h_2 \in \mathcal{S}^H_{m_1+m_2}$. Consequently, we obtain the well-defined bilinear maps

$$\mathcal{S}^H_{m_1} \times \mathcal{S}^H_{m_2} \to \mathcal{S}^H_{m_1+m_2}, \qquad \mathcal{F}^H_{m_1} \times \mathcal{F}^H_{m_2} \to \mathcal{F}^H_{m_1+m_2}.$$

For more detailed explanations, refer to [3, (12.14), (12.82), (13.7), (13.9)].

Proposition 4.1 (Beals-Greiner [3, Theorems 14.1 and 14.7]) *For $p \in \mathcal{S}^m_H$, $q \in \mathcal{S}^{m'}_H$ with*

$$p \sim \sum_{k \leq m} p_k, \quad q \sim \sum_{k \leq m'} q_k \quad (\textit{at each } \mathbb{P}), \quad p_k, q_k \in \mathcal{F}^H_k,$$

we have $p \circ q := \varsigma(\theta(p) \circ \theta(q)) \in \mathcal{S}^{m+m'}_H$ and

$$\begin{aligned} & p \circ q \sim \sum_{r \leq m+m'} (p \circ q)_r \quad (\textit{at each } \mathbb{P}), \quad (p\circ q)_r \in \mathcal{F}^H_r, \\ & (p\circ q)_{m+m'}(\mathbb{P},\xi) = (p_m \# q_{m'})(\mathbb{P},\xi), \\ & (p \circ q)_r(\mathbb{P},\xi) = \sum \frac{1}{A!\,B!\,G!}\,(D^G_x \mathbf{u}_{BC})(\mathbb{P}) \Big(\partial^{A+G}_\sigma p_k \,\#\, (D^A_x \partial^B_\sigma q_{k'})\sigma^C\Big)\Big|_{(x,\sigma)=(\mathbb{P},\xi)} \\ & \qquad \Big((\sigma(x,\xi) - \sigma^0(x,\xi))^B = \sum_{|C|=|B|} \mathbf{u}_{BC}(x)\, \sigma^0(x,\xi)^C\Big). \end{aligned}$$

The multi-indices A, B, … run only in the region given by $r = k + k' - |A|_H - |B|_H - |G|_H + |C|_H$ and $-|C|_H + |G|_H + |B|_H \geq |B| = |C|$, so that the summation is finite.

Let us derive the formula (3.9) from the proposition.

Proposition 4.2 *We have*

$$\varsigma(\sharp\nabla^{\overset{0,*}{\wedge_H}}_X)(x,\sigma) = \langle {}^t v^\bullet(x)\, iX_x, \sigma\rangle + \sum \omega(\sharp\nabla)^\alpha_\beta(X_x) \cdot e^{\bar\alpha}_\mathbb{C} \wedge e^{\bar\beta}_\mathbb{C} \vee, \tag{4.1}$$

$$\partial_{x_j} \varsigma(\sharp\nabla^{\overset{0,*}{\wedge_H}}_X)(x,\sigma)\Big|_{x=0} = \langle \partial_{x_j} {}^t v^\bullet(x) iX_x\Big|_{x=0}, \sigma\rangle - \frac{1}{4} F(\sharp\nabla;\wedge)(X,\partial/\partial x_j)(\mathbb{P}). \tag{4.2}$$

Proof. We have ${}^\sharp\nabla_X^{\wedge_H^{0,*}} = X + \sum \omega({}^\sharp\nabla)^\alpha_\beta(X)\cdot e^{\bar\alpha}_{\mathbb C}\wedge e^{\bar\beta}_{\mathbb C}\vee$ and, referring to (2.8), we have

$$
\begin{aligned}
&\varsigma(X)(\mathbb P(x),\mathcal T^x_{\mathbb P}(\xi_{\mathbb P}))(\mathcal T^x_{\mathbb P}(u_{\mathbb P})) = X\Big(e^{i\langle \exp^{-1}_{\mathbb P(x)}(x'),\mathcal T^x_{\mathbb P}(\xi_{\mathbb P})\rangle}\,\mathcal T^{x'}_{\mathbb P(x)}(\mathcal T^x_{\mathbb P}(u_{\mathbb P}))\Big)\Big|_{x'=0}\\
&= X\Big(e^{i\langle \exp^{-1}_{\mathbb P(x)}(x'),\mathcal T^x_{\mathbb P}(\xi_{\mathbb P})\rangle}\,\mathcal T^{y}_{\mathbb P}(u_{\mathbb P})\Big)\Big|_{x'=0,y=x}\qquad (\mathbb P(y)=\mathbb P(x)(x'))\\
&= \sum X_j\frac{\partial}{\partial y_j}\Big(e^{i\langle \exp^{-1}_{\mathbb P(x)}(x'),\mathcal T^x_{\mathbb P}(\xi_{\mathbb P})\rangle}\Big)\Big|_{x'=0}\mathcal T^x_{\mathbb P}(u_{\mathbb P})\\
&= \sum X_j v^{jk}(x)\frac{\partial}{\partial x'_k}\Big(e^{i\langle \exp^{-1}_{\mathbb P(x)}(x'),\mathcal T^x_{\mathbb P}(\xi_{\mathbb P})\rangle}\Big)\Big|_{x'=0}\mathcal T^x_{\mathbb P}(u_{\mathbb P})\\
&= \sum X(x)_j v^{jk}(x) i(\mathcal T^x_{\mathbb P}(\xi_{\mathbb P}))_k\cdot \mathcal T^x_{\mathbb P}(u_{\mathbb P})\\
&= \langle {}^t v^\bullet(x) iX_x,\mathcal T^x_{\mathbb P}(\xi_{\mathbb P})\rangle\cdot \mathcal T^x_{\mathbb P}(u_{\mathbb P}).
\end{aligned}
$$

Hence, we get (4.1), which, together with (1.1) and (2.11), yields (4.2) certainly. ∎

Another proof of the formula (3.9). Let us check it only in the case of (3.10) with $g=h=1$. Further we assume $X=X^H$, $Y=Y^H$ (i.e., $m_X=m_Y=1$) to simplify the argument. We have

$$
\begin{aligned}
&\sum_A \frac{1}{A!}\Big(\partial^A_\sigma p_m \,\#\, D^A_x q_{m'}\Big)\Big|_{(x,\sigma)=(0,\xi)}\\
&= \sum_k \frac{1}{k!}\Big(\partial_{\sigma_{j_1}}\cdots\partial_{\sigma_{j_k}} p_m \,\#\, D_{x_{j_1}}\cdots D_{x_{j_k}} q_{m'}\Big)\Big|_{(x,\sigma)=(0,\xi)}\\
&= \sum \frac{1}{k!}\frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x,\eta\rangle}\,\partial_{\xi_{j_1}}\cdots\partial_{\xi_{j_k}}\langle iX,\xi\rangle^m\Big|_{\xi\Rightarrow\xi+\eta}\\
&\qquad\qquad D_{x_{j_1}}\cdots D_{x_{j_k}}\Big(\varsigma({}^\sharp\nabla_Y^{\wedge_H^{0,*}})(x,\sigma)\Big)^{m'}\Big|_{(x,\sigma)=(0,\sigma^0(x,\xi))}dxd\eta
\end{aligned}
$$

and (4.2) implies

$$
\begin{aligned}
&D_{x_j}\Big(\varsigma({}^\sharp\nabla_Y^{\wedge_H^{0,*}})(x,\sigma)\Big)^{m'}\Big|_{(x,\sigma)=(0,\sigma^0(x,\xi))}\\
&= -im'\,\langle iY,\sigma^0(x,\xi)\rangle^{m'-1}\Big\{i\langle(\partial_{x_j}{}^t v^\bullet(x)Y_x)\Big|_{x=0},\sigma^0(x,\xi)\rangle-\frac14 F({}^\sharp\nabla;\wedge)(Y,\partial/\partial x_j)\Big\}.
\end{aligned}
$$

Hence, recalling the definition (3.7), we know that the top grading part of $(p\circ q)(\mathbb P,\xi)$ is equal to

$$
\begin{aligned}
&\sum k!\binom{m,m'}{k,k}\frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x,\eta\rangle}\,\langle iX,\xi+\eta\rangle^{m-k}\langle iY,\sigma^0(x,\xi)\rangle^{m'-k}dxd\eta\\
&\qquad\qquad\cdot\Big(\frac14 F({}^\sharp\nabla;\wedge)(X,Y)\Big)^k\\
&= \sum k!\binom{m,m'}{k,k}e^{-\frac{\xi_0}{2i}de^0(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\langle iX,\xi\rangle^{m-k}\langle iY,\xi'\rangle^{m'-k}\Big|_{\xi'=\xi}\Big(\frac14 F({}^\sharp\nabla;\wedge)(X,Y)\Big)^k\\
&= e^{-\frac14 F({}^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})-\frac{\xi_0}{2i}de^0(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\langle iX,\xi\rangle^{m}\langle iY,\xi'\rangle^{m'}\Big|_{\xi'=\xi}\\
&= e^{-\frac14 \mathscr F({}^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\langle iX,\xi\rangle^{m}\langle iY,\xi'\rangle^{m'}\Big|_{\xi'=\xi}.
\end{aligned}
$$

The first equality is shown as follows:

$$
\begin{aligned}
&\frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \langle iX, \xi+\eta\rangle^m \langle iY, \sigma^0(x,\xi)\rangle^{m'} dx d\eta \\
&= \sum_{a+b=m} \frac{m!}{a!\,b!} \langle iX,\xi\rangle^a \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \langle iX,\eta\rangle^b \langle iY,\sigma^0(x,\xi)\rangle^{m'} dx d\eta \\
&= \sum_{a+b=m} \frac{m!}{a!\,b!} \langle iX,\xi\rangle^a \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \langle iX,D_x\rangle^b \langle iY,\sigma^0(x,\xi)\rangle^{m'} dx d\eta \\
&= \sum_{a+b=m} \frac{m!}{a!\,b!} \langle iX,\xi\rangle^a \sum_{b\le m'} \langle iX,D_x\rangle^b \langle iY,\sigma^0(x,\xi)\rangle^{m'} \Big|_{x=0} \\
&= \sum_{a+b=m} \frac{m!}{a!\,b!} \langle iX,\xi\rangle^a \sum_{b\le m'} b! \binom{m'}{b} \langle iY,\xi\rangle^{m'-b} \left(\langle iX,D_x\rangle \langle iY,\sigma^0(x,\xi)\rangle\right)^b \Big|_{x=0} \\
&= \sum_{b=0}^{\max(m,m')} b! \binom{m,m'}{b,b} \langle iX,\xi\rangle^{m-b} \langle iY,\xi\rangle^{m'-b} \left(\frac{\xi_0}{2i} de^0(X,Y)\right)^b \\
&= e^{-\frac{\xi_0}{2i} de^0(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi'})} \langle iX,\xi\rangle^m \langle iY,\xi'\rangle^{m'} \Big|_{\xi'=\xi}.
\end{aligned}
$$

Hence we get (3.9). ∎

To express the composition of symbols $\in \mathcal{SC}_H^\infty$ explicitly, added to Proposition 4.1 thus we need to calculate the differentials $D_x^A \partial_\sigma^B q_{k'}$. Such a calculation is easy in principle and accordingly so is to compute exactly at least the top grading part of the composition. But, as was stated in Introduction, it will be hard to summarize them in a clear and concise formula.

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