CR conformal Laplacian and some invariants on contact Riemannian manifolds

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Abstract

On a contact Riemannian manifold, we study the heat kernel associated with a non-elliptic CR conformal Laplacian which controls the conformal behavior of the scalar curvature of hermitian Tanno connection, particularly deriving a formula for the asymptotic expansion coefficients on the basis of adiabatic expansion theory. One can compute them by using only basic calculus added to the formula. Consequently, associated Green function and zeta function are explicitly understood, and some CR conformal invariants are obtained.

Keywords: hermitian Tanno connection; CR conformal Laplacian; asymptotic expansion; adiabatic expansion; conformal invariant

0 Introduction

Let M be a compact manifold of dimension $2n+1$ equipped with a contact form θ, i.e., $\theta \wedge (d\theta)^n \neq 0$. We have hence the Reeb vector field ξ, which satisfies $\theta(\xi) = 1$ and $\mathcal{L}_\xi \theta = 0$. In addition, let us equip M with a Riemannian metric g and a $(1,1)$-tensor field J satisfying $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X,Y)$ and $J^2 X = -X + \theta(X)\xi$ for any vector fields X, Y. (In this paper we adopt such a notation as $d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$.) To the manifold $M = (M, \theta, \xi, g, J)$ called a contact Riemannian manifold we will attach the hermitian Tanno connection ∇ ([7, §1]) defined as: Let ∇^g be the Levi-Civita connection and ${}^*\nabla$ be the Tanno connection ([12]) defined by ${}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X \theta)(Y)\xi$. Then we define ∇ as its

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hermitian part, i.e.,

$$\nabla_X Y = \begin{cases} {}^*\nabla_X(f\xi) & (Y = f\xi), \\ \dfrac{1}{2}({}^*\nabla_X Y - J\,{}^*\nabla_X JY) & (Y \in \Gamma(HM)), \end{cases}$$

where we put $HM = \ker\theta$. In terms of the Tanno tensor $\mathcal{Q}$ defined by $\mathcal{Q}(X,Y) = (\nabla^g_Y J)(X) + (\nabla^g_Y \theta)(JX)\xi + \theta(X)J\nabla^g_Y \xi$, it is described as $\nabla_X Y = {}^*\nabla_X Y - \frac{1}{2}J\mathcal{Q}(Y,X)$ too. Canonically we decompose the complexified bundles $\mathbb{C}HM = HM \otimes \mathbb{C}$, etc., into $\mathbb{C}HM = H_{1,0}M \oplus H_{0,1}M$, $\mathbb{C}H^*M = H^{1,0}M \oplus H^{0,1}M$ ($H_{1,0}M := \{W \in \mathbb{C}HM \mid JW = iW\}$, etc.). Notice that, if the almost complex structure J is integrable, i.e., $[\Gamma(H_{1,0}M), \Gamma(H_{1,0}M)] \subset \Gamma(H_{1,0}M)$, then both ∇ and ${}^*\nabla$ coincide with the Tanaka-Webster connection.

In this paper, we consider the non-elliptic Laplacian

$$\Box^\theta = \frac{1}{2}\Delta_H + \frac{n}{4(n+1)}S(\nabla),$$

which will play an important role in studying the CR conformal behavior of the scalar curvature $S(\nabla)$. Here Δ_H denotes the sublaplacian $d_H^* d_H$, where d_H is the exterior differentiation d followed by the natural projection to $\Gamma(H^*M)$. Though the elliptic theory does not work, in §2 we will carry out an investigation into the heat kernel, which will be applied also to some basic subjects subsequently. In §3 we will indeed describe explicitly the behavior of the Green function $G^\theta(P,P')$ when $P \to P'$.

Consequently, in §4 and §5 we show that the coefficient $a_n^\theta(P)$ of the asymptotic expansion $e^{-t\Box^\theta}(P,P) \sim \sum_{k=0}^\infty t^{-(n+1)+k} a_k^\theta(P)$ (given in Theorem 2.2.3) is a CR conformal scalar invariant of weight $2n$, i.e., $a_n^{e^{2f}\theta}(P) = e^{-2nf(P)}a_n^\theta(P)$, and the integral of $a_{n+1}^\theta(P)$ over M is a global CR conformal invariant. In §5, properties of the zeta function will be summarized as well. In the Riemannian case these results are proved by Parker-Rosenberg [8, Theorems 3.1 and 2.1] and Branson-Ørsted [2, Corollary 3.7] (for the integral of $a_{n+1}^\theta(P)$). In a way similar to [8], Stanton [10, Theorems 2.2 and 3.3] proved these results in the integrable contact Riemannian case. Our research in the general contact Riemannian case follows also the idea in [8].

In terms of the Kohn-Rossi Laplacian $\Box_H = \bar\partial_H^* \bar\partial_H$ where $\bar\partial_H$ is d followed by the projection to $\Gamma(H^{0,1}M)$, the Laplacian $\Box^\theta$ is described as

$$\Box^\theta = \Box_H - \sqrt{-1}\,\frac{n}{2}\xi + \frac{n}{4(n+1)}S(\nabla) \tag{0.1}$$

(refer to Proposition 1.2). The author ([7], [3]) studied the heat kernel $e^{-t\Box_H}$ associated with $\Box_H$ $(= \bar\partial_H^*\bar\partial_H + \bar\partial_H\bar\partial_H^*)$ acting on (p,q)-forms $(0 < q < n)$ on the basis of the adiabatic expansion theory developed in [6]. Though $\Box^\theta$ acts on functions, yet the theory, which is a key tool, works well. It exhibits its ability particularly for studying such abnormal Laplacians, and particularly offers a striking formula for the coefficients $a_k^\theta(P)$, Using only a basic knowledge of calculus added to the formula, one can describe them explicitly up to an arbitrarily high order. Also to the Riemannian case ([8]), the theory can be applied and such a formula for the asymptotic expansion coefficients will be derived. This argument will be developed fully elsewhere.

We take a local unitary frame $\xi_\bullet = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{\bar 1}, \ldots, \xi_{\bar n})$ of $\mathbb{C}TM$ ($\xi_0 := \xi$, $\xi_{\bar\alpha} := \overline{\xi_\alpha} \in H_{0,1}M$, $g(\xi_\alpha, \xi_{\bar\beta}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) and the dual frame $\theta^\bullet = (\theta^0, \theta^1, \ldots, \theta^n, \theta^{\bar 1}, \ldots, \theta^{\bar n})$ ($\theta^0 := \theta$). As usual the Greek indices α, β, ... vary from 1 to n, the block Latin indices A, B, ... vary in $\{0, 1, \ldots, n, \bar 1, \ldots, \bar n\}$ and the symbol $\sum$ may be omitted (in an unusual manner). It is known (refer to [7, Lemma 1.2]) that the hermitian connection is described as $\nabla\xi = 0$, $\nabla\xi_\beta = \xi_\alpha \cdot \omega^\alpha_\beta$, $\nabla\xi_{\bar\beta} = \xi_{\bar\alpha} \cdot \omega^{\bar\alpha}_{\bar\beta}$, $\omega^{\bar\alpha}_{\bar\beta} = -\omega^\beta_\alpha$.

1 CR conformal change of contact Riemannian structure and CR conformal Laplacian

Let us recall that the (pseudohermitian) scalar curvatures of $\nabla = \nabla^\theta$, etc., are defined as $S(\nabla) = g(F(\nabla)(\xi_\beta, \xi_{\bar\beta})\xi_\alpha, \xi_{\bar\alpha})$ ($F(\nabla)(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$), etc. Referring to [7, Lemma 1.2], [12] and [1], then we have:

Proposition 1.1 *We have*

$$(1.1)\quad e^{2f} S({}^*\nabla^{e^{2f}\theta}) - S({}^*\nabla^\theta) = e^{2f} S(\nabla^{e^{2f}\theta}) - S(\nabla^\theta) = 2(n+1)(\Delta_H^\theta f - |d_H f|^2_\theta),$$
$$S({}^*\nabla^\theta) - S(\nabla^\theta) = \frac{1}{4} \sum |\mathcal{Q}(\xi_\alpha, \xi_\beta)|^2_\theta .$$

In particular, we have $e^{2f}(S({}^*\nabla^{e^{2f}\theta}) - S(\nabla^{e^{2f}\theta})) = S({}^*\nabla^\theta) - S(\nabla^\theta)$, that is, the difference $S({}^*\nabla^\theta) - S(\nabla^\theta)$ is a CR conformal (scalar) invariant of weight 2.

By setting $\phi^{2/n} = e^{2f}$, (1.1) for ∇ is equivalent to

$$S(\nabla^{e^{2f}\theta}) = \phi^{1-q} \cdot 2b_n \Box^\theta \phi \qquad (q = b_n = 2 + \frac{2}{n}).$$

Changing θ to $e^{2f}\theta$, hence, transforms $\Box^\theta$ to

$$(1.2)\qquad \Box^{e^{2f}\theta} = e^{-(n+2)f} \circ \Box^\theta \circ e^{nf}.$$

Proposition 1.2 (cf. Lee ([5, Theorem 2.3])) *We have $\Box_H + \overline{\Box}_H = \Delta_H$, $\Box_H - \overline{\Box}_H = \sqrt{-1}\, n\xi$, $\frac{1}{2}\Delta_H = \Box_H - \sqrt{-1}\, \frac{n}{2}\xi$ and the formula* (0.1)*, where we set $\overline{\Box}_H = \partial_H^* \partial_H$.*

Proof. Let us show the second equality. We have the Weitzenböck-type formula $\Box_H = -\sum(\xi_\alpha\xi_{\bar\alpha} - \nabla_{\xi_\alpha}\xi_{\bar\alpha})$ ([7, Proposition 1.3]). As for the torsion $T(\nabla)$ we have $T(\nabla)(\xi_\alpha, \xi_{\bar\beta}) = i\delta_{\alpha\beta}\,\xi$. Hence, we obtain $\Box_H - \overline{\Box}_H = \sum T(\nabla)(\xi_\alpha, \xi_{\bar\alpha}) = \sqrt{-1}\, n\xi$. ∎

Proposition 1.3 (cf. [7, Remark 2.2]) *The Laplacian $\Box^\theta$ is a self-adjoint hypoelliptic real operator and has Weitzenböck-type expression*

$$\Box^\theta = -\sum \xi_\alpha \xi_{\bar\alpha} - \sum \omega^\alpha_\beta(\xi_\alpha)\xi_{\bar\beta} - \sqrt{-1}\,\frac{n}{2}\xi + \frac{n}{4(n+1)}S(\nabla). \tag{1.3}$$

In addition, there is a constant $C > 0$ such that

$$\|\varphi\|_{s+1} \leq C\left\{\|\Box^\theta\varphi\|_s + \|\varphi\|\right\} \quad (\varphi \in C^\infty(M)), \tag{1.4}$$

where $\|\cdot\|_s$ is the Sobolev norm of order s.

Proof. (1.3) is obvious. As for the hypoellipticity and the estimate (1.4): We know (see [7, Remark 2.2]) that those hold for the Kohn-Rossi Laplacian $\Box_H$ acting on (p,q)-forms when $0 < q < n$. Though $\Box^\theta$ acts on functions, those still hold for it because of the existence of the term $\sqrt{-1}\,\frac{n}{2}\xi$. ∎

2 The heat equation for the CR conformal Laplacian

This section is devoted to the study of fundamental solution of the initial value problem for the heat equation

$$\left(\frac{\partial}{\partial t} + \Box^\theta\right)\phi = 0, \quad \lim_{t\to 0}\phi(t) = \varphi \quad (\varphi \in C^\infty(M)), \tag{2.1}$$

where the convergence is in the L^2-norm. The author ([7]) discussed the problem relative to the Kohn-Rossi Laplacian $\Box_H$ acting on forms. Following the argument ([7, Theorems 2.1, 2.3 and 5.3]), we will show the unique existence theorem (Theorem 2.1.1), the asymptotic expansion theorem (Theorem 2.2.3) for $\Box^\theta$. A formula for the asymptotic expansion coefficients is also derived.

2.1 On the existence of heat kernel

Theorem 2.1.1 (cf. [7, Theorem 2.1]) *The initial value problem (2.1) has a unique heat kernel $e^{-t\Box^\theta}(P,P')$. As to the initial condition, added to $\lim_{t\to 0}\int dV_\theta(P')\, e^{-t\Box^\theta}(P,P')\varphi(P') = \varphi(P)$, we have $\lim_{t\to 0}\int dV_\theta(P)\,\varphi(P)e^{-t\Box^\theta}(P,P') = \varphi(P')$, where dV_θ is the volume element, i.e., $dV_\theta = \theta\wedge(d\theta)^n/n!$.*

The proof is similar to that of [7, Theorem 2.1] (for $\Box_H$ acting on forms). It follows from Proposition 1.3 that this can be shown by functional analysis method. But, for the study of asymptotic expansion of $e^{-t\Box^\theta}$, it will be more desirable to actually construct it by Levi's iteration method. It is crucial to find out an appropriate first approximation, which is introduced carefully below.

We fix a point P^0 and take local unitary frames $\xi_\bullet$, $\theta^\bullet$ neat P^0 which are ∇-parallel along the ∇-geodesics from P^0. In addition, let $z = (z_0, z_1, \cdots, z_n)$ be ∇-normal coordinates centered at P^0 defined by $\exp^\nabla(\xi_\bullet(P^0)\cdot z_\bullet(P)) = P$: to be precise, first we set $\xi_0^{\mathbb{R}} = \xi$, $\xi_\alpha^{\mathbb{R}} = (\xi_\alpha + \xi_{\bar\alpha})/\sqrt{2}$, $\xi_{n+\alpha}^{\mathbb{R}} = J\xi_\alpha^{\mathbb{R}}$, next take real ∇-normal coordinates $x = (x_0, x_1, \cdots, x_{2n})$ centered at P^0 satisfying $\exp^\nabla(\xi_\bullet^{\mathbb{R}}(P^0)\cdot x(P)) = P$ and put $z_0 = x_0$, $z_\alpha = (x_\alpha + ix_{n+\alpha})/\sqrt{2}$, $z_{\bar\alpha} = \bar z_\alpha$. The frames $(\partial/\partial z) = (\partial/\partial z_\bullet) = (\partial/\partial z_0, \partial/\partial z_1, \cdots, \partial/\partial z_{\bar 1}, \cdots)$, $(dz) = (dz_\bullet) = (dz_0, dz_1, \cdots, dz_{\bar 1}, \cdots)$ are then defined as $\partial/\partial z_0 = \partial/\partial x_0$, $\partial/\partial z_\alpha = (\partial/\partial x_\alpha - i\partial/\partial x_{n+\alpha})/\sqrt{2}$, etc. (The symbols z and $z_\bullet$ originally mean $(z_0, z_1, \ldots, z_n)$ and $(z_0, z_1, \ldots, z_n, z_{\bar 1}, \ldots, z_{\bar n})$ respectively. But they may be used indiscriminately.) Then, by [7, Proposition 2.4], we have

$$\xi_\bullet = (\partial/\partial z_\bullet)\cdot V_\bullet(z) \;\; (\text{i.e., } \xi_A = \sum V_{BA}(z)\,\partial/\partial z_B), \quad \theta^\bullet = (dz_\bullet)\cdot V^\bullet(z), \tag{2.1.1}$$

$$\begin{aligned} V^{BA}(z) = \delta_{BA} &+ \sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum z_{A_1}\cdots z_{A_\ell} \frac{\partial^{\ell-1} T(\nabla)^A_{A_1}(\partial/\partial z_B)}{\partial z_{A_2}\cdots\partial z_{A_\ell}}(0) \\ &+ \sum_{\ell=2}^{\infty} \frac{\ell-1}{(\ell+1)!} \sum z_{A_1}\cdots z_{A_\ell} \frac{\partial^{\ell-2} F(\nabla)^A_{A_1}(\partial/\partial z_{A_2}, \partial/\partial z_B)}{\partial z_{A_3}\cdots\partial z_{A_\ell}}(0), \end{aligned} \tag{2.1.2}$$

where we put $T(\nabla)(\xi_B, X) = \xi_A \cdot T(\nabla)^A_B(X)$, $F(\nabla)(X,Y)\xi_B = \xi_A\cdot F(\nabla)^A_B(X,Y)$. Also the connection coefficients $\omega^\alpha_\beta(\partial/\partial z_A) := g(\nabla_{\partial/\partial z_A}\xi_\beta, \xi_{\bar\alpha})$ are formally expanded as

$$\omega^\alpha_\beta(\partial/\partial z_A)(z) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum z_{A_1}\cdots z_{A_\ell} \frac{\partial^{\ell-1} F(\nabla)^\alpha_\beta(\partial/\partial z_A, \partial/\partial z_{A_1})}{\partial z_{A_2}\cdots\partial z_{A_\ell}}(0). \tag{2.1.3}$$

Hence, the coefficients of the Taylor expansions of V^{BA}, V_{BA}, $\omega^\alpha_\beta(\partial/\partial z_A)$ are all expressed as polynomials made of

$$\begin{aligned} \mathcal{R}_{A_1A_2A_3A_4A_5\cdots A_\ell} &= \frac{\partial^{\ell-4} g(F(\nabla)((\partial/\partial z_{A_3}, \partial/\partial z_{A_4})\partial/\partial z_{A_2}, \partial/\partial z_{A_1})}{\partial z_{A_5}\cdots\partial z_{A_\ell}}(P^0), \\ \mathcal{T}_{A_1A_2A_3A_4\cdots A_\ell} &= \frac{\partial^{\ell-3} g(T(\nabla)(\partial/\partial z_{A_2}, \partial/\partial z_{A_3}), \partial/\partial z_{A_1})}{\partial z_{A_4}\cdots\partial z_{A_\ell}}(P^0). \end{aligned} \tag{2.1.4}$$

Indeed, for example we have (by [7, Corollary 2.5])

$$\begin{aligned} \theta &= dz_0 + dz_\beta\cdot z_{\bar\beta}\frac{-i}{2} + dz_{\bar\beta}\cdot z_\beta\frac{i}{2} + O(|z|^2), \\ \theta^\alpha &= dz_\alpha + dz_0\cdot z_{\bar\gamma}\frac{-\mathcal{T}_{\bar\alpha 0\bar\gamma}}{2} + dz_{\bar\beta}\cdot\left\{z_0\frac{\mathcal{T}_{\bar\alpha 0\bar\beta}}{2} + z_{\bar\gamma}\frac{\mathcal{T}_{\bar\alpha\bar\gamma\bar\beta}}{2}\right\} + O(|z|^2), \\ \xi &= \partial/\partial z_0 + \partial/\partial z_\alpha\cdot z_{\bar\gamma}\frac{\mathcal{T}_{\bar\alpha 0\bar\gamma}}{2} + \partial/\partial z_{\bar\alpha}\cdot z_\gamma\frac{\mathcal{T}_{\alpha 0\gamma}}{2} + O(|z|^2), \\ \xi_\beta &= \partial/\partial z_\beta + \partial/\partial z_0\cdot z_{\bar\beta}\frac{i}{2} + \partial/\partial z_{\bar\alpha}\cdot\left\{z_0\frac{-\mathcal{T}_{\alpha 0\beta}}{2} + z_\gamma\frac{-\mathcal{T}_{\alpha\gamma\beta}}{2}\right\} + O(|z|^2), \end{aligned}$$

which say that the structure of M near P^0 is roughly approximated by the standard structure of the Heisenberg group $H_n = \mathbb{R}\times\mathbb{C}^n$ near the origin. Here H_n is the Lie group, whose element is denoted by $z = (z_0, z_1, \dots, z_n) = (z_0, z_\blacktriangle)$, with the group action $zz' = (z_0 + z'_0 + \mathrm{Im} \sum z_\alpha z'_{\bar\alpha}, z_\blacktriangle + z'_\blacktriangle)$, and with the standard contact Riemannian structure $(\theta_H, \xi^H, J^H, g^H : \xi^H_\bullet, \theta^\bullet_H)$ given by

$$\theta_H = dz_0 + dz_\beta \cdot z_{\bar\beta}\frac{-i}{2} + dz_{\bar\beta} \cdot z_\beta \frac{i}{2}, \quad \theta^\alpha_H = dz_\alpha,$$
$$\xi^H = \partial/\partial z_0, \quad \xi^H_\beta = \partial/\partial z_\beta + \partial/\partial z_0 \cdot z_{\bar\beta}\frac{i}{2}.$$

In terms of the transition functions of frames, this is described as

$$\xi^H_\bullet = (\partial/\partial z_\bullet) \cdot E(-z), \quad \theta^\bullet_H = (dz_\bullet) \cdot {}^tE(z),$$
$$E(z) := \begin{pmatrix} 1 & z_{\bar 1}\frac{-i}{2} \cdots & z_1 \frac{i}{2} \cdots \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}.$$

Notice that the two kinds of coordinates $z - z'$ and $z'^{-1}z$ near z' are related to each other as

$$\begin{aligned} &(z'^{-1}z)_\bullet = E(z')(z - z')_\bullet = E(z)(z - z')_\bullet, \\ &(z - z')_\bullet = E(-z')(z'^{-1}z)_\bullet = E(-z)(z'^{-1}z)_\bullet. \end{aligned} \tag{2.1.5}$$

Now, the almost complex structure J^H is integrable and H_n is a typical strictly pseudoconvex CR manifold. Further we have $\omega^\alpha_\beta = 0$ and the CR conformal Laplacian is simplified to $\mathbf{L} = -\sum \xi^H_\alpha \xi^H_{\bar\alpha} - \sqrt{-1}\,\frac{n}{2}\,\xi^H$. It is evident from the work of Stanton [9] (see also [7, Lemma 2.6] and [11, (1.8)]) that the problem (2.1) with $\varphi \in C^\infty_0(H_n)$ has a unique fundamental solution $\mathbf{r}(t, z, z')$ defined by

$$\mathbf{r}(t, z, z') = r_t(z'^{-1}z), \quad r_t(z) := \int_{-\infty}^{\infty} ds\, e^{-is\cdot(2z_0/t)}\, \Phi_t(s, z_\blacktriangle),$$
$$\Phi_t(s, z_\blacktriangle) := \frac{1}{(2\pi t)^{n+1}} \Big(\frac{s}{\sinh s}\Big)^n \exp\Big(-\frac{|z_\blacktriangle|^2 s}{t \tanh s}\Big).$$

Here $\Phi_t(s, z_\blacktriangle)$, $r_t(z)$ are rapidly decreasing with respect to the variables $z_\blacktriangle$, z respectively.

Remark 2.1.2 *Stanton [9] showed that, if $-n < a < n$, then the problem* (2.1) *relative to the operator $\mathbf{L}_a = -\sum \xi^H_\alpha \xi^H_{\bar\alpha} - \sqrt{-1}\,\frac{n-a}{2}\,\xi^H$ on H_n has a unique fundamental solution $r^a_t(z'^{-1}z)$ with $r^a_t(z) := \int_{-\infty}^{\infty} ds\, e^{-is\cdot(2z_0/t)}\, \Phi_t(s, z_\blacktriangle) e^{-as}$. On H_n the problem relative to the Kohn-Rossi Laplacian $\square_H = \mathbf{L}_{n-2q}$ acting on (p, q)-forms $(0 < q < n$, i.e., $-n < n - 2q < n)$ has, thus, a fundamental solution $\sum \theta^{I\bar K}_H(z) \boxtimes \theta^{\bar I K}_H(z') \cdot r^{n-2q}_t(z'^{-1}z)$ (refer to [7, Lemma 2.6] and [11]). Relying on it, the author actually constructed the heat kernel $e^{-t\square_H}$ on M. The reader will, hence, agree that $e^{-t\square^\theta}$ can be constructed similarly.*

Referring to the argument following [7, Lemma 2.6], let us take the explanation further. Near P^0 we want to adopt a first approximation more geometric than $\mathbf{r}(t,z,z')$. Let $\Theta : U^0 \times U^0 \to H_n$, $(P',P) \mapsto \Theta(P',P)$, be a ∇-normal coordinate system with respect to $\xi_\bullet$ defined on a neighborhood U^0 of P^0, i.e., $\exp^\nabla(\xi_\bullet(P')\cdot\Theta(P',P)) = P$. Set $O(\Theta)^k_H = O(|\Theta(P',P)|^k_H)$ ($|z|_H := \{z_0^2 + |z_\blacktriangle|^4 + |z_{\bar\blacktriangle}|^4\}^{1/4}$). Then, in a way similar to [7, Lemma 2.6], we can show that, near P^0,

$$\Box^\theta = \mathbf{L}^\Theta + \sum_{A\neq 0, B\neq 0} O(\Theta)^1_H \frac{\partial}{\partial\Theta_A}\frac{\partial}{\partial\Theta_B} + \sum_{B\neq 0} O(\Theta)^2_H \frac{\partial}{\partial\Theta_0}\frac{\partial}{\partial\Theta_B}$$
$$+ O(\Theta)^1_H O(\Theta)^2_H \frac{\partial}{\partial\Theta_0}\frac{\partial}{\partial\Theta_0} + \sum_{B\neq 0} O(\Theta)^0_H \frac{\partial}{\partial\Theta_B} + O(\Theta)^1_H \frac{\partial}{\partial\Theta_0} + O(\Theta)^0_H,$$

where $\mathbf{L}^\Theta$ is $\mathbf{L}$ calculated in the coordinates $\Theta = \Theta(P',P)$. This asserts that, near P^0, $r_t(\Theta(P',P))$, which is certainly more geometric, approximates the heat kernel closely enough for constructing it by iteration method. An appropriate first approximation $r(t,P,P')$ is now given as follows: We cover M by a finite number of small open sets U^j centered at P^j. Each U^j is equipped with unitary frames $\xi^j_\bullet$, $\theta^\bullet_j$ which are ∇-parallel along the ∇-geodesics from P^j and the ∇-normal coordinate system Θ^j with respect to $\xi^j_\bullet$. Let ϕ_j be nonnegative C^∞ functions such that $\{\phi_j^2\}$ is a partition of unity subordinate to the cover $\{U^j\}$. We set

$$r(t,P,P') = \sum \phi_j(P)\,\phi_j(P')\,r_t(\Theta^j(P',P)).$$

The heat kernel is then constructed as follows: We set $q(t,P,P') = (\frac{\partial}{\partial t} + \Box^\theta)r(t,P,P')$ and $q^1 = q$, $q^2 = q\#q^1$, $q^3 = q\#q^2$, $\cdots$ ($\# = \#_\theta$), where the convolution $h_1\#_\theta h_2$ of functions is defined by $(h_1\#_\theta h_2)(t,P,P') = \int_0^t ds \int_M dV_\theta(Q)\, h_1(t-s,P,Q)h_2(s,Q,P')$. The sum

$$p = \sum_{k=0}^{\infty} (-1)^k r\#q^k \quad (r\#q^0 := r)$$

is now a unique fundamental solution of (2.1). Referring to [7, Proof of Theorems 2.1 and 2.3], here we present an estimate of $R_{k_0}(p) := \sum_{k\geq k_0}(-1)^k r\#q^k$: For every integer $m \geq 0$ and multi-indices $\mathbb{A} = (A_1, A_2, \ldots, A_{|\mathbb{A}|})$, $\mathbb{A}'$, and, further, for every integer $\ell = 0$ or $\ell \geq 2n+2$, there exists a constant $C(k_0,m,\mathbb{A},\mathbb{A}',\ell) > 0$ such that, on $(0,T_0]\times M\times M$,

$$\text{(2.1.6)} \qquad \sum_{U^j\ni P,\, U^{j'}\ni P'} \left| (\partial/\partial t)^m \xi^j_{\mathbb{A},P}\xi^{j'}_{\mathbb{A}',P'} R_{k_0}(p)(t,P,P') \right|$$
$$\leq C(k_0,m,\mathbb{A},\mathbb{A}',\ell)\, t^{(k_0-|\mathbb{A}|_H-|\mathbb{A}'|_H)/2-m+\ell/2-(n+1)}\delta(P',P)^{-\ell},$$

where we set $\xi^j_{\mathbb{A}} = \xi^j_{A_1} \cdots \xi^j_{A_{|\mathbb{A}|}}$, $|\mathbb{A}|_H = 2\#\{A_j = 0\} + \#\{A_j \neq 0\}$, and, taking a neighborhood U of the diagonal set in $M \times M$, we set

$$\delta(P', P) = \begin{cases} \min\limits_{j\,:\,P,P' \in U^j} \left(|\Theta^j(P', P)|_H, |\Theta^j(P, P')|_H\right) & ((P', P) \in U), \\ 1 \quad \text{(otherwise)}. \end{cases}$$

There are other various estimates similar to those in [7, Proposition 4.2]. All of them are proved in a similar way as in [7].

2.2 On the asymptotic expansion coefficients of the heat kernel

We want to investigate the behavior $e^{-t\Box^\theta}$ when $t \to 0$. First, let us localize the argument near the point P^0. For the sake of distinction, here the contact form, etc., on M are denoted by θ_M, etc. Let us consider the Heisenberg group $H_n = (H_n, w)$ with the standard contact form $\tilde{\theta}_H \,(= dw_0 + dw_\beta \cdot w_{\bar\beta} \frac{-i}{2} + dw_{\bar\beta} \cdot w_\beta \frac{i}{2})$, etc., and identify a small neighborhood U of the origin with a small neighborhood $U^0\,(\subset M)$ of P^0 via the ∇^M-normal coordinate map $U^0 \ni P \mapsto w = z(P) \in U$. Referring to [7, §3], we may take a new contact Riemannian structure (θ, ξ, J, g) and the associated hermitian Tanno connection ∇ on H_n which coincide with those of M near $0\,(= P^0)$ and with those of (H_n, w) a little apart from 0. The space H_n equipped with the structure and the connection, denoted by $H_n(P^0)$, may be assumed further to satisfy: The ∇-normal coordinates centered at the origin are globally defined, i.e., $z : H_n(P^0) \cong H_n$, $w \mapsto z = z(w)$, $(\exp^\nabla((\partial/\partial w_\bullet)_0 \cdot z_\bullet(w)) = w)$. Thus $H_n(P^0)$ has two kinds of global coordinates, w and z. The ∇-parallel frames $\theta^\bullet$, $\xi_\bullet$ are also assumed to be given globally and we regard the formulas (2.1.1)-(2.1.3) as the ones on $H_n(P^0)$. We denote the CR conformal Laplacian on $H_n(P^0)$ by $\Box^\theta_{H(P^0)}$. Then we have:

Proposition 2.2.1 (cf. [7, Theorem 3.4]) *The initial value problem* (2.1) *on* $H_n(P^0)$ $(\varphi \in C^\infty_0(H_n(P^0)))$ *has a unique heat kernel* $e^{-t\Box^\theta_{H(P^0)}}(z, z')$. *As to the initial condition, added to* $\lim_{t\to 0} \int dV_\theta(z')\, e^{-t\Box^\theta_{H(P^0)}}(z, z')\varphi(z') = \varphi(z)$, *we have* $\lim_{t\to 0} \int dV_\theta(z)\, \varphi(z) e^{-t\Box^\theta_{H(P^0)}}(z, z') = \varphi(z')$.

Careful argument for the proof is required because $H_n(P^0)$ is not compact. Once an appropriate first approximation is find out, such a heat kernel is constructed in a way similar to §2.1, however. (Refer to [7, §4], which was devoted to the construction of the Kohn-Rossi heat kernel on $H_n(P^0)$.)

Let $\Theta^M(z', z)$ be a ∇-normal coordinate system with respect to $\xi_\bullet$ which coincides with that on M near the origin $(= P^0)$, and let us set $\Theta^H(z', z) = w(z')^{-1} w(z)$. In addition, let $\tilde\rho_M(w)$, $\tilde\rho_H(w)$ be non-negative C^∞-functions such that $\{\tilde\rho^2_M(w), \tilde\rho^2_H(w)\}$ is a partition of unity subordinate to the cover $\{\{w \in H_n(P^0) \mid |w| < 2r\}, \{w \in H_n(P^0) \mid |w| > r\}\}$, where $r > 0$ is sufficiently small. Then, setting $\rho_M(z) = \tilde{\rho}_M(w(z))$, etc., we adopt

$$r(t, z, z') = \rho_M(z)\rho_M(z')\, r_t(\Theta^M(z', z)) + \rho_H(z)\rho_H(z')\, r_t(\Theta^H(z', z)) \tag{2.2.1}$$

as a first approximation. Notice that the support of $q(t, z, z') := (\frac{\partial}{\partial t} + \Box^\theta_{H(P^0)})r(t, z, z')$ may be assumed to be contained in $\{(z, z') \mid |z|_H \le r_0\}$. Now, by setting $q^1 = q$, $q^2 = q\#q^1$, $q^3 = q\#q^2$, $\cdots$ $(\# = \#_\theta)$, the sum

$$p = \sum_{k=0}^{\infty} (-1)^k r\#q^k \quad (r\#q^0 := r)$$

is a unique fundamental solution of (2.1) on $H_n(P^0)$. Referring to [7, Proposition 4.2], here we present also an estimate of $R_{k_0}(p) := \sum_{k \ge k_0} (-1)^k r\#q^k$: For every integer $m \ge 0$ and multi-indices $\mathbb{A}$, $\mathbb{A}'$, and, further, for every integer $\ell = 0$ or $\ell \ge 2n + 2$, there exists a constant $C(k_0, m, \mathbb{A}, \mathbb{A}', \ell) > 0$ such that, on $(0, T_0] \times H_n(P^0) \times H_n(P^0)$,

$$\begin{aligned} &\left|(\partial/\partial t)^m \xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\, R_{k_0}(p)(t, z, z')\right| \\ &\qquad \le C(k_0, m, \mathbb{A}, \mathbb{A}', \ell)\, t^{(k_0 - |\mathbb{A}|_H - |\mathbb{A}'|_H)/2 - m + \ell/2 - (n+1)} \delta(z', z)^{-\ell}, \end{aligned} \tag{2.2.2}$$

where we set $\delta(z', z) = |w(z')^{-1}w(z)|_H$. There are also other same estimates as those in [7, Proposition 4.2].

Duhamel's principle works properly because of (2.1.6) and (2.2.2) so that it is enough to investigate the behavior of $e^{-t\Box^\theta_{H(P^0)}}$ at $(0, 0)$ when $t \to 0$. On the basis of the adiabatic expansion theory, which is a key tool for our study as stated in §0, we will investigate the latter. Though abruptly, let us consider the transformation of $H_n(P^0)$ defined by $z \mapsto \iota_\varepsilon(z) = (\varepsilon z_0, \varepsilon^{1/2} z_1, \ldots, \varepsilon^{1/2} z_n)$, $\varepsilon > 0$, which induces a new contact Riemannian structure $(\theta^\bullet_{(\varepsilon)}, \xi^{(\varepsilon)}_\bullet, g^{(\varepsilon)}, J^{(\varepsilon)}) := (\iota^*_\varepsilon \theta^\bullet_\varepsilon, \iota^*_\varepsilon \xi^\varepsilon_\bullet, \iota^*_\varepsilon g^\varepsilon, \iota^*_\varepsilon J^\varepsilon)$ with

$$\theta^A_\varepsilon := \varepsilon^{-|A|_H/2}\theta^A, \quad \xi^\varepsilon_A := \varepsilon^{|A|_H/2}\xi_A, \quad g^\varepsilon := \sum \theta^A_\varepsilon \otimes \theta^{\bar{A}}_\varepsilon, \quad J^\varepsilon \xi^\varepsilon_\alpha := i\xi^\varepsilon_\alpha.$$

(Thus $\theta \Rightarrow \theta_\varepsilon$ is a kind of CR conformal change.) Obviously (2.1.1) produces

$$\begin{aligned} \xi^{(\varepsilon)}_\bullet &= (\partial/\partial z_\bullet) \cdot V^{(\varepsilon)}_\bullet, \quad V^{(\varepsilon)}_{BA}(z) := \varepsilon^{(|A|_H - |B|_H)/2} V_{BA}(\iota_\varepsilon(z)), \\ \theta^\bullet_{(\varepsilon)} &= (dz_\bullet) \cdot V^\bullet_{(\varepsilon)}, \quad V^{BA}_{(\varepsilon)}(z) := \varepsilon^{(|B|_H - |A|_H)/2} V^{BA}(\iota_\varepsilon(z)). \end{aligned} \tag{2.2.3}$$

To the structure $(\theta^\bullet_\varepsilon, \xi^\varepsilon_\bullet, g^\varepsilon, J^\varepsilon)$ the hermitian Tanno connection $\nabla^\varepsilon := \nabla$ and the Laplacian $\Box^{\theta,\varepsilon}_{H(P^0)} := \varepsilon\Box^\theta_{H(P^0)}$ are attached. Those for the structure $(\theta^\bullet_{(\varepsilon)}, \xi^{(\varepsilon)}_\bullet, g^{(\varepsilon)}, J^{(\varepsilon)})$ are $\nabla^{(\varepsilon)} := \iota^*_\varepsilon \nabla^\varepsilon$ and $\Box^{\theta,(\varepsilon)}_{H(P^0)} := \iota^*_\varepsilon \Box^{\theta,\varepsilon}_{H(P^0)}$ The coordinates z are then $\nabla^{(\varepsilon)}$-normal coordinates centered at 0 with $(\partial/\partial z_\bullet)_0 = \xi^{(\varepsilon)}_\bullet(0)$, and $\xi^{(\varepsilon)}_\bullet$ is $\nabla^{(\varepsilon)}$-parallel along the

$\nabla^{(\varepsilon)}$-geodesics sz $(0 \le s < \infty)$ as well. In addition, the heat kernels are described as

$$e^{-t\Box^{\theta,\varepsilon}_{H(P^0)}}(z,z') = \varepsilon^{n+1} e^{-t\varepsilon\Box^{\theta}_{H(P^0)}}(z,z'),$$
$$e^{-t\Box^{\theta,(\varepsilon)}_{H(P^0)}}(z,z') = \varepsilon^{n+1} e^{-t\varepsilon\Box^{\theta}_{H(P^0)}}(\iota_\varepsilon(z),\iota_\varepsilon(z')).$$

Beware that $\Box^{\theta,(\varepsilon)}_{H(P^0)}$ and $e^{-t\Box^{\theta,(\varepsilon)}_{H(P^0)}}$ act on C^∞-functions on $(H_n, dV_{\theta_{(\varepsilon)}})$. We define $\Box^{\theta}_{(\varepsilon)}$, $e^{-t\Box^{\theta}_{(\varepsilon)}}$ as the operators acting on C^∞-functions on (H_n, dV_{θ_H}). Since just only the volume element $dV_{\theta_{(\varepsilon)}}(z) = dV_{\theta_H}(z)\det V^\bullet(\iota_\varepsilon(z))$ is changed to $dV_{\theta_H}(z)$, certainly we have $\Box^{\theta}_{(\varepsilon)} = \Box^{\theta,(\varepsilon)}_{H(P^0)}$, but the heat kernel is changed to

$$e^{-t\Box^{\theta}_{(\varepsilon)}}(z,z') = \varepsilon^{n+1} e^{-t\varepsilon\Box^{\theta}_{H(P^0)}}(\iota_\varepsilon(z),\iota_\varepsilon(z'))\det V^\bullet(\iota_\varepsilon(z')). \tag{2.2.4}$$

On the other hand, the formula (1.3) induces **adiabatic Weitzenböck-type formula**

$$\Box^{\theta}_{(\varepsilon)} = -\sum \xi^{(\varepsilon)}_{\alpha}\xi^{(\varepsilon)}_{\bar\alpha} - \sum \omega^{\alpha}_{\beta}(\xi^{(\varepsilon)}_{\alpha})\xi^{(\varepsilon)}_{\bar\beta} - \sqrt{-1}\,\frac{n}{2}\xi^{(\varepsilon)} + \frac{n}{4(n+1)}S(\nabla^{(\varepsilon)}), \tag{2.2.5}$$
$$\omega^{\alpha}_{\beta}(\xi^{(\varepsilon)}_{\alpha})(z) = \varepsilon^{1/2}\omega^{\alpha}_{\beta}(\xi_\alpha)(\iota_\varepsilon(z)), \quad S(\nabla^{(\varepsilon)})(z) = \varepsilon^{2/2}S(\nabla)(\iota_\varepsilon(z)).$$

Hence, by (2.1.2), (2.1.3) and (2.2.3), the operator $\Box^{\theta}_{(\varepsilon)}$ can be extended smoothly up to $\varepsilon^{1/2} = 0$ and has a formal power series expansion

$$\Box^{\theta}_{(\varepsilon)} = \sum_{m=0}^{\infty} \varepsilon^{m/2}\,\Box^{\theta}_{m/2}, \quad \Box^{\theta}_{0/2} = \mathbf{L},$$

which we call the **adiabatic expansion of $\Box^\theta$ at** P^0. Crucially the coefficients can be described explicitly up to an arbitrarily high order by using only basic calculus. Suggested by the equality $(\frac{\partial}{\partial t} + \Box^{\theta}_{(\varepsilon)})e^{-t\Box^{\theta}_{(\varepsilon)}} = 0$, let us construct now a formal power series

$$\mathfrak{p}^{\theta}_{(\varepsilon)}(t,z,z') = \sum_{m=0}^{\infty} \varepsilon^{m/2}\,\mathfrak{p}^{\theta}_{m/2}(t,z,z')$$

so as to satisfy $(\frac{\partial}{\partial t} + \Box^{\theta}_{(\varepsilon)})\mathfrak{p}^{\theta}_{(\varepsilon)} = 0$. Namely, we define it inductively by

$$\begin{aligned}
&\mathfrak{p}^{\theta}_{0/2}(t,z,z') = \mathbf{r}(t,z,z'),\\
&\mathfrak{p}^{\theta}_{m/2}(t,z,z') = -\Big(\mathfrak{p}^{\theta}_{0/2}\#\sum_{m_1+m_2=m}^{m_1>0}\Box^{\theta}_{m_1/2}\mathfrak{p}^{\theta}_{m_2/2}\Big)(t,z,z')\\
&\quad = \sum_{\sum m_\ell = m}^{m_1,\ldots,m_k>0}(-1)^k\Big(\mathfrak{p}^{\theta}_{0/2}\#\,\Box^{\theta}_{m_1/2}\mathfrak{p}^{\theta}_{0/2}\#\,\cdots\#\,\Box^{\theta}_{m_k/2}\mathfrak{p}^{\theta}_{0/2}\Big)(t,z,z') \quad (m>0),
\end{aligned}$$

where $\# := \#_{\theta_H}$. The function $\mathfrak{p}^{\theta}_{m/2}(t,z,z')$ is well-defined and, as is expected from the construction, we have:

Proposition 2.2.2 (cf. [7, Proposition 6.1]) *The double form* $p^\theta_{(\varepsilon)}(t,z,z') := e^{-t\square^\theta_{(\varepsilon)}}(z,z')$ *can be extended smoothly up to* $\varepsilon^{1/2}=0$*. As to the Taylor expansion*

$$p^\theta_{(\varepsilon)}(t,z,z') = \sum_{0\le m<m_*} \varepsilon^{m/2}\, p^\theta_{m/2}(t,z,z') + \varepsilon^{m_*/2}\, p^\theta_{m_*/2}(\varepsilon^{1/2},t,z,z'),$$

we have $p^\theta_{m/2}(t,z,z') = \mathfrak{p}^\theta_{m/2}(t,z,z')$ $(0\le m<m_*)$.

In [7, §6], the proposition for $\square_H$ (acting on forms), i.e., [7, Proposition 6.1], was proved. It is easy to alter the argument to fit it for $\square^\theta$. Refer also to Lemma 3.1.1, Propositions 3.1.2 and 3.5.2, which offer more detailed information on $\mathfrak{p}^\theta_{m/2}(t,z,z')$ and $p^\theta_{m_*/2}(\varepsilon^{1/2},t,z,z')$.

Let us set

$$\mathcal{P}^\theta_{(\varepsilon)}(t,z,z') := \mathfrak{p}^\theta_{(\varepsilon)}(t,z,z')\det V_\bullet(\iota_\varepsilon(z')) = \sum_{m=0}^\infty \varepsilon^{m/2}\,\mathcal{P}^\theta_{m/2}(t,z,z').$$

Then the formula (2.2.4) and Proposition 2.2.2 induce a formal power series expansion

$$\varepsilon^{n+1}\, e^{-t\varepsilon\square^\theta_{H(P^0)}}(\iota_\varepsilon(z),\iota_\varepsilon(z')) = \sum_{m=0}^\infty \varepsilon^{m/2}\,\mathcal{P}^\theta_{m/2}(t,z,z'),$$

which yields the asymptotic expansion

$$e^{-t\square^\theta_{H(P^0)}}(0,0) \sim \sum_{m=0}^\infty t^{-(n+1)+m/2}\,\mathcal{P}^\theta_{m/2}(1,0,0).$$

Further, considering the series expansion

$$\begin{aligned}\varepsilon^{(n+1)+(|\mathbb{A}|_H+|\mathbb{A}'|_H)/2}&\Big((\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'} e^{-t\varepsilon\square^\theta_{H(P^0)}}\Big)(\iota_\varepsilon(z),\iota_\varepsilon(z'))\\ &= \sum_{m=0}^\infty \varepsilon^{m/2}\,(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\mathcal{P}^\theta_{m/2}(t,z,z')\end{aligned}$$

and recalling Duhamel's principle, in general we have:

Theorem 2.2.3 (cf. [7, Theorems 2.3 and 5.3]) *There is an asymptotic expansion*

$$(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'} e^{-t\square^\theta}(P^0,P^0) \sim \sum_{m\ge-(|\mathbb{A}|_H+|\mathbb{A}'|_H)} t^{-(n+1)+m/2}\, a^\theta_{m/2}(P^0:\mathbb{A},\mathbb{A}')$$

when $t\to0$*. If we set* $\mathcal{P}^\theta_{m/2}(t,z,z':\mathbb{A},\mathbb{A}') = (\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'}\mathcal{P}^\theta_{m/2}(t,z,z')$*, then*

$$a^\theta_{m/2}(P^0:\mathbb{A},\mathbb{A}') = \mathcal{P}^\theta_{(m+|\mathbb{A}|_H+|\mathbb{A}'|_H)/2}(1,0,0:\mathbb{A},\mathbb{A}'), \tag{2.2.6}$$

which vanishes when m *is odd. In particular, we have*

$$a^\theta_{m/2}(P^0) := a^\theta_{m/2}(P^0:\emptyset,\emptyset) = \mathfrak{p}^\theta_{m/2}(1,0,0),\quad a^\theta_{0/2}(P^0) = \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^\infty ds\left(\frac{s}{\sinh s}\right)^n.$$

In general, (2.2.6) *is expressed as a universal polynomial made of* (2.1.4)*, which can be described explicitly by using only a basic knowledge of calculus.*

It is easily shown in a similar manner as [7, §6.5] (see also Lemma 3.3.1) that

$$\mathcal{P}^{\theta}_{m/2}(t,z,z' : \mathbb{A},\mathbb{A}') = (-1)^{m+|\mathbb{A}|_H+|\mathbb{A}'|_H}\mathcal{P}^{\theta}_{m/2}(t,\hat{z},\hat{z}' : \mathbb{A},\mathbb{A}'), \tag{2.2.7}$$

where, for $z = (z_0, z_\blacktriangle)$, we set $\hat{z} = (z_0, -z_\blacktriangle)$. This ascertains that (2.2.6) vanishes when m is odd.

3 On the Green function

Since $e^{-t\Box^\theta}$ is a compact self-adjoint real operator (by Theorem 2.2.3), the spectrum of $\Box^\theta$ consists of eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ $(\to +\infty)$ with $\sum_{j=1}^\infty e^{-t\lambda_j} < \infty$. Let us take corresponding real eigenfunctions $\phi_1, \phi_2, \cdots$ which form a complete orthonormal basis of $L^2(M, dV_\theta)$. Then we have

$$\begin{aligned} \Box^\theta &= \sum \lambda_i \phi_i \otimes \phi_i \left(= \sum \lambda_i \phi_i \otimes \phi_i^{*(\theta)}\right), \\ e^{-t\Box^\theta}(P,P') &= \sum e^{-t\lambda_j}\phi_j(P)\,\phi_j(P') \\ &= (e^{-t\Box^\theta})^- + (e^{-t\Box^\theta})^0 + (e^{-t\Box^\theta})^+ := \sum_{\lambda_i<0} + \sum_{\lambda_i=0} + \sum_{\lambda_i>0}. \end{aligned} \tag{3.1}$$

We are interested in the Green function (or the Green operator) $G^\theta = G^\theta(P,P')$, which is characterized by $G^\theta \circ \pi_\theta = 0$ and $\pi_\theta + \Box^\theta \circ G^\theta = I$ where $\pi_\theta : C^\infty(M) \to \ker \Box^\theta$ is the canonical projection, and described as

$$\begin{aligned} G^\theta &= \sum_{\lambda_i \neq 0} \lambda_i^{-1}\phi_i \otimes \phi_i = G^\theta_- + G^\theta_+ := \sum_{\lambda_i<0} + \sum_{\lambda_i>0}, \\ G^\theta_\pm(P,P') &= \sum_{\lambda_i \gtrless 0} \lambda_i^{-1}\phi_i(P)\,\phi_i(P'), \quad G^\theta_+(P,P') = \int_0^\infty dt\,(e^{-t\Box^\theta})^+(P,P'). \end{aligned}$$

Lemma 3.1 *The kernel $G^\theta_-(P,P')$ is smooth on $M \times M$ and $G^\theta_+(P,P')$ is smooth outside the diagonal set of $M \times M$.*

Proof. We examine $G^\theta_+(P,P')$. By (2.1.6), for any $\delta > 0$ and integers k and k', $|(\partial/\partial P)^k(\partial/\partial P')^{k'}e^{-t\Box^\theta}(P,P')|$ is bounded on $(0,1] \times \{(P,P') \mid \delta(P',P) \ge \delta\}$ where $(\partial/\partial P)^k$ is a differentiation of order k at P, and

$$\begin{aligned} \int_0^1 dt\,(e^{-t\Box^\theta})^+(P,P') &= \int_0^1 dt\,e^{-t\Box^\theta}(P,P') \\ &\quad + \sum_{\lambda_i<0}\lambda_i^{-1}(e^{-\lambda_i}-1)\phi_i(P)\,\phi_i(P') - \sum_{\lambda_i=0}\phi_i(P)\,\phi_i(P'). \end{aligned}$$

Hence, $(e^{-t\Box^\theta})^+(P,P')$ is certainly integrable on $(0,1]$ when $P \neq P'$ and the integral is smooth on $\{(P,P') \mid P \neq P'\}$. On the other hand, by (1.4), for any k, there is a constant $C>0$ such that $\|\varphi\|_k^2 \le C\sum_{\ell=0}^k \left\|(\Box^\theta)^\ell\varphi\right\|_0^2$ for any $\varphi\in C^\infty(M)$. Let us take $a>0$ such that $\lambda_i>a$ for all $\lambda_i>0$. Then, for any integers ℓ, ℓ', there is $C'>0$ such that $\left\|(\Box_P^\theta)^\ell(\Box_{P'}^\theta)^{\ell'}(e^{-t\Box^\theta})^+\right\|_0 = \left\|\sum_{\lambda_i>0}\lambda_i^{\ell+\ell'}e^{-t\lambda_i}\phi_i\,\phi_i\right\|_0 \le Ce^{-ta}$ when $t\ge 1$. Hence, by the Sobolev lemma, for any k, k', there exists $C>0$ such that

$$\left|(\partial/\partial P)^k(\partial/\partial P')^{k'}(e^{-t\Box^\theta})^+(P,P')\right| \le Ce^{-ta} \quad (t\ge 1). \tag{3.2}$$

Thus $(e^{-t\Box^\theta})^+(P,P')$ is integrable on $[1,\infty)$ and the integral is smooth on $M\times M$. ∎

The purpose in the section is to describe explicitly the behavior of $G^\theta(P,P')$ when $P\to P'$. We follow the work by Parker-Rosenberg ([8, Theorem 2.2]) in the Riemannian case. By virtue of the argument relying on the adiabatic expansion theory in §2.2, our result is much explicit, however. (So will be also in the Riemannian case as stated in §0.) Recall that we have set $|z|_H=\{z_0^2+|z_\blacktriangle|^4+|z_{\bar\blacktriangle}|^4\}^{1/4}$, where z is the ∇-normal coordinates centered at P^0.

Theorem 3.2 *Let z^0 belong to $\{z \mid |z|_H=1\}$. Then $\mathfrak{p}^\theta_{m/2}(t,z^0,0)$ $(0\le m<2n)$ are integrable on $(0,\infty)$ and the Green function has an expansion*

$$G^\theta(\iota_\varepsilon(z^0),0)=\sum_{0\le m<2n}\varepsilon^{-n+m/2}\int_0^\infty dt\,\mathfrak{p}^\theta_{m/2}(t,z^0,0)-a_n^\theta(P^0)\log\varepsilon+O(1), \tag{3.3}$$

where $O(1)$ is a bounded function on $\{\varepsilon^{1/2}>0\}$ which satisfies $(\partial/\partial\varepsilon^{1/2})^iO(1)=O(\varepsilon^{-i/2})$ for any i. In addition, we have

$$\varepsilon^{-n+m/2}\int_0^\infty dt\,\mathfrak{p}^\theta_{m/2}(t,z^0,0)=\int_0^\infty dt\,\mathfrak{p}^\theta_{m/2}(t,\iota_\varepsilon(z^0),0), \tag{3.4}$$

$$\int_0^\infty dt\,\mathfrak{p}^\theta_{m/2}(t,z^0,0)=(-1)^m\int_0^\infty dt\,\mathfrak{p}^\theta_{m/2}(t,\hat z^0,0) \quad (\hat z^0:=(z^0_0,-z^0_\blacktriangle)). \tag{3.5}$$

The proof requires a long argument. In general, let us denote by O_∞ a smooth function defined on a neighborhood of $z=0$. By Lemma 3.1 (and its proof) and Duhamel's principle,

$$\begin{aligned} G^\theta(z,0) &= \int_0^\infty dt\,(e^{-t\Box^\theta})^+(z,0)+O_\infty=\int_0^1 dt\,(e^{-t\Box^\theta})^+(z,0)+O_\infty \\ &= \int_0^1 dt\,e^{-t\Box^\theta}(z,0)+O_\infty=\int_0^1 dt\,e^{-t\Box^\theta_{H(P^0)}}(z,0)+O_\infty. \end{aligned}$$

Further, by (2.2.4),

$$\int_0^1 dt\,e^{-t\Box^\theta_{H(P^0)}}(\iota_\varepsilon(z^0),0)=\varepsilon^{-n}\int_0^{1/\varepsilon}dt\,e^{-t\Box^\theta_{(\varepsilon)}}(z^0,0). \tag{3.6}$$

We will summarize some properties of $p^\theta_{(\varepsilon)}(t,z^0,0)=e^{-t\Box^\theta_{(\varepsilon)}}(z^0,0)$ $(0<t<\varepsilon^{-1})$ in §3.1 and then prove the theorem in §3.2.

3.1 On the function $p^\theta_{(\varepsilon)}(t,z,z') = e^{-t\square^\theta_{(\varepsilon)}}(z,z')$ $(0 < t < \varepsilon^{-1})$

By recalling the definition (refer also to the proof of [7, Lemma 6.9]), it is expressed as $p^\theta_{(\varepsilon)} = \sum_{k=0}^\infty (-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)} = \sum_{0\le k<k_0} (-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)} + R_{k_0}(p^\theta_{(\varepsilon)})$ $(\# = \#_{\theta_H})$ with

$$\text{(3.1.1)} \quad \begin{aligned} r_{(\varepsilon)}(t,z,z') &:= \varepsilon^{n+1} r(t\varepsilon, \iota_\varepsilon(z), \iota_\varepsilon(z')) \det V^\bullet(\iota_\varepsilon(z')) \\ &= t^{-n-1} \sum \rho_\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z',z)), \end{aligned}$$

$$\text{(3.1.2)} \quad \begin{aligned} q_{(\varepsilon)}(t,z,z') &:= \varepsilon^{n+1} q(t\varepsilon, \iota_\varepsilon(z), \iota_\varepsilon(z')) \det V^\bullet(\iota_\varepsilon(z')) = \sum_{b\ge 1} \varepsilon^{b/2} q_{b,(\varepsilon)}(t,z,z'), \\ q_{b,(\varepsilon)}(t,z,z') &= t^{-n-2+b/2} \sum \rho_\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))\, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z',z)), \end{aligned}$$

$$\text{(3.1.3)} \quad (-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)} = (-1)^k \sum_{b_i\ge 1} \varepsilon^{\sum b_i/2} r_{(\varepsilon)} \# q_{b_1,(\varepsilon)} \# \cdots \# q_{b_k,(\varepsilon)}.$$

Here all the summations are finite. We set $\circ = M$ or H, and $\rho_\circ(z',z)$ is a function with support in $\{(z',z) \in H_n \times H_n \mid |w(z')| < 2r, |w(z)| < 2r\}$ $(\circ = M)$ or in $\{(z',z) \in H_n \times H_n \mid |w(z')| > r, |w(z)| > r\}$ $(\circ = H)$, and $|\xi_{\mathbb{A},z}\xi_{\mathbb{A}',z'}\rho_H(z',z)|$ is bounded for every $(\mathbb{A},\mathbb{A}')$. In particular, $\rho_\circ(z',z)$ appearing in (3.1.2) have supports contained in $\{(z,z') \mid |z|_H \le r_0\}$ (see (2.2.1) around). Further, $\mathcal{K}(\Theta)$ are rapidly decreasing functions and we set $\Theta^{\circ(\varepsilon)}(z',z) = \iota_{1/\varepsilon}\Theta^\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))$. There have appeared many $\rho_\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))$, etc., which are different from each other, so that $q_{b,(\varepsilon)} = t^{-n-2+b/2} \sum \rho_{\circ,j}(\iota_\varepsilon(z'), \iota_\varepsilon(z))\, \mathcal{K}_j(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z',z))$, etc., might be reasonable expressions. Now, in fact, Proposition 2.2.2 is guaranteed by:

Lemma 3.1.1 (cf. [7, Lemma 6.9 and §6.4]) *Each $(-1)^k(r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t,z,z')$ can be extended smoothly up to $[0,\varepsilon_0^{1/2}] \times (0,\infty) \times H_n \times H_n$ $(\ni (\varepsilon^{1/2},t,z,z'))$ and has a series expansion*

$$(-1)^k r_{(\varepsilon)} \# q^k_{(\varepsilon)} = \sum_{k\le m<m_*} \varepsilon^{m/2} p^{\theta,k}_{m/2} + \varepsilon^{m_*/2} p^{\theta,k}_{m_*/2}(\varepsilon^{1/2}), \quad p^{\theta,0}_{0/2} = \mathbf{r}(t,z,z').$$

In addition,

$$\text{(3.1.4)} \quad p^\theta_{m/2}(t,z,z') = \sum_{0\le k\le m} p^{\theta,k}_{m/2}(t,z,z') = \mathfrak{p}^\theta_{m/2}(t,z,z') \quad (0 \le m < m_*),$$

$$\text{(3.1.5)} \quad \begin{aligned} p^{\theta,k}_{m_*/2}(\varepsilon^{1/2},t,z,z') = \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{m_*-1}} d\sigma_{m_*} \\ \cdot\, (\partial/\partial\varepsilon^{1/2})^{m_*} (-1)^k (r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t,z,z')\Big|_{\varepsilon^{1/2} \Rightarrow \sigma_{m_*}\varepsilon^{1/2}} \end{aligned}$$

and

$$p^\theta_{m_*/2}(\varepsilon^{1/2},t,z,z') := \sum_{0\le k<m_*} p^{\theta,k}_{m_*/2}(\varepsilon^{1/2},t,z,z') + \varepsilon^{-m_*/2} R_{m_*}(p_{(\varepsilon)})(t,z,z')$$

can be extended smoothly also up to $\varepsilon^{1/2} = 0$.

The proof, which we wish to leave to the readers, is similar to that in [7]. In the proof of the lemma we may assume that t is in a bounded region chosen independently of ε. But, for the study of (3.6), it is necessary to consider their behavior when $0 < t < \varepsilon^{-1}$.

Proposition 3.1.2 *There is a formula*

$$\mathfrak{p}^{\theta}_{m/2}(t\varepsilon, \iota_\varepsilon(z), \iota_\varepsilon(z')) = \varepsilon^{-(n+1)+m/2}\mathfrak{p}^{\theta}_{m/2}(t, z, z'). \tag{3.1.6}$$

In addition, for any $\delta_i > 0$ $(i = 1, 2)$, $N > 0$ and $(\mathbb{A}, \mathbb{A}')$, there exists a constant $B_{m,k}$ $(= B_{m,k,(\mathbb{A},\mathbb{A}')}) > 0$ and $b_N > 0$ such that, on the domain $\{(\varepsilon^{1/2}, t, z, z') \mid \varepsilon > 0, 0 < t < \varepsilon^{-1}, |z|_H \le \delta_1, |z'^{-1}z|_H \ge \delta_2\}$,

$$\begin{aligned} &\left|(\partial/\partial z)^{\mathbb{A}}(\partial/\partial z')^{\mathbb{A}'} p^{\theta,k}_{m/2}(\varepsilon^{1/2}, t, z, z')\right| \\ &\qquad \le B_{m,k}\begin{cases} b_N\, t^N & (0 < t \le 1), \\ t^{-(n+1)+m/2-(|\mathbb{A}|_H+|\mathbb{A}'|_H)/2} & (1 \le t < \varepsilon^{-1}). \end{cases} \end{aligned} \tag{3.1.7}$$

This holds canonically also in the case $\varepsilon^{1/2} = 0$ and becomes an estimate for $p^{\theta,k}_{m/2}(t, z, z')$.

This proposition will be proved in §3.3 and §3.5.

3.2 The proof of Theorem 3.2

By (3.1.7) with $\varepsilon^{1/2} = 0$ and (3.1.4), $\mathfrak{p}^{\theta}_{m/2}(t, z^0, 0)$ $(m < 2n)$ are certainly integrable on $(0, \infty)$. The formulas (3.4) and (3.5) come from (3.1.6) and (2.2.7). Referring to Lemma 3.1.1, we have

$$\begin{aligned} \varepsilon^{-n}\int_0^{1/\varepsilon} dt\, e^{-t\Box^{\theta}_{(\varepsilon)}}(z^0, 0) &= \sum_{m=0}^{2n} \varepsilon^{-n+m/2}\int_0^{1/\varepsilon} dt\, \mathfrak{p}^{\theta}_{m/2}(t, z^0, 0) \\ &+ \varepsilon^{1/2}\int_0^{1/\varepsilon} dt \sum_{0\le k<2n+1} p^{\theta,k}_{(2n+1)/2}(\varepsilon^{1/2}, t, z^0, 0) + \int_0^{1/\varepsilon} dt\, \varepsilon^{-n}R_{2n+1}(p_{(\varepsilon)})(t, z^0, 0). \end{aligned}$$

For the proof of (3.3), it is enough to prove the following:

Proposition 3.2.1 *When $\varepsilon^{1/2} \to 0$,*

$$\int_0^{1/\varepsilon} dt\, \mathfrak{p}^{\theta}_{m/2}(t, z^0, 0) = \begin{cases} \displaystyle\int_0^{\infty} dt\, \mathfrak{p}^{\theta}_{m/2}(t, z^0, 0) + O(\varepsilon^{n-m/2}) & (m < 2n), \\ -a^{\theta}_n(P^0)\log\varepsilon + O(1) & (m = 2n), \end{cases} \tag{3.2.1}$$

$$\varepsilon^{1/2}\int_0^{1/\varepsilon} dt \sum_{0\le k<2n+1} p^{\theta,k}_{(2n+1)/2}(\varepsilon^{1/2}, t, z^0, 0) = O(1), \tag{3.2.2}$$

$$\int_0^{1/\varepsilon} dt\, \varepsilon^{-n}R_{2n+1}(p_{(\varepsilon)})(t, z^0, 0) = O(1). \tag{3.2.3}$$

Proof of (3.2.1). By (3.1.6), in the case $m/2 < n$,

$$\frac{\partial}{\partial \varepsilon^{1/2}} \int_1^{1/\varepsilon} dt\, \mathfrak{p}^\theta_{m/2}(t, z^0, 0) = -2\varepsilon^{-3/2} \mathfrak{p}^\theta_{m/2}(1/\varepsilon, z^0, 0)$$
$$= -2\varepsilon^{n-m/2-1/2} \mathfrak{p}^\theta_{m/2}(1, \iota_\varepsilon(z^0), 0),$$
$$\int_1^{1/\varepsilon} dt\, \mathfrak{p}^\theta_{m/2}(t, z^0, 0) - \int_1^{\infty} dt\, \mathfrak{p}^\theta_{m/2}(t, z^0, 0)$$
$$= \varepsilon^{1/2} \int_0^1 d\sigma\, \frac{\partial \int_1^{1/\varepsilon} dt\, \mathfrak{p}^\theta_{m/2}(t, z^0, 0)}{\partial \varepsilon^{1/2}} \Big|_{\varepsilon^{1/2} \Rightarrow \sigma\varepsilon^{1/2}} = O(\varepsilon^{n-m/2})$$

and, in the case $m/2 = n$,

$$\mathfrak{p}^\theta_n(1, \iota_s(z^0), 0) = \mathfrak{p}^\theta_n(1, 0, 0) + s^{1/2} \int_0^1 d\sigma\, \frac{\partial \mathfrak{p}^\theta_n(1, \iota_s(z^0), 0)}{\partial s^{1/2}} \Big|_{s^{1/2} \Rightarrow s^{1/2}\sigma}$$
$$= \mathfrak{p}^\theta_n(1, 0, 0) + O(s^{1/2}) \quad (s \to 0),$$
$$\mathfrak{p}^\theta_n(t, z^0, 0) = t^{-1} \mathfrak{p}^\theta_n(1, \iota_{1/t}(z^0), 0) = t^{-1} \mathfrak{p}^\theta_n(1, 0, 0) + O(t^{-3/2}) \quad (t \to \infty),$$
$$\int_1^{1/\varepsilon} dt\, \mathfrak{p}^\theta_n(t, z^0, 0) = -\mathfrak{p}^\theta_n(1, 0, 0) \log \varepsilon + O(1) = -a^\theta_n(P^0) \log \varepsilon + O(1).$$

It is easy to show that $(\partial/\partial \varepsilon^{1/2})^i \int_1^{1/\varepsilon} dt\, \mathfrak{p}^\theta_{m/2}(t, z^0, 0) = O(\varepsilon^{n-m/2-i/2})$, etc., for any i. Thus we obtain (3.2.1). ∎

Proof of (3.2.2). By (3.1.7), we have $\int_0^1 dt \sum_{0 \le k < 2n+1} p^{\theta,k}_{(2n+1)/2}(\varepsilon^{1/2}, t, z^0, 0) = O(1)$ and

$$\left| \varepsilon^{1/2} \int_1^{1/\varepsilon} dt \sum_{0 \le k < 2n+1} p^{\theta,k}_{(2n+1)/2}(\varepsilon^{1/2}, t, z^0, 0) \right| \le C_1 \varepsilon^{1/2} \int_1^{1/\varepsilon} dt\, t^{-(n+1)+(2n+1)/2} \le C_2.$$

It is also easy to check the differentials of the left hand side of (3.2.2) by referring to the expression (3.1.5). ∎

Proof of (3.2.3). By (2.2.2), we have

$$\left| \int_0^{1/\varepsilon} dt \varepsilon^{-n} R_{2n+1}(\mathfrak{p}_{(\varepsilon)})(t, z^0, 0) \right| = \left| \varepsilon \int_0^{1/\varepsilon} dt R_{2n+1}(p)(t\varepsilon, \iota_\varepsilon(z^0), 0) \right|$$
$$\le C(2n+1, 0, \emptyset, \emptyset, 0)\, \varepsilon \int_0^{1/\varepsilon} dt\, (t\varepsilon)^{(2n+1)/2-(n+1)} \le C,$$

and the differentials are also estimated as are desired. ∎

3.3 The proof of (3.1.6)

Considering the differentiations of (2.2.5) by $\varepsilon^{1/2}$, we can show:

Lemma 3.3.1 (cf. [7, Proposition 5.2]) *We have*

$$\Box^{\theta}_{m/2} = \sum_{2+|\mathbb{C}|_H=|\mathbb{B}|_H+m}^{|\mathbb{B}|=0,1,2} \Box^{\theta}_{m/2}(\mathbb{B},\mathbb{C}) \cdot z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}} \quad (z^{\mathbb{C}} := z_{C_1}\cdots z_{C_{|\mathbb{C}|}}),$$

where $\Box^{\theta}_{m/2}(\mathbb{B},\mathbb{C})$ *is a constant.*

By recalling the definition, (3.1.6) with $m = 0$ is correct. If (3.1.6) with $m = m$ holds, then, by Lemma 3.3.1,

$$\begin{aligned}(\Box^{\theta}_{m_1/2}\mathfrak{p}^{\theta}_{m/2})(t\varepsilon, \iota_\varepsilon(z), \iota_\varepsilon(z')) &= \varepsilon^{m_1/2-1}\,\Box^{\theta}_{m_1/2}(\mathfrak{p}^{\theta}_{m/2}(t\varepsilon, \iota_\varepsilon(z), \iota_\varepsilon(z')))\\ &= \varepsilon^{-(n+2)+(m_1+m)/2}\,(\Box^{\theta}_{m_1/2}\mathfrak{p}^{\theta}_{m/2})(t, z, z').\end{aligned}$$

Hence, (3.1.6) can be shown inductively.

3.4 Change of variables : preparation for the proof of (3.1.7)

We will examine the integrand of (3.1.5). For the sake of the differentiation by $\varepsilon^{1/2}$, it will be better to regard $(-1)^k(r_{(\varepsilon)}\#q^k_{(\varepsilon)})(t,z,z')$ as a function of the variables $(t, \Theta^{\circ(\varepsilon)}(z',z), z')$, etc. We will describe the features of the changes of variables $(z',z) \rightleftharpoons (z', \Theta^{\circ(\varepsilon)})$, etc., which has been examined roughly in [7, §6.3]. Since we intend to estimate $p^{\theta,k}_{m/2}(\varepsilon^{1/2}, t, z, z')$ when $0 < t < \varepsilon^{-1}$, we will describe the changes more in detail here. Referring to (2.2.1) around, we assume that the domain of the function $\Theta^\circ(z',z)$ is $\{(z',z) \in H_n \times H_n \mid |w(z')| < 2r, |w(z)| < 2r\}$ (if $\circ = M$), $\{(z',z) \in H_n \times H_n \mid |w(z')| > r, |w(z)| > r\}$ (if $\circ = H$).

Lemma 3.4.1 *The changes* $(z', z'^{-1}z) \rightleftharpoons (z', \Theta^\circ(z',z))$ *have quasi-bounded finite sum expressions*

$$\begin{aligned}(z',\Theta^\circ)_A &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |A|_H} (z', z'^{-1}z)^{(\mathbb{C},\mathbb{D})} h(z', z'^{-1}z) = (z', z'^{-1}z)_A + \sum_{|(\mathbb{C},\mathbb{D})|_H > |A|_H},\\ (z', z'^{-1}z)_B &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |B|_H} (z',\Theta^\circ)^{(\mathbb{C},\mathbb{D})} h(z',\Theta^\circ) = (z',\Theta^\circ)_B + \sum_{|(\mathbb{C},\mathbb{D})|_H > |B|_H},\end{aligned}$$

where the functions h are different from each other in general. The modifier "quasi-bounded" means that each function $h(z', z'^{-1}z)$ (and each $h(z',\Theta^\circ)$) is quasi-bounded in the sense: it is bounded and each (high order) differential is described as $\frac{\partial h(z',z'^{-1}z)}{\partial(z',z'^{-1}z)_B} = \sum (z',z'^{-1}z)^{(\mathbb{C},\mathbb{D})} h(z',z'^{-1}z)$, *etc., where again each $h(z',z'^{-1}z)$ on the right hand side is bounded and so forth, successively. A quasi-bounded $h(z',z'^{-1}z)$ is also quasi-bounded as a function of (z',Θ°), and so is the converse. Further, also the (high order) differentials* $\frac{\partial(z',\Theta^\circ)_A}{\partial(z',z'^{-1}z)_B}$, $\frac{\partial(z',z'^{-1}z)_B}{\partial(z',\Theta^\circ)_A}$, *etc., have quasi-bounded finite sum expressions.*

Proof. In the case $\circ = M$ the lemma is obvious because the domain of Θ^M is bounded (see also [7, (2.15)]). In the case $\circ = H$, Θ^H has a quasi-bounded finite sum expression

$$\begin{aligned}\Theta^H_A(z',z) &= (w(z')^{-1}w(z))_A = w(z'^{-1}z)_A + \sum z'_C(z'^{-1}z)_D h(z,z'^{-1}z)\\ &= (z'^{-1}z)_A + \sum_{|(\mathbb{C},\mathbb{D})|_H > |A|_H} (z',z'^{-1}z)^{(\mathbb{C},\mathbb{D})} h(z',z'^{-1}z).\end{aligned}$$

Each (high order) differential of $w(z)$ by z is bounded ([7, (3.13)]). Note that a differential of $w(z')^{-1}w(z) = w(z')^{-1}w(z' + E(-z')(z'^{-1}z))$ by $(z', z'^{-1}z)$ has such a divergent part as $(z', z'^{-1}z)^{(\mathbb{C},\mathbb{D})}$ in general. Similarly, $(z'^{-1}z) = E(z')(w^{-1}(w(z')\Theta^H) - z')$ also has such an expression. ∎

By (2.1.5), the changes $(z', z) \rightleftharpoons (z', z'^{-1}z)$ also have quasi-bounded finite sum expressions

$$\begin{aligned}(z', z'^{-1}z)_A &= (z', E(z')(z - z'))_A = \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |A|_H} (z',z)^{(\mathbb{C},\mathbb{D})} h(z',z),\\ (z',z)_B &= (z', z' + E(-z')(z'^{-1}z))_B = \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |B|_H} (z',z'^{-1}z)^{(\mathbb{C},\mathbb{D})} h(z',z'^{-1}z),\end{aligned}$$

which, together with Lemma 3.4.1, imply such expressions

$$\begin{aligned}(z',\Theta^\circ)_A &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |A|_H} (z',z)^{(\mathbb{C},\mathbb{D})} h(z',z),\\ (z',z)_B &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |B|_H} (z',\Theta^\circ)^{(\mathbb{C},\mathbb{D})} h(z',\Theta^\circ).\end{aligned}$$

Hence we obtain:

Proposition 3.4.2 (cf. [7, §6.3]) *The changes $(z', z) \rightleftharpoons (z', \Theta^{\circ(\varepsilon)})$ has quasi-bounded finite sum expressions*

$$\begin{aligned}(z',\Theta^{\circ(\varepsilon)})_A &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |A|_H} \varepsilon^{|(\mathbb{C},\mathbb{D})|_H/2 - |A|_H/2} (z',z)^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z'), \iota_\varepsilon(z)),\\ (z',z)_B &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |B|_H} \varepsilon^{|(\mathbb{C},\mathbb{D})|_H/2 - |B|_H/2} (z',\Theta^{\circ(\varepsilon)})^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}).\end{aligned}$$

Here, "quasi-bounded" means that functions $h(z',z)$ (not $h(\iota_\varepsilon(z'),\iota_\varepsilon(z))$), etc., are quasi-bounded. A quasi-bounded $h(z',z)$ is also quasi-bounded as a function of $(z',\Theta^{\circ(\varepsilon)})$, and so is the converse. The (high order) differentials also have quasi-bounded finite sum expressions such as

$$\begin{aligned}\frac{\partial(z',\Theta^{\circ(\varepsilon)})_A}{\partial(z',z)_B} &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |A|_H - |B|_H} \varepsilon^{|(\mathbb{C},\mathbb{D})|_H/2 - (|A|_H - |B|_H)/2}\\ &\qquad \times (z',z)^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z'), \iota_\varepsilon(z)),\\ \frac{\partial(z',z)_B}{\partial(z',\Theta^{\circ(\varepsilon)})_A} &= \sum_{|(\mathbb{C},\mathbb{D})|_H \geq |B|_H - |A|_H} \varepsilon^{|(\mathbb{C},\mathbb{D})|_H/2 - (|B|_H - |A|_H)/2}\\ &\qquad \times (z',\Theta^{\circ(\varepsilon)})^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}).\end{aligned}$$

Further, so have the other changes, fixing one of the variables and alternating the other with some variable, such as $(z', z) \rightleftharpoons (z, \Theta^{\circ(\varepsilon)})$, $(z', \Theta^{\circ(\varepsilon)}) \rightleftharpoons (z, \Theta^{\circ(\varepsilon)})$, *etc., and also their composites*

3.5 The proof of (3.1.7)

We want to differentiate by $\varepsilon^{1/2}$ the functions $\rho_\circ(\iota_\varepsilon(z'), \iota_\varepsilon(z))\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z', z))$ appearing in (3.1.1), (3.1.2). By Proposition 3.4.2, we have:

Lemma 3.5.1 *Let* $\mathcal{K}(\Theta)$ *be a rapidly decreasing function and* $h(z', z)$ *be a quasi-bounded function* (*defined on the region of* Θ°). *Then we have a quasi-bounded finite sum expression*

$$
\begin{aligned}
(3.5.1)\quad & (\partial/\partial\varepsilon^{1/2})^m (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z', z))) \\
&= \sum_{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H) - m \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - m/2} \\
&\qquad \times z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})),
\end{aligned}
$$

(*that is, the functions* $h(z', z)$ *on the right hand side are quasi-bounded*). *Also we have such expressions*

$$
\begin{aligned}
(3.5.2)\quad & z'^{\mathbb{C}'} (\partial/\partial z')^{\mathbb{B}'} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum_{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H) - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H) \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum_{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H) - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H) \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times h(\iota_\varepsilon(z'), \iota_\varepsilon(z))\, z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} ((\Theta^{\circ(\varepsilon)})^{\mathbb{D}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})).
\end{aligned}
$$

Further, we have similar expressions, replacing $z^{\mathbb{C}}(\partial/\partial z)^{\mathbb{B}}$ *with* $z^{\mathbb{C}}(\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{B}}$, *etc.*

Proof. As for (3.5.1): Using the changes $(z', z) \rightleftharpoons (z, \Theta^{\circ(\varepsilon)})$, we have

$$
\begin{aligned}
&(\partial/\partial\varepsilon^{1/2}) h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) = \sum \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - 1)/2} z^{\mathbb{D}} h_1(\iota_\varepsilon(z'), \iota_\varepsilon(z)) z'^{\mathbb{C}} \\
&= \sum \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - 1)/2} z^{\mathbb{D}} h_1(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \\
&\qquad \times \varepsilon^{(|(\mathbb{C}_1,\mathbb{D}_1)|_H - |\mathbb{C}|_H)/2} (z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C}_1,\mathbb{D}_1)} h_2(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \\
&= \sum_{|(\mathbb{C},\mathbb{D})|_H - 1 \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - 1)/2} z^{\mathbb{C}} h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}}
\end{aligned}
$$

and

$$
(\partial/\partial\varepsilon^{1/2}) \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z', z)) = \frac{\partial\Theta_A^{\circ(\varepsilon)}}{\partial\varepsilon^{1/2}} \cdot \frac{\partial}{\partial\Theta_A^{\circ(\varepsilon)}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})
$$

$$
\begin{aligned}
&= \sum \varepsilon^{(|(\mathbb{C}_1,\mathbb{D}_1)|_H - |A|_H - 1)/2} (z', \Theta^{\circ(\varepsilon)})^{(\mathbb{C}_1,\mathbb{D}_1)} h_1(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \frac{\partial}{\partial \Theta_A^{\circ(\varepsilon)}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}) \\
&= \sum \varepsilon^{(|(\mathbb{C}_2,\mathbb{D}_2)|_H - |\mathbb{B}_2|_H - 1)/2} \\
&\qquad \times z'^{\mathbb{C}_2} (\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{B}_2} (h_2(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}_2} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&\overset{(*)}{=} \sum \varepsilon^{(|(\mathbb{C}'_3,\mathbb{C}_3,\mathbb{D}_3)|_H - |\mathbb{B}_3|_H - 1)/2} \\
&\qquad \times z^{\mathbb{C}_3} (\partial/\partial z)^{\mathbb{B}_3} (z'^{\mathbb{C}'_3} h_3(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}_3} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum \varepsilon^{(|(\mathbb{C}_4,\mathbb{D}_4,\mathbb{C}_3,\mathbb{D}_3)|_H - |\mathbb{B}_3|_H - 1)/2} z^{\mathbb{C}_3} (\partial/\partial z)^{\mathbb{B}_3} ((z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C}_4,\mathbb{D}_4)} \\
&\qquad \times h_4(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) h_3(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}_3} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum_{|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H - 1 \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H - 1)/2} \\
&\qquad \times z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})),
\end{aligned}
$$

where, at $\overset{(*)}{=}$, we use the changes $(z', z) \rightleftharpoons (z', \Theta^{\circ(\varepsilon)})$. Hence, (3.5.1) with $m = 1$ is correct and, for general m, it can be shown inductively. As for (3.5.2): Again, using the changes $(z', z) \rightleftharpoons (z, \Theta^{\circ(\varepsilon)})$, we have

$$
\begin{aligned}
& z'^{\mathbb{C}'} (\partial/\partial z')^{\mathbb{B}'} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= z'^{\mathbb{C}'} \sum \varepsilon^{|(\mathbb{C}'_1,\mathbb{D}'_1)|_H/2 - (|\mathbb{D}''_1|_H - |\mathbb{B}'|_H)/2} (z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C}'_1,\mathbb{D}'_1)} h_1(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \\
&\qquad \times (\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{D}''_1} (h(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= z'^{\mathbb{C}'} \sum \varepsilon^{|(\mathbb{C}'_2,\mathbb{D}'_2)|_H/2 - (|\mathbb{D}''_2|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times (\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{D}''_2} ((z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C}'_2,\mathbb{D}'_2)} h_2(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= z'^{\mathbb{C}'} \sum \varepsilon^{|(\mathbb{C}'_3,\mathbb{D}'_3)|_H/2 - (|\mathbb{D}''_2|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times (\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{D}''_2} ((z', \Theta^{\circ(\varepsilon)})^{(\mathbb{C}'_3,\mathbb{D}'_3)} h_3(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum \varepsilon^{(|(\mathbb{C}'_4,\mathbb{D}'_4)|_H - |\mathbb{D}''_4|_H)/2 - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times z'^{\mathbb{C}'_4} (\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{D}''_4} (h_4(\iota_\varepsilon(z'), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}'_4} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\
&= \sum_{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H) - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H) \geq 0} \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - (|(\mathbb{C}',\mathbb{D}')|_H - |\mathbb{B}'|_H)/2} \\
&\qquad \times z^{\mathbb{C}} (\partial/\partial z)^{\mathbb{B}} (h_3(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \cdot (\Theta^{\circ(\varepsilon)})^{\mathbb{D}} \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})).
\end{aligned}
$$

Thus we get the first expression. The second one is shown by

$$
\begin{aligned}
\varepsilon^{-|\mathbb{B}_1|_H/2} (\partial/\partial z)^{\mathbb{B}_1} (h(\iota_\varepsilon(z'), \iota_\varepsilon(z))) &= \varepsilon^{|(\mathbb{C}_1,\mathbb{D}_1)|_H/2} z'^{\mathbb{C}_1} z^{\mathbb{D}_1} h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \\
&= \varepsilon^{|(\mathbb{C}_2,\mathbb{D}_2)|_H/2} (z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C}_2,\mathbb{D}_2)} h_2(\iota_\varepsilon(z'), \iota_\varepsilon(z)),
\end{aligned}
$$

etc. Other expressions are similarly shown. ∎

In general, let $\mathcal{K}(\Theta)$ be rapidly decreasing, $h(z', z) = h_\circ(z', z)$ be a quasi-bounded function (defined on the domain of Θ°) and let us set

$$
k^b(\varepsilon^{1/2}, t, z, z') = t^{-(n+2)+b/2} h(\iota_\varepsilon(z'), \iota_\varepsilon(z)) \, \mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)}(z', z)),
$$

called an (ε)-kernel of type b. Then, referring to (3.1.3) and (3.5.1), we have a finite sum expression

$$(3.5.3)\quad (\partial/\partial\varepsilon^{1/2})^m(-1)^k(r_{(\varepsilon)}\# q^k_{(\varepsilon)})(t,z,z') \\ = \sum \varepsilon^{(\sum_{i>0} b_i - m_*)/2 + \sum(|(\mathbb{C}_j,\mathbb{D}_j)|_H - |\mathbb{B}_j|_H - m_j)/2} \\ \times z^{\mathbb{C}_0}(\partial/\partial z)^{\mathbb{B}_0} k^{b_0+|\mathbb{D}_0|_H}(\varepsilon^{1/2})\#\cdots\# z^{\mathbb{C}_k}(\partial/\partial z)^{\mathbb{B}_k} k^{b_k+|\mathbb{D}_k|_H}(\varepsilon^{1/2}),$$

where $b_0 = 2$ and $\# = \#_{\theta_H}$. Here we take non-negative integers $m_*, m_0, \ldots .m_k$ satisfying $m_* + \sum m_j = m$ and $\sum_{i>0} b_i - m_* \geq 0$, take, for each m_j, a finite number of $z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2})$ ($|(\mathbb{C}_j,\mathbb{D}_j)|_H - |\mathbb{B}_j|_H - m_j \geq 0$), and sum up all the terms made of them, where the quasi-bounded functions $h_j(z',z) = h_{\circ j}(z',z)$ appearing in the (ε)-kernels $k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}, z, z')$ ($j \geq 1$) have supports contained in $\{(z',z) \in H_n \times H_n \mid |z|_H \leq r_0\}$.

Proposition 3.5.2 *Suppose $k^{b_i}(\varepsilon^{1/2})$ are (ε)-kernels of types $b_i\,(\geq 1)$, where the supports of the corresponding quasi-bounded functions $h_i(z',z)$ ($i \geq 2$) are contained in $\{(z',z) \in H_n \times H_n \mid |z|_H \leq r_0\}$. Then $z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2})\#\cdots\# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2})$ is well-defined, and there exists constants $B(\ell) > 0$ ($\ell = 0$ or $\ell \geq 2n+2$) and $C > 0$ such that, when $0 < t < \varepsilon^{-1}$,*

$$(3.5.4)\quad \left|(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2})\#\cdots\# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z')\right| \\ \leq B(\ell)\, t^{\sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2 + \sum b_i/2 + \ell/2 - (n+2)} \delta^{(\varepsilon)}(z',z)^{-\ell} \sum |\iota_{1/t}(z)^{\mathbb{E}}|,$$

$$(3.5.5)\quad \left\|(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2})\#\cdots\# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z')\right\|_{L^1(z')} \\ \leq C\, t^{\sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2 + \sum b_i/2 - 1} \sum |\iota_{1/t}(z)^{\mathbb{E}}|,$$

where we set $\delta^{(\varepsilon)}(z',z) = \varepsilon^{-1/2}|w(\iota_\varepsilon(z'))^{-1}w(\iota_\varepsilon(z))|_H$ (see (2.2.2)). Here $\sum |\iota_{1/t}(z)^{\mathbb{E}}|$ is a finite sum independent of ℓ, which can be chosen to be 1 when $(\mathbb{C}_i,\mathbb{B}_i) = (\emptyset,\emptyset)$ for all i. The norm $\|\cdots\|_{L^1(z')}$ is the L^1-norm with respect to the variable z', and another one $\|\cdots\|_{L^1(z)}$ is similarly estimated too. All the estimates hold canonically up to $\varepsilon = 0$ (hence, $\varepsilon^{-1} = \infty$ and $\delta^{(0)}(z',z) = |z'^{-1}z|_H$).

Proof. Let us show $(3.5.4)_{\ell=0}$ and (3.5.5) by inductive argument. By Lemma 3.5.1,

$$(3.5.6)\quad z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1}(t^{-(n+2)+b_1/2} h_1(\iota_\varepsilon(z'), \iota_\varepsilon(z))\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\ = \sum \varepsilon^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - (|\mathbb{C}_1|_H - |\mathbb{B}_1|_H)/2} t^{-(n+2)+b_1/2} \\ \times h(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}) z^{\mathbb{C}}(\partial/\partial\Theta^{\circ(\varepsilon)})^{\mathbb{B}}((\Theta^{\circ(\varepsilon)})^{\mathbb{D}}\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})) \\ = \sum_{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H) - (|\mathbb{C}_1|_H - |\mathbb{B}_1|_H) \geq 0} (t\varepsilon)^{(|(\mathbb{C},\mathbb{D})|_H - |\mathbb{B}|_H)/2 - (|\mathbb{C}_1|_H - |\mathbb{B}_1|_H)/2} \\ \times t^{(|\mathbb{C}_1|_H - |\mathbb{B}_1|_H)/2 + b_1/2 - (n+2)} \iota_{1/t}(z)^{\mathbb{C}} \\ \times h(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)})(\partial/\partial\iota_{1/t}\Theta^{\circ(\varepsilon)})^{\mathbb{B}}((\iota_{1/t}\Theta^{\circ(\varepsilon)})^{\mathbb{D}}\mathcal{K}(\iota_{1/t}\Theta^{\circ(\varepsilon)})).$$

Hence, $(3.5.4)_{\ell=0}$ with $j=1$ holds. As for the volume element $dV = dV_{\theta_H} = (\sqrt{-1})^{n^2}$ $dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n \wedge dz_{\bar{1}} \wedge \cdots \wedge dz_{\bar{n}}$, we have

$$\begin{aligned} t^{-(n+2)} dV(z') &= t^{-(n+2)} dV(\Theta^{\circ(\varepsilon)}) \\ &\quad \times \det\Big(\sum \varepsilon^{|(\mathbb{C},\mathbb{D})|_H/2-(|\mathbb{A}|_H-|\mathbb{B}|_H)/2} (z, \Theta^{\circ(\varepsilon)})^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z), \iota_\varepsilon \Theta^{\circ(\varepsilon)})\Big)_{AB} \\ &= t^{-1} dV(\iota_{1/t}\Theta^{\circ(\varepsilon)}) \cdot \sum (t\varepsilon)^{|(\mathbb{C},\mathbb{D})|_H/2} (\iota_{1/t}(z), \iota_{1/t}\Theta^{\circ(\varepsilon)})^{(\mathbb{C},\mathbb{D})} h(\iota_\varepsilon(z), \iota_\varepsilon\Theta^{\circ(\varepsilon)}). \end{aligned}$$

Thus, also (3.5.5) with $j=1$ holds. Next, by (3.5.2) and integration by parts, and then by (3.5.6),

$$\begin{aligned} &(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2}) \# \cdots \# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z') \\ &= \sum_{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H) - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H) \geq 0} \varepsilon^{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H)/2 - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2} \\ &\qquad \times z^{\mathbb{C}} k^{b_1 + |\mathbb{D}_1|_H - |\mathbb{A}|_H}(\varepsilon^{1/2}) \# k^{b_2+|\mathbb{D}_2|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j + |\mathbb{D}_j|_H}(\varepsilon^{1/2}) \\ &= \sum_{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H) - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H) \geq 0} \varepsilon^{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H)/2 - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2} \\ &\qquad \times z^{\mathbb{C}} k^{b_1 + |\mathbb{D}_1|_H}(\varepsilon^{1/2}) \# k^{b_2+|\mathbb{D}_2|_H - |\mathbb{A}|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j + |\mathbb{D}_j|_H}(\varepsilon^{1/2}), \end{aligned}$$

so that

$$\begin{aligned} &(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2}) \# \cdots \# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z') \\ &= \sum \varepsilon^{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H)/2 - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2} z^{\mathbb{C}} \\ &\quad \times \Big\{ \int_0^{t/2} ds \int dV(z'')(k^{b_1+|\mathbb{D}_1|_H - |\mathbb{A}|_H}(\varepsilon^{1/2}))(t-s,z,z'') \\ &\qquad\qquad \times (k^{b_2+|\mathbb{D}_2|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \\ &\qquad + \int_{t/2}^{t} ds \int dV(z'')(k^{b_1+|\mathbb{D}_1|_H}(\varepsilon^{1/2}))(t-s,z,z'') \\ &\qquad\qquad \times (k^{b_2+|\mathbb{D}_2|_H - |\mathbb{A}|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \Big\}. \end{aligned}$$

Hence, $(3.5.4)_{\ell=0}$ and (3.5.5) in the general case are proved inductively in a manner similar to the proofs of [7, $(4.15)_{\ell=0}$ and (4.14)]. Indeed, if they hold when $j = 1, \ldots, j-1$, then

$$\begin{aligned} (3.5.7)\ &\Big|(z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2}) \# \cdots \# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z')\Big| \\ &\leq \sum \varepsilon^{(|\mathbb{C}|_H + \sum |\mathbb{D}_i|_H - |\mathbb{A}|_H)/2 - \sum(|\mathbb{C}_i|_H - |\mathbb{B}_i|_H)/2} |z^{\mathbb{C}}| \\ &\quad \times \Big\{ \int_0^{t/2} ds \max_{z''} \Big|(k^{b_1+|\mathbb{D}_1|_H - |\mathbb{A}|_H}(\varepsilon^{1/2}))(t-s,z,z'')\Big| \\ &\qquad\qquad \times \Big\|(k^{b_2+|\mathbb{D}_2|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z')\Big\|_{L^1(z'')} \end{aligned}$$

$$
\begin{aligned}
&+\int_{t/2}^{t} ds \left\| (k^{b_1+|\mathbb{D}_1|_H}(\varepsilon^{1/2}))(t-s,z,z'') \right\|_{L^1(z'')} \\
&\qquad \times \max_{z''} \left| (k^{b_2+|\mathbb{D}_2|_H-|\mathbb{A}|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \right| \Big\} \\
&\le \sum \varepsilon^{(|\mathbb{C}|_H+\sum|\mathbb{D}_i|_H-|\mathbb{A}|_H)/2-\sum(|\mathbb{C}_i|_H-|\mathbb{B}_i|_H)/2} |z^{\mathbb{C}}| \\
&\quad \times \Big\{ \int_0^{t/2} ds\, B(0)(t-s)^{(b_1+|\mathbb{D}_1|_H-|\mathbb{A}|_H)/2-(n+2)} C s^{\sum_{i>1}(b_i+|\mathbb{D}_i|_H)/2-1} \\
&\quad + \int_{t/2}^{t} ds\, C(t-s)^{(b_1+|\mathbb{D}_1|_H)/2-1} B(0) s^{(b_2+|\mathbb{D}_2|_H-|\mathbb{A}|_H)/2+\sum_{i>2}(b_i+|\mathbb{D}_i|_H)/2-(n+2)} \Big\} \\
&\le \sum |z^{\mathbb{C}}| \varepsilon^{(|\mathbb{C}|_H+\sum|\mathbb{D}_i|_H-|\mathbb{A}|_H)/2-\sum(|\mathbb{C}_i|_H-|\mathbb{B}_i|_H)/2} t^{\sum(b_i+|\mathbb{D}_i|_H)/2-|\mathbb{A}|_H/2-(n+2)} \\
&\quad \times B(0) C \Big\{ \int_0^{1/2} d\sigma\, (1-\sigma)^{(b_1+|\mathbb{D}_1|_H-|\mathbb{A}|_H)/2-(n+2)} \sigma^{\sum_{i>1}(b_i+|\mathbb{D}_i|_H)/2-1} \\
&\qquad + \int_{1/2}^{1} d\sigma\, (1-\sigma)^{(b_1+|\mathbb{D}_1|_H)/2-1} \sigma^{(b_2+|\mathbb{D}_2|_H-|\mathbb{A}|_H)/2+\sum_{i>2}(b_i+|\mathbb{D}_i|_H)/2-(n+2)} \Big\} \\
&\le B'(0)|z^{\mathbb{C}}| \cdot (t\varepsilon)^{(|\mathbb{C}|_H+\sum|\mathbb{D}_i|_H-|\mathbb{A}|_H)/2-\sum(|\mathbb{C}_i|_H-|\mathbb{B}_i|_H)/2} \\
&\qquad\qquad \times t^{\sum(|\mathbb{C}_i|_H-|\mathbb{B}_i|_H)/2+\sum b_i/2-(n+2)-|\mathbb{C}|_H/2}.
\end{aligned}
$$

Thus $(3.5.4)_{\ell=0}$ with $j=j$ holds. $(3.5.4)_{\ell\ge 2n+2}$ is also proved similarly (refer to [7, $(4.15)_{\ell\ge 2n+2}$]). Indeed, we take a constant $\gamma \ge 1$ such that the inequality $|w'^{-1}w|_H \le \gamma(|w''^{-1}w|_H + |w'^{-1}w''|_H)$ holds, and put $\tilde{\delta} = \delta^{(\varepsilon)}(z',z)$ and $B_{\tilde{\delta}/2\gamma} = B_{\tilde{\delta}/2\gamma}(z) = \{z'' \in H_n \mid \delta^{(\varepsilon)}(z,z'') < \tilde{\delta}/2\gamma\}$ (hence, if $z'' \in B_{\tilde{\delta}/2\gamma}$, then $\delta^{(\varepsilon)}(z'',z') \ge \tilde{\delta}/2\gamma$). Instead of (3.5.7), under a similar inductive assumption here we consider

$$
\begin{aligned}
&\left| (z^{\mathbb{C}_1}(\partial/\partial z)^{\mathbb{B}_1} k^{b_1}(\varepsilon^{1/2}) \# \cdots \# z^{\mathbb{C}_j}(\partial/\partial z)^{\mathbb{B}_j} k^{b_j}(\varepsilon^{1/2}))(t,z,z') \right| \\
&\le \sum \varepsilon^{(|\mathbb{C}|_H+\sum|\mathbb{D}_i|_H-|\mathbb{A}|_H)/2-\sum(|\mathbb{C}_i|_H-|\mathbb{B}_i|_H)/2} |z^{\mathbb{C}}| \\
&\quad \times \Big\{ \int_0^{t/2} ds \max_{z'' \in H_n - B_{\tilde{\delta}/2\gamma}} \left| (k^{b_1+|\mathbb{D}_1|_H-|\mathbb{A}|_H}(\varepsilon^{1/2}))(t-s,z,z'') \right| \\
&\qquad\qquad \times \left\| (k^{b_2+|\mathbb{D}_2|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \right\|_{L^1(z'' \in H_n - B_{\tilde{\delta}/2\gamma})} \\
&\quad + \int_0^{t/2} ds \left\| (k^{b_1+|\mathbb{D}_1|_H-|\mathbb{A}|_H}(\varepsilon^{1/2}))(t-s,z,z'') \right\|_{L^1(z'' \in B_{\tilde{\delta}/2\gamma})} \\
&\qquad\qquad \times \max_{z'' \in B_{\tilde{\delta}/2\gamma}} \left| (k^{b_2+|\mathbb{D}_2|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \right| \\
&\quad + \int_{t/2}^{t} ds \max_{z'' \in H_n - B_{\tilde{\delta}/2\gamma}} \left| (k^{b_1+|\mathbb{D}_1|_H}(\varepsilon^{1/2}))(t-s,z,z'') \right| \\
&\qquad \times \left\| (k^{b_2+|\mathbb{D}_2|_H-|\mathbb{A}|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \right\|_{L^1(z'' \in H_n - B_{\tilde{\delta}/2\gamma})} \\
&\quad + \int_{t/2}^{t} ds \left\| (k^{b_1+|\mathbb{D}_1|_H}(\varepsilon^{1/2}))(t-s,z,z'') \right\|_{L^1(z'' \in B_{\tilde{\delta}/2\gamma})} \\
&\qquad\qquad \times \max_{z'' \in B_{\tilde{\delta}/2\gamma}} \left| (k^{b_2+|\mathbb{D}_2|_H-|\mathbb{A}|_H}(\varepsilon^{1/2}) \# \cdots \# k^{b_j+|\mathbb{D}_j|_H}(\varepsilon^{1/2}))(s,z'',z') \right| \Big\}
\end{aligned}
$$

$$\leq \sum |z^{\mathbb{E}}| \int_0^{t/2} ds \Big\{ B(\ell)(t-s)^{(|\mathbb{D}_1|_H - |\mathbb{F}|_H + b_1)/2 + \ell/2 - (n+2)} \tilde{\delta}^{-\ell} C s^{\sum_{i\geq 2}(|\mathbb{D}_i|_H + b_i)/2 - 1}$$
$$+ C(t-s)^{(|\mathbb{D}_1|_H - |\mathbb{F}|_H + b_1)/2 - 1} B(\ell) s^{\sum_{i\geq 2}(|\mathbb{D}_i|_H + b_i)/2 + \ell/2 - (n+2)} \tilde{\delta}^{-\ell} \Big\}$$
$$+ \sum |z^{\mathbb{E}}| \int_{t/2}^{t} ds \Big\{ B(\ell)(t-s)^{(|\mathbb{D}_1|_H + b_1)/2 + \ell/2 - (n+2)} \tilde{\delta}^{-\ell} C s^{\sum_{i\geq 2}(|\mathbb{D}_i|_H - |\mathbb{F}_i|_H + b_i)/2 - 1}$$
$$+ C(t-s)^{(|\mathbb{D}_1|_H + b_1)/2 - 1} B(\ell) s^{\sum_{i\geq 2}(|\mathbb{D}_i|_H - |\mathbb{F}_i|_H + b_i)/2 + \ell/2 - (n+2)} \tilde{\delta}^{-\ell} \Big\},$$

which can be estimated desirably. ∎

Proof of (3.1.7). By (3.5.3) and (3.5.4),

$$\Big| (\partial/\partial z)^{\mathbb{A}} (\partial/\partial z')^{\mathbb{A}'} (\partial/\partial \varepsilon^{1/2})^m (-1)^k (r_{(\varepsilon)} \# q^k_{(\varepsilon)})(t, z, z') \Big|$$
$$\leq B \sum \varepsilon^{(\sum_{i>0} b_i - m_*)/2 + \sum(|(\mathbb{C}_j, \mathbb{D}_j)|_H - |\mathbb{B}_j|_H - m_j)/2 + |(\mathbb{A}, \mathbb{A}')|_H/2}$$
$$\times\, t^{\sum(|\mathbb{C}_j|_H - |\mathbb{B}_j|_H)/2 + \sum(b_j + |\mathbb{D}_j|_H)/2 + \ell/2 - (n+2)} \delta^{(\varepsilon)}(z', z)^{-\ell} \sum |\iota_{1/t}(z)^{\mathbb{E}}|$$
$$= t^{m/2 - (n+1) - |(\mathbb{A}, \mathbb{A}')|_H/2 + \ell/2} \delta^{(\varepsilon)}(z', z)^{-\ell} \sum |\iota_{1/t}(z)^{\mathbb{E}}|$$
$$\times\, B \sum (t\varepsilon)^{(\sum_{i>0} b_i - m_*)/2 + \sum(|(\mathbb{C}_j, \mathbb{D}_j)|_H - |\mathbb{B}_j|_H - m_j)/2 + |(\mathbb{A}, \mathbb{A}')|_H/2}.$$

Hence, referring to the formula (3.1.5), we obtain (3.1.7). ∎

4 CR conformal Green function and the local coefficient $a_n^\theta(P)$

In this section, following the idea due to Parker-Rosenberg [8, Theorem 2.1], we will show that the coefficient $a_n^\theta(P)$ is a pointwise CR conformal (scalar) invariant of weight $2n$, that is, $dV_\theta(P)\, a_n^\theta(P)$ is such an invariant of weight -2.

Let us consider the $k/(2n+2)$-density bundle $L^k = |\wedge^{2n+1} T^*M|^{k/(2n+2)}$ with the trivialization $i_{\theta,k} : L^k \cong M \times \mathbb{R}$, $|dV_\theta(P)|^{k/(2n+2)} \mapsto (P, 1)$. By (1.2), we obtain a Laplacian $\Box^{[\theta]} : \Gamma(L^n) \to \Gamma(L^{n+2})$ which depends only on the CR conformal class, and a commutative diagram

$$\begin{array}{ccc} \Gamma(L^n) & \xrightarrow{\Box^{[\theta]}} & \Gamma(L^{n+2}) \\ {\scriptstyle i_{\theta,n}}\downarrow\cong & & \cong\downarrow{\scriptstyle i_{\theta,n+2}} \\ C^\infty(M) & \xrightarrow{\Box^\theta} & C^\infty(M) \end{array}$$

for each θ in the CR conformal class. On the other hand, in general it is not correct that an operator $\tilde{G}^\theta : \Gamma(L^{n+2}) \to \Gamma(L^n)$ which makes the diagram

$$\begin{array}{ccc} \Gamma(L^{n+2}) & \xrightarrow{\tilde{G}^\theta} & \Gamma(L^n) \\ {\scriptstyle i_{\theta,n+2}}\downarrow\cong & & \cong\downarrow{\scriptstyle i_{\theta,n}} \\ C^\infty(M) & \xrightarrow{G^\theta} & C^\infty(M) \end{array}$$

commutative depends only on the class. But we have:

Proposition 4.1 *For each $e^{2f}\theta$, let us consider a function $G_{e^{2f}\theta}$ which makes the diagram*

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{G_{e^{2f}\theta}} & C^\infty(M) \\ {\scriptstyle i_{\theta,n+2}\circ i^{-1}_{e^{2f}\theta,n+2}}\Big\downarrow\cong & & \cong\Big\downarrow{\scriptstyle i_{\theta,n}\circ i^{-1}_{e^{2f}\theta,n}} \\ C^\infty(M) & \xrightarrow{G^\theta} & C^\infty(M) \end{array}$$

commutative, that is, set $G_{e^{2f}\theta}(P,P') = e^{-n(f(P)+f(P'))}G^\theta(P,P')$. Then

$$G^{e^{2f}\theta}(P,P') = G_{e^{2f}\theta}(P,P') + O_\infty, \tag{4.1}$$

which means that, modulo O_∞ the function $\tilde{G}^\theta$ depends only on the class $[\theta]$. In particular, if $\ker\Box^\theta = \{0\}$, then $G^{e^{2f}\theta}(P,P') = G_{e^{2f}\theta}(P,P')$, that is, the function $\tilde{G}^\theta$ depends only on the class in the original meaning.

Proof. We have

$$\begin{aligned} &\ker\Box^\theta \cong \ker\Box^{e^{2f}\theta},\ \phi\mapsto e^{-nf}\phi = (i_{e^{2f}\theta,n}\circ i^{-1}_{\theta,n})\phi, \\ &(\ker\Box^\theta)^{\perp(\theta)} \cong (\ker\Box^{e^{2f}\theta})^{\perp(e^{2f}\theta)},\ \phi\mapsto e^{-(n+2)f}\phi = (i_{e^{2f}\theta,n+2}\circ i^{-1}_{\theta,n+2})\phi \end{aligned} \tag{4.2}$$

and $\Box^{e^{2f}\theta}\circ G_{e^{2f}\theta} = I - e^{-(n+2)f}\circ\pi_\theta\circ e^{(n+2)f}$, which imply $\Box^{e^{2f}\theta}\circ G_{e^{2f}\theta} = \Box^{e^{2f}\theta}\circ G^{e^{2f}\theta} = I$ on $(\ker\Box^{e^{2f}\theta})^{\perp(e^{2f}\theta)}$, i.e., $(G_{e^{2f}\theta} - G^{e^{2f}\theta})(\ker\Box^{e^{2f}\theta})^{\perp(e^{2f}\theta)} \subset \ker\Box^{e^{2f}\theta}$. Putting $\Box^{e^{2f}\theta} = \sum\tilde{\lambda}_i\tilde{\phi}_i\otimes\tilde{\phi}_i$ in a manner similar to (3.1), hence we have

$$(G_{e^{2f}\theta} - G^{e^{2f}\theta})\Big|_{(\ker\Box^{e^{2f}\theta})^{\perp(e^{2f}\theta)}} = \sum_{\tilde{\lambda}_i=0}\tilde{\phi}_i(P)\otimes\int d\tilde{V}(Q)\,G_{e^{2f}\theta}(P',Q)\,\tilde{\phi}_i(Q),$$

where $d\tilde{V} := dV_{e^{2f}\theta}$. On the other hand, obviously we have

$$(G_{e^{2f}\theta} - G^{e^{2f}\theta})\Big|_{\ker\Box^{e^{2f}\theta}} = \sum_{\tilde{\lambda}_i=0}\int d\tilde{V}(Q)\,G_{e^{2f}\theta}(P,Q)\,\tilde{\phi}_i(Q)\otimes\tilde{\phi}_i(P').$$

Thus, we obtain

$$\begin{aligned} &G_{e^{2f}\theta}(P,P') - G^{e^{2f}\theta}(P,P') \\ &= \sum_{\tilde{\lambda}_i=0}\left\{\tilde{\phi}_i(P)\int d\tilde{V}(Q)\,G_{e^{2f}\theta}(P',Q)\,\tilde{\phi}_i(Q) + \tilde{\phi}_i(P')\int d\tilde{V}(Q)\,G_{e^{2f}\theta}(P,Q)\,\tilde{\phi}_i(Q)\right\} \\ &\quad - \sum_{\tilde{\lambda}_i=\tilde{\lambda}_j=0}\tilde{\phi}_i(P)\,\tilde{\phi}_j(P')\iint d\tilde{V}(Q)\,d\tilde{V}(Q')\,G_{e^{2f}\theta}(Q,Q')\,\tilde{\phi}_i(Q)\,\tilde{\phi}_j(Q'), \end{aligned}$$

which is certainly smooth. ∎

Theorem 4.2 *We have $a_n^{e^{2f}\theta}(P^0) = e^{-2nf(P^0)}a_n^\theta(P^0)$, that is, the coefficient $a_n^\theta(P^0)$ is a CR conformal (scalar) invariant of weight $2n$.*

Proof. Changing θ to $\tilde{\theta} = e^{2f}\theta$ transforms $\xi_\bullet$ to $\tilde{\xi}_\bullet = \xi_\bullet \cdot U_\bullet$ with $\tilde{\xi}_0 = e^{-2f}\{\xi_0 - 2i\xi_{\bar\beta}(f)\xi_\beta + 2i\xi_\beta(f)\xi_{\bar\beta}\}$ and $\tilde{\xi}_\alpha = e^{-f}\xi_\alpha$. Let $\tilde{z}$ be $\nabla^{\tilde\theta}$-normal coordinates defined by $\exp^{\nabla^{\tilde\theta}}(\tilde{\xi}_\bullet(P^0)\cdot \tilde{z}_\bullet(P)) = P$. We take a curve $c(\varepsilon) = \iota_\varepsilon(z^0)$ $(0 \le \varepsilon \le 1)$ in the coordinates z, and set $\tilde{c}(\varepsilon) = (\tilde{z}\circ z^{-1})(c(\varepsilon))$, $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon) = |\tilde{c}(\varepsilon)|_H^2$ and $\tilde{c}(\varepsilon) = \iota_{\tilde\varepsilon}(\tilde{c}(\varepsilon)^0)$. By (4.1) and (3.3), the function $G^{\tilde\theta}$ can be described in the coordinates z, $\tilde{z}$ as

$$
\begin{aligned}
(4.3)\qquad G^{\tilde\theta}(c(\varepsilon),0) &= e^{-n(f(c(\varepsilon))+f(P^0))}G^\theta(c(\varepsilon),0) + O_\infty \\
&= \sum_{0\le m<2n} \varepsilon^{-n+m/2}\int_0^\infty dt\, \mathfrak{p}^{(\tilde\theta)}_{m/2}(t,z^0,0) - e^{-2nf(P^0)}a_n^\theta(P^0)\log\varepsilon + O(1),
\end{aligned}
$$

$$
(4.4)\qquad G^{\tilde\theta}(\tilde{c}(\varepsilon),0) = \sum_{0\le m<2n} \tilde{\varepsilon}^{-n+m/2}\int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,\tilde{c}(\varepsilon)^0,0) - a_n^{\tilde\theta}(P^0)\log\tilde{\varepsilon} + \tilde{O}(1),
$$

where $O_\infty = O_\infty(\varepsilon^{1/2})$ is a function smooth up to $\varepsilon^{1/2} = 0$. (We could describe the functions $\mathfrak{p}^{(\tilde\theta)}_{m/2}(t,z^0,0)$ concretely.) If we change (4.4) to an expansion relative to the variable ε, we obtain the expansion (4.3). We examine the coordinate change. Let us set $\tilde{c}_A(\varepsilon) = (\tilde{z}\circ z^{-1})_A(\iota_\varepsilon(z^0)) = \sum \tilde{C}_{AB}(\varepsilon^{1/2})\cdot\iota_\varepsilon(z^0)_B$. Then $\tilde{C}_\bullet(0) = \frac{\partial(\tilde{z}\circ z^{-1})_\bullet}{\partial z_\bullet}(0) = U_\bullet(0)^{-1}$ because of $\tilde{\xi}_\bullet = \xi_\bullet\cdot U_\bullet$ and (2.1.1) for $\xi_\bullet$ and $\tilde{\xi}_\bullet$. Hence, $\tilde{c}_\bullet(\varepsilon) = \iota_{e^{2f(P^0)}\varepsilon}(z^0 + \varepsilon^{1/2}O_\infty)_\bullet$, $\tilde{\varepsilon} = \varepsilon\{e^{2f(P^0)} + \varepsilon^{1/2}O_\infty\}$ and

$$
\begin{aligned}
&\tilde{\varepsilon}^{-n+m/2}\int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,\tilde{c}(\varepsilon)^0,0) = \int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,\tilde{c}(\varepsilon),0) \\
&= \varepsilon^{-n+m/2}\cdot e^{(-2n+m)f(P^0)}\int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,z^0+\varepsilon^{1/2}O_\infty,0) \\
&= \varepsilon^{-n+m/2}\cdot e^{(-2n+m)f(P^0)}\Big\{\int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,z^0,0) + \varepsilon^{1/2}O_\infty\Big\}.
\end{aligned}
$$

Note that, to obtain the last expression, we use (3.1.7). Consequently we have

$$
\begin{aligned}
G^{\tilde\theta}(\tilde{c}(\varepsilon),0) = &\sum_{0\le m<2n} \varepsilon^{-n+m/2}\cdot e^{(-2n+m)f(P^0)}\Big\{\int_0^\infty dt\, \mathfrak{p}^{\tilde\theta}_{m/2}(t,z^0,0) + \varepsilon^{1/2}O_\infty\Big\} \\
&- a_n^{\tilde\theta}(P^0)\Big\{\log\varepsilon + \log\big(e^{2f(P^0)} + \varepsilon^{1/2}O_\infty\big)\Big\} + O(1),
\end{aligned}
$$

which, compared with (4.3), implies $a_n^{\tilde\theta}(P^0) = e^{-2nf(P^0)}a_n^\theta(P^0)$. ∎

5 The global coefficient $\int_M dV_\theta(P)a^\theta_{n+1}(P)$ and the zeta function

In the Riemannian case Parker-Rosenberg [8, §3 and §5] investigated the zeta function to get global conformal invariants. We will present here the corresponding study in the contact Riemannian case.

First, we will show that $\int_M dV_\theta(P)a^\theta_{n+1}(P)$ is a CR conformal invariant, by computing the variation of the functional $f \mapsto \int dV_{e^{2f}\theta}(P)a_k^{e^{2f}\theta}(P)$. It will be troublesome that the volume element varies together, so that we consider the operator

$$\Box^{(\theta,f)} := e^{(n+1)f} \circ \Box^{e^{2f}\theta} \circ e^{-(n+1)f} = e^{-f} \circ \Box^\theta \circ e^{-f} \quad \text{acting on } L^2(M, dV_\theta).$$

If we express $\Box^{e^{2f}\theta}$ acting on $L^2(M, dV_{e^{2f}\theta})$ as $\Box^{e^{2f}\theta} = \sum \tilde\lambda_i\, \tilde\phi_i \otimes \tilde\phi_i^{*(e^{2f}\theta)}$, then we have $\Box^{(\theta,f)} = \sum \tilde\lambda_i\, e^{(n+1)f}\tilde\phi_i \otimes (e^{(n+1)f}\tilde\phi_i)^{*(\theta)}$. Further, the heat kernel $e^{-t\Box^{(\theta,f)}}(P,P') = e^{(n+1)(f(P)+f(P'))}e^{-t\Box^{e^{2f}\theta}}(P,P')$ has an asymptotic expansion

$$e^{-t\Box^{(\theta,f)}}(P,P) \sim \sum_{k=0}^\infty t^{-(n+1)+k} a_k^{(\theta,f)}(P), \quad a_k^{(\theta,f)}(P) = e^{(2n+2)f(P)}a_k^{e^{2f}\theta}(P)$$

and, hence, $\int dV_\theta(P)a_k^{(\theta,f)}(P) = \int dV_{e^{2f}\theta}(P)a_k^{e^{2f}\theta}(P)$. Thus, it is enough to compute the variation of $f \mapsto \int dV_\theta(P)a_k^{(\theta,f)}(P)$. The study in §2 (with some extra argument) implies that the functions $e^{-t\Box^{(\theta,\delta f)}}(P,P')$ and $a_k^{(\theta,\delta f)}(P)$ are also smooth with respect to δ near $\delta = 0$ and there is an asymptotic expansion

$$\frac{d}{d\delta}\Big|_{\delta=0} e^{-t\Box^{(\theta,\delta f)}}(P,P) \sim \sum_{m=0}^\infty t^{-(n+1)+m/2} \frac{d}{d\delta}\Big|_{\delta=0} a_k^{(\theta,\delta f)}(P).$$

Hence, it follows that $\int_0^1 dt\, t^{s-1}\int dV_\theta(P)\frac{d}{d\delta}\Big|_{\delta=0} e^{-t\Box^{(\theta,\delta f)}}(P,P)$ is analytic for $\operatorname{Re} s \gg 0$ and has a meromorphic continuation to $\mathbb{C}$ with only simple poles at $s = n+1-k$ $(k = 0, 1, \dots)$ and

$$\text{(5.1)} \quad \operatorname*{Res}_{s=n+1-k} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P) \frac{d}{d\delta}\Big|_{\delta=0} e^{-t\Box^{(\theta,\delta f)}}(P,P) = \frac{d}{d\delta}\Big|_{\delta=0} \int_M dV_\theta(P)\, a_k^{(\theta,\delta f)}(P).$$

By calculating the left hand side, we know:

Theorem 5.1 *We have*

$$\frac{d}{d\delta}\Big|_{\delta=0} \int_M dV_\theta(P)\, a_k^{(\theta,\delta f)}(P) = 2(n+1-k) \int_M dV_\theta(P)\, f(P) a_k^\theta(P).$$

In particular, $\int_M dV_\theta(P)a^\theta_{n+1}(P)$ *is a CR conformal invariant.*

Proof. It follows from an argument similar to [8, Lemma 1.1] that there is a variational formula

$$\frac{d}{d\delta}\Big|_{\delta=0} e^{-t\Box^{(\theta,\delta f)}}(P,P') = -\int_0^t ds \int_M dV_\theta(Q)\, f(Q)\Big\{e^{-(t-s)\Box^\theta}(P,Q)\, \frac{\partial}{\partial s} e^{-s\Box^\theta}(Q,P') + e^{-(t-s)\Box^\theta}(P',Q)\, \frac{\partial}{\partial s} e^{-s\Box^\theta}(Q,P)\Big\},$$

which implies that

$$\begin{aligned}
&\int_0^1 dt\, t^{s-1} \int dV_\theta(P)\, \frac{d}{d\delta}\Big|_{\delta=0} e^{-t\Box^{(\theta,\delta f)}}(P,P) \\
&= -2 \int_0^1 dt\, t^{s-1} \int dV_\theta(P) \int_0^t ds \int dV_\theta(Q)\, f(Q)\, e^{-(t-s)\Box^\theta}(P,Q)\, \frac{\partial}{\partial s} e^{-s\Box^\theta}(Q,P) \\
&= -2 \int_0^1 dt\, t^{s} \int dV_\theta(P)\, f(P)\, \frac{\partial}{\partial t} e^{-t\Box^\theta}(P,P).
\end{aligned}$$

Since (by Theorem 2.2.3 with an argument added) there is an asymptotic expansion

$$\frac{\partial}{\partial t} e^{-t\Box^\theta}(P,P) \sim \sum_{k=0}^{\infty} (k-(n+1)) t^{-(n+1)+k-1} a_k^\theta(P),$$

the theorem is shown. ∎

Next, let us consider the zeta function associated with $\Box^\theta$

$$\zeta(\theta:s) = \sum_{\lambda_i \neq 0} |\lambda_i|^{-s} = \sum_{\lambda_i<0} |\lambda_i|^{-s} + \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P).$$

Theorem 5.2 *The function $\zeta(\theta:s)$ is analytic for $\mathrm{Re}\, s \gg 0$ and has a meromorphic continuation to $\mathbb{C}$ with only simple poles at $s = n+1, n, \cdots, 1$ and*

$$\begin{aligned}
\operatorname*{Res}_{s=n+1-k} \zeta(\theta:s) &= \frac{1}{\Gamma(n+1-k)} \int_M dV_\theta(P)\, a_k^\theta(P) \quad (n+1-k = n+1, n, \ldots, 1), \\
\operatorname*{Res}_{s=n+1} \zeta(\theta:s) &= \frac{\mathrm{Vol}(M, dV_\theta)}{n!\,(2\pi)^{n+1}} \int_{-\infty}^{\infty} ds \left(\frac{s}{\sinh s}\right)^n, \\
\zeta(\theta:-k) &= \begin{cases} \displaystyle\int_M dV_\theta(P)\, a_{n+1}^\theta(P) - \#\{\lambda_i = 0\} \quad (k=0), \\ \displaystyle(-1)^k k! \int_M dV_\theta(P)\, a_{n+1+k}^\theta(P) + \Big(1-(-1)^k\Big) \sum_{\lambda_i<0} |\lambda_i|^k \quad (k \geq 1) \end{cases}
\end{aligned}$$

and, thus we have

$$\int_M dV_\theta(P)\, a_k^\theta(P) = \begin{cases} \Gamma(n+1-k) \displaystyle\operatorname*{Res}_{s=n+1-k} \zeta(\theta:s) \quad (k = 0, 1, \cdots, n), \\ \zeta(\theta:0) + \#\{\lambda_i = 0\} \quad (k = n+1), \\ \displaystyle\frac{(-1)^{k-n-1}}{(k-n-1)!} \Big\{ \zeta(\theta:n+1-k) \\ \qquad \displaystyle- \sum_{\lambda_i<0} \Big(1-(-1)^{k-n-1}\Big) |\lambda_i|^{k-n-1} \Big\} \quad (k \geq n+2). \end{cases}$$

Proof. Since, by (3.2), there are constants $a > 0$ and $C > 0$ such that

$$\int_1^\infty dt \left| t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P) \right| \leq \int_1^\infty dt\, t^{\mathrm{Re}\, s-1} C e^{-at},$$

the function $\int_1^\infty dt\, t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P)$ is analytic on $\mathbb{C}$. Besides, $1/\Gamma(s)$ is analytic on $\mathbb{C}$ and $\Gamma(s)$ is a meromorphic function having only simple poles at $s = n+1-k$ ($k = n+1, n+2, \ldots$) with $\mathrm{Res}_{s=n+1-k}\, \Gamma(s) = (-1)^{k-n-1}/(k-n-1)!$. Hence, $\zeta(\theta : s)$ is analytic for $\mathrm{Re}\, s \gg 0$ and has a meromorphic continuation to $\mathbb{C}$ with only simple poles at $s = n+1-k$ ($k = 0, 1, \ldots, n$) and, referring to (5.1),

$$\begin{aligned}
\operatorname*{Res}_{s=n+1-k} \zeta(\theta : s) &= \frac{1}{\Gamma(n+1-k)} \operatorname*{Res}_{s=n+1-k} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P) \\
&= \frac{1}{\Gamma(n+1-k)} \operatorname*{Res}_{s=n+1-k} \int_0^1 dt\, t^{s-1} \Big\{ \int_M dV_\theta(P)\, e^{-t\Box^\theta}(P,P) - \sum_{\lambda_j \le 0} e^{-t\lambda_j} \Big\} \\
&= \frac{1}{\Gamma(n+1-k)} \int_M dV_\theta(P)\, a_k^\theta(P).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\zeta(\theta : 0) &= \#\{\lambda_i < 0\} + \frac{1}{\mathrm{Res}_{s=0}\, \Gamma(s)} \operatorname*{Res}_{s=0} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P) \\
&= -\#\{\lambda_i = 0\} + \operatorname*{Res}_{s=0} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P)\, e^{-t\Box^\theta}(P,P) \\
&= -\#\{\lambda_i = 0\} + \int_M dV_\theta(P)\, a_{n+1}^\theta(P)
\end{aligned}$$

and, for $-N = -1, -2, \ldots$,

$$\begin{aligned}
\zeta(\theta : -N) &= \sum_{\lambda_i < 0} |\lambda_i|^N + \frac{1}{\mathrm{Res}_{s=-N}\, \Gamma(s)} \operatorname*{Res}_{s=-N} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P)\, (e^{-t\Box^\theta})^+(P,P) \\
&= \sum_{\lambda_i < 0} (|\lambda_i|^N - \lambda_i^N) + \frac{N!}{(-1)^N} \operatorname*{Res}_{s=-N} \int_0^1 dt\, t^{s-1} \int_M dV_\theta(P)\, e^{-t\Box^\theta}(P,P) \\
&= \sum_{\lambda_i < 0} \Big(1 - (-1)^N\Big) |\lambda_i|^N + \frac{N!}{(-1)^N} \int_M a_{n+1+N}(\theta : P)
\end{aligned}$$

∎

Corollary 5.3 *$\#\{\lambda_i < 0\}$, $\dim \ker \Box^\theta$ and $\zeta(\theta : 0)$ are CR conformal invariants.*

Proof. By (4.2), $\dim \ker \Box^\theta$ is such an invariant, and so is $\zeta(\theta : 0)$ because of Theorems 5.1 and 5.2. In addition, since $\Box^{(\theta, \delta f)} = \sum \tilde{\lambda}_i^\delta \tilde{\phi}_i^\delta \otimes \tilde{\phi}_i^\delta$ ($\delta \ge 0$) form a holomorphic family (in perturbation theory), $\#\{\tilde{\lambda}_i^\delta < 0\}$ is continuous with respect to δ so that $\#\{\lambda_i < 0\}$ is also such an invariant. ∎

Parker-Rosenberg [8, §5] considered also the differential of zeta function at $s = 0$ and the functional determinant. Unlike in the Riemannian case, we may not withdraw any invariant from the investigation of them in our case. Note that M is odd dimensional and we have:

Proposition 5.4 (cf. [8, Proposition 5.1 and Theorem 5.3]) *We have*

$$\frac{d}{d\delta}\Big|_{\delta=0}\dot\zeta(e^{2\delta f}\theta:0) = \int_M dV_\theta(P)\,2f(P)\Big\{a^\theta_{n+1}(P) - \sum_{\lambda_i=0}|\phi_i(P)|^2\Big\},$$
$$\frac{d}{d\delta}\Big|_{\delta=0}\det\Box^{e^{2\delta f}\theta} = -\det\Box^\theta\int_M dV_\theta(P)\,2f(P)\,a^\theta_{n+1}(P),$$

where ϕ_i are given at (3.1).

References

[1] D. E. Blair and S. Dragomir, Pseudohermitian geometry on contact Riemannian manifolds, Rend. Mat. Appl. (7) **22**(2002), 275-341.

[2] T. P. Branson and B. Ørsted, Conformal indices of Riemannian manifolds, Compositio Math. **60**(1986), 261-293.

[3] R. Imai and M. Nagase, The second term in the asymptotics of Kohn-Rossi heat kernel on contact Riemannian manifolds, preprint.

[4] D. Jerison and J. M. Lee, The Yamabe problem on CR manifolds, J. Diff. Geom. **25**(1987), 167-197.

[5] J. M. Lee, The Fefferman metric and pseudohermitian invariants, Trans. Amer. Math. Soc. **296**(1)(1986), 411-429.

[6] M. Nagase, Twistor spaces and the general adiabatic expansions, J. Funct. Anal. **251**(2007), 680-737.

[7] M. Nagase, The heat equation for the Kohn-Rossi Laplacian on contact Riemannian manifolds, preprint.

[8] T. Parker and S. Rosenberg, Invariants of conformal Laplacians, J. Diff. Geom. **25**(1987), 199-222.

[9] N. K. Stanton, The heat equation for the $\bar\partial$-Neumann problem in a strictly pseudoconvex Siegel domain, J. d'Analyse Math. **38**(1980), 67-112.

[10] N. K. Stanton, Spectral invariants of CR manifolds, Michigan Math. J. **36**(1989), 267-288.

[11] N. K. Stanton and D. S. Tartakoff, The heat equation for the $\bar\partial_b$-Laplacian, Comm. Partial Differential Equations **9**(7)(1984), 597-686.

[12] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. **314**(1)(1989), 349-379.