On the curvature of the Fefferman metric of contact Riemannian manifolds

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Abstract

It is known that a contact Riemannian manifold carries a generalized Fefferman metric on a circle bundle over the manifold. We compute the curvature of the metric explicitly in terms of a modified Tanno connection on the underlying manifold. In particular, the scalar curvature descends to the pseudohermitian scalar curvature multiplied by a certain constant. This is an answer to a problem considered by Blair-Dragomir.

1 Introduction

Let (M, θ) be a $(2n+1)$-dimensional contact manifold with a contact form θ. There is a unique vector field ξ such that $\xi \rfloor \theta = 1$ and $\xi \rfloor d\theta = 0$. Let us equip M with a Riemannian metric g and a $(1,1)$-tensor field J which satisfy $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X, Y)$ and $J^2 X = -X + \theta(X)\xi$ for any vector fields X, Y. We set $H = \ker\theta$, $H_\pm = \{X \in H \otimes \mathbb{C} \mid JX = \pm iX\}$. In this paper we adopt such a notation as $(\omega_1 \wedge \cdots \wedge \omega_k)(X_1, \ldots, X_k) = \det(\omega_i(X_j))$ for 1-forms ω_i and vectors X_j, and, hence, $d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$. To study the contact Riemannian manifold (M, θ, g, J), Tanno ([10]) introduced a generalized Tanaka-Webster connection ${}^*\nabla$, called the Tanno connection in this paper, given by

$$ {}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X \xi + (\nabla^g_X \theta)(Y)\xi $$

(∇^g is the Levi-Civita connection of g), whose action does not commute with that of the almost complex structure J in general, however. In fact, he showed

$$ ({}^*\nabla_X J)Y = \mathcal{Q}(Y, X) := (\nabla^g_X J)Y + (\nabla^g_X \theta)(JY)\,\xi + \theta(Y)J\nabla^g_X \xi. $$

The author ([7]) considered a modified Tanno connection ∇, called the hermitian Tanno connection, defined by

$$ \nabla_X Y = {}^*\nabla_X Y - \frac{1}{2}J\mathcal{Q}(Y, X) = \begin{cases} {}^*\nabla_X(f\xi) & : Y = f\xi \quad (f \in C^\infty(M)), \\ \frac{1}{2}\Big({}^*\nabla_X Y - J\,{}^*\nabla_X JY\Big) & : Y \in \Gamma(H), \end{cases} $$

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so that $\nabla J = 0$. This has been profitably employed by the author et al. in investigating the subjects relating to the Kohn-Rossi Laplacian, the CR conformal Laplacian and Bochner type tensors, etc., on contact Riemannian manifolds ([7], [5] (with Imai), [8], [9] (with Sasaki)).

In this paper, our study by means of the connection focuses on a generalized Fefferman metric $G = G_\theta$ (cf. (2.4)) of the contact Riemannian manifold M, i.e., a Lorentz metric on the total space of a canonical $U(1)$-bundle $\pi : F(M) \to M$, introduced by Barletta-Dragomir in [1, §6]: recall that the ordinary one ([4], [6]) is restricted to the case where J is integrable, i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$. After preliminaries in §2 through §4, we will present an explicit description of the curvature $F(\nabla^G)$ of the Levi-Civita connection ∇^G of G in §5. In particular, the following formula for the scalar curvature will be confirmed in the last paragraph.

Theorem 1.1 *We have*

$$s(\nabla^G) = \frac{2(2n+1)}{n+1}\,\pi^* s^\nabla,$$

where s^∇ is the pseudohermitian scalar curvature of ∇.

If J is integrable, in other words, if the Tanno tensor $\mathcal{Q}$ vanishes (cf. [10, Proposition 2.1]), then the connections ${}^*\nabla$, ∇ and the Tanaka-Webster connection coincide (cf. [10, Proposition 3.1], [7, Lemma 1.1], [3, §1.2]), and accordingly the generalized Fefferman metric also coincides with the ordinary one (cf. the comment following (2.4)). The theorem is thus a generalization of Lee's result [6, Theorem 6.2] and is an answer to the problem remaining in Blair-Dragomir's paper [2, Remark 5]. The author is uncertain whether the Chern-Moser normal form theory employed by the easier proof of [6, Theorem 6.2] has improved enough to be applicable to the non-integrable case. In this paper we intend to calculate the curvature directly as Lee did for the proof of [6, Theorem 6.6]. It is rather simplified by considering the concept of hermitian Tanno connection, the formulas (2.7) and a derived connection $\pi_*\nabla^G$ (cf. §3).

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2 The connections ${}^*\nabla$ and ∇, and the (generalized) Fefferman metric of contact Riemannian manifolds

First, let us collect some properties of the connections for quick reference. Refer to [10], [7], [9] for more detailed explanation. We have ${}^*\nabla\theta = \nabla\theta = 0$, ${}^*\nabla g = \nabla g = 0$, $T({}^*\nabla)(Z, W) = 0$, $T({}^*\nabla)(Z, \overline{W}) = ig(Z, \overline{W})\xi$, $T(\nabla)(Z, W) = [J, J](Z, W)/4 := (-[Z, W] + [JZ, JW] - J[JZ, W] - J[Z, JW])/4$, $T(\nabla)(Z, \overline{W}) = ig(Z, \overline{W})\xi$ $(Z, W \in \Gamma(H_+))$, where $T({}^*\nabla)$, etc., are the torsion tensors. If we set ${}^*\tau X = T({}^*\nabla)(\xi, X)$, etc., then ${}^*\tau = \tau$ and $\tau \circ J + J \circ \tau = 0$. In this paper, a local frame $\xi_\bullet = (\xi_0 = \xi, \xi_1, \ldots, \xi_n, \xi_{\bar 1}, \ldots, \xi_{\bar n})$ $(\xi_{\bar\alpha} := \overline{\xi_\alpha} \in H_-)$ of the bundle $TM \otimes \mathbb{C} = \mathbb{C}\xi \oplus H_+ \oplus H_-$ is always assumed to be unitary, i.e., $g(\xi_\alpha, \xi_\beta) = 0$, $g(\xi_\alpha, \xi_{\bar\beta}) = \delta_{\alpha\beta}$ $(1 \le \alpha, \beta \le n)$, and its dual frame is denoted by $\theta^\bullet = (\theta^0 = \theta, \theta^1, \ldots, \theta^n, \theta^{\bar 1}, \ldots, \theta^{\bar n})$. As usual the Greek indices α, β, … vary from 1 to n, the block Latin indices A, B, … vary in $\{0, 1, \ldots, n, \bar 1, \ldots, \bar n\}$, and the summation symbol $\sum$ will be omitted in an unusual manner. We have

$$\tau = \xi_\alpha \otimes \theta^{\bar\gamma} \cdot \tau^\alpha_{\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\gamma \cdot \tau^{\bar\alpha}_\gamma \quad (\tau^{\bar\alpha}_\gamma = \tau^{\bar\gamma}_\alpha),$$
$$\mathcal{Q} = \xi_\alpha \otimes \theta^{\bar\beta} \otimes \theta^{\bar\gamma} \cdot \mathcal{Q}^\alpha_{\bar\beta\bar\gamma} + \xi_{\bar\alpha} \otimes \theta^\beta \otimes \theta^\gamma \cdot \mathcal{Q}^{\bar\alpha}_{\beta\gamma} \quad (\mathcal{Q}^{\bar\alpha}_{\beta\gamma} = -\mathcal{Q}^{\bar\beta}_{\alpha\gamma} = -\mathcal{Q}^{\bar\beta}_{\gamma\alpha} - \mathcal{Q}^{\bar\gamma}_{\alpha\beta}).$$

If we set ${}^*\nabla\xi_B = \xi_A \cdot \omega({}^*\nabla)^A_B$, $\nabla\xi_B = \xi_A \cdot \omega(\nabla)^A_B$, then

$$\omega({}^*\nabla)^\alpha_\beta = \omega(\nabla)^\alpha_\beta, \quad \omega({}^*\nabla)^{\bar\alpha}_{\bar\beta} = \omega(\nabla)^{\bar\alpha}_{\bar\beta}, \quad \omega({}^*\nabla)^{\bar\alpha}_\beta(\xi_\gamma) = -\frac{i}{2}\mathcal{Q}^{\bar\alpha}_{\beta\gamma}, \quad \omega({}^*\nabla)^\alpha_{\bar\beta}(\xi_{\bar\gamma}) = \frac{i}{2}\mathcal{Q}^\alpha_{\bar\beta\bar\gamma}$$

and the others vanish. Let us mention briefly also the pseudohermitian Ricci curvature $\mathrm{Ric}^\nabla(X,Y) := \sum g(F(\nabla)(X,Y)\xi_\nu, \xi_{\bar\nu})$, the pseudohermitian scalar curvature $s^\nabla := \sum \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\alpha})$ and the ordinary ones $\mathrm{Ric}(\nabla)(X,Y) := \mathrm{tr}_{TM}(Z \mapsto F(\nabla)(Z,Y)X) = \sum g(F(\nabla)(\xi_\nu, Y)X, \xi_{\bar\nu}) + \sum g(F(\nabla)(\xi_{\bar\nu}, Y)X, \xi_\nu)$, etc.

Proposition 2.1 (cf. [9, Propositions 1.1 and 1.2]) *We have*

$$\mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) = F(\nabla)^\nu_{\nu\alpha\bar\beta} = F(\nabla^g)^\nu_{\nu\alpha\bar\beta} - \left\{\frac{1}{4}\mathcal{Q}^{\bar\mu}_{\nu\alpha}\mathcal{Q}^\mu_{\bar\nu\bar\beta} + \tau^{\bar\nu}_\alpha\tau^\nu_{\bar\beta}\right\} + \frac{2n+1}{4}\delta_{\alpha\beta},$$
$$\mathrm{Ric}^\nabla(\xi_\alpha, \xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar\nu}}\mathcal{Q})^{\bar\beta}_{\alpha\nu}, \quad \mathrm{Ric}^\nabla(\xi_\alpha, \xi) = (\nabla_{\xi_{\bar\mu}}\tau)^{\bar\alpha}_\mu + \frac{i}{2}\tau^\nu_{\bar\mu}\mathcal{Q}^{\bar\alpha}_{\nu\mu},$$
$$\mathrm{Ric}^{{}^*\nabla}(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) + \frac{1}{4}\mathcal{Q}^{\bar\mu}_{\nu\alpha}\mathcal{Q}^\mu_{\bar\nu\bar\beta},$$
$$\mathrm{Ric}^{{}^*\nabla}(\xi_\alpha, \xi_\beta) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_\beta), \quad \mathrm{Ric}^{{}^*\nabla}(\xi_\alpha, \xi) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi),$$
$$s^{{}^*\nabla} = s^\nabla + \frac{1}{4}\sum \left|\mathcal{Q}^{\bar\mu}_{\nu\alpha}\right|^2$$

and $\mathrm{Ric}^\nabla(\overline{X}, \overline{Y}) = -\overline{\mathrm{Ric}^\nabla(X,Y)}$*, etc. In addition, we have*

$$\mathrm{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}^\nabla(\xi_\alpha, \xi_{\bar\beta}) - \frac{1}{4}\mathcal{Q}^{\bar\alpha}_{\nu\mu}\mathcal{Q}^\beta_{\bar\mu\bar\nu},$$
$$\mathrm{Ric}(\nabla)(\xi_\alpha, \xi_\beta) = \frac{i}{2}(\nabla_{\xi_{\bar\nu}}\mathcal{Q})^{\bar\beta}_{\nu\alpha} + i(n-1)\tau^{\bar\alpha}_\beta, \quad \mathrm{Ric}(\nabla)(\xi_\alpha, \xi) = (\nabla_{\xi_{\bar\mu}}\tau)^{\bar\alpha}_\mu,$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_{\bar\beta}) + \frac{1}{4}\Big(\mathcal{Q}^{\bar\mu}_{\nu\alpha}\mathcal{Q}^\mu_{\bar\nu\bar\beta} - \mathcal{Q}^{\bar\alpha}_{\nu\mu}\mathcal{Q}^\beta_{\bar\mu\bar\nu}\Big),$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_\beta) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi_\beta) + \frac{i}{2}(\nabla_{\xi_{\bar\nu}}\mathcal{Q})^{\bar\alpha}_{\nu\beta},$$
$$\mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi) = \mathrm{Ric}(\nabla)(\xi_\alpha, \xi) - \frac{i}{2}\tau^\mu_{\bar\nu}\mathcal{Q}^{\bar\alpha}_{\mu\nu},$$
$$s({}^*\nabla) = s(\nabla) = 2s^\nabla$$

and $\mathrm{Ric}(\nabla)(\xi, Y) = 0$, $\mathrm{Ric}(\nabla)(\overline{X}, \overline{Y}) = \overline{\mathrm{Ric}(\nabla)(X,Y)}$*, etc.*

Next, let us recall the definition of a generalized Fefferman metric introduced by Barletta-Dragomir [1, §6]. The canonical bundle $\pi_0 : K(M) := \{\omega \in \wedge^{n+1}T^*M \otimes \mathbb{C} \mid X\rfloor\omega = 0\ (X \in H_-)\} \to M$ carries a natural tautologous $(n+1)$-form Υ on $K(M)$, whose value at $\omega \in K(M)$ is the lift to $K(M)$ of ω itself. We set $K^0(M) = \{\omega \in K(M) \mid \omega \neq 0\}$ and consider the canonical $U(1)$-bundle $\pi : F(M) := K^0(M)/\mathbb{R}_+ \to M$. There is a natural embedding

$$\iota_\theta : F(M) \to K(M), \quad \iota_\theta([\omega]) = \frac{1}{\sqrt{\lambda}}\,\omega,$$

where $\lambda \in C^\infty(M,\mathbb{R}_+)$ is uniquely defined by $i^{n^2} n!\, \theta \wedge (\xi \rfloor \omega) \wedge (\xi \rfloor \overline{\omega}) = \lambda\, \theta \wedge (d\theta)^n$. This induces a differential form

$$\mathcal{Z} = \iota_\theta^* \Upsilon \ \in \Gamma(\wedge^{n+1} T^* F(M) \otimes \mathbb{C}).$$

Given a local unitary frame $\theta^\bullet$ of $T^*M \otimes \mathbb{C}$ over an open set U, there is a local trivialization

$$F(M)|U \ \cong\ U \times [0, 2\pi), \quad [\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n](p) \cdot e^{i\varphi} \leftrightarrow (p, \varphi), \tag{2.1}$$

via which $\mathcal{Z}_{[\omega]} = e^{i\varphi([\omega])} \pi^* (\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_{[\omega]}$ (cf. [1, Lemma 4]). The local forms $\rho_{[\omega]} := e^{i\varphi([\omega])} \pi^*(\theta^1 \wedge \cdots \wedge \theta^n)_{[\omega]}$ on $F(M)|U$ determine all together a global n-form ρ on $F(M)$, which satisfies $\mathcal{Z} = \pi^*\theta \wedge \rho$ and $V \rfloor \rho = 0$ for any lift V of ξ to $F(M)$. [1, Lemma 5] indicates that a global n-form ρ satisfying the conditions is actually unique.

Proposition 2.2 (cf. Barletta-Dragomir [1, Proposition 3], Blair-Dragomir [2, §4.2])
(1) *There is a unique real* 1*-form* σ *on* $F(M)$ *such that*

$$d\mathcal{Z} = i\,(n+2)\sigma \wedge \mathcal{Z} + e^{i\varphi}\, \pi^* \mathcal{W}, \tag{2.2}$$

$$\sigma \wedge d\rho \wedge \overline{\rho} = \mathrm{tr}(d\sigma)\, i\sigma \wedge (\pi^*\theta) \wedge \rho \wedge \overline{\rho}, \tag{2.3}$$

where $\mathcal{W}$ *is the* $(n+2)$*-form on* M *given by*

$$\mathcal{W} = \frac{i}{2}\, \theta \wedge \sum (-1)^\alpha \theta^1 \wedge \cdots \wedge \Big(\mathcal{Q}^\alpha_{\bar\beta\bar\gamma} \theta^{\bar\beta} \wedge \theta^{\bar\gamma} \Big) \wedge \cdots \wedge \theta^n \quad (\textit{on } U)$$

and, for a 2-form $\Phi = i\, \Phi_{\alpha\bar\beta}\, \pi^*\theta^\alpha \wedge \pi^*\theta^{\bar\beta} + \cdots$ *on* $F(M)$*, we set* $\mathrm{tr}(\Phi) = \Phi_{\alpha\bar\alpha}$*.*
(2) *On* $F(M)|U$*, the* 1*-form* σ *is expressed as*

$$\sigma = \frac{1}{n+2} \Big\{ d\varphi + \pi^* \Big(i\omega(\nabla)^\alpha_\alpha - \frac{s^\nabla}{2(n+1)} \theta \Big) \Big\}.$$

Proof. Let us verify (2). We set $\Upsilon_0 = \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$. Since $d\theta = i\theta^\alpha \wedge \theta^{\bar\alpha}$ and $d\theta^\alpha = \theta^\beta \wedge \omega({}^*\nabla)^\alpha_\beta + \theta^{\bar\beta} \wedge \omega({}^*\nabla)^\alpha_{\bar\beta} + \theta \wedge \tau^\alpha = \omega(\nabla)^\alpha_\beta(\xi_{\bar\gamma})\, \theta^\beta \wedge \theta^{\bar\gamma} + \frac{i}{2} \mathcal{Q}^\alpha_{\bar\beta\bar\gamma}\, \theta^{\bar\beta} \wedge \theta^{\bar\gamma} + \cdots$,

$$\begin{aligned} d\,\Upsilon_0 &= d\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n + \theta \wedge \sum (-1)^\alpha \theta^1 \wedge \cdots \wedge d\theta^\alpha \wedge \cdots \wedge \theta^n \\ &= \theta \wedge \sum (-1)^\alpha \theta^1 \wedge \cdots \wedge \Big\{ \omega(\nabla)^\alpha_\alpha(\xi_{\bar\gamma})\, \theta^\alpha \wedge \theta^{\bar\gamma} + \frac{i}{2} \mathcal{Q}^\alpha_{\bar\beta\bar\gamma}\, \theta^{\bar\beta} \wedge \theta^{\bar\gamma} \Big\} \wedge \cdots \wedge \theta^n \\ &= -\omega(\nabla)^\alpha_\alpha \wedge \Upsilon_0 + \mathcal{W}. \end{aligned}$$

Hence, for any $f \in C^\infty(M, \mathbb{R})$, the global real 1-form σ on $F(M)$ defined by

$$\sigma = \frac{1}{n+2} \Big\{ d\varphi + \pi^*\, i\omega(\nabla)^\alpha_\alpha \Big\} + \pi^*(f\theta) \quad (\text{on } F(M)|U)$$

satisfies (2.2). In addition, we have

$$\begin{aligned} \sigma \wedge d\rho \wedge \overline{\rho} &= \sigma \wedge \Big(i d\varphi \wedge \rho - \pi^* \omega(\nabla)^\alpha_\alpha(\xi) \pi^*\theta \wedge \rho \Big) \wedge \overline{\rho} \\ &= -i\pi^* f \cdot d\varphi \wedge (\pi^*\theta) \wedge \rho \wedge \overline{\rho} \end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(d\sigma) &= \pi^*\Big\{-i\Big(\frac{i}{n+2}\,d\,\omega(\nabla)^\alpha_\alpha + df\wedge\theta + f\,d\theta\Big)(\xi_\gamma,\xi_{\bar\gamma})\Big\}\\ &= \pi^*\Big(\frac{1}{n+2}d\,\omega(\nabla)^\alpha_\alpha(\xi_\gamma,\xi_{\bar\gamma}) + nf\Big) = \pi^*\Big(\frac{s^\nabla}{n+2}+nf\Big),\\ \operatorname{tr}(d\sigma)\,i\sigma\wedge(\pi^*\theta)\wedge\rho\wedge\bar\rho &= \frac{i}{n+2}\pi^*\Big(\frac{s^\nabla}{n+2}+nf\Big)\cdot d\varphi\wedge(\pi^*\theta)\wedge\rho\wedge\bar\rho.\end{aligned}$$

Consequently, (2.3) also holds for σ with $f = -s^\nabla/2(n+1)(n+2)$. ∎

Now, the **(generalized) Fefferman metric** of the contact Riemannian manifold (M,θ,g,J) is the Lorentz metric G_θ on $F(M)$ (cf. [1, (60)]) given by

$$G_\theta = \frac{1}{2}(\pi^*\theta^\alpha\otimes\pi^*\theta^{\bar\alpha} + \pi^*\theta^{\bar\alpha}\otimes\pi^*\theta^\alpha) + (\pi^*\theta\otimes\sigma + \sigma\otimes\pi^*\theta), \tag{2.4}$$

which certainly coincides with the ordinary one (cf. [4], [6]) in the case J is integrable (i.e., $\mathcal{Q}=0$). One finds its systematic study in [1], [2]. For example, it is invariant of weight -2 under the CR conformal change $\theta \Rightarrow e^{2f}\theta$ (together with canonical changes of unitary frames $\xi_\bullet$ and $\theta^\bullet$), i.e., $G_{e^{2f}\theta} = e^{2f}G_\theta$ ([2, Theorem 11]), which is the contact Riemannian analogue of Lee's result [6, Theorem 3.8]. As stated in the introduction, Theorem 1.1 is that of his another result [6, Theorem 6.2].

Last, let us introduce an assertion, which is obvious but plays an important role in the study of the curvature.

Proposition 2.3 *The $\mathfrak{u}(1)$-valued 1-form $i(n+2)\sigma \in \Gamma(\mathfrak{u}(1)\otimes T^*F(M))$ is an Ehresmann-type connection on the $U(1)$-bundle $F(M)$ over M. That is, it satisfies the invariance conditions $i(n+2)\sigma\Big(\frac{d(R_{e^{it\varphi}}([\omega]))}{dt}\Big|_{t=0}\Big) = i\varphi$ and $R^*_{e^{i\varphi}}\sigma = \sigma\,(=\operatorname{Ad}(e^{-i\varphi})\sigma)$, where $R_{e^{i\varphi}}$ is the right action of $e^{i\varphi}\in U(1)$ on $F(M)$.*

Via the trivialization (2.1), the horizontal lift of $X\in TM$ is written as

$$\pi^*_{\mathcal{H}}X = X - i\Big\{\omega(\nabla)^\alpha_\alpha(X) + \frac{i\,s^\nabla\,\theta(X)}{2(n+1)}\Big\}\partial/\partial\varphi$$

and the dual frame of the local frame $(\pi^*\theta,\pi^*\theta^1,\dots,\pi^*\theta^n,\pi^*\theta^{\bar 1},\dots,\pi^*\theta^{\bar n},\sigma)$ is

$$(N := \pi^*_{\mathcal{H}}\xi, \pi^*_{\mathcal{H}}\xi_1,\dots,\pi^*_{\mathcal{H}}\xi_n,\pi^*_{\mathcal{H}}\xi_{\bar 1},\dots,\pi^*_{\mathcal{H}}\xi_{\bar n},\Sigma := (n+2)\partial/\partial\varphi). \tag{2.5}$$

The curvature 2-form $F(i(n+2)\sigma)\in\Gamma(\mathfrak{u}(1)\otimes\wedge^2T^*F(M))$ is expressed as

$$\begin{aligned}&F(i(n+2)\sigma) = d(i(n+2)\sigma) = i(n+2)\,\pi^*\mathcal{F}(\sigma),\\ &\mathcal{F}(\sigma) := \frac{i}{n+2}\Big(\mathrm{Ric}^\nabla + \frac{i\,d(s^\nabla\theta)}{2(n+1)}\Big) = \overline{\mathcal{F}(\sigma)}\ \in\Gamma(\wedge^2T^*M)\end{aligned} \tag{2.6}$$

and it will be obvious that the horizontal and vertical components of the bracket $[\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y]$ are expressed as

$$[\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y]_{\mathcal{H}} = \pi^*_{\mathcal{H}}[X,Y],\quad [\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y]_{\mathcal{V}} = -\mathcal{F}(\sigma)(X,Y)\,\Sigma. \tag{2.7}$$

3 The Levi-Civita connection ∇^G and a derived connection $\pi_*\nabla^G$

In this section, we will offer an explicit expression of the connection form of ∇^G. Since ∇^G is invariant under $U(1)$-action, it descends to a connection $\pi_*\nabla^G$ on M, which is well defined by $(\pi_*\nabla^G)_X Y = \pi_*(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y)$.

Proposition 3.1 *The torsion tensor of $\pi_*\nabla^G$ vanishes and we have*

$$(3.1)\quad (\pi_*\nabla^G)_X Y = {}^*\nabla_X Y + \frac{1}{2}\, g(X, JY)\xi + \theta(Y)\tau X \\ - \theta(X)\Big(\mathcal{F}^\sigma(Y) + \mathcal{F}(\sigma)(Y,\xi)\,\xi\Big) - \theta(Y)\Big(\mathcal{F}^\sigma(X) + \mathcal{F}(\sigma)(X,\xi)\,\xi\Big),$$

where $\mathcal{F}^\sigma(Y)$ is the vector defined by $g(Z, \mathcal{F}^\sigma(Y)) = \mathcal{F}(\sigma)(Z, Y)$ for any vector Z.

Proof. By definition,

$$g({}^*\nabla_X Y, Z) = g(\nabla^g_X Y, Z) - g(\theta(Y)\,\tau X, Z) + g(g(\tau X, Y)\xi, Z) \\ + \frac{1}{2}\Big\{ - g(\theta(X)JY, Z) - g(\theta(Y)JX, Z) - g(g(X, JY)\xi, Z)\Big\},$$

which, together with (2.7), produces the formula (3.1). Indeed, for a vector Z with $Z_0 := \theta(Z)\xi = 0$,

$$\begin{aligned} &g((\pi_*\nabla^G)_X Y, Z) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y, \pi^*_{\mathcal{H}}Z) \\ &= \pi^*_{\mathcal{H}}XG(\pi^*_{\mathcal{H}}Y, \pi^*_{\mathcal{H}}Z) + \pi^*_{\mathcal{H}}YG(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Z) - \pi^*_{\mathcal{H}}ZG(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y) \\ &\qquad + G([\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y], \pi^*_{\mathcal{H}}Z) + G([\pi^*_{\mathcal{H}}Z, \pi^*_{\mathcal{H}}X], \pi^*_{\mathcal{H}}Y) - G(\pi^*_{\mathcal{H}}X, [\pi^*_{\mathcal{H}}Y, \pi^*_{\mathcal{H}}Z]) \\ &= g(\nabla^g_X Y, Z) + \frac{1}{2}\Big\{Zg(X_0, Y_0) - g([Z, X], Y_0) - g(X_0, [Z, Y])\Big\} \\ &\qquad - \mathcal{F}(\sigma)(Z, X)\theta(Y) - \mathcal{F}(\sigma)(Z, Y)\theta(X) \\ &= g(\nabla^g_X Y, Z) + \frac{1}{2}\Big\{ - g(\theta(X)JY, Z) - g(\theta(Y)JX, Z)\Big\} \\ &\qquad - \theta(X)\mathcal{F}(\sigma)(Z, Y) - \theta(Y)\mathcal{F}(\sigma)(Z, X) \\ &= g({}^*\nabla_X Y, Z) + \theta(Y)g(\tau X, Z) - \theta(X)\mathcal{F}(\sigma)(Z, Y) - \theta(Y)\mathcal{F}(\sigma)(Z, X), \end{aligned}$$

and

$$(3.2)\quad \begin{aligned} &2g((\pi_*\nabla^G)_X Y, \xi) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Y, \Sigma) \\ &= \pi^*_{\mathcal{H}}XG(\pi^*_{\mathcal{H}}Y, \Sigma) + \pi^*_{\mathcal{H}}YG(\pi^*_{\mathcal{H}}X, \Sigma) - \Sigma G(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y) \\ &\qquad + G([\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y], \Sigma) + G([\Sigma, \pi^*_{\mathcal{H}}X], \pi^*_{\mathcal{H}}Y) - G(\pi^*_{\mathcal{H}}X, [\pi^*_{\mathcal{H}}Y, \Sigma]) \\ &= X\theta(Y) + Y\theta(X) + \theta([X, Y]) \\ &= 2g({}^*\nabla_X Y, \xi) - g(T({}^*\nabla)(X - X_0, Y), \xi) = 2g({}^*\nabla_X Y, \xi) - g(JX, Y). \end{aligned}$$

Thus we obtain (3.1). It is easy to show $T(\pi_*\nabla^G) = 0$. ∎

Proposition 3.2

(1) *Set* $(\pi_*\nabla^G)\xi_B = \xi_A \cdot \omega(\pi_*\nabla^G)^A_B$. *Then* $\omega(\pi_*\nabla^G)^{\bar{A}}_{\bar{B}} = \overline{\omega(\pi_*\nabla^G)^A_B}$ *and*

$$\omega(\pi_*\nabla^G)^\alpha_\beta = \omega({}^*\nabla)^\alpha_\beta + \mathcal{F}(\sigma)(\xi_\beta, \xi_{\bar{\alpha}})\theta, \quad \omega(\pi_*\nabla^G)^{\bar{\alpha}}_\beta = \omega({}^*\nabla)^{\bar{\alpha}}_\beta + \mathcal{F}(\sigma)(\xi_\beta, \xi_\alpha)\theta,$$
$$\omega(\pi_*\nabla^G)^0_\beta = \frac{i}{2}\,\theta^{\bar{\beta}},$$
$$\omega(\pi_*\nabla^G)^\alpha_0 = -\mathcal{F}(\sigma)(\xi_{\bar{\alpha}}, \xi_\gamma)\theta^\gamma - \Big(\mathcal{F}(\sigma)(\xi_{\bar{\alpha}}, \xi_{\bar{\gamma}}) - \tau^\alpha_{\bar{\gamma}}\Big)\theta^{\bar{\gamma}} - 2\mathcal{F}(\sigma)(\xi_{\bar{\alpha}}, \xi)\theta,$$
$$\omega(\pi_*\nabla^G)^0_0 = 0.$$

(2) *Denote* (2.5) *by* $(W_0, W_1, \ldots, W_{\bar{1}}, \ldots, W_{2n+1})$ *and set* $\nabla^G W_B = W_A \cdot \omega(\nabla^G)^A_B$. *Then* $\omega(\nabla^G)^{\bar{A}}_{\bar{B}} = \overline{\omega(\nabla^G)^A_B}$ $(\bar{0} := 0,\ \overline{2n+1} := 2n+1)$ *and*

$$\omega(\nabla^G)^\alpha_\beta = \pi^*\omega(\pi_*\nabla^G)^\alpha_\beta + i\,\delta_{\alpha\beta}\,\sigma, \qquad \omega(\nabla^G)^{\bar{\alpha}}_\beta = \pi^*\omega(\pi_*\nabla^G)^{\bar{\alpha}}_\beta,$$
$$\omega(\nabla^G)^{(N)}_\beta = \pi^*\omega(\pi_*\nabla^G)^0_\beta, \qquad \omega(\nabla^G)^\alpha_{(N)} = \pi^*\omega(\pi_*\nabla^G)^\alpha_0,$$
$$\omega(\nabla^G)^{(\Sigma)}_\beta = -\frac{1}{2}\,\overline{\pi^*\omega(\pi_*\nabla^G)^\beta_0}, \qquad \omega(\nabla^G)^\alpha_{(\Sigma)} = -2\,\overline{\pi^*\omega(\pi_*\nabla^G)^0_\alpha},$$
$$\omega(\nabla^G)^{(N)}_{(N)} = \omega(\nabla^G)^{(\Sigma)}_{(N)} = \omega(\nabla^G)^{(N)}_{(\Sigma)} = \omega(\nabla^G)^{(\Sigma)}_{(\Sigma)} = 0,$$

where we put $\omega(\nabla^G)^{(N)}_B = \omega(\nabla^G)^0_B$, $\omega(\nabla^G)^{(\Sigma)}_B = \omega(\nabla^G)^{2n+1}_B$, *etc.*

Remark: The formulas in (2) agree with those of [6, Proposition 6.5] in the case J is integrable.

Proof. (1) follows from Proposition 3.1. As for (2): We have

$$\begin{aligned}
\omega(\nabla^G)^\alpha_\beta(\pi^*_{\mathcal{H}}\xi_C) &= 2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_C}\pi^*_{\mathcal{H}}\xi_\beta, \pi^*_{\mathcal{H}}\xi_{\bar{\alpha}}) \\
&= \pi^* g((\pi_*\nabla^G)_{\xi_C}\xi_\beta, \xi_{\bar{\alpha}}) = \pi^*\omega(\pi_*\nabla^G)^\alpha_\beta(\xi_C), \\
\omega(\nabla^G)^\alpha_\beta(\Sigma) &= 2G(\nabla^G_\Sigma\pi^*_{\mathcal{H}}\xi_\beta, \pi^*_{\mathcal{H}}\xi_{\bar{\alpha}}) = 2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_\beta}\Sigma, \pi^*_{\mathcal{H}}\xi_{\bar{\alpha}}) \\
&= -2G(\nabla^G_{\pi^*_{\mathcal{H}}\xi_\beta}\pi^*_{\mathcal{H}}\xi_{\bar{\alpha}}, \Sigma) = g(J\xi_\beta, \xi_{\bar{\alpha}}) = i\delta_{\alpha\beta}.
\end{aligned}$$

In the last line, (3.2) was applied. These yield the formula for $\omega(\nabla^G)^\alpha_\beta$. The others can be shown similarly. ∎

4 The curvature $F(\pi_*\nabla^G)$

A straightforward computation based on Proposition 3.1 leads to the following formula.

Proposition 4.1 *We have*

$$\begin{aligned}
F(\pi_*\nabla^G)(X,Y)Z = {} & F({}^*\nabla)(X,Y)Z \\
& + g(X,JZ)\Big\{\frac{1}{2}\mathcal{F}^\sigma(Y) + \frac{1}{2}\theta(Y)\mathcal{F}^\sigma(\xi) + \frac{1}{2}\mathcal{F}(\sigma)(Y,\xi)\,\xi - \frac{1}{2}\tau Y\Big\} \\
& - g(Y,JZ)\Big\{\frac{1}{2}\mathcal{F}^\sigma(X) + \frac{1}{2}\theta(X)\mathcal{F}^\sigma(\xi) + \frac{1}{2}\mathcal{F}(\sigma)(X,\xi)\,\xi - \frac{1}{2}\tau X\Big\} \\
& - \theta(T({}^*\nabla)(X,Y))\Big\{\mathcal{F}^\sigma(Z) + \mathcal{F}(\sigma)(Z,\xi)\,\xi\Big\}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\, g(X, \mathcal{Q}(Z,Y))\, \xi + \frac{1}{2}\, g(Y, \mathcal{Q}(Z,X))\, \xi - \frac{1}{2}\, g(Z, JT({}^*\nabla)(X,Y))\, \xi \\
&+ \theta(X)\Big\{ ({}^*\nabla_Y \mathcal{F}^\sigma)(Z) + ({}^*\nabla_Y \mathcal{F}(\sigma))(Z,\xi)\, \xi - \frac{1}{2}\, \mathcal{F}(\sigma)(JY, Z)\, \xi \Big\} \\
&- \theta(Y)\Big\{ ({}^*\nabla_X \mathcal{F}^\sigma)(Z) + ({}^*\nabla_X \mathcal{F}(\sigma))(Z,\xi)\, \xi - \frac{1}{2}\, \mathcal{F}(\sigma)(JX, Z)\, \xi \Big\} \\
&+ \theta(Z)\Big\{ ({}^*\nabla_Y \mathcal{F}^\sigma)(X) + ({}^*\nabla_Y \mathcal{F}(\sigma))(X,\xi)\, \xi - \frac{1}{2}\, \mathcal{F}(\sigma)(JY, X)\, \xi \\
&\qquad - ({}^*\nabla_X \mathcal{F}^\sigma)(Y) - ({}^*\nabla_X \mathcal{F}(\sigma))(Y,\xi)\, \xi + \frac{1}{2}\, \mathcal{F}(\sigma)(JX, Y)\, \xi \\
&\qquad + ({}^*\nabla_X \tau)Y - ({}^*\nabla_Y \tau)X + \tau T({}^*\nabla)(X,Y) \\
&\qquad - \mathcal{F}^\sigma(T({}^*\nabla)(X,Y)) - \mathcal{F}(\sigma)(T({}^*\nabla)(X,Y), \xi)\, \xi \Big\} \\
&+ \theta(X)\theta(Z)\Big\{ \mathcal{F}^\sigma(\mathcal{F}^\sigma(Y)) + \mathcal{F}(\sigma)(\mathcal{F}^\sigma(Y), \xi)\, \xi + \mathcal{F}(\sigma)(Y,\xi)\, \mathcal{F}^\sigma(\xi) \\
&\qquad\qquad - \mathcal{F}^\sigma(\tau Y) - \mathcal{F}(\sigma)(\tau Y, \xi)\, \xi \Big\} \\
&- \theta(Y)\theta(Z)\Big\{ \mathcal{F}^\sigma(\mathcal{F}^\sigma(X)) + \mathcal{F}(\sigma)(\mathcal{F}^\sigma(X), \xi)\, \xi + \mathcal{F}(\sigma)(X,\xi)\, \mathcal{F}^\sigma(\xi) \\
&\qquad\qquad - \mathcal{F}^\sigma(\tau X) - \mathcal{F}(\sigma)(\tau X, \xi)\, \xi \Big\}.
\end{aligned}
$$

Corollary 4.2 *We have*

$$
\begin{aligned}
&\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha, \xi_{\bar\beta}) = \mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_{\bar\beta}) + i\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\beta}), \\
&\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha, \xi_\beta) = \mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi_\beta), \\
&\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha, \xi) = \mathrm{Ric}({}^*\nabla)(\xi_\alpha, \xi) - ({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu}, \xi_\alpha) \\
&\qquad - ({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu, \xi_\alpha) + i\mathcal{F}(\sigma)(\xi_\alpha, \xi), \\
&\mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi_\beta) = \mathrm{Ric}({}^*\nabla)(\xi, \xi_\beta) + i\mathcal{F}(\sigma)(\xi_\beta, \xi) - ({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu}, \xi_\beta) \\
&\qquad - ({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu, \xi_\beta) + ({}^*\nabla_{\xi_\nu}\tau)^\nu_\beta + ({}^*\nabla_{\xi_{\bar\nu}}\tau)^{\bar\nu}_\beta - 2({}^*\nabla_{\xi_\beta}\tau)^\nu_\nu, \\
&\mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi) = \mathrm{Ric}({}^*\nabla)(\xi, \xi) - 2({}^*\nabla_{\xi_\nu}\mathcal{F}(\sigma))(\xi_{\bar\nu}, \xi) - 2({}^*\nabla_{\xi_{\bar\nu}}\mathcal{F}(\sigma))(\xi_\nu, \xi) \\
&\qquad - 2\mathcal{F}(\sigma)(\xi_\nu, \xi_\mu)\mathcal{F}(\sigma)(\xi_{\bar\mu}, \xi_{\bar\nu}) - 2\mathcal{F}(\sigma)(\xi_\nu, \xi_{\bar\mu})\mathcal{F}(\sigma)(\xi_\mu, \xi_{\bar\nu}) \\
&\qquad + 2\mathcal{F}(\sigma)(\xi_{\bar\nu}, \tau_\nu) + 2\mathcal{F}(\sigma)(\xi_\nu, \tau_{\bar\nu}) - 2g(\tau\xi_\nu, \tau\xi_{\bar\nu})
\end{aligned}
$$

and $\mathrm{Ric}(\pi_*\nabla^G)(\overline{X}, \overline{Y}) = \overline{\mathrm{Ric}(\pi_*\nabla^G)(X,Y)}$. (*Note that* $\mathrm{Ric}({}^*\nabla)(\xi, \xi_\beta) = \mathrm{Ric}({}^*\nabla)(\xi, \xi) = 0$.) *The scalar curvature of* $\pi_*\nabla^G$ *is*

$$
s(\pi_*\nabla^G) = \frac{2n+1}{n+1}\, s^\nabla + \mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi). \tag{4.1}
$$

Proof. The formulas for the Ricci curvatures follow from Proposition 4.1 (or Proposition 3.2(1)). As for (4.1): Referring also to Proposition 2.1 and (2.6), we have

$$
\begin{aligned}
s(\pi_*\nabla^G) &= \mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha, \xi_{\bar\alpha}) + \mathrm{Ric}(\pi_*\nabla^G)(\xi_{\bar\alpha}, \xi_\alpha) + \mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi) \\
&= s({}^*\nabla) + 2i\mathcal{F}(\sigma)(\xi_\alpha, \xi_{\bar\alpha}) + \mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi) \\
&= 2s^\nabla - \frac{s^\nabla}{n+1} + \mathrm{Ric}(\pi_*\nabla^G)(\xi, \xi).
\end{aligned}
$$

∎

5 The curvature $F(\nabla^G)$ and the proof of Theorem 1.1

Since $F(\nabla^G)$ is also invariant under $U(1)$-action, it descends to a tensor $\pi_* F(\nabla^G) \in \Gamma(TM \otimes T^*M \otimes T^*M \otimes T^*M)$, which is well defined by $(\pi_* F(\nabla^G))(X,Y)Z = \pi_*(F(\nabla^G)(\pi_{\mathcal{H}}^* X, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z)$.

Theorem 5.1 *We have*

$$
\begin{aligned}
&F(\nabla^G)(\pi_{\mathcal{H}}^* X, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z \\
&\qquad = \pi_{\mathcal{H}}^*\Big((\pi_* F(\nabla^G))(X,Y)Z\Big) + \Big(F(\nabla^G)(\pi_{\mathcal{H}}^* X, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z\Big)_{\mathcal{V}},
\end{aligned}
$$

$$
\begin{aligned}
(5.1)\ (\pi_* F(\nabla^G))(X,Y)Z &= F(\pi_*\nabla^G)(X,Y)Z + \mathcal{F}(\sigma)(X,Y)JZ \\
&+ \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Z,Y) - g(\tau Z, Y)\Big\}JX - \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Z,X) - g(\tau Z, X)\Big\}JY \\
&+ \frac{1}{2}\mathcal{F}(\sigma)(Z,\xi)\Big\{\theta(Y)JX - \theta(X)JY\Big\} \\
&+ \frac{1}{2}\theta(Z)\Big\{\mathcal{F}(\sigma)(Y,\xi)JX - \mathcal{F}(\sigma)(X,\xi)JY\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(5.2)\ \Big(F(\nabla^G)(\pi_{\mathcal{H}}^* X, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z\Big)_{\mathcal{V}} = \frac{1}{2}\Big\{&((\pi_*\nabla^G)_X \mathcal{F}(\sigma))(Z,Y) - ((\pi_*\nabla^G)_Y \mathcal{F}(\sigma))(Z,X) \\
&+ ((\pi_*\nabla^G)_X \mathcal{F}(\sigma))(Z,Y_0) - ((\pi_*\nabla^G)_X \mathcal{F}(\sigma))(Z_0,Y) \\
&- ((\pi_*\nabla^G)_Y \mathcal{F}(\sigma))(Z,X_0) + ((\pi_*\nabla^G)_Y \mathcal{F}(\sigma))(Z_0,X) \\
&+ \mathcal{F}(\sigma)(\xi,X)((\pi_*\nabla^G)_Y\theta)(Z) - \mathcal{F}(\sigma)(\xi,Y)((\pi_*\nabla^G)_X\theta)(Z) \\
&+ \mathcal{F}(\sigma)(Z,\xi)\{((\pi_*\nabla^G)_X\theta)(Y) - ((\pi_*\nabla^G)_Y\theta)(X)\} \\
&- \theta(X)\,\mathcal{F}(\sigma)(Z,(\pi_*\nabla^G)_Y\xi) + \theta(Y)\,\mathcal{F}(\sigma)(Z,(\pi_*\nabla^G)_X\xi) \\
&- \theta(Z)\{\mathcal{F}(\sigma)((\pi_*\nabla^G)_X\xi, Y) - \mathcal{F}(\sigma)((\pi_*\nabla^G)_Y\xi, X)\} \\
&- g(((\pi_*\nabla^G)_X\tau)Z, Y) + g(((\pi_*\nabla^G)_Y\tau)Z, X)\Big\}\Sigma
\end{aligned}
$$

and

$$
\begin{aligned}
F(\nabla^G)(\pi_{\mathcal{H}}^* X, \pi_{\mathcal{H}}^* Y)\Sigma &= \pi_{\mathcal{H}}^*\Big\{((\pi_*\nabla^G)_X J)Y - ((\pi_*\nabla^G)_Y J)X\Big\} \\
&\quad + \frac{1}{2}\Big\{\mathcal{F}(\sigma)(JY,X) - \mathcal{F}(\sigma)(JX,Y) \\
&\qquad + \theta(X)\,\mathcal{F}(\sigma)(JY,\xi) - \theta(Y)\,\mathcal{F}(\sigma)(JX,\xi)\Big\}\Sigma, \\
F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^* Y)\pi_{\mathcal{H}}^* Z &= \pi_{\mathcal{H}}^*\Big\{-((\pi_*\nabla^G)_Y J)Z\Big\} \\
&\quad + \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Y,JZ) + \mathcal{F}(\sigma)(Y_0,JZ) + g(\tau Y, JZ)\Big\}\Sigma, \\
F(\nabla^G)(\Sigma, \pi_{\mathcal{H}}^* Y)\Sigma &= \pi_{\mathcal{H}}^*\Big\{-Y + \theta(Y)\xi\Big\},
\end{aligned}
$$

where we set $Y_0 = \theta(Y)\xi$ as before.

Proof. By Proposition 3.2,

$$
\nabla^G_{\pi_{\mathcal{H}}^* Y}\pi_{\mathcal{H}}^* Z = \pi_{\mathcal{H}}^*\Big((\pi_*\nabla^G)_Y Z\Big) + \sigma\Big(\nabla^G_{\pi_{\mathcal{H}}^* Y}\pi_{\mathcal{H}}^* Z\Big)\Sigma,
$$

$$(5.3)\quad \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big) = G(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z, N) = -G(\pi^*_{\mathcal{H}}Z, \nabla^G_{\pi^*_{\mathcal{H}}Y}N)$$
$$= -\frac{1}{2}\theta^{\alpha}(Z)\,\omega(\nabla^G)^{\bar\alpha}_{(N)}(\pi^*_{\mathcal{H}}Y) - \frac{1}{2}\theta^{\bar\alpha}(Z)\,\omega(\nabla^G)^{\alpha}_{(N)}(\pi^*_{\mathcal{H}}Y)$$
$$= \frac{1}{2}\mathcal{F}(\sigma)(Z,Y) + \frac{1}{2}\Big\{\mathcal{F}(\sigma)(Z,Y_0) - \mathcal{F}(\sigma)(Z_0,Y)\Big\} - \frac{1}{2}g(\tau Z, Y)$$

and

$$\nabla^G_{\pi^*_{\mathcal{H}}X}\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z = \nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y Z\Big) + \nabla^G_{\pi^*_{\mathcal{H}}X}\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)\Sigma$$
$$= \pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_X(\pi_*\nabla^G)_Y Z\Big) + \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)\pi^*_{\mathcal{H}}JX$$
$$+ \Big\{\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y Z\Big)\Big) + (\pi^*_{\mathcal{H}}X)\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)\Big\}\Sigma,$$
$$\nabla^G_{[\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y]}\pi^*_{\mathcal{H}}Z = \nabla^G_{\pi^*_{\mathcal{H}}[X,Y]}\pi^*_{\mathcal{H}}Z - \mathcal{F}(\sigma)(X,Y)\nabla^G_{\Sigma}\pi^*_{\mathcal{H}}Z$$
$$= \pi^*_{\mathcal{H}}\Big\{(\pi_*\nabla^G)_{[X,Y]}Z - \mathcal{F}(\sigma)(X,Y)JZ\Big\} + \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}[X,Y]}\pi^*_{\mathcal{H}}Z\Big)\Sigma.$$

Hence, we have

$$(\pi_* F(\nabla^G))(X,Y)Z = F(\pi_*\nabla^G)(X,Y)Z + \mathcal{F}(\sigma)(X,Y)JZ$$
$$+ \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)JX - \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Z\Big)JY,$$
$$\Big(F(\nabla^G)(\pi^*_{\mathcal{H}}X, \pi^*_{\mathcal{H}}Y)\pi^*_{\mathcal{H}}Z\Big)_{\mathcal{V}}$$
$$= \Big\{\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y Z\Big)\Big) - \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_X Z\Big)\Big)$$
$$+ (\pi^*_{\mathcal{H}}X)\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big) - (\pi^*_{\mathcal{H}}Y)\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}Z\Big) - \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}[X,Y]}\pi^*_{\mathcal{H}}Z\Big)\Big\}\Sigma,$$

which, together with (5.3), imply (5.1) and (5.2). Since

$$\nabla^G_{\pi^*_{\mathcal{H}}X}\nabla^G_{\pi^*_{\mathcal{H}}Y}\Sigma = \pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_X JY\Big) + \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}X}\pi^*_{\mathcal{H}}JY\Big)\Sigma,$$
$$\nabla^G_{[\pi^*_{\mathcal{H}}X,\pi^*_{\mathcal{H}}Y]}\Sigma = \nabla^G_{\pi^*_{\mathcal{H}}[X,Y]}\Sigma - \mathcal{F}(\sigma)(X,Y)\nabla^G_{\Sigma}\Sigma = \pi^*_{\mathcal{H}}(J[X,Y]),$$
$$\nabla^G_{\Sigma}\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z = \nabla^G_{\Sigma}\pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y Z\Big) + \nabla^G_{\Sigma}\sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}Z\Big)\Sigma = \pi^*_{\mathcal{H}}\Big(J(\pi_*\nabla^G)_Y Z\Big),$$
$$\nabla^G_{\pi^*_{\mathcal{H}}Y}\nabla^G_{\Sigma}\pi^*_{\mathcal{H}}Z = \pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y JZ\Big) + \sigma\Big(\nabla^G_{\pi^*_{\mathcal{H}}Y}\pi^*_{\mathcal{H}}JZ\Big)\Sigma$$
$$= \pi^*_{\mathcal{H}}\Big((\pi_*\nabla^G)_Y JZ\Big) + \frac{1}{2}\Big\{\mathcal{F}(\sigma)(JZ,Y) + \mathcal{F}(\sigma)(JZ,Y_0) - g(\tau JZ, Y)\Big\}\Sigma,$$
$$\nabla^G_{[\Sigma,\pi^*_{\mathcal{H}}Y]}\pi^*_{\mathcal{H}}Z = 0,$$
$$\nabla^G_{\Sigma}\nabla^G_{\pi^*_{\mathcal{H}}Y}\Sigma = \nabla^G_{\Sigma}\pi^*_{\mathcal{H}}JY = \pi^*_{\mathcal{H}}J^2Y = -\pi^*_{\mathcal{H}}Y + \pi^*_{\mathcal{H}}\theta(Y)\xi,$$
$$\nabla^G_{\pi^*_{\mathcal{H}}Y}\nabla^G_{\Sigma}\Sigma = \nabla^G_{[\Sigma,\pi^*_{\mathcal{H}}Y]}\Sigma = 0,$$

the others can be shown similarly. ∎

Corollary 5.2 *We have*

$$\mathrm{Ric}(\nabla^G)(\pi^*_{\mathcal{H}}Z, \pi^*_{\mathcal{H}}Y) = \pi^*\Big\{\mathrm{Ric}(\pi_*\nabla^G)(Z,Y) + \frac{1}{2}\Big(g(\tau Z, JY) + g(\tau Y, JZ)\Big)$$

$$+\frac{1}{2}\Big(\mathcal{F}(\sigma)(JZ,Y)+\mathcal{F}(\sigma)(JY,Z)+\mathcal{F}(\sigma)(Z_0,JY)+\mathcal{F}(\sigma)(Y_0,JZ)\Big)\Big\},$$

$$\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*Z,\Sigma)=\pi^*\Big\{-2i\,\theta(Z)\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\alpha})\Big\},$$

$$\mathrm{Ric}(\nabla^G)(\Sigma,\Sigma)=2n.$$

Proof. We have

$$\begin{aligned}
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*Y)&=2G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha,\pi_{\mathcal{H}}^*Y)\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})\\
&\quad+2G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha},\pi_{\mathcal{H}}^*Y)\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*\xi_\alpha)+G(F(\nabla^G)(\pi_{\mathcal{H}}^*\xi,\pi_{\mathcal{H}}^*Y)\pi_{\mathcal{H}}^*Z,\Sigma)\\
&\quad+G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*Y)\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*\xi),\\
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*Z,\Sigma)&=-2G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*\xi_\alpha)\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})\\
&\quad-2G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})\pi_{\mathcal{H}}^*Z,\pi_{\mathcal{H}}^*\xi_\alpha)-G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*\xi,\Sigma)\pi_{\mathcal{H}}^*Z),\\
\mathrm{Ric}(\nabla^G)(\Sigma,\Sigma)&=-2G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*\xi_\alpha)\Sigma,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})-2G(F(\nabla^G)(\Sigma,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})\Sigma,\pi_{\mathcal{H}}^*\xi_\alpha).
\end{aligned}$$

Hence, by Theorem 5.1, we obtain the formulas. ∎

Last, Corollary 5.2 implies Theorem 1.1 as follows.

Proof of Theorem 1.1. We have

$$\begin{aligned}
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})&=\pi^*\Big\{\mathrm{Ric}(\pi_*\nabla^G)(\xi_\alpha,\xi_{\bar\alpha})+i\,\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\alpha})\Big\},\\
\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha},\pi_{\mathcal{H}}^*\xi_\alpha)&=\pi^*\Big\{\mathrm{Ric}(\pi_*\nabla^G)(\xi_{\bar\alpha},\xi_\alpha)+i\,\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\alpha})\Big\},\\
\mathrm{Ric}(\nabla^G)(\Sigma,N)=\mathrm{Ric}(\nabla^G)(N,\Sigma)&=\pi^*\Big\{-2i\,\mathcal{F}(\sigma)(\xi_\alpha,\xi_{\bar\alpha})\Big\}.
\end{aligned}$$

Referring also to (4.1), we know

$$\begin{aligned}
s(\nabla^G)\;&=\;2\,\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_\alpha,\pi_{\mathcal{H}}^*\xi_{\bar\alpha})+2\,\mathrm{Ric}(\nabla^G)(\pi_{\mathcal{H}}^*\xi_{\bar\alpha},\pi_{\mathcal{H}}^*\xi_\alpha)\\
&\quad+\mathrm{Ric}(\nabla^G)(N,\Sigma)+\mathrm{Ric}(\nabla^G)(\Sigma,N)\\
&=\;2\,\pi^*\Big\{s(\pi_*\nabla^G)-\mathrm{Ric}(\pi_*\nabla^G)(\xi,\xi)\Big\}=\frac{2(2n+1)}{n+1}\,\pi^*s^\nabla.
\end{aligned}$$

∎

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